Publ. RIMS, Kyoto Univ.
44 (2008), 1169–1197

Jacquet Modules of Principal Series Generated by the Trivial K-Type

By

Noriyuki Abe*

Abstract

We propose a new approach to the study of the Jacquet module of a Harish-Chandra module of a real semisimple Lie group. Using this method, we investigate the structure of the Jacquet module of a principal series representation generated by the trivial K-type.

§1. Introduction

Let G be a real semisimple Lie group. By Casselman's subrepresentation theorem, any irreducible admissible representation $\mathbb{X}$ is realized as a subrepresentation of a certain non-unitary principal series representation. Such an embedding is a powerful tool to study an irreducible admissible representation but the subrepresentation theorem dose not tell us how it can be realized.

Casselman [Cas80] introduced the Jacquet module $J(\mathbb{X})$ of $\mathbb{X}$. This important object retains all information of embedding given by the subrepresentation theorem. For example, Casselman's subrepresentation theorem is equivalent to $J(\mathbb{X}) \neq 0$. However the structure of $J(\mathbb{X})$ is very intricate and difficult to determine. We remark that if G has real rank one, then Collingwood [Col85] has computed the detailed structure of Jacquet modules.

In this paper we give generators of the Jacquet module of a principal series representation generated by the trivial K-type. This representation is

Communicated by M. Kashiwara. Received December 17, 2007. Revised April 18, 2008, May 8, 2008.
2000 Mathematics Subject Classification(s): 22E47.
*Graduate School of Mathematical Sciences, the University of Tokyo, 3–8–1 Komaba, Meguro-ku, Tokyo 153–8914, Japan.
e-mail: `abenori@ms.u-tokyo.ac.jp`

deeply related to the harmonic analysis on the Riemannian symmetric space G/K [Hel62, KKM+78]. Let $\mathbb{Z}$ be the ring of integers, $\mathfrak{g}_0$ the Lie algebra of G, θ a Cartan involution of $\mathfrak{g}_0$, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ the Iwasawa decomposition of $\mathfrak{g}_0$, $\mathfrak{m}_0$ the centralizer of $\mathfrak{a}_0$ in $\mathfrak{k}_0$, W the little Weyl group for $(\mathfrak{g}_0, \mathfrak{a}_0)$, $e \in W$ the unit element of W, Σ the restricted root system for $(\mathfrak{g}_0, \mathfrak{a}_0)$, $\mathfrak{g}_{0,\alpha}$ the root space for $\alpha \in \Sigma$, Σ^+ the positive system of Σ such that $\mathfrak{n}_0 = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_{0,\alpha}$, $\rho = \sum_{\alpha\in\Sigma^+}(\dim \mathfrak{g}_{0,\alpha}/2)\alpha$, $\mathcal{P} = \{\sum_{\alpha\in\Sigma^+} n_\alpha\alpha \mid n_\alpha \in \mathbb{Z}\}$, $\mathcal{P}^+ = \{\sum_{\alpha\in\Sigma^+} n_\alpha\alpha \mid n_\alpha \in \mathbb{Z},\ n_\alpha \geq 0\}$. For a Lie algebra $\mathfrak{h}$, let $U(\mathfrak{h})$ be the universal enveloping algebra of $\mathfrak{h}$.

For a Harish-Chandra module $\mathbb{X}$, the Jacquet module $J(\mathbb{X})$ of $\mathbb{X}$ is defined by

$$J(\mathbb{X}) = \left\{ x \in \varprojlim_k \mathbb{X}/\mathfrak{n}_0^k\mathbb{X} \;\middle|\; \dim U(\mathfrak{a}_0)x < \infty \right\}.$$

In this paper we prove the following theorem.

Theorem 1.1 (Theorem 4.10, Proposition 5.1). *Let $\lambda \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}_0, \mathbb{C})$ and $I(\lambda)$ be the unique principal series representation with an infinitesimal character λ generated by the trivial K-type. Assume that λ is regular. Set $\mathcal{W}(w) = \{w' \in W \mid w\lambda - w'\lambda \in 2\mathcal{P}^+\}$ for $w \in W$. Then there exist generators $\{v_w \mid w \in W\}$ of $J(I(\lambda))$ such that*

$$\begin{cases} (H - (\rho + w\lambda)(H))v_w \in \sum_{w'\in\mathcal{W}(w)} U(\mathfrak{g})v_{w'} \text{ for all } H \in \mathfrak{a}_0, \\ Xv_w \in \sum_{w'\in\mathcal{W}(w)} U(\mathfrak{g})v_{w'} \text{ for all } X \in \mathfrak{m}_0 \oplus \theta(\mathfrak{n}_0). \end{cases}$$

Hence v_w is a lowest weight vector of $J(I(\lambda))/\sum_{w'\in\mathcal{W}(w)} U(\mathfrak{g})v_{w'}$.

Recall the definition of generalized Verma modules. For $\mu \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}_0, \mathbb{C})$, let $\mathbb{C}_\mu$ be the one-dimensional representation of $\mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \theta(\mathfrak{n}_0)$ defined by $\mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \theta(\mathfrak{n}_0) \to \mathfrak{a}_0 \to \mathbb{C}$, where the first map is the projection to the direct summand and the second map is μ. Then the generalized Verma module $M(\mu)$ is defined by $M(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{m}_0\oplus\mathfrak{a}_0\oplus\theta(\mathfrak{n}_0))} \mathbb{C}_{\mu+\rho}$.

We enumerate $W = \{w_1, w_2, \ldots, w_r\}$ in such a way that $\mathrm{Re}\, w_1\lambda \geq \mathrm{Re}\, w_2\lambda \geq \cdots \geq \mathrm{Re}\, w_r\lambda$. Set $V_i = \sum_{j\geq i} U(\mathfrak{g})v_{w_j}$. Then by Theorem 1.1 we have a surjective map $M(w_i\lambda) \to V_i/V_{i+1}$ where $M(w_i\lambda)$ is the generalized Verma module. This map is isomorphic. Namely we can prove the following theorem.

Theorem 1.2 (Theorem 5.5). *There exists a filtration $J(I(\lambda)) = V_1 \supset V_2 \supset \cdots \supset V_{r+1} = 0$ of $J(I(\lambda))$ such that $V_i/V_{i+1} \simeq M(w_i\lambda)$. Moreover if $w\lambda - \lambda \notin 2\mathcal{P}$ for $w \in W \setminus \{e\}$ then $J(I(\lambda)) \simeq \bigoplus_{w\in W} M(w\lambda)$.*

This theorem does not need the assumption that λ is regular. If G is split and $I(\lambda)$ is irreducible, Collingwood [Col91] proved Theorem 1.2.

For example, we obtain the following for $\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{R})$: Choose a basis $\{H, E_+, E_-\}$ of $\mathfrak{g}_0$ such that $\mathbb{R}H = \mathfrak{a}_0$, $\mathbb{R}E_+ = \mathfrak{n}_0$, $[H, E_\pm] = \pm E_\pm$ and $\theta(E_+) = -E_-$. Then $\Sigma^+ = \{\alpha\}$ where $\alpha(H) = 1$. Let $\lambda = r\alpha$ for $r \in \mathbb{C}$. We may assume $\operatorname{Re} r \geq 0$. By Theorems 1.1 and 1.2, we have an exact sequence

$$0 \longrightarrow M(-r\alpha) \longrightarrow J(I(r\alpha)) \longrightarrow M(r\alpha) \longrightarrow 0.$$

Consider the case where λ is integral, i.e., $2r \in \mathbb{Z}$. If $r \notin \mathbb{Z}$ then this sequence splits by Theorem 1.2. On the other hand, if $r \in \mathbb{Z}$ then by a direct calculation using the method introduced in this paper we can show it does not split. Notice that $I(r\alpha)$ is irreducible if and only if $r \in \mathbb{Z}$. Then we have the following; if λ is integral then $J(I(\lambda))$ is isomorphic to the direct sum of generalized Verma modules if and only if $I(\lambda)$ is reducible.

We summarize the content of this paper. In Section 2, we prove our main theorem for the case $G = SL(2, \mathbb{R})$. We do not need this section later, but it serves as a prototype for the arguments that follow. We begin to treat the general case from Section 3 on. In Section 3 we show fundamental properties of Jacquet modules and introduce a certain extension of the universal enveloping algebra. We construct special elements in the Jacquet module in Section 4. In Section 5 we prove our main theorem in the case of a regular infinitesimal character using the result of Section 4. We complete the proof in Section 6 using the translation principle.

Notation

Throughout this paper we use the following notation. As usual we denote the ring of integers, the set of non-negative integers, the set of positive integers, the real number field and the complex number field by $\mathbb{Z}, \mathbb{Z}_{\geq 0}, \mathbb{Z}_{>0}, \mathbb{R}$ and $\mathbb{C}$ respectively. Let $\mathfrak{g}_0$ be a real semisimple Lie algebra. Fix a Cartan involution θ of $\mathfrak{g}_0$. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{s}_0$ be the decomposition of $\mathfrak{g}_0$ into the $+1$ and -1 eigenspaces for θ. Take a maximal abelian subspace $\mathfrak{a}_0$ of $\mathfrak{s}_0$ and let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ be the corresponding Iwasawa decomposition of $\mathfrak{g}_0$. Set $\mathfrak{m}_0 = \{X \in \mathfrak{k}_0 \mid [H, X] = 0 \text{ for all } H \in \mathfrak{a}_0\}$. Then $\mathfrak{p}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ is a minimal parabolic subalgebra of $\mathfrak{g}_0$. Write $\mathfrak{g}$ for the complexification of $\mathfrak{g}_0$ and $U(\mathfrak{g})$ for the universal enveloping algebra of $\mathfrak{g}$. We use analogous notation for other Lie algebras.

Set $\mathfrak{a}^* = \operatorname{Hom}_{\mathbb{C}}(\mathfrak{a}, \mathbb{C})$ and $\mathfrak{a}_0^* = \operatorname{Hom}_{\mathbb{R}}(\mathfrak{a}_0, \mathbb{R})$. Let $\Sigma \subset \mathfrak{a}^*$ be the restricted root system for $(\mathfrak{g}, \mathfrak{a})$ and $\mathfrak{g}_\alpha$ the root space for $\alpha \in \Sigma$. Let Σ^+ be the positive root system determined by $\mathfrak{n}$, i.e., $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$. Σ^+ determines the set of simple roots $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$. We define a total order on $\mathfrak{a}_0^*$ by the following; for $c_i, d_i \in \mathbb{R}$ we define $\sum_i c_i\alpha_i > \sum_i d_i\alpha_i$ if and only if there exists an integer k such that $c_1 = d_1, \dots, c_k = d_k$ and $c_{k+1} > d_{k+1}$. Let $\{H_1, H_2, \dots, H_l\}$ be the dual basis of $\{\alpha_i\}$. Write W for the little Weyl group for $(\mathfrak{g}_0, \mathfrak{a}_0)$ and e for the unit element of W. Set $\mathcal{P} = \{\sum_{\alpha\in\Sigma^+} n_\alpha\alpha \mid n_\alpha \in \mathbb{Z}\}$, $\mathcal{P}^+ = \{\sum_{\alpha\in\Sigma^+} n_\alpha\alpha \mid n_\alpha \in \mathbb{Z}_{\geq 0}\}$ and $\mathcal{P}^{++} = \mathcal{P}^+ \setminus \{0\}$. Let m be the dimension of $\mathfrak{n}$. Fix a basis $E_1, E_2, \dots, E_m$ of $\mathfrak{n}$ such that each E_i is a restricted root vector. Let β_i be a restricted root such that $E_i \in \mathfrak{g}_{\beta_i}$. For $\mathbf{n} = (\mathbf{n}_i) \in \mathbb{Z}_{\geq 0}^m$ we denote $E_1^{\mathbf{n}_1} E_2^{\mathbf{n}_2} \cdots E_m^{\mathbf{n}_m}$ by $E^{\mathbf{n}}$.

For $x = (x_1, x_2, \dots, x_n) \in \mathbb{Z}_{\geq 0}^n$, we write $|x| = x_1 + x_2 + \cdots + x_n$ and $x! = x_1!\, x_2! \cdots x_n!$.

For a $\mathbb{C}$-algebra R, let $M(r, r', R)$ be the space of $r \times r'$ matrices with entries in R and $M(r, R) = M(r, r, R)$. Write $1_r \in M(r, R)$ for the identity matrix.

§2. The Case $SL(2, \mathbb{R})$

Definition 2.1 (Jacquet module [Cas80]). Let $\mathbb{X}$ be a $U(\mathfrak{g})$-module. Define modules $\widehat{J}(\mathbb{X})$ and $J(\mathbb{X})$ by

$$\widehat{J}(\mathbb{X}) = \varprojlim_k \mathbb{X}/\mathfrak{n}^k\mathbb{X},$$
$$J(\mathbb{X}) = \widehat{J}(\mathbb{X})_{\mathfrak{a}\text{-finite}} = \{x \in \widehat{J}(\mathbb{X}) \mid \dim U(\mathfrak{a})x < \infty\}.$$

We call $J(\mathbb{X})$ the Jacquet module of $\mathbb{X}$.

Remark 2.2. In some articles, e.g., Wallach [Wal88, 4.1.5], the definition of the Jacquet module is different from what we give here. These Jacquet modules are dual to each other (cf. Matumoto [Mat90, Corollary 4.7.4]).

In this section, let $G = SL(2, \mathbb{R})$.

Take a basis H, E_+, E_- of $\mathfrak{sl}(2, \mathbb{R})$ such that $[H, E_\pm] = \pm E_\pm$, $[E_+, E_-] = H$, $\mathfrak{a}_0 = \mathbb{R}H$, $\mathfrak{n}_0 = \mathbb{R}E_+$ and $\theta(E_+) = -E_-$. Fix $\lambda \in \mathbb{C}\setminus\{0\}$ such that $\operatorname{Re}\lambda \geq 0$. Let $I(\lambda)$ be the unique principal series representation generated by the trivial K-type with an infinitesimal character λ. Then $I(\lambda)$ has a generator u_λ and the following relations:

$$(H^2 - H + 2E_+E_-)u_\lambda = (\lambda^2 - 1/4)u_\lambda,$$
$$(E_+ - E_-)u_\lambda = 0.$$

The first relation says that $I(\lambda)$ has an infinitesimal character λ ($(H^2 - H + 2E_+E_-)$ is the Casimir element of $\mathfrak{sl}(2, \mathbb{C})$) and the second relation says that u_λ belongs to its trivial K-type.

