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On the First Homology of the Groups of Foliation Preserving Diffeomorphisms for Foliations with Singularities of Morse Type

By

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Abstract

Let $\mathbf{R}^n$ be an n-dimensional Euclidean space and $\mathcal{F}_\varphi$ be the foliation defined by levels of a Morse function $\varphi : \mathbf{R}^n \to \mathbf{R}$. We determine the first homology of the identity component of the foliation preserving diffeomorphism group of $(\mathbf{R}^n, \mathcal{F}_\varphi)$. Then we can apply it to the calculation of the first homology of the foliation preserving diffeomorphism groups for codimension one compact foliations with singularities of Morse type.

§1. Introduction and Statement of Results

Let $\mathcal{F}$ be a C^∞-foliation (with or without singularities) on a C^∞-manifold M. Let $D^\infty(M, \mathcal{F})$ denote the group of all foliation preserving C^∞-diffeomorphisms of $(M, \mathcal{F})$ which are isotopic to the identity through foliation preserving C^∞-diffeomorphisms with compact support. In [2], we have studied the structure of the first homology of $D^\infty(M, \mathcal{F})$ for compact Hausdorff foliations $\mathcal{F}$, and we have that it describes the holonomy structures of isolated singular leaves of $\mathcal{F}$. Here the first homology group $H_1(G)$ of a group G is defined as the quotient of G by its commutator subgroup.

In this paper we treat the foliation preserving C^∞-diffeomorphism groups for codimension one foliations with singularities.

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Let $\mathbf{R}^n$ be an n-dimensional Euclidean space and $\varphi_r : \mathbf{R}^n \to \mathbf{R}$ the Morse function of index r defined by

$$\varphi_r(x, y) = -(x_1)^2 - \cdots - (x_r)^2 + (y_1)^2 + \cdots + (y_{n-r})^2,$$

where $(x, y) = (x_1, \cdots, x_r, y_1, \cdots, y_{n-r})$ is a coordinate of $\mathbf{R}^n = \mathbf{R}^r \times \mathbf{R}^{n-r}$. Let $\mathcal{F}_{\varphi_r}$ be the foliation defined by levels of φ_r, that is, $\mathcal{F}_{\varphi_r}$ has leaves of the form $L_c = \varphi_r^{-1}(c)$ $(c \in \mathbf{R})$. Note that $\mathcal{F}_{\varphi_r}$ has the only one singular leaf $L_0 = \varphi_r^{-1}(0)$ through the origin. Any foliation preserving C^∞-diffeomorphism $f : (\mathbf{R}^n, \mathcal{F}_{\varphi_r}) \to (\mathbf{R}^n, \mathcal{F}_{\varphi_r})$ induces a homeomorphism h of the leaf space $\mathbf{R}^n/\mathcal{F}_{\varphi_r}$. For the case $r \neq 1, n-1$, $\mathbf{R}^n/\mathcal{F}_{\varphi_r}$ is homeomorphic to the real line $\mathbf{R}$ or the half line $\mathbf{R}_{\geq 0}$ via φ_r, but for the case $r = 1, n-1$, $\mathbf{R}^n/\mathcal{F}_{\varphi_r}$ is homeomorphic to a space obtained from a disjoint union of two or three real lines by identifying their closed half lines suitably. Note that each real line is diffeomorphic to $\mathbf{R}$ via φ_r. Then h restricted to each real line is a C^∞-diffeomorphism of itself. We have the following equation

$$\varphi_r(f(x, y)) = h(\varphi_r(x, y))$$

on the domains of $\mathbf{R}^n$ which are saturated sets of a transverse curve. Remark that h depends on only f and coincides with the identity if and only if f is a leaf preserving C^∞-diffeomorphism. For the case $r = 0, n$, any foliation preserving C^∞-diffeomorphism f induces a $\mathbf{Z}_2$-equivariant C^∞-diffeomorphism h of $\mathbf{R}$. For the case $r \neq 0, n$, note that if L_0 divides $\mathbf{R}^n$ into only two parts, the domain is the whole $\mathbf{R}^n$, and at most three kinds of C^∞-diffeomorphisms of the real line are defined generally.

We call that f has a compact support if h has a compact support and f satisfies a certain condition on the leaf direction (see §2). By $D_c^\infty(\mathbf{R}^n, \mathcal{F}_{\varphi_r})$ we denote the group of all foliation preserving C^∞-diffeomorphisms of $(\mathbf{R}^n, \mathcal{F}_{\varphi_r})$ which are isotopic to the identity through foliation preserving C^∞-diffeomorphisms with compact support.