By the relations, we have $(H^2 - H + 2E_+^2)u_\lambda = (\lambda^2 - 1/4)u_\lambda$. Put $\overline{u_\lambda} = u_\lambda + \mathfrak{n}I(\lambda) \in I(\lambda)/\mathfrak{n}I(\lambda)$. Then we have $(H-(\lambda+1/2))(H-(-\lambda+1/2))\overline{u_\lambda} = 0$. Hence the dimension of $I(\lambda)/\mathfrak{n}I(\lambda)$ is 2 and the eigenvalues of H are $\pm\lambda + 1/2$.

Put $u_1 = (H - (-\lambda + 1/2))u_\lambda$, $u_2 = (H - (\lambda + 1/2))u_\lambda$, $u = {}^t(u_1, u_2)$. Then $\{u_1 + \mathfrak{n}I(\lambda), u_2 + \mathfrak{n}I(\lambda)\}$ is a basis of $I(\lambda)/\mathfrak{n}I(\lambda)$. We have

$$(H1_2 - Q)u = \frac{1}{\lambda}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} E_+^2 u \quad \text{where} \quad Q = \begin{pmatrix} \lambda + 1/2 & 0 \\ 0 & -\lambda + 1/2 \end{pmatrix}.$$

Put

$$R = \frac{1}{\lambda}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then we have $(H1_2 - Q - RE_+^2)u = 0$. Define a $\mathbb{C}$-algebra $\widehat{\mathcal{E}}(\mathfrak{n})$ by $\widehat{\mathcal{E}}(\mathfrak{n}) = \varprojlim_k U(\mathfrak{n})/\mathfrak{n}^k U(\mathfrak{n})$. In this case, we have $\widehat{\mathcal{E}}(\mathfrak{n}) = \mathbb{C}[[E_+]]$. This is a complete local ring with the maximal ideal $E_+\mathbb{C}[[E_+]] = \mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n})$. Notice that the $\mathbb{C}$-algebra $\widehat{\mathcal{E}}(\mathfrak{n})$ acts on $\widehat{J}(I(\lambda))$.

The crucial fact is the following lemma. The lemma says that the action of H can be expressed by an upper triangular matrix.

Lemma 2.3. *There exist $L \in 1_2 + M(2, \mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))$ and $T \in M(2, U(\mathfrak{n}))$ which have the following properties.*

(1) *We have $(H1_2 - Q - RE_+^2)L = L(H1_2 - Q - T)$.*

(2) *If $\lambda \notin \mathbb{Z}_{>0}$, then $T = 0$.*

(3) *If $\lambda \in \mathbb{Z}_{>0}$, then $T = T_\lambda E_+^{2\lambda}$ where*

$$T_\lambda = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$$

for some $t \in \mathbb{C}$.

Notice that L is invertible since $\det L \in 1 + \mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n})$ is invertible. (Recall that $\widehat{\mathcal{E}}(\mathfrak{n})$ is a local ring with the maximal ideal $\mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n})$.) Take L as in the lemma and put $v = {}^t(v_1, v_2) = L^{-1}u$. Since $(H1_2 - Q - RE_+^2)u = 0$, we have $(H1_2 - Q - T)v = 0$ by condition (1). The action of H on v is easier than that on u (for example, v_2 is an H-eigenvector) since $Q + T$ is an upper triangular matrix.

Proof of Lemma 2.3. From condition (1), we have

$$[H1_2, L] - [Q, L] = RE_+^2 L - LT. \tag{2.1}$$

The calculation of the left hand side is not so difficult. In fact, the term $[H1_2, L]$ can be calculated using the H-weight decomposition, and $[Q, L]$ calculated easily since Q is a diagonal matrix.

Thus, take $L_k, T_k \in M(2, \mathbb{C})$ such that $L = \sum_k L_k E_+^{2k}$, $T = \sum_k T_k E_+^{2k}$. Moreover, define $a_k, b_k, c_k, d_k \in \mathbb{C}$ by

$$L_k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}.$$

Then the left hand side of (2.1) is

$$\sum_{k=0}^{\infty} \begin{pmatrix} 2ka_k & 2(k-\lambda)b_k \\ 2(k+\lambda)c_k & 2kd_k \end{pmatrix} E_+^{2k}$$

On the other hand, the right hand side of (2.1) is not so easy. We only expand the right hand side of (2.1) into a power series. Namely, the right hand side of (2.1) is

$$\sum_{k=0}^{\infty} \left(RL_{k-1} - \sum_{l=0}^{k} L_l T_{k-l} \right) E_+^{2k},$$

where $L_{-1} = 0$. By the conditions, we have $T_0 = 0$ and $L_0 = 1_2$. Hence the right hand side of (2.1) is

$$\sum_{k=0}^{\infty} \left(RL_{k-1} - \sum_{l=1}^{k-1} L_l T_{k-l} - T_k \right) E_+^{2k}.$$

Therefore $(H - Q - RE_+^2)L = L(H - Q - T)$ is equivalent to

$$\begin{pmatrix} 2ka_k & 2(k-\lambda)b_k \\ 2(k+\lambda)c_k & 2kd_k \end{pmatrix} + T_k = RL_{k-1} - \sum_{l=1}^{k-1} L_l T_{k-l} \tag{2.2}$$

for all $k > 0$. (The case of $k = 0$ is automatically satisfied.)

We take L_k and T_k inductively. Assume that we have already chosen $L_0, \dots, L_{k-1}$ and $T_0, \dots, T_{k-1}$. Then the right hand side of (2.2) is determined. If $\lambda \neq k$, then we can take a_k, b_k, c_k, d_k such that $T_k = 0$. If $\lambda = k$, then we can take a_k, b_k, c_k, d_k such that

$$T_k = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$$

for some $t \in \mathbb{C}$. (Recall that we assume $\operatorname{Re} \lambda \geq 0$ and $\lambda \neq 0$.) Hence the lemma is proved. □

Take L as in the lemma and put $v = {}^t(v_1, v_2) = L^{-1}u$. As already mentioned, $(H1_2 - Q - T)v = 0$. In particular, we have $Hv_2 = (-\lambda + 1/2)v_2$. Moreover, since $(H - (\lambda + 1/2))v_1 = tE_+^{2\lambda}v_2$ for some $t \in \mathbb{C}$, we have $(H - (\lambda + 1/2))^2 v_1 = t(H - (\lambda + 1/2))E_+^{2\lambda}v_2 = tE_+^{2\lambda}(H - (-\lambda + 1/2))v_2 = 0$. Namely, v_1 has a generalized H-eigenvalue $\lambda + 1/2$.

We prove that v_1 and v_2 generate $\widehat{J}(I(\lambda))$ as an $\widehat{\mathcal{E}}(\mathfrak{n})$-module. By Nakayama's lemma, it is sufficient to prove that $v_1 + \mathfrak{n}\widehat{J}(I(\lambda))$ and $v_2 + \mathfrak{n}\widehat{J}(I(\lambda))$ generate $\widehat{J}(I(\lambda))/\mathfrak{n}\widehat{J}(I(\lambda))$. However, since $L \in 1_2 + M(2, \mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))$, we have $v_i \equiv u_i \pmod{\mathfrak{n}\widehat{J}(I(\lambda))}$. Since the set of images of u_1, u_2 in $I(\lambda)/\mathfrak{n}I(\lambda)$ is a basis of $I(\lambda)/\mathfrak{n}I(\lambda) \simeq \widehat{J}(I(\lambda))/\mathfrak{n}\widehat{J}(I(\lambda))$, $v_1 + \mathfrak{n}\widehat{J}(I(\lambda))$ and $v_2 + \widehat{J}(I(\lambda))$ generate $\widehat{J}(I(\lambda))/\mathfrak{n}\widehat{J}(I(\lambda))$.

Now we prove the following lemma.

Lemma 2.4.

(1) $J(I(\lambda)) = U(\mathfrak{g})v_1 + U(\mathfrak{g})v_2$.

(2) $E_-v_2 = 0$.

(3) $E_-v_1 \in U(\mathfrak{g})v_2$.

From Lemma 2.4, v_2 is a lowest weight vector in $J(I(\lambda))$ and v_1 is a lowest weight vector in $J(I(\lambda))/U(\mathfrak{g})v_2$.

To prove these formulae, we use the projection to a generalized H-eigenspace. The justification will be done in §3 (Corollary 3.9). Recall that the generalized H-eigenvalue of v_1 (resp. v_2) is $\lambda + 1/2$ (resp. $-\lambda + 1/2$).

Proof of Lemma 2.4. First we prove (1). Let $v \in J(I(\lambda))$ be a generalized eigenvector of H with a generalized eigenvalue μ. Since v_1, v_2 generate $\widehat{J}(I(\lambda))$ as an $\widehat{\mathcal{E}}(\mathfrak{n})$-module, there exist $B^{(1)} = \sum_k B_k^{(1)}E_+^k \in \widehat{\mathcal{E}}(\mathfrak{n})$ and $B^{(2)} = \sum_k B_k^{(2)}E_+^k \in \widehat{\mathcal{E}}(\mathfrak{n})$ such that $v = B^{(1)}v_1 + B^{(2)}v_2$. Applying the projection to the generalized H-eigenspace with an eigenvalue μ, we have $v = B^{(1)}_{\mu-\lambda-1/2}E_+^{\mu-\lambda-1/2}v_1 + B^{(2)}_{\mu+\lambda-1/2}E_+^{\mu+\lambda-1/2}v_2$ (Here $E_+^t = 0$ if $t \notin \mathbb{Z}_{\geq 0}$). This implies $v \in U(\mathfrak{g})v_1 + U(\mathfrak{g})v_2$.

We will calculate E_-v_1 and E_-v_2. Roughly speaking, these equations are induced from $(E_+ - E_-)u_\lambda = 0$ and the projection to a generalized H-eigenspace. Take $A^{(1)}, A^{(2)} \in \widehat{\mathcal{E}}(\mathfrak{n})$ such that $u_\lambda = A^{(1)}v_1 + A^{(2)}v_2$. Take $A_k^{(i)} \in \mathbb{C}$ such that $A^{(i)} = \sum A_k^{(i)}E_+^k$. From $(E_- - E_+)u_\lambda = 0$, we have

$$0 = A_0^{(1)} E_- v_1 + \sum_{k=1}^{\infty} A_k^{(1)} E_- E_+^k v_1 - \sum_{k=0}^{\infty} A_k^{(1)} E_+^{k+1} v_1 + A_0^{(2)} E_- v_2 + \sum_{k=1}^{\infty} A_k^{(2)} E_- E_+^k v_2 - \sum_{k=0}^{\infty} A_k^{(2)} E_+^{k+1} v_2.$$

Recall that v_1 (resp. v_2) has a generalized H-eigenvalue $\lambda+1/2$ (resp. $-\lambda+1/2$). Hence the generalized H-eigenvalue of each term is the following:

$$\begin{gathered}\lambda + 1/2 - 1, \quad \lambda + 1/2 - 1 + k, \quad \lambda + 1/2 + (k+1), \\ -\lambda + 1/2 - 1, -\lambda + 1/2 - 1 + k, -\lambda + 1/2 + (k+1).\end{gathered}$$

Using the projection to the generalized H-eigenspace with an eigenvalue $-\lambda + 1/2 - 1$, we get $A_0^{(2)} E_- v_2 = 0$ since $\operatorname{Re}\lambda \geq 0$.

Next, we project both sides to the generalized H-eigenspace with an eigenvalue $\lambda + 1/2 - 1$. If $\lambda \notin \mathbb{Z}/2$, then $A_0^{(1)} E_- v_1 = 0$. If $\lambda \in \mathbb{Z}/2$, we get

$$0 = A_0^{(1)} E_- v_1 + A_{2\lambda}^{(2)} E_- E_+^{2\lambda} v_2 - A_{2\lambda-2}^{(2)} E_+^{2\lambda-1} v_2.$$

Hence we have $A_0^{(1)} E_- v_1 \in U(\mathfrak{g}) v_2$.

We must prove that $A_0^{(1)}, A_0^{(2)} \neq 0$. Since $u_\lambda = A^{(1)} v_1 + A^{(2)} v_2$, we have $u_\lambda \equiv A_0^{(1)} v_1 + A_0^{(2)} v_2 \pmod{\mathfrak{n}\widehat{J}(I(\lambda))}$. Moreover, since $L \in 1 + M(2, \mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))$, we have $u_i \equiv v_i \pmod{\mathfrak{n}\widehat{J}(I(\lambda))}$. Hence we have $u_\lambda \equiv A_0^{(1)} u_1 + A_0^{(2)} u_2 \pmod{\mathfrak{n}\widehat{J}(I(\lambda))}$. By the definition of u_1, u_2, $u_\lambda = (1/2\lambda) u_1 + (-1/2\lambda) u_2$. Since $\{u_1 + \mathfrak{n} I(\lambda), u_2 + \mathfrak{n} I(\lambda)\}$ is a basis of $I(\lambda)/\mathfrak{n} I(\lambda) \simeq \widehat{J}(I(\lambda))/\mathfrak{n}\widehat{J}(I(\lambda))$, we have $A_0^{(1)} = 1/2\lambda$ and $A_0^{(2)} = -1/2\lambda$. These are nonzero. □

Recall the definition of Verma module. For $\mu \in \mathbb{C}$, let $\mathbb{C}_\mu$ be the 1-dimensional representation of $\mathbb{C}H + \mathbb{C}E_-$ such that $Hv = \mu v$ and $E_- v = 0$ for $v \in \mathbb{C}_\mu$. Put $M(\mu) = U(\mathfrak{g}) \otimes_{U(\mathbb{C}H + \mathbb{C}E_-)} \mathbb{C}_{\mu+1/2}$ and $V = U(\mathfrak{g}) v_2$. Since v_2 is a lowest weight vector of $J(I(\lambda))$ and v_1 is a lowest weight vector of $J(I(\lambda))/V$, we have surjective maps $M(-\lambda) \to V$ and $M(\lambda) \to J(I(\lambda))/V$.

To prove that these maps are isomorphism, we use the Osborne conjecture, which is now a theorem [HS83a]. Using the Osborne conjecture, we can calculate the character of $J(I(\lambda))$ from the character of $I(\lambda)$ (see Section 5). As a consequence, we have $\Theta(J(I(\lambda))) = \Theta(M(\lambda)) + \Theta(M(-\lambda))$ where Θ denotes the character. Hence we have $V \simeq M(-\lambda)$ and $J(I(\lambda))/V = M(\lambda)$.