The purpose of this paper is to determine the first homology of $D_c^\infty(\mathbf{R}^n, \mathcal{F}_{\varphi_r})$ and apply this result to codimension one compact foliations with singularities of Morse type.

Then we have the following.

Theorem 1.1. $H_1(D_c^\infty(\mathbf{R}^n, \mathcal{F}_{\varphi_r})) \cong \begin{cases} \mathbf{R} \times S^1 & \textit{if } n = 2 \textit{ and } r = 0, n \\ \mathbf{R} & \textit{otherwise.} \end{cases}$

Remark 1.2 (Fukui [5]). Let $D_c^\infty(\mathbf{R}^n, 0)$ be the group of all C^∞-diffeomorphisms of $\mathbf{R}^n$ preserving the origin, which are isotopic to the identity through C^∞-diffeomorphisms preserving the origin with compact support. Then we have $H_1(D_c^\infty(\mathbf{R}^n,0)) \cong \mathbf{R}$.

Remark 1.3 (Abe-Fukui [1], Abe-Fukui-Miura [4]). Let $\mathcal{H}_{LIP}(\mathbf{R}^n,0)$ be the identity component of the group of all Lipschitz homeomorphisms of $\mathbf{R}^n$ preserving the origin with compact support under the *compact open Lipschitz topology.* Then we have that $\mathcal{H}_{LIP}(\mathbf{R}^n,0)$ is perfect. On the other hand, let $L_c(\mathbf{R}^n,0)$ be the identity component of the group of all Lipschitz homeomorphisms of $\mathbf{R}^n$ preserving the origin with compact support under the *compact open topology.* Then we have that $H_1(L_c(\mathbf{R}^2,0))$ has continuous moduli.

Let M be an n-dimensional compact manifold without boundary and $\mathcal{F}$ a codimension one foliation on M. A point $p \in M$ is called a singularity of Morse type if there is a coordinate neighborhood $(U,(x,y))$ ($x = (x_1,\cdots,x_r)$, $y=(y_1,\cdots,y_{n-r})$) around p where it is defined a Morse function $\varphi_r : U \to \mathbf{R}$, $\varphi_r(x,y) = -(x_1)^2 - \cdots - (x_r)^2 + (y_1)^2 + \cdots + (y_{n-r})^2$ such that $\mathcal{F}\mid_U$ is given by levels of φ_r. For $r \neq 0,n$, we have a conical leaf given by $\varphi_r(x,y)=0$ in U. Such a singular leaf is called a separatrix leaf L_p through p. For $r=0,n$, a singular leaf is a singleton. Let denote by $\mathcal{S}(\mathcal{F})$ the set of such singular leaves of $\mathcal{F}$.

We suppose the following assumption (A):

(1) Any $L \in \mathcal{S}(\mathcal{F})$ has only one singularity and

(2) For each $L \in \mathcal{S}(\mathcal{F})$, there is a compact saturated neighborhood V_L of L in M such that

(i) $\mathcal{S}(\mathcal{F}) \cap V_L = L$,

(ii) for distinct $L, L' \in \mathcal{S}(\mathcal{F})$, V_L and $V_{L'}$ are disjoint, and

(iii) $\mathcal{F}\mid_{V_L}$ is given by levels of $\tilde{\varphi}$, where $\tilde{\varphi} : V_L \to \mathbf{R}$ is a Morse function with $\tilde{\varphi}\mid_{V_L\cap U} = \varphi_r \mid_{V_L \cap U}$ for some r $(0 \le r \le n)$.

Such examples are given by Morse functions from M to $\mathbf{R}$ or S^1 in which no two critical points lie at the same level.

By $D^\infty(M^n,\mathcal{F})$ we denote the group of all foliation preserving C^∞-diffeomorphisms of $(M^n,\mathcal{F})$, which are isotopic to the identity through foliation preserving C^∞-diffeomorphisms. Then we have

Theorem 1.4. *Let $\mathcal{F}$ be a codimension one C^∞ foliation with singularities of Morse type satisfying* (A) *on a compact manifold M. We suppose that all leaves of $\mathcal{F}$ are compact and have no holonomy. Then we have*

$$H_1(D^\infty(M^n,\mathcal{F})) \cong \begin{cases} \mathbf{R}^k \times (S^1)^\ell & \text{if } n=2 \\ \mathbf{R}^k & \text{if } n \ge 3, \end{cases}$$

where k is the number of singularities of $\mathcal{F}$ and ℓ is the number of singularities of index 0 *and n.*

The paper is organized as follows. In §2, we analyze the behavior of the differentials of foliation preserving diffeomorphisms of $(\mathbf{R}^n, \mathcal{F}_{\varphi_r})$ at the origin. This plays a key role to prove Theorem 1.1. In §3, we prove Theorem 1.1 using the theorems of Sternberg [10], Abe-Fukui [2], Thurston [11], Sergeraert [8], and Theorem 2.6 (due to Tsuboi [12]). In §4 we have an application of Theorem 1.1 to codimension one compact foliations with singularities of Morse type.

§2. Preliminaries

Let $\mathcal{F}_r$ be the foliation of $\mathbf{R}^n$ defined by levels of the Morse function φ_r with singularity of index r and $\mathcal{G}_r$ be the one dimensional foliation of $\mathbf{R}^n$ defined by the gradient flow of φ_r. We write φ in stead of φ_r.

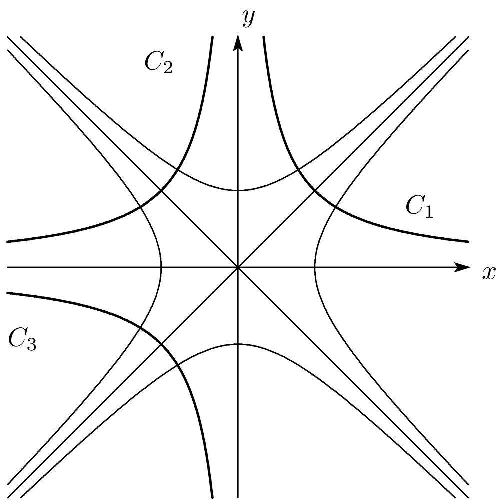
We assume that $r \neq 0, n$. Any foliation preserving C^∞-diffeomorphism $f : (\mathbf{R}^n, \mathcal{F}_r) \to (\mathbf{R}^n, \mathcal{F}_r)$ satisfies the equation

$$\varphi(f(x, y)) = h(\varphi(x, y))$$

on a domain of $\mathbf{R}^n$, where h is a certain C^∞-diffeomorphism of $\mathbf{R}$ preserving the origin, which is uniquely determined by f. Remark that the above relation holds on the whole $\mathbf{R}^n$ when $r \geq 2$ and $n - r \geq 2$. The separatrix $L_0 = \varphi^{-1}(0)$ divides $\mathbf{R}^n$ into two, three or four components. Indeed, the number of the components is four for $n = 2$ and $r = 1$, three for $n \geq 3$ and $r = 1$ or $r = n-1$, and two otherwise. Take at most three regular leaves C_i of $\mathcal{G}_r$ whose union meets all leaves of $\mathcal{F}_r$ like the figure below. Denote by $\#\{C_i\}$ the number of such C_i. Then we have that $\#\{C_i\} = 3$ for $n = 2$ and $r = 1$, $\#\{C_i\} = 2$ for $n \geq 3$ and $r = 1$ or $n = r - 1$, and $\#\{C_i\} = 1$ otherwise. Any foliation preserving C^∞-diffeomorphism f induces at most three C^∞-diffeomorphisms h_i of $\mathbf{R}$ via C_i.

For the case $r = 0, n$, each regular leaf of $\mathcal{G}_r$ meets all leaves of $\mathcal{F}_r$ except the singular leaf. In this case, we see using the holonomy map for $\mathcal{F}_r$ that any foliation preserving C^∞-diffeomorphism f with compact support induces a $\mathbf{Z}_2$-equivariant C^∞-diffeomorphism h of $\mathbf{R}$ with compact support.

Now we define the support compactness of a foliation preserving C^∞-diffeomorphism of $(\mathbf{R}^n, \mathcal{F}_r)$ for $r \neq 0, n$. We take a saturated neighborhood U' of L_0. Then we can take a closed neighborhood U'' of the origin 0 satisfying that $U' \cap \partial U''$ is not empty and is contained in a union of regular leaves of $\mathcal{G}_r$. Remark that $\mathcal{F}_r$ is a product foliation on the intersection $U = U' \cap (U'')^c$ of



U' and the complement $(U'')^c$ of U'' by corresponding each point $p = (x, y)$ of U to a pair of $\varphi(x, y) \in \mathbf{R}$ and the point of L_0 where the regular leaf of $\mathcal{G}_r$ passing through p intersects with L_0. Then any foliation preserving C^∞-diffeomorphism f has the form $(f^1(u), f^2(u, v)) (\in \mathbf{R} \times (L_0 \cap U))$ on U, where (u, v) is a foliated coordinate of U such that $u = \textit{constant}$ gives a leaf of $\mathcal{F}_r$. We fix the coordinate.