In the rest of this paper, we generalize these arguments to the general case.

§3. Jacquet Modules and Fundamental Properties

We now treat the general case. Set $\widehat{\mathcal{E}}(\mathfrak{n}) = \varprojlim_k U(\mathfrak{n})/\mathfrak{n}^k U(\mathfrak{n})$. For a projective system $\{M_k, \varphi_k \colon M_k \to M_{k-1}\}_{k\in\mathbb{Z}_{\geq 0}}$, we construct the projective limit as $\varprojlim_k M_k = \{(m_k)_{k\in\mathbb{Z}_{\geq 0}} \mid m_k \in M_k,\ \varphi_k(m_k) = m_{k-1}\}$.

Proposition 3.1.

(1) *The* $\mathbb{C}$*-algebra* $\widehat{\mathcal{E}}(\mathfrak{n})$ *is right and left Noetherian.*

(2) *The* $\mathbb{C}$*-algebra* $\widehat{\mathcal{E}}(\mathfrak{n})$ *is flat over* $U(\mathfrak{n})$*.*

(3) *If* $\mathbb{X}$ *is a finitely generated* $U(\mathfrak{n})$*-module then* $\varprojlim_k \mathbb{X}/\mathfrak{n}^k\mathbb{X} = \widehat{\mathcal{E}}(\mathfrak{n}) \otimes_{U(\mathfrak{n})} \mathbb{X}$*.*

(4) *Let* $S = (S_k)_{k\in\mathbb{Z}_{\geq 0}}$ *be an element of* $M(r, \mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))$ *and* $(a_n) \in \mathbb{C}^{\mathbb{Z}_{\geq 0}}$*. Define* $\sum_{n=0}^{\infty} a_n S^n = (\sum_{n=0}^{k} a_n S_k^n)_k$*. Then* $\sum_{n=0}^{\infty} a_n S^n \in M(r, \widehat{\mathcal{E}}(\mathfrak{n}))$*.*

Proof. Since Stafford and Wallach [SW82, Theorem 2.1] show that $\mathfrak{n}U(\mathfrak{n}) \subset U(\mathfrak{n})$ satisfies the Artin-Rees property, the usual argument of the proof for commutative rings can be applied to prove (1), (2) and (3). (4) is obvious. □

Corollary 3.2. *Let* S *be an element of* $M(r, \widehat{\mathcal{E}}(\mathfrak{n}))$ *such that* $S - 1_r \in M(r, \mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))$*. Then* S *is invertible.*

Proof. Set $T = 1_r - S$. By Proposition 3.1, $R = \sum_{n=0}^{\infty} T^n \in M(r, \widehat{\mathcal{E}}(\mathfrak{n}))$. Then $SR = RS = 1_r$. □

We can prove the following proposition using the methods of Goodman and Wallach [GW80, Lemma 2.2]. For the sake of completeness we give a proof.

Proposition 3.3. *Let* $\mathbb{X}$ *be a* $U(\mathfrak{a}\oplus\mathfrak{n})$*-module which is finitely generated as a* $U(\mathfrak{n})$*-module. Assume that every element of* $\mathbb{X}$ *is* $\mathfrak{a}$*-finite. For* $\mu \in \mathfrak{a}^*$ *set*

$$\mathbb{X}_\mu = \{x \in \mathbb{X} \mid \text{For all } H \in \mathfrak{a} \text{ there exists a positive integer } N \text{ such that } (H - \mu(H))^N x = 0\}.$$

Then

$$\widehat{J}(\mathbb{X}) \simeq \prod_{\mu\in\mathfrak{a}^*} \mathbb{X}_\mu.$$

Proof. Since the action of $\mathfrak{n}$ increases an $\mathfrak{a}$-weight and $\mathbb{X}$ is a finitely generated $U(\mathfrak{n})$-module, there exists a positive integer k_μ for all μ such that $\mathfrak{n}^{k_\mu}\mathbb{X} \cap \mathbb{X}_\mu = 0$. We fix such a k_μ.

For $k \in \mathbb{Z}_{>0}$ put $S_k = \{\mu \in \mathfrak{a}^* \mid \mathbb{X}_\mu \neq 0,\ \mathbb{X}_\mu \not\subset \mathfrak{n}^k\mathbb{X}\}$. Since $\mathbb{X}$ is finitely generated, $\dim \mathbb{X}/\mathfrak{n}^k\mathbb{X} < \infty$. Therefore S_k is a finite set. Define a map $\varphi\colon \prod_{\mu\in\mathfrak{a}^*} \mathbb{X}_\mu \to \widehat{J}(\mathbb{X})$ by

$$\varphi((x_\mu)_{\mu\in\mathfrak{a}^*}) = \left(\sum_{\mu\in S_k} x_\mu \pmod{\mathfrak{n}^k\mathbb{X}}\right)_k.$$

First we show that φ is injective. Assume $\varphi((x_\mu)_{\mu\in\mathfrak{a}^*}) = 0$. We have $\sum_{\mu\in S_k} x_\mu \in \mathfrak{n}^k\mathbb{X}$ for all $k \in \mathbb{Z}_{>0}$. Since $\mathfrak{n}^k\mathbb{X}$ is $\mathfrak{a}$-stable and S_k is a finite set, $x_\mu \in \mathfrak{n}^k\mathbb{X}$ for all $\mu \in \mathfrak{a}^*$ and $k \in \mathbb{Z}_{>0}$. In particular, we have $x_\mu \in \mathbb{X}_\mu \cap \mathfrak{n}^{k_\mu}\mathbb{X} = 0$.

We have to show that φ is surjective. Let $x = (x_k \pmod{\mathfrak{n}^k\mathbb{X}})_k$ be an element of $\widehat{J}(\mathbb{X})$. Since every element of $\mathbb{X}$ is $\mathfrak{a}$-finite, we have $\mathbb{X} = \bigoplus_{\mu\in\mathfrak{a}^*} \mathbb{X}_\mu$. Let $p_\mu\colon \mathbb{X} \to \mathbb{X}_\mu$ be the projection. Notice that if $i, i' \geq k_\mu$ then $p_\mu(x_i) = p_\mu(x_{i'})$. Hence we have $\varphi((p_\mu(x_{k_\mu}))_{\mu\in\mathfrak{a}^*}) = x$. □

We define an $(\mathfrak{a}\oplus\mathfrak{n})$-representation structure of $U(\mathfrak{n})$ by $(H + X)(u) = Hu - uH + Xu$ for $H \in \mathfrak{a}$, $X \in \mathfrak{n}$, $u \in U(\mathfrak{n})$. Then $U(\mathfrak{n})$ is a $U(\mathfrak{a}\oplus\mathfrak{n})$-module. By Proposition 3.3, $\widehat{\mathcal{E}}(\mathfrak{n}) = \prod_{\mu\in\mathfrak{a}^*} U(\mathfrak{n})_\mu$. The following results are corollaries of Proposition 3.3.

Corollary 3.4. *The linear map*

$$\begin{array}{ccc} \mathbb{C}[[X_1, X_2, \dots, X_m]] & \longrightarrow & \widehat{\mathcal{E}}(\mathfrak{n}) \\ \sum_{\mathbf{n}\in\mathbb{Z}^m_{\geq 0}} a_\mathbf{n} X^\mathbf{n} & \longmapsto & \left(\sum_{|\mathbf{n}|\leq k} a_\mathbf{n} E^\mathbf{n} \pmod{\mathfrak{n}^k U(\mathfrak{n})}\right)_k \end{array}$$

is bijective, where $X^\mathbf{n} = X_1^{\mathbf{n}_1} X_2^{\mathbf{n}_2} \cdots X_m^{\mathbf{n}_m}$ *for* $\mathbf{n} = (\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_m) \in \mathbb{Z}^m_{\geq 0}$.

Proof. By the Poincaré-Birkhoff-Witt theorem, $\{E^\mathbf{n} \mid \sum_i \mathbf{n}_i\beta_i = \mu\}$ is a basis of $U(\mathfrak{n})_\mu$. This implies the corollary since $\widehat{\mathcal{E}}(\mathfrak{n}) = \prod_{\mu\in\mathfrak{a}^*} U(\mathfrak{n})_\mu$. □

We denote the image of $\sum_{\mathbf{n}\in\mathbb{Z}^m_{\geq 0}} a_\mathbf{n} X^\mathbf{n}$ under the map in Corollary 3.4 by $\sum_{\mathbf{n}\in\mathbb{Z}^m_{\geq 0}} a_\mathbf{n} E^\mathbf{n}$.

Corollary 3.5. *Let $\mathbb{X}$ be a $U(\mathfrak{g})$-module which is finitely generated as a $U(\mathfrak{n})$-module. Assume that all elements are $\mathfrak{a}$-finite. Then $J(\mathbb{X}) = \mathbb{X}$.*

Proof. This follows from the following equation.

$$J(\mathbb{X}) = \widehat{J}(\mathbb{X})_{\mathfrak{a}\text{-finite}} = \left(\prod_{\mu\in\mathfrak{a}^*} \mathbb{X}_\mu\right)_{\mathfrak{a}\text{-finite}} = \bigoplus_{\mu\in\mathfrak{a}^*} \mathbb{X}_\mu = \mathbb{X}.$$

□

Put $\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) = \widehat{\mathcal{E}}(\mathfrak{n}) \otimes_{U(\mathfrak{n})} U(\mathfrak{g})$. We can define a $\mathbb{C}$-algebra structure of $\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})$ by

$$\begin{aligned}
(f \otimes 1)(1 \otimes u) &= f \otimes u, \\
(1 \otimes u)(1 \otimes u') &= 1 \otimes (uu'), \\
(f \otimes 1)(f' \otimes 1) &= (ff') \otimes 1, \\
(1 \otimes X)(f \otimes 1) &= \sum_{\mathbf{n} \in \mathbb{Z}^r_{\geq 0}} \frac{1}{\mathbf{n}!} \frac{\partial^{|\mathbf{n}|}}{\partial E^{\mathbf{n}}} f \otimes {}^t((\operatorname{ad}(E))^{\mathbf{n}})(X),
\end{aligned}$$

where $u, u' \in U(\mathfrak{g})$, $X \in \mathfrak{g}$, $f, f' \in \widehat{\mathcal{E}}(\mathfrak{n})$, ${}^t((\operatorname{ad}(E))^{\mathbf{n}}) = (-\operatorname{ad}(E_m))^{\mathbf{n}_m} \cdots (-\operatorname{ad}(E_1))^{\mathbf{n}_1}$ and

$$\frac{\partial^{|\mathbf{n}|}}{\partial E^{\mathbf{n}}} \left(\sum_{\mathbf{m} \in \mathbb{Z}^m_{\geq 0}} a_{\mathbf{m}} E^{\mathbf{m}} \right) = \sum_{\mathbf{m} \in \mathbb{Z}^m_{\geq 0}} a_{\mathbf{m}} \frac{\mathbf{m}!}{(\mathbf{m}-\mathbf{n})!} E^{\mathbf{m}-\mathbf{n}}.$$

It is not difficult to see that this definition is independent of a choice of a basis $\{E_i\}$ and its order. However, we do not use it. So we omit the proof.

Notice that $\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \otimes_{U(\mathfrak{g})} \mathbb{X} \simeq \widehat{\mathcal{E}}(\mathfrak{n}) \otimes_{U(\mathfrak{n})} \mathbb{X}$ as an $\widehat{\mathcal{E}}(\mathfrak{n})$-module for a $U(\mathfrak{g})$-module $\mathbb{X}$. By Proposition 3.1, $\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})$ is flat over $U(\mathfrak{g})$. Notice that if $\mathfrak{b}$ is a subalgebra of $\mathfrak{g}$ which contains $\mathfrak{n}$ then $\widehat{\mathcal{E}}(\mathfrak{n}) \otimes_{U(\mathfrak{n})} U(\mathfrak{b})$ is a subalgebra of $\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})$. Put $\widehat{\mathcal{E}}(\mathfrak{b},\mathfrak{n}) = \widehat{\mathcal{E}}(\mathfrak{n}) \otimes_{U(\mathfrak{n})} U(\mathfrak{b})$.

Let $\mathbb{X}$ be a $U(\mathfrak{a} \oplus \mathfrak{n})$-module such that $\mathbb{X} = \bigoplus_{\mu \in \mathfrak{a}^*} \mathbb{X}_\mu$. Set

$$V = \left\{ (x_\mu)_\mu \in \prod_{\mu \in \mathfrak{a}^*} \mathbb{X}_\mu \;\middle|\; \text{there exists an element } \nu \in \mathfrak{a}_0^* \text{ such that } x_\mu = 0 \text{ for } \operatorname{Re} \mu < \nu \right\}.$$

Then we can define an $\mathfrak{a}$-module homomorphism

$$\varphi \colon \widehat{\mathcal{E}}(\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{n}) \otimes_{U(\mathfrak{a} \oplus \mathfrak{n})} \mathbb{X} \simeq \left(\prod_{\mu \in \mathfrak{a}^*} U(\mathfrak{n})_\mu \right) \otimes_{U(\mathfrak{n})} \left(\bigoplus_{\mu' \in \mathfrak{a}^*} \mathbb{X}_{\mu'} \right) \to V$$

by $\varphi((f_\mu)_{\mu \in \mathfrak{a}^*} \otimes (x_{\mu'})_{\mu' \in \mathfrak{a}^*}) = (\sum_{\mu + \mu' = \lambda} f_\mu x_{\mu'})_{\lambda \in \mathfrak{a}^*}$. Notice that the composition of the maps $\mathbb{X} \to \widehat{\mathcal{E}}(\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{n}) \otimes_{U(\mathfrak{a} \oplus \mathfrak{n})} \mathbb{X} \to V$ is equal to the inclusion map $\mathbb{X} \hookrightarrow V$.