Definition 2.1. A foliation preserving C^∞-diffeomorphism f is said to have a compact support if $f = id$ outside of a saturated neighborhood of L_0 and for $r \neq 0, n$, $f^2(u, v) = v$ on the intersection of the saturated neighborhood of L_0 and the complement of a neighborhood of 0.

Let $D_c^\infty(\mathbf{R}^n, \mathcal{F}_r)$ denote the group of all foliation preserving C^∞-diffeomorphisms of $(\mathbf{R}^n, \mathcal{F}_r)$, which are isotopic to the identity through foliation preserving C^∞-diffeomorphisms with compact support. Let $D_c^\infty(\mathbf{R}, 0)$ denote the group of all C^∞-diffeomorphisms of $\mathbf{R}$ preserving the origin, which are isotopic to the identity through C^∞-diffeomorphisms preserving the origin with compact support. Let $D_{\mathbf{Z_2},c}^\infty(\mathbf{R})$ denote the group of all $\mathbf{Z}_2$-equivariant C^∞-diffeomorphisms of $\mathbf{R}$, which are isotopic to the identity through C^∞-$\mathbf{Z}_2$-equivariant diffeomorphisms with compact support.

Set $G = D_c^\infty(\mathbf{R}^n, \mathcal{F}_r)$ and set $G_0 = D_{\mathbf{Z_2},c}^\infty(\mathbf{R})$, and $G_1 = D_c^\infty(\mathbf{R}, 0)$, $G_2 = \{(h_1, h_2) \in D_c^\infty(\mathbf{R}, 0) \times D_c^\infty(\mathbf{R}, 0) \mid h_1 = h_2 \text{ on } \mathbf{R}_{\geq 0}\}$, and $G_3 = \{(h_1, h_2, h_3) \in D_c^\infty(\mathbf{R}, 0) \times D_c^\infty(\mathbf{R}, 0) \times D_c^\infty(\mathbf{R}, 0) \mid h_1 = h_2 \text{ on } \mathbf{R}_{\geq 0},\ h_2 = h_3 \text{ on } \mathbf{R}_{\leq 0}\}$.

Remark that $h'_i(0) = h'_j(0)$ for any i, j, where $h'_i(0)$ is the differential of h_i at the origin 0.

Let $f \in G$. Then, by taking suitable regular leaves C_i of $\mathcal{G}_r$, f induces a $\mathbf{Z}_2$-equivariant C^∞-diffeomorphism $h \in G_0$, and a C^∞-diffeomorphism $h \in G_1$, a pair $(h_1, h_2) \in G_2$ and a triple $(h_1, h_2, h_3) \in G_3$ according to $r = 0, n$, and $\#\{C_i\} = 1, 2$ and 3 respectively. Then we define the map $d_1 : G \to G_i$ by $d_1(f) = h$ when $r = 0, n$, $d_1(f) = h$ when $\#\{C_i\} = 1$, $d_1(f) = (h_1, h_2)$ when $\#\{C_i\} = 2$, and $d_1(f) = (h_1, h_2, h_3)$ when $\#\{C_i\} = 3$ respectively, for any $f \in G$. Then we have the following lemma.

Lemma 2.2. *d_1 is a surjective homomorphism.*

Proof. It is clear that d_1 is a homomorphism. First we prove the surjectivity of d_1 for the case $\#\{C_i\} = 1$. Take $h \in G_1$. Then there is a C^∞-mapping $g : \mathbf{R} \to \mathbf{R}$ satisfying that $h(t) = g(t)t$ for $t \in \mathbf{R}$. Note that $g(t) > 0$ for any $t \in \mathbf{R}$, $g(0) = h'(0)$ and $g(t) = 1$ on the complement of the support of h. Hence $\sqrt{g(t)}$ also becomes a C^∞-mapping of $\mathbf{R}$ to $\mathbf{R}$. Then we define a C^∞-mapping $\tilde{h} : \mathbf{R}^n \to \mathbf{R}^n$ by

$$\tilde{h}(x, y) = \sqrt{g(\varphi(x, y))}(x, y) \quad \text{for } (x, y) \in \mathbf{R}^r \times \mathbf{R}^{n-r} = \mathbf{R}^n.$$

Then we see that $\tilde{h}$ is a C^∞-diffeomorphism and satisfies the equation

$$\varphi \circ \tilde{h}(x, y) = h \circ \varphi(x, y)$$

for $(x, y) \in \mathbf{R}^r \times \mathbf{R}^{n-r} = \mathbf{R}^n$, hence it is foliation preserving. By modifying $\tilde{h}$ along the leaves on the intersection of a saturated neighborhood of L_0 and the complement of a neighborhood of 0 appropriately, we may assume $\tilde{h} \in G$. Hence d_1 is surjective. When $r = 0, n$, it is proved similarly as in the proof of the case $\#\{C_i\} = 1$ by noting $g(-t) = g(t)$ $(t \in \mathbf{R})$ for any $h \in G_0$.

Next we prove the surjectivity of d_1 for the case $\#\{C_i\} = 2$. Take $(h_1, h_2) \in G_2$. As in the proof of the case $\#\{C_i\} = 1$, we can take $\tilde{h}_i$ on the union U_i of leaves intersecting C_i for each h_i $(i = 1, 2)$. Note that since $h_1 = h_2$ on $\mathbf{R}_{\geq 0}$, $\tilde{h}_1 = \tilde{h}_2$ on $U_1 \cap U_2$. Therefore we can define $\tilde{h} \in G$ by $\tilde{h} = \tilde{h}_i$ on U_i. Then we have $d_1(\tilde{h}) = (h_1, h_2) \in G_2$.

It is proved similarly for the case $\#\{C_i\} = 3$. This completes the proof. □

Next we define a map $d_2 : G \to GL_+(n, \mathbf{R})$ as follows. For any $f \in G$, we take $h \in G_i (i = 0, 1)$, $(h_1, h_2) \in G_2$ or $(h_1, h_2, h_3) \in G_3$ as above and consider lifts $\tilde{h}$ or $\tilde{h}_i$ of h or (h_1, h_2), (h_1, h_2, h_3) as in Lemma 2.2, and take the differential of $f \circ \tilde{h}^{-1}$ or $f \circ \tilde{h}_i^{-1}$ at the origin 0. Then we define the map

$d_2 : G \to GL_+(n, \mathbf{R})$ by $d_2(f) = d(f \circ \tilde{h}^{-1})(0)$ or $d(f \circ \tilde{h}_i^{-1})(0)$. Remark that the differential of $\tilde{h}_i$ at the origin 0 is equal to $\sqrt{h_i'(0)} I_n$ and $h_i'(0) = h_j'(0)$, where I_n denotes the unit matrix.

Lemma 2.3. *d_2 is a homomorphism.*

Proof. For $f_1, f_2 \in G$, we take $d_1(f_1), d_1(f_2)$ and their lifts $\tilde{h}_1, \tilde{h}_2$ as above. Since $d\tilde{h}_1(0) = \sqrt{d_1(f_1)'(0)} I_n$ and $d\tilde{h}_2(0) = \sqrt{d_1(f_2)'(0)} I_n$, we have

$$\begin{aligned} d(f_2 \circ f_1) &= d_2(f_2 \circ f_1 \circ (\tilde{h}_2 \circ \tilde{h}_1)^{-1}) \\ &= df_2(0) \cdot df_1(0) \cdot d\tilde{h}_1^{-1}(0) \cdot d\tilde{h}_2^{-1}(0) \\ &= df_2(0) \cdot df_1(0) \cdot \frac{1}{\sqrt{d_1(f_1)'(0)}} I_n \cdot \frac{1}{\sqrt{d_1(f_2)'(0)}} I_n \\ &= df_2(0) \cdot \frac{1}{\sqrt{d_1(f_2)'(0)}} I_n \cdot df_1(0) \cdot \frac{1}{\sqrt{d_1(f_1)'(0)}} I_n \\ &= d_2(f_2 \circ \tilde{h}_2^{-1}) \cdot d_2(f_1 \circ \tilde{h}_1^{-1}) \\ &= d(f_2) \circ d(f_1). \end{aligned}$$

Thus d_2 is a homomorphism. This completes the proof. □

Let $SO(r, n-r) = \{A \in M(n, \mathbf{R})\,;\, {}^t A I_{r,n-r} A = I_{r,n-r}\,,\ \det A = 1\}$ and $SO(r, n-r)_0$ be its connected component containing the unit matrix, where $I_{r,n-r} = \begin{pmatrix} -I_r & 0 \\ 0 & I_{n-r} \end{pmatrix}$.