We consider the case $\mathbb{X} = U(\mathfrak{g})$. Define an $(\mathfrak{a} \oplus \mathfrak{n})$-module structure of $U(\mathfrak{g})$ by $(H + X)(u) = Hu - uH + Xu$ for $H \in \mathfrak{a}$, $X \in \mathfrak{n}$, $u \in U(\mathfrak{g})$. We have a map

$$\varphi\colon \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \to \left\{ (P_\mu)_{\mu\in\mathfrak{a}^*} \in \prod_{\mu\in\mathfrak{a}^*} U(\mathfrak{g})_\mu \;\middle|\; \begin{array}{l}\text{there exists an element}\\ \nu \in \mathfrak{a}_0^* \text{ such that } P_\mu = 0 \text{ for } \operatorname{Re}\mu < \nu\end{array} \right\}.$$

We write $\varphi(P) = (P^{(\mu)})_{\mu\in\mathfrak{a}^*}$. Put $P^{(H,z)} = \sum_{\mu(H)=z} P^{(\mu)}$ for $z \in \mathbb{C}$ and $H \in \mathfrak{a}$ such that $\operatorname{Re}\alpha(H) > 0$ for all $\alpha \in \Sigma^+$. By the condition on H, the right hand side is a finite sum.

Proposition 3.6 and 3.7 follow at once from the definition.

Proposition 3.6. *Let $\mathbb{X}$ be a $U(\mathfrak{a}\oplus\mathfrak{n})$-module such that $\mathbb{X}$ is finitely generated as a $U(\mathfrak{n})$-module and each element is $\mathfrak{a}$-finite. Let $\varphi\colon \widehat{\mathcal{E}}(\mathfrak{a}\oplus\mathfrak{n},\mathfrak{n})\otimes_{U(\mathfrak{a}\oplus\mathfrak{n})} \mathbb{X} \to \prod_{\mu\in\mathfrak{a}^*}\mathbb{X}_\mu$ be the $\mathfrak{a}$-module homomorphism defined as above. Then φ coincides with the map given in Proposition 3.3. In particular φ is isomorphic.*

Proposition 3.7.

(1) *We have $(PQ)^{(\lambda)} = \sum_{\mu+\mu'=\lambda} P^{(\mu)}Q^{(\mu')}$ for $P, Q \in \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})$ and $\lambda \in \mathfrak{a}^*$. (Since $P^{(\mu)} = 0$ (resp. $Q^{(\mu')} = 0$) if $\operatorname{Re}\mu$ (resp. $\operatorname{Re}\mu'$) is sufficiently small, the right hand side is a finite sum.)*

(2) *We have*
$$\left(\sum_{\mathbf{n}\in\mathbb{Z}_{\geq 0}^m} a_{\mathbf{n}}E^{\mathbf{n}}\right)^{(\lambda)} = \sum_{\sum_i \mathbf{n}_i\beta_i=\lambda} a_{\mathbf{n}}E^{\mathbf{n}}$$
for $\lambda \in \mathfrak{a}^*$.

Proposition 3.8. *Let $\mathbb{X}$ be a $U(\mathfrak{g})$-module which is finitely generated as a $U(\mathfrak{n})$-module. We take generators $v_1, v_2, \dots, v_n$ of an $\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})$-module $\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})\otimes_{U(\mathfrak{g})}\mathbb{X}$ and set $V = \sum_i U(\mathfrak{g})v_i \subset \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})\otimes_{U(\mathfrak{g})}\mathbb{X}$. Define a surjective map $\psi\colon \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})\otimes_{U(\mathfrak{g})} V \to \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})\otimes_{U(\mathfrak{g})}\mathbb{X}$ by $\psi(f\otimes v) = fv$. Assume that there exist weights $\lambda_i \in \mathfrak{a}^*$ and a positive integer N such that $(H-\lambda_i(H))^N v_i = 0$ for all $H \in \mathfrak{a}$ and $1 \leq i \leq n$. Let $\varphi\colon \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})\otimes_{U(\mathfrak{g})} V \to \prod_{\mu\in\mathfrak{a}^*} V_\mu$ be the map defined as above. Then there exists a unique map $\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})\otimes_{U(\mathfrak{g})}\mathbb{X} \to \prod_{\mu\in\mathfrak{a}^*} V_\mu$ such that the diagram*

$$\begin{array}{ccc}
\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})\otimes_{U(\mathfrak{g})} V & \xrightarrow{\varphi} & \prod_{\mu\in\mathfrak{a}^*} V_\mu \\
\downarrow{\scriptstyle\psi} & \nearrow & \\
\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})\otimes_{U(\mathfrak{g})}\mathbb{X} & &
\end{array}$$

is commutative.

Proof. Set $\widehat{\mathbb{X}} = \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \otimes_{U(\mathfrak{g})} \mathbb{X}$ and $\widehat{V} = \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \otimes_{U(\mathfrak{g})} V$. Take $f^{(i)} \in \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})$ and $v^{(i)} \in V$ such that $\psi(\sum_i f^{(i)} \otimes v^{(i)}) = 0$. We have to show $\varphi(\sum_i f^{(i)} \otimes v^{(i)}) = 0$. Since $\widehat{V} = \widehat{\mathcal{E}}(\mathfrak{n}) \otimes_{U(\mathfrak{n})} V$, we may assume $f^{(i)} \in \widehat{\mathcal{E}}(\mathfrak{n})$. We can write $f^{(i)} = (f^{(i)}_\mu)_{\mu\in\mathfrak{a}^*}$ by the isomorphism $\widehat{\mathcal{E}}(\mathfrak{n}) \simeq \prod_{\mu\in\mathfrak{a}^*} U(\mathfrak{n})_\mu$. Since $V = \bigoplus_{\mu'\in\mathfrak{a}^*} V_{\mu'}$, we can write $v^{(i)} = \sum_{\mu'\in\mathfrak{a}^*} v^{(i)}_{\mu'}$, $v^{(i)}_{\mu'} \in V_{\mu'}$. We have to show $\sum_i \sum_{\mu+\mu'=\lambda} f^{(i)}_\mu v^{(i)}_{\mu'} = 0$ for all $\lambda \in \mathfrak{a}^*$. Since $\mathbb{X}$ is a finitely generated $U(\mathfrak{n})$-module we have $\widehat{\mathbb{X}} = \varprojlim_k \mathbb{X}/\mathfrak{n}^k\mathbb{X} = \varprojlim_k \widehat{\mathbb{X}}/\mathfrak{n}^k\widehat{\mathbb{X}}$. It is sufficient to prove $\sum_i \sum_{\mu+\mu'=\lambda} f^{(i)}_\mu v^{(i)}_{\mu'} \in \mathfrak{n}^k\widehat{\mathbb{X}}$ for all $k \in \mathbb{Z}_{>0}$.

Fix $\lambda \in \mathfrak{a}^*$ and $k \in \mathbb{Z}_{>0}$. We can choose an element $\nu \in \mathfrak{a}_0^*$ such that $\bigoplus_{\operatorname{Re}\mu\geq\nu} U(\mathfrak{n})_\mu \subset \mathfrak{n}^k U(\mathfrak{n})$. Then $0 = \psi(\sum_i f^{(i)} \otimes v^{(i)}) \equiv \sum_i \sum_{\operatorname{Re}\mu<\nu} f^{(i)}_\mu v^{(i)}_{\mu'}$ (mod $\mathfrak{n}^k\widehat{\mathbb{X}}$). Notice that the following two sets are finite.

$$\{\mu \mid \operatorname{Re}(\mu) < \nu \text{ and there exists an integer } i \text{ such that } f^{(i)}_\mu \neq 0\},$$
$$\{\mu' \mid \text{there exists an integer } i \text{ such that } v^{(i)}_{\mu'} \neq 0\}.$$

This implies $\sum_i \sum_{\mu+\mu'=\lambda} f^{(i)}_\mu v^{(i)}_{\mu'} \in \mathfrak{n}^k\widehat{\mathbb{X}}$. □

The following result is a corollary of Proposition 3.8.

Corollary 3.9. *In the setting of Proposition* 3.8, *we have the following. Let P_i $(1 \leq i \leq n)$ be elements of $\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})$ such that $\sum_{i=1}^n P_i v_i = 0$. Then $\sum_i P_i^{(\lambda-\lambda_i)} v_i = 0$ for all $\lambda \in \mathfrak{a}^*$.*

§4. Construction of Special Elements

Let Λ be a subset of $\mathcal{P}$. Put $\Lambda^+ = \Lambda \cap \mathcal{P}^+$ and $\Lambda^{++} = \Lambda \cap \mathcal{P}^{++}$. We define vector spaces $U(\mathfrak{g})_\Lambda, U(\mathfrak{n})_\Lambda, \widehat{\mathcal{E}}(\mathfrak{n})_\Lambda$ and $\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})_\Lambda$ by

$$U(\mathfrak{g})_\Lambda = \{P \in U(\mathfrak{g}) \mid P^{(\mu)} = 0 \text{ for all } \mu \notin \Lambda\},$$
$$U(\mathfrak{n})_\Lambda = \{P \in U(\mathfrak{n}) \mid P^{(\mu)} = 0 \text{ for all } \mu \notin \Lambda\},$$
$$\widehat{\mathcal{E}}(\mathfrak{n})_\Lambda = \{P \in \widehat{\mathcal{E}}(\mathfrak{n}) \mid P^{(\mu)} = 0 \text{ for all } \mu \notin \Lambda\},$$
$$\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})_\Lambda = \{P \in \widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n}) \mid P^{(\mu)} = 0 \text{ for all } \mu \notin \Lambda\}.$$

Put $(\mathfrak{n}U(\mathfrak{n}))_\Lambda = \mathfrak{n}U(\mathfrak{n}) \cap U(\mathfrak{n})_\Lambda$ and $(\mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))_\Lambda = \mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}) \cap \widehat{\mathcal{E}}(\mathfrak{n})_\Lambda$.

We assume that Λ is a subgroup of $\mathfrak{a}^*$. Then $U(\mathfrak{g})_\Lambda, U(\mathfrak{n})_\Lambda, \widehat{\mathcal{E}}(\mathfrak{n})_\Lambda$ and $\widehat{\mathcal{E}}(\mathfrak{g},\mathfrak{n})_\Lambda$ are $\mathbb{C}$-algebras. Let $\mathbb{X}$ be a $U(\mathfrak{g})_\Lambda$-module which is finitely generated as a $U(\mathfrak{n})_\Lambda$-module. Let $u_1, u_2, \dots, u_N$ be generators of $\mathbb{X}$ as a $U(\mathfrak{n})_\Lambda$-module. Put $u = {}^t(u_1, u_2, \dots, u_N)$, $\overline{\mathbb{X}} = \mathbb{X}/(\mathfrak{n}U(\mathfrak{n}))_\Lambda \mathbb{X}$ and $\overline{u} = u + (\mathfrak{n}U(\mathfrak{n}))_\Lambda \mathbb{X}^N \in \overline{\mathbb{X}}^N$. The module $\overline{\mathbb{X}}$ has an $\mathfrak{a}$-module structure induced from that of $\mathbb{X}$. Since $\mathbb{X}$ is a finitely generated $U(\mathfrak{n})_\Lambda$-module, we have $\dim \overline{\mathbb{X}} < \infty$. Let $\lambda_1, \lambda_2, \dots, \lambda_r \in \mathfrak{a}^*$ ($\operatorname{Re}\lambda_1 \ge \operatorname{Re}\lambda_2 \ge \cdots \ge \operatorname{Re}\lambda_r$) be eigenvalues of $\mathfrak{a}$ on $\overline{\mathbb{X}}$ with multiplicities. We can choose a basis $\overline{v_1}, \overline{v_2}, \dots, \overline{v_r}$ of $\overline{\mathbb{X}}$ and a linear map $\overline{Q}\colon \mathfrak{a} \to M(r, \mathbb{C})$ such that

$$\begin{cases} H\overline{v} = \overline{Q}(H)\overline{v} \text{ for all } H \in \mathfrak{a}, \\ \overline{Q}(H)_{ii} = \lambda_i(H) \text{ for all } H \in \mathfrak{a}, \\ \text{if } i > j \text{ then } \overline{Q}(H)_{ij} = 0 \text{ for all } H \in \mathfrak{a}, \\ \text{if } \lambda_i \ne \lambda_j \text{ then } \overline{Q}(H)_{ij} = 0 \text{ for all } H \in \mathfrak{a}, \end{cases}$$

where $\overline{v} = {}^t(\overline{v_1}, \overline{v_2}, \dots, \overline{v_r})$. Take $\overline{C} \in M(N, r, \mathbb{C})$ and $\overline{D} \in M(r, N, \mathbb{C})$ such that $\overline{v} = \overline{D}\overline{u}$ and $\overline{u} = \overline{C}\overline{v}$.

Set $\widehat{\mathbb{X}} = \widehat{\mathcal{E}}(\mathfrak{g}, \mathfrak{n})_\Lambda \otimes_{U(\mathfrak{g})_\Lambda} \mathbb{X}$.

Theorem 4.1. *We use the above notation. There exist matrices* $C \in M(N, r, \widehat{\mathcal{E}}(\mathfrak{n})_\Lambda)$ *and* $D \in M(r, N, \widehat{\mathcal{E}}(\mathfrak{n})_\Lambda)$ *such that the following conditions hold*:

- *There exists a linear map* $Q\colon \mathfrak{a} \to M(r, U(\mathfrak{n})_\Lambda)$ *such that*

$$\begin{cases} Hv = Q(H)v \textit{ for all } H \in \mathfrak{a}, \\ Q(H) - \overline{Q}(H) \in M(r, (\mathfrak{n}U(\mathfrak{n}))_\Lambda) \textit{ for all } H \in \mathfrak{a}, \\ \textit{if } \lambda_i - \lambda_j \notin \Lambda^+ \textit{ then } Q(H)_{ij} = 0 \textit{ for all } H \in \mathfrak{a}, \\ \textit{if } \lambda_i - \lambda_j \in \Lambda^+ \textit{ then } [H', Q(H)_{ij}] = (\lambda_i - \lambda_j)(H')Q(H)_{ij} \textit{ for all } H, H' \in \mathfrak{a}, \\ \textit{if } i > j \textit{ then } Q(H)_{ij} = 0 \textit{ for all } H \in \mathfrak{a}. \end{cases}$$

  where $v = Du$.

- *We have* $u = CDu$.
- *We have* $C - \overline{C} \in M(N, r, (\mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))_\Lambda)$ *and* $D - \overline{D} \in M(r, N, (\mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))_\Lambda)$.

Remark 4.2. Assume that Theorem 4.1 holds.

(1) Fix $H \in \mathfrak{a}$. Then $Q(H)_{ii} - \lambda_i(H) = Q(H)_{ii} - \overline{Q}(H)_{ii} \in (\mathfrak{n}U(\mathfrak{n}))_\Lambda$. However we have $[H', Q(H)_{ii} - \lambda_i(H)] = [H', Q(H)_{ii}] = (\lambda_i - \lambda_i)(H')Q(H)_{ii} = 0$ for all $H' \in \mathfrak{a}$. Hence we have $Q(H)_{ii} = \lambda_i(H)$.