Lemma 2.4. $\operatorname{Im} d_2 = SO(r, n-r)_0$.

Proof. Take $f \in G$ and a lift $\tilde{h}$ of $d_1(f)$ as in Lemma 2.2. Since $d_1(f) = d_1(\tilde{h})$, $k = f \circ \tilde{h}^{-1}$ is leaf preserving. Thus we have the equation

$$\varphi(k(x, y)) = \varphi(x, y) \tag{1}$$

for $(x, y) \in \mathbf{R}^r \times \mathbf{R}^{n-r}$. Then we have that $d_2(f) \in SO(r, n-r)_0$ by differentiating (1) with respect to x and y and taking the limit as x and y tend to 0.

Conversely, any $A \in SO(r, n-r)_0$ acts on $\mathbf{R}^n$ mapping each leaf of $\mathcal{F}_r$ to itself. Then by deforming A outside of a saturated neighborhood of L_0 to the identity and adjusting along the leaves on the intersection of a neighborhood of

L_0 and the complement of a neighborhood of 0 appropriately, we have $f_A \in G$ satisfying $f_A = A$ on a neighborhood of 0. This completes the proof. $\square$

Let $d : G \to G_i \times SO(r, n-r)_0$ be the map defined by $d(f) = (d_1(f), d_2(f))$ for any $f \in G$. Then we have the following from Lemmas 2.2, 2.3 and 2.4.

Corollary 2.5. *d is a surjective homomorphism.*

Proof. Take $(h, A) \in G_i \times SO(r, n-r)_0$. From Lemma 2.2, we have $\tilde{h} \in G$ with $d_1(\tilde{h}) = h$. For $A \in SO(r, n-r)_0$, we construct $f_A \in G$ as in the proof of Lemma 2.4. Then since $d_2(\tilde{h}) = I_n$ and f_A is leaf preserving, the composition $f_A \circ \tilde{h}$ satisfies $d(f_A \circ \tilde{h}) = (h, A)$. This completes the proof. $\square$

Let $\mathcal{F}_0$ be the product foliation of $\mathbf{R}^m$ with leaves of form $\{\{x\} \times \mathbf{R}^{m-q}\}$, where (x, y) is a coordinate of $\mathbf{R}^m = \mathbf{R}^q \times \mathbf{R}^{m-q}$. By $D^\infty_{L,c}(\mathbf{R}^m, \mathcal{F}_0)$ we denote the group of leaf preserving C^∞-diffeomorphisms of $(\mathbf{R}^m, \mathcal{F}_0)$ which are isotopic to the identity through leaf preserving C^∞-diffeomorphisms with compact support.

T.Tsuboi [12] (and T.Rybicki [7]) proved the following by looking at the proofs in Herman [6] and Thurston [11].

Theorem 2.6 (Theorem 1.1 of [12]). *$D^\infty_{L,c}(\mathbf{R}^m, \mathcal{F}_0)$ is perfect.*

§3. Proof of Theorem 1.1

From Corollary 2.5, for each $i (i = 0, 1, 2, 3)$ there is a short exact sequence

$$1 \longrightarrow \ker d \longrightarrow G \xrightarrow{d} G_i \times SO(r, n-r)_0 \longrightarrow 1.$$

Thus we have the following exact sequence of homology groups:

$$\ker d/[\ker d, G] \longrightarrow H_1(G) \xrightarrow{d_*} H_1(G_i \times SO(r, n-r)_0) \longrightarrow 1.$$

First we shall prove $\ker d/[\ker d, G] = 0$.

Proposition 3.1. $\ker d = [\ker d, G]$.