(2) We can prove $(H - \lambda_i(H))^{r-i+1} v_i = 0$ for all $H \in \mathfrak{a}$ by backward induction on i. If $i = r$, then the claim follows from $Hv = Q(H)v$ and $Q(H)_{ij} = 0$ for $i > j$. Assume $i < r$. Then from $Hv = Q(H)v$ and (1), we have $(H - \lambda_i(H))v_i = \sum_{j>i} Q(H)_{ij} v_j$. Since $[H, Q(H)_{ij}] = (\lambda_i - \lambda_j)(H)Q(H)_{ij}$, we have $(H - \lambda_i(H))^{r-i+1} = \sum_{j>i} Q(H)_{ij}(H - \lambda_j(H))^{r-i} v_j = 0$.

For the proof we need some lemmas. Put $w = \overline{D}u \in \widehat{\mathbb{X}}^r$.

Lemma 4.3. *For $H \in \mathfrak{a}$ there exists a matrix $R \in M(r, (\mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))_\Lambda)$ such that $Hw = (\overline{Q}(H) + R)w$ in $\widehat{\mathbb{X}}^r$.*

Proof. Since $w \pmod{((\mathfrak{n}U(\mathfrak{n}))_\Lambda \mathbb{X})^r} = \overline{v}$, we have $Hw - \overline{Q}(H)w \in ((\mathfrak{n}U(\mathfrak{n}))_\Lambda \mathbb{X})^r$. Hence there exists a matrix $R_1 \in M(N, r, (\mathfrak{n}U(\mathfrak{n}))_\Lambda)$ such that $Hw - \overline{Q}(H)w = R_1 u$. Similarly we can choose a matrix $S \in M(N, (\mathfrak{n}U(\mathfrak{n}))_\Lambda)$ which satisfies $u = \overline{C}w + Su$. Put $R = R_1(1-S)^{-1}\overline{C}$. Then $(H - \overline{Q}(H) - R)w = R_1 u - R_1(1-S)^{-1}\overline{C}w = 0$. □

Lemma 4.4. *Let $\lambda \in \mathbb{C}$ and $Q_0, R_0 \in M(r, \mathbb{C})$. Assume that Q_0 is an upper triangular matrix. Then there exist matrices $L_0, T_0 \in M(r, \mathbb{C})$ such that*

$$\begin{cases} \lambda L_0 - [Q_0, L_0] = T_0 + R_0, \\ \textit{if } (Q_0)_{ii} - (Q_0)_{jj} \neq \lambda \textit{ then } (T_0)_{ij} = 0. \end{cases}$$

Proof. Since $(Q_0)_{ij} = 0$ for $i > j$, we have

$$\begin{aligned} (\lambda L_0 - [Q_0, L_0])_{ij} &= \lambda (L_0)_{ij} - \sum_{k=1}^{r} ((Q_0)_{ik}(L_0)_{kj} - (L_0)_{ik}(Q_0)_{kj}) \\ &= \lambda (L_0)_{ij} - \sum_{k=i}^{r} (Q_0)_{ik}(L_0)_{kj} + \sum_{k=1}^{j} (L_0)_{ik}(Q_0)_{kj} \\ &= (\lambda - ((Q_0)_{ii} - (Q_0)_{jj}))(L_0)_{ij} - \sum_{k=i+1}^{r} (Q_0)_{ik}(L_0)_{kj} \\ &\quad + \sum_{k=1}^{j-1} (L_0)_{ik}(Q_0)_{kj}. \end{aligned}$$

Hence we can choose $(L_0)_{ij}$ and $(T_0)_{ij}$ inductively on $(j - i)$. □

Lemma 4.5. *Let H be an element of $\mathfrak{a}$ such that $\alpha(H) > 0$ for all $\alpha \in \Sigma^+$. Let $Q_0 \in M(r, \mathbb{C})$, $R \in M(r, (\mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))_\Lambda)$. Assume $(Q_0)_{ij} = 0$ for $i > j$. Set $\mathcal{L}^{++} = \{\lambda(H) \mid \lambda \in \Lambda^{++}\}$. Then there exist matrices $L \in M(r, \widehat{\mathcal{E}}(\mathfrak{n})_\Lambda)$ and $T \in M(r, (\mathfrak{n}U(\mathfrak{n}))_\Lambda)$ such that*

$$\begin{cases} L \in 1_r + M(r, (\mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))_\Lambda), \\ (H1_r - Q_0 - T)L = L(H1_r - Q_0 - R), \\ \textit{if } (Q_0)_{ii} - (Q_0)_{jj} \notin \mathcal{L}^{++} \textit{ then } T_{ij} = 0, \\ \textit{if } (Q_0)_{ii} - (Q_0)_{jj} \in \mathcal{L}^{++} \textit{ then } [H, T_{ij}] = ((Q_0)_{ii} - (Q_0)_{jj})T_{ij}. \end{cases}$$

Proof. Set $\mathcal{L} = \{\lambda(H) \mid \lambda \in \Lambda\}$ and $\mathcal{L}^+ = \{\lambda(H) \mid \lambda \in \Lambda^+\}$. Put $f(\mathbf{n}) = \sum_i \mathbf{n}_i\beta_i$ for $\mathbf{n} = (\mathbf{n}_i) \in \mathbb{Z}^m$. Set $\widetilde{\Lambda} = \{\mathbf{n} \in \mathbb{Z}^m_{\geq 0} \mid f(\mathbf{n}) \in \Lambda\}$. We define an order on $\widetilde{\Lambda}$ by $\mathbf{n} < \mathbf{n}'$ if and only if $f(\mathbf{n}) < f(\mathbf{n}')$.

By Corollary 3.4, we can write $R = \sum_{\mathbf{n}\in\widetilde{\Lambda}} R_{\mathbf{n}} E^{\mathbf{n}}$ where $R_{\mathbf{n}} \in M(r, \mathbb{C})$. We have $R_{\mathbf{0}} = 0$ where $\mathbf{0} = (0, \dots, 0) \in \widetilde{\Lambda}$ since $R \in M(r, (\mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))_\Lambda)$. We have to show the existence of L and T. Write $L = 1_r + \sum_{\mathbf{n}\in\widetilde{\Lambda}} L_{\mathbf{n}} E^{\mathbf{n}}$ and $T = \sum_{\mathbf{n}\in\widetilde{\Lambda}} T_{\mathbf{n}} E^{\mathbf{n}}$. By the conditions of L and T, we have $T_{\mathbf{0}} = L_{\mathbf{0}} = 0$. Then $(H1_r - Q_0 - T)L = L(H1_r - Q_0 - R)$ is equivalent to

$$f(\mathbf{n})(H)L_{\mathbf{n}} - [Q_0, L_{\mathbf{n}}] = T_{\mathbf{n}} + S_{\mathbf{n}} - R_{\mathbf{n}},$$

where $S_{\mathbf{n}}$ is defined by

$$\sum_{\mathbf{n}\in\widetilde{\Lambda}} S_{\mathbf{n}} E^{\mathbf{n}} = T(L - 1_r) - (L - 1_r)R.$$

By Proposition 3.6, the above equation is equivalent to

$$\sum_{f(\mathbf{n})=\mu} S_{\mathbf{n}} E^{\mathbf{n}} = \sum_{f(\mathbf{k})+f(\mathbf{l})=\mu} T_{\mathbf{k}} L_{\mathbf{l}} E^{\mathbf{k}} E^{\mathbf{l}} - \sum_{f(\mathbf{k})+f(\mathbf{l})=\mu} L_{\mathbf{k}} R_{\mathbf{l}} E^{\mathbf{k}} E^{\mathbf{l}}$$

for all $\mu \in \mathfrak{a}^*$. $S_{\mathbf{n}}$ can be defined from the data $\{T_{\mathbf{k}} \mid \mathbf{k} < \mathbf{n}\}$, $\{L_{\mathbf{k}} \mid \mathbf{k} < \mathbf{n}\}$ and $\{R_{\mathbf{k}} \mid \mathbf{k} < \mathbf{n}\}$.

Now we prove the existence of L and T. We choose $L_{\mathbf{n}}$ and $T_{\mathbf{n}}$ which satisfy

$$\begin{cases} L_{\mathbf{0}} = 0, \quad T_{\mathbf{0}} = 0, \\ f(\mathbf{n})(H)L_{\mathbf{n}} - [Q_0, L_{\mathbf{n}}] = T_{\mathbf{n}} + S_{\mathbf{n}} - R_{\mathbf{n}}, \\ \text{if } (Q_0)_{ii} - (Q_0)_{jj} \neq f(\mathbf{n})(H) \text{ then } (T_{\mathbf{n}})_{ij} = 0. \end{cases}$$

By Lemma 4.4, we can choose such $L_{\mathbf{n}}$ and $T_{\mathbf{n}}$ inductively. Put $L = 1_r + \sum_{\mathbf{n}\in\widetilde{\Lambda}} L_{\mathbf{n}} E^{\mathbf{n}}$ and $T = \sum_{\mathbf{n}\in\widetilde{\Lambda}} T_{\mathbf{n}} E^{\mathbf{n}}$. Since $f(\mathbf{n})(H) \neq (Q_0)_{ii} - (Q_0)_{jj}$ implies $(T_{\mathbf{n}})_{ij} = 0$, we have

$$\begin{aligned}[H, T_{ij}] &= \sum_{\mathbf{n}\in\widetilde{\Lambda}} (f(\mathbf{n})(H))(T_{\mathbf{n}})_{ij} E^{\mathbf{n}} \\ &= \sum_{\mathbf{n}\in\widetilde{\Lambda},\ f(\mathbf{n})(H)=(Q_0)_{ii}-(Q_0)_{jj}} (f(\mathbf{n})(H))(T_{\mathbf{n}})_{ij} E^{\mathbf{n}} = ((Q_0)_{ii} - (Q_0)_{jj})T_{ij},\end{aligned}$$

Hence L and T satisfy the conditions of the lemma. □

Proof of Theorem 4.1. We can choose positive integers $\zeta = (\zeta_i) \in \mathbb{Z}^l_{>0}$

such that

$$\{\alpha \in \Lambda^{++} \mid \alpha(\textstyle\sum_s \zeta_s H_s) = (\lambda_i - \lambda_j)(\sum_s \zeta_s H_s)\}$$
$$= \begin{cases} \{\lambda_i - \lambda_j\} & (\lambda_i - \lambda_j \in \Lambda^{++}), \\ \emptyset & (\lambda_i - \lambda_j \notin \Lambda^{++}). \end{cases}$$

The existence of such ζ is shown by Oshima [Osh84, Lemma 2.3]. Set $X = \sum_s \zeta_s H_s$. Notice that $(\lambda_i - \lambda_j)(X) \in \{\mu(X) \mid \mu \in \Lambda^{++}\}$ if and only if $\lambda_i - \lambda_j \in \Lambda^{++}$. By Lemma 4.5, there exist $T \in M(r, (\mathfrak{n}U(\mathfrak{n}))_\Lambda)$ and $L \in M(r, \widehat{\mathcal{E}}(\mathfrak{n})_\Lambda)$ such that

$$\begin{cases} L \in 1_r + M(r, (\mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))_\Lambda), \\ (X1_r - \overline{Q}(X) - T)L = L(X1_r - \overline{Q}(X) - R), \\ \text{if } \lambda_i - \lambda_j \notin \Lambda^{++} \text{ then } T_{ij} = 0, \\ \text{if } \lambda_i - \lambda_j \in \Lambda^{++} \text{ then } [X, T_{ij}] = (\lambda_i - \lambda_j)(X)T_{ij}. \end{cases}$$

Let $S \in M(N, (\mathfrak{n}U(\mathfrak{n}))_\Lambda)$ such that $u = \overline{C}w + Su$. Put $C = (1-S)^{-1}\overline{C}L^{-1}$, $D = L\overline{D}$ and $v = (v_1, v_2, \ldots, v_r) = Du = Lw$. Then $CDu = (1-S)^{-1}\overline{C}L^{-1}L\overline{D}u = (1-S)^{-1}\overline{C}w = u$. Moreover, we have $(X1_r - \overline{Q}(X) - T)v = 0$.

Assume $i < j$. Then $\operatorname{Re}\lambda_i - \operatorname{Re}\lambda_j \leq 0$. Hence $\lambda_i - \lambda_j \notin \Lambda^{++}$. So we have $T_{ij} = 0$. From this fact and $(X1_r - \overline{Q}(X) - T)v = 0$, we have $(X - \lambda_i(X))^{r-i+1}v_i = 0$ (see Remark 4.2 (2)).

We construct the map Q. Fix a positive integer k such that $1 \leq k \leq l$. We can choose a matrix $R_k \in M(r, (\mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n}))_\Lambda)$ such that $H_k w = (\overline{Q}(H_k) + R_k)w$ by Lemma 4.3. Set $T_k = H_k 1_r - \overline{Q}(H_k) - L(H_k 1_r - \overline{Q}(H_k) - R_k)L^{-1}$. Then we have $(H_k 1_r - \overline{Q}(H_k) - T_k)v = 0$, i.e.,

$$H_k v_i - \sum_{j=1}^{r} \overline{Q}(H_k)_{ij} v_j - \sum_{j=1}^{r} (T_k)_{ij} v_j = 0$$

for each $i = 1, 2, \ldots, r$. By Corollary 3.9, we have

$$H_k v_i - \sum_{j=1}^{r} \overline{Q}(H_k)_{ij} v_j - \sum_{j=1}^{r} (T_k)_{ij}^{(X, (\lambda_i - \lambda_j)(X))} v_j = 0. \tag{4.1}$$

Define $T'_k \in M(r, (\mathfrak{n}U(\mathfrak{n}))_\Lambda)$ by $(T'_k)_{ij} = (T_k)_{ij}^{(X, (\lambda_i - \lambda_j)(X))}$. Since $(T_k)_{ij}^{(\mu)} = 0$ for $\mu \notin \Lambda^{++}$, we have

$$\begin{aligned}(T_k)_{ij}^{(X,(\lambda_i-\lambda_j)(X))} &= \sum_{\mu\in\Lambda^{++},\ \mu(X)=(\lambda_i-\lambda_j)(X)} (T_k)_{ij}^{(\mu)} \\ &= \begin{cases}(T_k)_{ij}^{(\lambda_i-\lambda_j)} & (\lambda_i-\lambda_j\in\Lambda^{++}),\\ 0 & (\lambda_i-\lambda_j\notin\Lambda^{++})\end{cases}\end{aligned}$$

by the condition on ζ. In particular $[H,(T'_k)_{ij}] = (\lambda_i-\lambda_j)(H)(T'_k)_{ij}$ for all $H\in\mathfrak{a}$.

Put $Q(\sum x_sH_s) = \overline{Q}(\sum x_sH_s) + \sum x_sT'_s$ for $(x_1,x_2,\dots,x_l)\in\mathbb{C}^l$. Then we have $(H-Q(H))v = 0$ by (4.1). Recall that we have $T_{ij}=0$ for $i<j$. Hence we have $(T'_s)_{ij}=0$ for $i<j$. Moreover, by the condition on $\overline{Q}$, we have $\overline{Q}(H)_{ij}=0$ for $i<j$ and $H\in\mathfrak{a}$. Hence we have $Q(H)_{ij}=0$ for $i<j$ and $H\in\mathfrak{a}$. Therefore Q satisfies the conditions of the theorem. ☐

We apply Theorem 4.1 to a principal series representation. First we define the principal series representation which we will study.

Set $\rho=\sum_{\alpha\in\Sigma^+}(\dim\mathfrak{g}_\alpha/2)\alpha$. From the Iwasawa decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$ we have the decomposition into the direct sum

$$U(\mathfrak{g}) = \mathfrak{n}U(\mathfrak{a}\oplus\mathfrak{n})\oplus U(\mathfrak{a})\oplus U(\mathfrak{g})\mathfrak{k}.$$

Let χ_1 be the projection of $U(\mathfrak{g})$ to $U(\mathfrak{a})$ with respect to this decomposition and χ_2 an algebra automorphism of $U(\mathfrak{a})$ defined by $\chi_2(H)=H-\rho(H)$ for $H\in\mathfrak{a}$. We define $\chi\colon U(\mathfrak{g})^{\mathfrak{k}}\to U(\mathfrak{a})$ by $\chi=\chi_2\circ\chi_1$ where $U(\mathfrak{g})^{\mathfrak{k}}=\{u\in U(\mathfrak{g})\mid Xu=uX \text{ for all } X\in\mathfrak{k}\}$. Let $U(\mathfrak{a})^W$ be a subalgebra of $U(\mathfrak{a})$ consisting of W-invariant elements. It is known that the image of $U(\mathfrak{g})^{\mathfrak{k}}$ under χ is contained in $U(\mathfrak{a})^W$ and induces an isomorphism of algebra $U(\mathfrak{g})^{\mathfrak{k}}/(U(\mathfrak{g})^{\mathfrak{k}}\cap U(\mathfrak{g})\mathfrak{k})\to U(\mathfrak{a})^W$. For details, see Helgason [Hel62, Chapter X, §6.3].

Fix $\lambda\in\mathfrak{a}^*$. We can define an algebra homomorphism $U(\mathfrak{a})\to\mathbb{C}$ by $H\mapsto\lambda(H)$ for $H\in\mathfrak{a}$. We denote this map by the same letter λ. Put $\chi_\lambda=\lambda\circ\chi$. Define the $U(\mathfrak{g})$-module $I(\lambda)$ by

$$I(\lambda)=U(\mathfrak{g})/(U(\mathfrak{g})\operatorname{Ker}\chi_\lambda+U(\mathfrak{g})\mathfrak{k}).$$

By a result of Kostant [Kos75, Theorem 2.10.3], $I(\lambda)$ is isomorphic to the principal series representation generated by the trivial K-type with an infinitesimal character λ (see p. 1194). Set $u_\lambda = 1 \mod (U(\mathfrak{g})\operatorname{Ker}\chi_\lambda+U(\mathfrak{g})\mathfrak{k})\in I(\lambda)$ and $I(\lambda)_0=U(\mathfrak{g})_{2\mathcal{P}}u_\lambda$. Before applying Theorem 4.1 to $I(\lambda)_0$, we give some lemmas.

Lemma 4.6. *Let $u\in U(\mathfrak{g})_\mu$ where $\mu\in\mathfrak{a}^*$. Then there exists an element $x\in U(\mathfrak{g})\mathfrak{k}$ such that $u+x\in U(\mathfrak{a}\oplus\mathfrak{n})_{\mu+2\mathcal{P}}$.*

Proof. In general, for a Lie algebra $\mathfrak{c}$, let $\{U_n(\mathfrak{c})\}_{n\in\mathbb{Z}_{\geq 0}}$ be the standard filtration of $U(\mathfrak{c})$. Let n be the smallest integer such that $u \in U_n(\mathfrak{g})$. We will prove the existence of x by induction on n.

If $n = 0$ then the lemma is obvious. Assume $n > 0$. Set $\overline{\mathfrak{n}} = \theta(\mathfrak{n})$. First assume that $u \in U_n(\overline{\mathfrak{n}})_\mu$. We may assume that there exist a restricted root $\alpha \in \Sigma^+$, an element $u_0 \in U_{n-1}(\overline{\mathfrak{n}})_{\mu+\alpha}$ and a vector $E_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $u = u_0 E_{-\alpha}$. Set $E_\alpha = \theta(E_{-\alpha})$, $u_1 = u_0 E_\alpha$, $u_2 = E_\alpha u_0$ and $u_3 = u_1 - u_2$. Then $u + u_2 + u_3 = u + u_1 \in U(\mathfrak{g})\mathfrak{k}$, $u_1, u_2 \in U(\mathfrak{g})_{\mu+2\alpha}$ and $u_3 \in U_{n-1}(\mathfrak{g})_{\mu+2\alpha}$. Using the induction hypothesis we can choose an element $u_5 \in U(\mathfrak{a}\oplus\mathfrak{n})_{\mu+2\mathcal{P}}$ such that $u_3 - u_5 \in U(\mathfrak{g})\mathfrak{k}$. Again by the induction hypothesis we can choose an element $u_6 \in U(\mathfrak{a}\oplus\mathfrak{n})_{\mu+\alpha+2\mathcal{P}}$ such that $u_0 - u_6 \in U(\mathfrak{g})\mathfrak{k}$. Then $u + u_5 + E_\alpha u_6 \in U(\mathfrak{g})\mathfrak{k}$, $u_5 \in U(\mathfrak{a}\oplus\mathfrak{n})_{\mu+2\mathcal{P}}$ and $E_\alpha u_6 \in U(\mathfrak{a}\oplus\mathfrak{n})_{\mu+2\alpha+2\mathcal{P}}$.

Now assume that $u \in U_n(\mathfrak{g})_\mu$. Since $U_n(\mathfrak{g})_\mu = \sum_{k=0}^n \bigoplus_{\mu_1} U_{n-k}(\mathfrak{a}\oplus\mathfrak{n}\oplus\mathfrak{m})_{\mu-\mu_1} \otimes U_k(\overline{\mathfrak{n}})_{\mu_1}$, we may assume that $u = u'u''$ where $u' \in U_{n-k}(\mathfrak{a}\oplus\mathfrak{n}\oplus\mathfrak{m})_{\mu-\mu_1}$ and $u'' \in U_k(\overline{\mathfrak{n}})_{\mu_1}$. Take $x \in U(\mathfrak{g})\mathfrak{k}$ such that $x + u'' \in U(\mathfrak{a}\oplus\mathfrak{n})_{\mu_1+2\mathcal{P}}$ and put $x' = u'x \in U(\mathfrak{g})\mathfrak{k}$. Then we have $u + x' \in U(\mathfrak{a}\oplus\mathfrak{n}\oplus\mathfrak{m})_{\mu+2\mathcal{P}} = U(\mathfrak{a}\oplus\mathfrak{n})_{\mu+2\mathcal{P}} \oplus U(\mathfrak{a}\oplus\mathfrak{n})_{\mu+2\mathcal{P}}\mathfrak{m}$. Take $x'' \in U(\mathfrak{a}\oplus\mathfrak{n})_{\mu+2\mathcal{P}}\mathfrak{m}$ such that $u + x' - x'' \in U(\mathfrak{a}\oplus\mathfrak{n})_{\mu+2\mathcal{P}}$. Since $\mathfrak{m} \subset \mathfrak{k}$, $x' - x'' \in U(\mathfrak{g})\mathfrak{k} + U(\mathfrak{a}\oplus\mathfrak{n})_{\mu+2\mathcal{P}}\mathfrak{m} = U(\mathfrak{g})\mathfrak{k}$. Hence we have the lemma. □

Lemma 4.7. *The following formulae hold.*

(1) $U(\mathfrak{g})^{\mathfrak{k}} \subset U(\mathfrak{g})_{2\mathcal{P}}$. *In particular, we have* $\operatorname{Ker}\chi_\lambda \subset U(\mathfrak{g})_{2\mathcal{P}}$.

(2) $U(\mathfrak{a}\oplus\mathfrak{n}) \cap (\operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k}) \subset U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}} \cap (\operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k})$.

(3) $U(\mathfrak{a}\oplus\mathfrak{n}) \cap (U(\mathfrak{a}\oplus\mathfrak{n})\operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k}) \subset U(\mathfrak{a}\oplus\mathfrak{n})(U(\mathfrak{a}\oplus\mathfrak{n}) \cap (\operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k}))$.

(4) $U(\mathfrak{a}\oplus\mathfrak{n}) \cap (U(\mathfrak{g})\operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k}) = U(\mathfrak{a}\oplus\mathfrak{n})((U(\mathfrak{g})\operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k}) \cap U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}})$.

Proof. (1) Take a connected Lie group G such that Lie algebra of G is $\mathfrak{g}_0$ and G has a complexification. Let K be its maximal compact subgroup such that $\operatorname{Lie}(K) = \mathfrak{k}_0$. Since K is connected, $U(\mathfrak{g})^{\mathfrak{k}} = U(\mathfrak{g})^K = \{u \in U(\mathfrak{g}) \mid \operatorname{Ad}(k)u = u \text{ for all } k \in K\}$. Fix a maximal abelian subspace $\mathfrak{t}_0$ of $\mathfrak{m}_0$. Let K_{split} and A_{split} be the analytic subgroups with Lie algebras given as the intersections of $\mathfrak{k}_0$ and $\mathfrak{a}_0$ with $[Z_{\mathfrak{g}_0}(\mathfrak{t}_0), Z_{\mathfrak{g}_0}(\mathfrak{t}_0)]$ where $Z_{\mathfrak{g}_0}(\mathfrak{t}_0)$ is the centralizer of $\mathfrak{t}_0$ in $\mathfrak{g}_0$. Let F be the centralizer of A_{split} in K_{split}. Since $F \subset K$, we have $U(\mathfrak{g})^K \subset U(\mathfrak{g})^F$. On the other hand, we have $U(\mathfrak{g})^F \subset U(\mathfrak{g})_{2\mathcal{P}}$ (See Knapp [Kna02, Theorem 7.55] and Lepowsky [Lep75, Proposition 6.1, Proposition 6.4]). Hence (1) follows.

(2) Let $u \in \operatorname{Ker}\chi_\lambda$ and $x \in U(\mathfrak{g})^{\mathfrak{k}}$ such that $u + x \in U(\mathfrak{a} \oplus \mathfrak{n})$. We can write $u = \sum_\mu u_\mu$ where $u_\mu \in U(\mathfrak{g})_\mu$. By (1), we have $u_\mu = 0$ for $\mu \notin 2\mathcal{P}$. Let $\mu \in 2\mathcal{P}$. By Lemma 4.6, there exists an element $y_\mu \in U(\mathfrak{g})^{\mathfrak{k}}$ such that $u_\mu + y_\mu \in U(\mathfrak{a}\oplus\mathfrak{n})_{\mu+2\mathcal{P}} = U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}}$. Put $y = \sum_\mu y_\mu$. Then $u+y \in U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}}$. Since $u+y \in U(\mathfrak{a}\oplus\mathfrak{n})$ and $x, y \in U(\mathfrak{g})^{\mathfrak{k}}$ we have $y = x$ by the Poincaré-Birkhoff-Witt theorem. Hence we have $u + x = u + y \in U(\mathfrak{a} \oplus \mathfrak{n})_{2\mathcal{P}}$.

(3) Let $\sum_i x_i u_i \in U(\mathfrak{a} \oplus \mathfrak{n}) \operatorname{Ker}\chi_\lambda$ where $x_i \in U(\mathfrak{a} \oplus \mathfrak{n})$ and $u_i \in \operatorname{Ker}\chi_\lambda$. We write $u_i = \sum_j z_j^{(i)} v_j^{(i)}$ where $z_j^{(i)} \in U(\mathfrak{a}\oplus\mathfrak{n})$ and $v_j^{(i)} \in U(\mathfrak{k})$. Let $y \in U(\mathfrak{g})^{\mathfrak{k}}$ and assume $\sum_i x_i u_i + y \in U(\mathfrak{a} \oplus \mathfrak{n})$. By the Poincaré-Birkhoff-Witt theorem, $\sum_i x_i u_i + y = \sum_{i,j} x_i z_j^{(i)} v_{j,0}^{(i)}$ where $v_{j,0}^{(i)}$ is the constant term of $v_j^{(i)}$. Since $u_i + \sum_j z_j^{(i)}(v_{j,0}^{(i)} - v_j^{(i)}) = \sum_j z_j^{(i)} v_{j,0}^{(i)} \in U(\mathfrak{a} \oplus \mathfrak{n})$, we have $\sum_i x_i u_i + y = \sum_i x_i(u_i + \sum_j z_j^{(i)}(v_{j,0}^{(i)} - v_j^{(i)})) \in U(\mathfrak{a}\oplus\mathfrak{n})(U(\mathfrak{a}\oplus\mathfrak{n}) \cap (\operatorname{Ker}\chi_\lambda + U(\mathfrak{g})^{\mathfrak{k}}))$.