Proof. Take any $f \in \ker d$. Note that f is leaf preserving. Take an expansion L_c defined by $L_c(x, y) = (cx, cy)$ for $(x, y) \in \mathbf{R}^n$, where $c > 1$. Then L_c is foliation preserving and we may assume that it is contained in G by modifying L_c outside of a neighborhood of 0 suitably. We consider the composition $f \circ L_c$. Then we have $d(f \circ L_c)(0) = cI_n$. From Theorem 1 of Sternberg [10], there exists $R \in D^\infty(\mathbf{R}^n, 0)$ satisfying that $dR(0) = I_n$ and

$$R^{-1} \circ f \circ L_c \circ R = L_c \tag{2}$$

on a neighborhood of 0, say V. We prove that R is leaf preserving on V. From (2), we have $\varphi \circ R(x, y) = \varphi \circ f \circ L_c \circ R \circ L_c^{-1}(x, y)$ for $(x, y) \in V$. Since f is leaf preserving and φ is a quadratic form, we have

$$\varphi(R(x, y)) = c^2 \left(\varphi \left(R \left(\frac{1}{c}x, \frac{1}{c}y \right) \right) \right) \tag{3}$$

for $(x, y) \in V$. From (3) and $dR(0) = I_n$, R has the following form

$$R(x, y) = (x, y)A(x, y),$$

where $A : V \to GL(n, \mathbf{R})$ is a C^∞-mapping which is 1-tangent to the constant mapping e_n $(e_n(x, y) = I_n)$ at 0. Furthermore we have

$$\varphi((x, y)A(x, y)) = \varphi \left((x, y)A \left(\frac{1}{c}x, \frac{1}{c}y \right) \right)$$

from (3), thus

$$\varphi((x, y)A(x, y)) = \varphi \left((x, y)A \left(\frac{1}{c^n}x, \frac{1}{c^n}y \right) \right)$$

for any positive integer n. By letting n tend to ∞, we have

$$\varphi((x, y)A(x, y)) = \varphi(x, y).$$

Hence we have $\varphi \circ R(x, y) = \varphi(x, y)$ for $(x, y) \in V$, so R is leaf preserving on V. Furthermore we may assume $R \in G$ by modifying R outside of V suitably. Therefore we have $f = [R, L_c]$ on V, where $R \in \ker d$ and $L_c \in G$.

Since $\mathcal{F}_r$ is a product foliation on the intersection of the complement of V and a neighborhood of L_0 and f has a compact support, $f \circ [R, L_c]^{-1}$ is written as a product of commutators of leaf preserving diffeomorphisms from Theorem 2.6. This completes the proof. □

Proof of Theorem 1.1 *continued.* From Proposition 3.1, we have

$$H_1(G) \cong H_1(SO(r, n-r)_0 \times G_i) \cong H_1(SO(r, n-r)_0) \times H_1(G_i).$$

We calculate $H_1(G_i)$ $(i = 0, 1, 2, 3)$. We have $H_1(G_i) \cong \mathbf{R}$ for $i = 0, 1$ from Abe-Fukui [2] and Fukui [5]. For $i = 2$, we define the map $d_3 : G_2 \to \mathbf{R}_{>0}$

by taking the differential of h_1 or h_2 at 0 for any $(h_1, h_2) \in G_2$. Note that $h_1'(0) = h_2'(0)$ because that $h_1 = h_2$ on $\mathbf{R}_{\geq 0}$. Then for any $(h_1, h_2) \in \ker d_3$, h_1 can be represented by a commutator of a diffeomorphism ϕ of the form $t \to ct\,(0 < c < 1)$ locally and an element in G_1 which is 1-tangent to the identity at 0, say ψ, on a neighborhood of 0 by the standard argument (cf. Fukui [5]). Then we may assume from the theorem of Thurston [11] that $h_2 \circ [\phi, \psi]^{-1}$ is equal to the identity on $\mathbf{R}_{\geq 0}$, and is C^∞-tangent to the identity at 0. From Sergeraert [8], $h_2 \circ [\phi, \psi]^{-1}$ can be represented by a product of commutators of diffeomorphisms which are equal to the identity on $\mathbf{R}_{\geq 0}$ and are C^∞-tangent to the identity at 0. Thus (h_1, h_2) can be represented by a product of commutators of elements in G_2. Hence we have $H_1(G_2) \cong \mathbf{R}$. It is also proved for $i = 3$ similarly.

Since $SO(r, n-r)_0$ is simple or semi-simple except for $n = 2$ and $r = 0$ or 2, we complete the proof. □

§4. Proof of Theorem 1.4

Let $p_1, \cdots, p_k$ be all singularities of $\mathcal{F}$. For each p_i, there are a compact saturated neighborhood V_i and a Morse function $\varphi_i : V_i \to \mathbf{R}$ such that

(i) $\mathcal{S}(\mathcal{F}) \cap V_i = L_{p_i}$, where L_{p_i} is the leaf passing through p_i,

(ii) $V_i \cap V_j = \emptyset$ if $i \neq j$, and

(iii) $\mathcal{F}\,|_{V_i}$ is given by levels of φ_i.