(4) Since $\operatorname{Ker}\chi_\lambda \subset U(\mathfrak{g})^{\mathfrak{k}}$, we have

$$U(\mathfrak{g}) \operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k} = U(\mathfrak{a}\oplus\mathfrak{n})(\operatorname{Ker}\chi_\lambda)U(\mathfrak{k}) + U(\mathfrak{g})\mathfrak{k} = U(\mathfrak{a}\oplus\mathfrak{n}) \operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k}.$$

By (2) and (3), we have

$$\begin{aligned} &U(\mathfrak{a} \oplus \mathfrak{n}) \cap (U(\mathfrak{g}) \operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k}) \\ &\qquad = U(\mathfrak{a} \oplus \mathfrak{n}) \cap (U(\mathfrak{a} \oplus \mathfrak{n}) \operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k}) \\ &\qquad \subset U(\mathfrak{a} \oplus \mathfrak{n})(U(\mathfrak{a} \oplus \mathfrak{n})_{2\mathcal{P}} \cap (\operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k})) \\ &\qquad \subset U(\mathfrak{a} \oplus \mathfrak{n})((U(\mathfrak{g}) \operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k}) \cap U(\mathfrak{a} \oplus \mathfrak{n})_{2\mathcal{P}}). \end{aligned}$$

This implies (4). □

Lemma 4.8.

(1) *We have* $I(\lambda)_0 = U(\mathfrak{a} \oplus \mathfrak{n})_{2\mathcal{P}} u_\lambda$.

(2) *The map* $p \otimes u \mapsto pu$ *induces an isomorphism* $U(\mathfrak{n}) \otimes_{U(\mathfrak{n})_{2\mathcal{P}}} I(\lambda)_0 \simeq I(\lambda)$.

Proof. (1) Since $\mathfrak{k}u_\lambda = 0$, this is obvious from Lemma 4.6.
(2) Let $I = U(\mathfrak{a}\oplus\mathfrak{n}) \cap (U(\mathfrak{g}) \operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k})$. We have $U(\mathfrak{a}\oplus\mathfrak{n}) \otimes_{U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}}} \mathbb{X} = U(\mathfrak{n}) \otimes_{U(\mathfrak{n})_{2\mathcal{P}}} \mathbb{X}$ for any $U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}}$-module $\mathbb{X}$ since $U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}} = U(\mathfrak{a}) \otimes U(\mathfrak{n})_{2\mathcal{P}}$.

By (1), we have $I(\lambda)_0 = U(\mathfrak{a} \oplus \mathfrak{n})_{2\mathcal{P}}/(I \cap U(\mathfrak{a} \oplus \mathfrak{n})_{2\mathcal{P}})$. Hence

$$\begin{aligned} U(\mathfrak{n}) \otimes_{U(\mathfrak{n})_{2\mathcal{P}}} I(\lambda)_0 &= U(\mathfrak{a} \oplus \mathfrak{n}) \otimes_{U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}}} I(\lambda)_0 \\ &= U(\mathfrak{a} \oplus \mathfrak{n}) \otimes_{U(\mathfrak{a}\oplus\mathfrak{n})_{2\mathcal{P}}} (U(\mathfrak{a} \oplus \mathfrak{n})_{2\mathcal{P}}/(I \cap U(\mathfrak{a} \oplus \mathfrak{n})_{2\mathcal{P}})) \\ &= U(\mathfrak{a} \oplus \mathfrak{n})/(U(\mathfrak{a} \oplus \mathfrak{n})(I \cap U(\mathfrak{a} \oplus \mathfrak{n})_{2\mathcal{P}})) \\ &= U(\mathfrak{a} \oplus \mathfrak{n})/I \\ &= I(\lambda) \end{aligned}$$

by Lemma 4.7 (4). □

Lemma 4.9. *Let* $\{U_n(\mathfrak{n})\}_{n\in\mathbb{Z}_{\geq 0}}$ *be the standard filtration of* $U(\mathfrak{n})$ *and* $U_n(\mathfrak{n})_{2\mathcal{P}} = U_n(\mathfrak{n}) \cap U(\mathfrak{n})_{2\mathcal{P}}$. *Set* $U_{-1}(\mathfrak{n}) = U_{-1}(\mathfrak{n})_{2\mathcal{P}} = 0$, $R = \operatorname{Gr} U(\mathfrak{n})_{2\mathcal{P}} = \bigoplus_n U_n(\mathfrak{n})_{2\mathcal{P}}/U_{n-1}(\mathfrak{n})_{2\mathcal{P}}$ *and* $R' = \operatorname{Gr} U(\mathfrak{n}) = \bigoplus_n U_n(\mathfrak{n})/U_{n-1}(\mathfrak{n})$.

(1) R' *is a finitely generated* R*-module.*

(2) $U(\mathfrak{n})$ *is a finitely generated* $U(\mathfrak{n})_{2\mathcal{P}}$*-module.*

(3) $U(\mathfrak{n})_{2\mathcal{P}}$ *is right and left Noetherian.*

(4) $I(\lambda)_0$ *is a finitely generated* $U(\mathfrak{n})_{2\mathcal{P}}$*-module.*

Proof. (1) Let $\Gamma = \{E^{\varepsilon} \mid \varepsilon \in \{0,1\}^m\}$. We denote the principal symbol of $u \in U(\mathfrak{n})$ by $\sigma(u) \in R'$. Notice that if $u \in U(\mathfrak{n})_{2\mathcal{P}}$ then $\sigma(u)$ is the principal symbol of u as an element of $U(\mathfrak{n})_{2\mathcal{P}}$.

We will prove that $\{\sigma(E) \mid E \in \Gamma\}$ generates R' as an R-module. Let $x \in R'$. We may assume that x is homogeneous, so there exists an element $u \in U(\mathfrak{n})$ such that $x = \sigma(u)$. Moreover we may assume that there exist non-negative integers $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ such that $u = E^{\mathbf{p}}$. Choose $\varepsilon_i \in \{0,1\}$ such that $\varepsilon_i \equiv p_i \pmod 2$. Set $q_i = (p_i - \varepsilon_i)/2 \in \mathbb{Z}_{\geq 0}$, $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_m)$. Then we have $x = \sigma(E^{\mathbf{p}}) = \sigma(E^{2\mathbf{q}})\sigma(E^{\varepsilon})$. Since $\sigma(E^{2\mathbf{q}}) \in R$, this implies that $\{\sigma(E) \mid E \in \Gamma\}$ generates R' as an R-module.

(2) This is a direct consequence of (1).

(3) By the Poincaré-Birkhoff-Witt theorem, R' is isomorphic to a polynomial ring. In particular R' is Noetherian. By the theorem of Eakin-Nagata and (1), we have R is Noetherian. This implies (3).

(4) Since $I(\lambda)$ is a finite-length $(\mathfrak{g}, K)$-module, $I(\lambda)$ is a finitely generated $U(\mathfrak{n})$-module by a theorem of Casselman-Osborne [CO78, 2.3 Theorem]. Since $U(\mathfrak{n})$ is a finitely generated $U(\mathfrak{n})_{2\mathcal{P}}$-module, $I(\lambda)$ is a finitely generated $U(\mathfrak{n})_{2\mathcal{P}}$-module. Hence $I(\lambda)_0$ is a finitely generated $U(\mathfrak{n})_{2\mathcal{P}}$-module by (3). □

We enumerate $W = \{w_1, w_2, \ldots, w_r\}$ such that $\operatorname{Re} w_1\lambda \geq \operatorname{Re} w_2\lambda \geq \cdots \geq \operatorname{Re} w_r\lambda$.

Theorem 4.10. *There exist matrices* $A \in M(1, r, \widehat{\mathcal{E}}(\mathfrak{n})_{2\mathcal{P}})$ *and* $B \in M(r, 1, \widehat{\mathcal{E}}(\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{n})_{2\mathcal{P}})$ *such that* $v_\lambda = Bu_\lambda \in (\widehat{\mathcal{E}}(\mathfrak{g}, \mathfrak{n}) \otimes_{U(\mathfrak{g})} I(\lambda))^r$ *satisfies the following conditions*:

- *There exists a linear map* $Q\colon \mathfrak{a} \to M(r, U(\mathfrak{n})_{2\mathcal{P}})$ *such that*

$$\begin{cases} Hv_\lambda = Q(H)v_\lambda \text{ for all } H \in \mathfrak{a}, \\ Q(H)_{ii} = (\rho + w_i\lambda)(H) \text{ for all } H \in \mathfrak{a}, \\ \text{if } w_i\lambda - w_j\lambda \notin 2\mathcal{P}^+ \text{ then } Q(H)_{ij} = 0 \text{ for all } H \in \mathfrak{a}, \\ \text{if } w_i\lambda - w_j\lambda \in 2\mathcal{P}^+ \text{ then } [H', Q(H)_{ij}] \\ \qquad\qquad = (w_i\lambda - w_j\lambda)(H')Q(H)_{ij} \text{ for all } H, H' \in \mathfrak{a}, \\ \text{if } i > j \text{ then } Q(H)_{ij} = 0 \text{ for all } H \in \mathfrak{a}. \end{cases}$$

- *We have* $u_\lambda = Av_\lambda$.
- *Let* $(v_1, v_2, \ldots, v_r) = v_\lambda$. *Then* $\{v_i \pmod{\mathfrak{n}I(\lambda)}\}$ *is a basis of* $I(\lambda)/\mathfrak{n}I(\lambda)$.

Proof. Let $u_1, u_2, \ldots, u_N$ be generators of $I(\lambda)_0$ as a $U(\mathfrak{n})_{2\mathcal{P}}$-module. These are also generators of $I(\lambda)$ as a $U(\mathfrak{n})$-module by Lemma 4.8. We choose matrices $E = {}^t(E_1, E_2, \ldots, E_N) \in M(N, 1, U(\mathfrak{a} \oplus \mathfrak{n})_{2\mathcal{P}})$ and $F = (F_1, F_2, \ldots, F_N) \in M(1, N, U(\mathfrak{n})_{2\mathcal{P}})$ such that ${}^t(u_1, u_2, \ldots, u_N) = Eu_\lambda$ and $u_\lambda = F\,{}^t(u_1, u_2, \ldots, u_N)$.

Notice that $U(\mathfrak{n})_{2\mathcal{P}} + \mathfrak{n}U(\mathfrak{n}) = U(\mathfrak{n})$. By Lemma 4.8,

$$\begin{aligned} I(\lambda)/\mathfrak{n}I(\lambda) &= (U(\mathfrak{n})/\mathfrak{n}U(\mathfrak{n})) \otimes_{U(\mathfrak{n})} I(\lambda) \\ &= (U(\mathfrak{n})/\mathfrak{n}U(\mathfrak{n})) \otimes_{U(\mathfrak{n})} U(\mathfrak{n}) \otimes_{U(\mathfrak{n})_{2\mathcal{P}}} I(\lambda)_0 \\ &= (U(\mathfrak{n})/\mathfrak{n}U(\mathfrak{n})) \otimes_{U(\mathfrak{n})_{2\mathcal{P}}} I(\lambda)_0 \\ &= ((U(\mathfrak{n})_{2\mathcal{P}} + \mathfrak{n}U(\mathfrak{n}))/\mathfrak{n}U(\mathfrak{n})) \otimes_{U(\mathfrak{n})_{2\mathcal{P}}} I(\lambda)_0 \\ &= (U(\mathfrak{n})_{2\mathcal{P}}/(\mathfrak{n}U(\mathfrak{n}) \cap U(\mathfrak{n})_{2\mathcal{P}})) \otimes_{U(\mathfrak{n})_{2\mathcal{P}}} I(\lambda)_0 \\ &= (U(\mathfrak{n})_{2\mathcal{P}}/(\mathfrak{n}U(\mathfrak{n}))_{2\mathcal{P}}) \otimes_{U(\mathfrak{n})_{2\mathcal{P}}} I(\lambda)_0 \\ &= I(\lambda)_0/(\mathfrak{n}U(\mathfrak{n}))_{2\mathcal{P}} I(\lambda)_0. \end{aligned}$$

On the other hand,

$$\begin{aligned} I(\lambda)/\mathfrak{n}I(\lambda) &= U(\mathfrak{g})/(\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k}) \\ &= (\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{a}) + U(\mathfrak{g})\mathfrak{k})/(\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k}) \\ &= U(\mathfrak{a})/((\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k}) \cap U(\mathfrak{a})). \end{aligned}$$

Since χ induces an algebra isomorphism $U(\mathfrak{g})^{\mathfrak{k}}/(U(\mathfrak{g})^{\mathfrak{k}} \cap U(\mathfrak{g})\mathfrak{k}) \to U(\mathfrak{a})^W$, χ_1 also induces an isomorphism $\operatorname{Ker}\chi_\lambda/(\operatorname{Ker}\chi_\lambda \cap U(\mathfrak{g})\mathfrak{k}) \to \sum_{p \in U(\mathfrak{a})^W}(\chi_2^{-1}(p) - \lambda(p))$. Hence by the definition of χ_1, we have

$$(\mathfrak{n}U(\mathfrak{g}) + \operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k}) = \mathfrak{n}U(\mathfrak{g}) + \left(\sum_{p \in U(\mathfrak{a})^W}(\chi_2^{-1}(p) - \lambda(p))\right) + U(\mathfrak{g})\mathfrak{k}.$$

By the Poincaré-Birkhoff-Witt theorem, we have

$$(\mathfrak{n}U(\mathfrak{g}) + \operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k}) \cap U(\mathfrak{a}) = \sum_{p\in U(\mathfrak{a})^W} (\chi_2^{-1}(p) - \lambda(p)).$$

Since $\operatorname{Ker}\chi_\lambda \subset U(\mathfrak{g})^{\mathfrak{k}}$ and $\mathfrak{a}$ normalizes $\mathfrak{n}$, we have $U(\mathfrak{g})\operatorname{Ker}\chi_\lambda = (\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{a}) + U(\mathfrak{g})\mathfrak{k})\operatorname{Ker}\chi_\lambda \subset \mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{a})\operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k} = U(\mathfrak{a})(\mathfrak{n}U(\mathfrak{g}) + \operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k})$. Hence we have

$$(\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\operatorname{Ker}\chi_\lambda + U(\mathfrak{g})\mathfrak{k}) \cap U(\mathfrak{a}) = \sum_{p\in U(\mathfrak{a})^W} U(\mathfrak{a})(\chi_2^{-1}(p) - \lambda(p))$$

The set of eigenvalues of $H \in \mathfrak{a}$ on $U(\mathfrak{a})/(\sum_{p\in U(\mathfrak{a})^W} U(\mathfrak{a})(\chi_2^{-1}(p) - \lambda(p)))$ is $\{(\rho + w\lambda)(H) \mid w \in W\}$ with multiplicities [Osh88, Proposition 2.8].