Let $\mathcal{G}$ be a one dimensional foliation of M transverse to $\mathcal{F}$ satisfying that $\mathcal{G}\,|_{V_i}$ is defined by the gradient flow of φ_i. For each i, take at most three regular leaves $C_{j(i)}$ of $\mathcal{G}$ whose union meets all leaves of $\mathcal{F}\,|_{V_i}$.

Let G_1 denote the group of germs at the origin 0 of elements of the group $D^\infty(\mathbf{R}, 0)$ whose elements are isotopic to the identity through C^∞-diffeomorphisms preserving the origin. We denote by G_0 the group of germs at the origin 0 of elements of the group $D^\infty_{\mathbf{Z_2}}(\mathbf{R})$ whose elements are isotopic to the identity through C^∞-$\mathbf{Z}_2$-equivariant diffeomorphisms. Set $G_2 = \{(h_1, h_2) \in G_1 \times G_1 \mid \hat{h}_1 = \hat{h}_2$ on $\mathbf{R}_{\geq 0}$, where $\hat{h}_1,\ \hat{h}_2$ are representatives of $h_1,\ h_2\}$ and $G_3 = \{(h_1, h_2, h_3) \in G_1 \times G_1 \times G_1 \mid \hat{h}_1 = \hat{h}_2$ on $\mathbf{R}_{\geq 0},\ \hat{h}_2 = \hat{h}_3$ on $\mathbf{R}_{\leq 0}$, where $\hat{h}_1,\ \hat{h}_2,\ \hat{h}_3$ are representatives of $h_1,\ h_2,\ h_3$ respectively$\}$.

Take any $f \in D^\infty(M^n, \mathcal{F})$. By taking the map d at each p_i as in §2 via regular leaves $C_{j(i)}$, we have the surjective homomorphism

$$\Psi : D^\infty(M^n, \mathcal{F}) \to \prod_{i=1}^{k} (G_{\ell(i)} \times SO(r_i, n - r_i)_0),$$

where $\ell(i) = 0, 1, 2, 3$. Then we have a short exact sequence:

$$1 \longrightarrow \ker \Psi \longrightarrow D^\infty(M^n, \mathcal{F}) \xrightarrow{\Psi} \prod_{i=1}^{k} (G_{\ell(i)} \times SO(r_i, n - r_i)_0) \longrightarrow 1.$$

Thus we have the following exact sequence of homology groups:

$$\ker \Psi / [\ker \Psi, D^\infty(M^n, \mathcal{F})] \longrightarrow H_1(D^\infty(M^n, \mathcal{F})) \xrightarrow{\Psi_*} H_1\left(\prod_{i=1}^{k} (G_{\ell(i)} \times SO(r_i, n - r_i)_0)\right) \longrightarrow 1.$$

Since any $g \in \ker \Psi$ is leaf preserving on a saturated neighborhood W of $\bigcup_{i=1}^k L_{p_i}$ and $\mathcal{F}$ restricted to each connected component of $W^c = M - W$ is a bundle foliation with compact fiber, any $g \in \ker \Psi$ can be decomposed as $g = g_1 \circ \cdots \circ g_k \circ g'$, where the support of each g_i is contained in $V_i \cap W$ and the support of g' is contained in a neighborhood of W^c. Thus from Proposition 3.1, Theorem 2.6 and Corollary 5.4 of [2], we have $\ker \Psi / [\ker \Psi, D^\infty(M^n, \mathcal{F})] = 0$. Hence we have

$$\begin{aligned} H_1(D^\infty(M^n, \mathcal{F})) &\cong H_1\left(\prod_{i=1}^{k} (G_{\ell(i)} \times SO(r_i, n - r_i)_0)\right) \\ &\cong \prod_{i=1}^{k} H_1(G_{\ell(i)} \times SO(r_i, n - r_i)_0). \end{aligned}$$

As in the proof of Theorem 1.1 continued, we can show $H_1(G_{\ell(i)}) \cong \mathbf{R}$ ($i = 0, 1, 2, 3$) using the results of Abe-Fukui [2], Fukui [5] and Sergeraert [8]. The proof follows from Theorem 1.1.

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