Put $\mathbb{X} = I(\lambda)_0$, $\Lambda = 2\mathcal{P}$ and apply Theorem 4.1. We take matrices $C \in M(N, r, \widehat{\mathcal{E}}(\mathfrak{n})_{2\mathcal{P}})$ and $D \in M(r, N, \widehat{\mathcal{E}}(\mathfrak{n})_{2\mathcal{P}})$ such that the conditions of Theorem 4.1 hold. Put $A = FC$, $B = CE$. Then A and B satisfy the conditions of the theorem (for the second condition on Q, see Remark 4.2 (1)). □

Remark 4.11. For all $H \in \mathfrak{a}$ we have $(H - (w_i\lambda + \rho)(H))^{r-i+1}v_i = 0$. See Remark 4.2 (2).

§5. Structure of Jacquet Modules (Regular Case)

In this section we assume that λ is regular, i.e., $w\lambda \neq \lambda$ for all $w \in W \setminus \{e\}$. Let $r = \#W$ and $v_\lambda = (v_1, v_2, \ldots, v_r) \in (\widehat{\mathcal{E}}(\mathfrak{g}, \mathfrak{n}) \otimes_{U(\mathfrak{g})} I(\lambda))^r$ as in Theorem 4.10. Set $\mathcal{W}(i) = \{j \mid w_i\lambda - w_j\lambda \in 2\mathcal{P}^+\}$ for each $i = 1, 2, \ldots, r$.

Proposition 5.1. *The vector v_i is a lowest weight vector of $J(I(\lambda))/\sum_{j\in\mathcal{W}(i)} U(\mathfrak{g})v_j$, i.e., we have $Xv_i \in \sum_{j\in\mathcal{W}(i)} U(\mathfrak{g})v_j$ for all $X \in \theta(\mathfrak{n}) \oplus \mathfrak{m}$.*

Let $A = {}^t(A^{(1)}, A^{(2)}, \ldots, A^{(r)})$ be as in Theorem 4.10 and $\overline{A} = {}^t(\overline{A^{(1)}}, \overline{A^{(2)}}, \ldots, \overline{A^{(r)}})$ an element of $M(r, 1, \mathbb{C})$ such that $A^{(i)} - \overline{A^{(i)}} \in \mathfrak{n}\widehat{\mathcal{E}}(\mathfrak{n})$.

Lemma 5.2. *We have $\overline{A^{(i)}} \neq 0$ for each $i = 1, 2, \ldots, r$.*

Proof. Put $\overline{I(\lambda)} = I(\lambda)/\mathfrak{n}I(\lambda) \simeq \widehat{J}(I(\lambda))/\mathfrak{n}\widehat{J}(I(\lambda))$. Since u_λ generates $I(\lambda)$, there exists $\overline{B} = (\overline{B^{(1)}}, \overline{B^{(2)}}, \ldots, \overline{B^{(r)}}) \in U(\mathfrak{g})^r$ such that $v_i - \overline{B^{(i)}}u_\lambda \in \mathfrak{n}\widehat{J}(I(\lambda))$. From $\mathfrak{k}u_\lambda = 0$ and the Iwasawa decomposition, we may assume $\overline{B^{(i)}} \in U(\mathfrak{a})$.

Put $\overline{u_\lambda} = u_\lambda \pmod{\mathfrak{n}\widehat{J}(I(\lambda))}$ and $\overline{v_i} = v_i \pmod{\mathfrak{n}I(\lambda)}$. Then we have $\overline{v_i} = \sum_j \overline{A^{(j)}}\,\overline{B^{(i)}}\overline{v_j}$.

We can choose $H \in \mathfrak{a}$ such that $(\rho + w_i\lambda)(H) \neq (\rho + w_j\lambda)(H)$ for all $i \neq j$ since λ is regular. Then we have $(H - (\rho + w_i\lambda)(H))^{r-i+1}v_i = 0$ (Remark 4.11). Since $\overline{v_i}$ is a basis of $\overline{I(\lambda)}$, we have $(H - (\rho + w_i\lambda)(H))\overline{v_i} = 0$. Hence for all $i = 1, \dots, r$ there exists a polynomial f_i such that $f_i(H)$ is a projection $\overline{I(\lambda)} = \bigoplus_j \mathbb{C}\overline{v_j} \to \mathbb{C}\overline{v_i}$. Since $\mathfrak{a}$ is abelian, we have $v_i = f_i(H)v_i = \sum_j \overline{A^{(j)}}\,\overline{B^{(i)}} f_i(H)\overline{v_j} = \overline{A^{(i)}}\,\overline{B^{(i)}}\overline{v_i}$. This implies $\overline{A^{(i)}} \neq 0$. □

Proof of Proposition 5.1. Put $f(\mathbf{n}) = \sum_i \mathbf{n}_i\beta_i$ for $\mathbf{n} = (\mathbf{n}_i) \in \mathbb{Z}^m$. Set $\widetilde{\Lambda} = \{\mathbf{n} \in \mathbb{Z}^m_{\geq 0} \mid f(\mathbf{n}) \in 2\mathcal{P}\}$. We write $A^{(j)} = \sum_{\mathbf{n}\in\widetilde{\Lambda}} A^{(j)}_{\mathbf{n}} E^{\mathbf{n}}$ where $A^{(j)}_{\mathbf{n}} \in M(r, 1, \mathbb{C})$. Let $\alpha \in \Sigma^+$ and $E_\alpha \in \mathfrak{g}_\alpha$. Since $\mathfrak{k}u_\lambda = 0$, we have $(\theta(E_\alpha) + E_\alpha)u_\lambda = 0$. Hence $(\theta(E_\alpha) + E_\alpha)\sum_j \sum_{\mathbf{n}} A^{(j)}_{\mathbf{n}} E^{\mathbf{n}} v_j = 0$.

By applying Corollary 3.9 we have

$$\sum_{j=1}^{r} \left(\sum_{\mathbf{n}\in\widetilde{\Lambda}} A^{(j)}_{\mathbf{n}} (\theta(E_\alpha) + E_\alpha) E^{\mathbf{n}} \right)^{(w_i\lambda - w_j\lambda - \alpha)} v_j = 0$$

for $i = 1, 2, \dots, r$. On one hand, if $w_i\lambda - w_j\lambda \notin 2\mathcal{P}_+$, then

$$\left(\sum_{\mathbf{n}\in\widetilde{\Lambda}} A^{(j)}_{\mathbf{n}} (\theta(E_\alpha) + E_\alpha) E^{\mathbf{n}} \right)^{(w_i\lambda - w_j\lambda - \alpha)} = 0.$$

On the other hand,

$$\left(\sum_{\mathbf{n}\in\widetilde{\Lambda}} A^{(i)}_{\mathbf{n}} (\theta(E_\alpha) + E_\alpha) E^{\mathbf{n}} \right)^{(-\alpha)} = A^{(i)}_{\mathbf{0}} \theta(E_\alpha).$$

Hence we have

$$A^{(i)}_{\mathbf{0}} \theta(E_\alpha) v_i \in \sum_{j\in\mathcal{W}(i)} U(\mathfrak{g}) v_j.$$

Since $A^{(i)}_{\mathbf{0}} = \overline{A^{(i)}} \neq 0$, we have

$$\theta(E_\alpha) v_i \in \sum_{j\in\mathcal{W}(i)} U(\mathfrak{g}) v_j.$$

Next let X be an element of $\mathfrak{m}$. By Corollary 3.9, we have

$$\sum_{j=1}^{r} \left(\sum_{\mathbf{n}\in\widetilde{\Lambda}} A^{(j)}_{\mathbf{n}} X E^{\mathbf{n}} \right)^{(w_i\lambda - w_j\lambda)} v_j = 0.$$

We can prove $Xv_i \in \sum_{j\in\mathcal{W}(i)} U(\mathfrak{g}) v_j$ by the same argument. □

Corollary 5.3. *Put $V(\lambda) = \sum_i U(\mathfrak{g})v_i \subset \widehat{\mathcal{E}}(\mathfrak{g}, \mathfrak{n}) \otimes_{U(\mathfrak{g})} I(\lambda)$. Then we have $V(\lambda) = J(I(\lambda))$.*

Proof. By Theorem 4.1, we have $u_\lambda = Av_\lambda$. Since $I(\lambda)$ is generated by u_λ, the module $\widehat{\mathcal{E}}(\mathfrak{g}, \mathfrak{n}) \otimes_{U(\mathfrak{g})} I(\lambda)$ is generated by $v_1, \ldots, v_r$ as an $\widehat{\mathcal{E}}(\mathfrak{g}, \mathfrak{n})$-module.

By Proposition 5.1, $V(\lambda)$ is finitely generated as a $U(\mathfrak{n})$-module. By applying Proposition 3.3, we see that the map $\widehat{\mathcal{E}}(\mathfrak{g}, \mathfrak{n}) \otimes_{U(\mathfrak{g})} V(\lambda) \to \prod_{\mu \in \mathfrak{a}^*} V(\lambda)_\mu$ is bijective. Hence $\widehat{\mathcal{E}}(\mathfrak{g}, \mathfrak{n}) \otimes_{U(\mathfrak{g})} V(\lambda) \to \widehat{\mathcal{E}}(\mathfrak{g}, \mathfrak{n}) \otimes_{U(\mathfrak{g})} I(\lambda)$ is injective by Proposition 3.8. This map is also surjective since $v_1, v_2, \ldots, v_r$ are generators of $\widehat{\mathcal{E}}(\mathfrak{g}, \mathfrak{n}) \otimes_{U(\mathfrak{g})} I(\lambda)$.

We have $\widehat{\mathcal{E}}(\mathfrak{g}, \mathfrak{n}) \otimes_{U(\mathfrak{g})} V(\lambda) = \widehat{\mathcal{E}}(\mathfrak{g}, \mathfrak{n}) \otimes_{U(\mathfrak{g})} I(\lambda)$. Since $I(\lambda)$ and $V(\lambda)$ are finitely generated as $U(\mathfrak{n})$-modules, we have

$$\widehat{\mathcal{E}}(\mathfrak{g}, \mathfrak{n}) \otimes_{U(\mathfrak{g})} I(\lambda) = \widehat{J}(I(\lambda)),$$
$$\widehat{\mathcal{E}}(\mathfrak{g}, \mathfrak{n}) \otimes_{U(\mathfrak{g})} V(\lambda) = \widehat{J}(V(\lambda)),$$

by Proposition 3.1. Hence we have $J(I(\lambda)) = J(V(\lambda)) = V(\lambda)$ by Corollary 3.5. □

Recall the definition of generalized Verma modules. Set $\overline{\mathfrak{p}} = \theta(\mathfrak{p})$ and $\overline{\mathfrak{n}} = \theta(\mathfrak{n})$.

Definition 5.4 (Generalized Verma module)**.** Let $\mu \in \mathfrak{a}^*$. Define the one-dimensional representation $\mathbb{C}_{\rho+\mu}$ of $\overline{\mathfrak{p}}$ by $(X + Y + Z)v = (\rho + \mu)(Y)v$ for $X \in \mathfrak{m}$, $Y \in \mathfrak{a}$, $Z \in \overline{\mathfrak{n}}$, $v \in \mathbb{C}_{\rho+\mu}$. We define a $U(\mathfrak{g})$-module $M(\mu)$ by

$$M(\mu) = U(\mathfrak{g}) \otimes_{U(\overline{\mathfrak{p}})} \mathbb{C}_{\rho+\mu}.$$

This is called the generalized Verma module.

Set $V_i = \sum_{j \geq i} U(\mathfrak{g})v_j$. By the universality of tensor products, any $U(\overline{\mathfrak{p}})$-module homomorphism $\mathbb{C}_{\rho+\mu} \to V$ can be uniquely extended to the $U(\mathfrak{g})$-module homomorphism $M(\mu) \to V$ for a $U(\mathfrak{g})$-module V. In particular, we have a surjective $U(\mathfrak{g})$-module homomorphism $M(w_i\lambda) \to V_i/V_{i+1}$. We shall show that V_i/V_{i+1} is isomorphic to the generalized Verma module using the theory of characters.

Let G be a connected Lie group such that $\mathrm{Lie}(G) = \mathfrak{g}_0$, K its maximal compact subgroup with its Lie algebra $\mathfrak{k}_0$, P the parabolic subgroup whose Lie algebra is $\mathfrak{p}_0$ and $P = MAN$ the Langlands decomposition of P where Lie algebra of M (resp. A, N) is $\mathfrak{m}_0$ (resp. $\mathfrak{a}_0$, $\mathfrak{n}_0$).

Since $I(w\lambda) = I(\lambda)$ for $w \in W$, we may assume that $\operatorname{Re}\lambda$ is dominant, i.e., $\operatorname{Re}\lambda(H_i) \geq 0$ for each $i = 1, 2, \ldots, l$. By a result of Kostant [Kos75, Theorem 2.10.3] , $I(\lambda)$ is isomorphic to the space of K-finite vectors of the non-unitary principal series representation $\operatorname{Ind}_P^G(1 \otimes \lambda)$. The character of this representation is calculated by Harish-Chandra (See Knapp [Kna01, Proposition 10.18]). Before we state it, we fix some notation. Let $H = TA$ be a maximally split Cartan subgroup, $\mathfrak{h}_0$ its Lie algebra, $T = H \cap M$, Δ the root system of H, Δ^+ the positive system compatible with Σ^+, Δ_I the set of imaginary roots, $\Delta_I^+ = \Delta^+ \cap \Delta_I$ and ξ_α the one-dimensional representation of H whose derivation is α for $\alpha \in \mathfrak{h}^*$. Under these notation, the distribution character $\Theta_G(I(\lambda))$ of $I(\lambda)$ is as follows;

$$\Theta_G(I(\lambda))(ta) = \frac{\sum_{w \in W} \xi_{\rho + w\lambda}(a)}{\prod_{\alpha \in \Delta^+ \setminus \Delta_I^+} |1 - \xi_\alpha(ta)|} \quad (t \in T,\ a \in A).$$

We will use the Osborne conjecture, which was proved by Hecht and Schmid [HS83a, Theorem 3.6]. To state it, we must define the character of $J(\mathbb{X})$ for a Harish-Chandra module $\mathbb{X}$. Recall that $J(\mathbb{X})$ is an object of the category $\mathcal{O}'_P$, i.e.,

(1) the actions of $M \cap K$ and $\mathfrak{g}$ are compatible,

(2) $J(\mathbb{X})$ splits under $\mathfrak{a}$ into the direct sum of generalized weight spaces, each of them being a Harish-Chandra module for MA,

(3) $J(\mathbb{X})$ is $U(\overline{\mathfrak{n}})$- and $Z(\mathfrak{g})$-finite

(See Hecht and Schmid [HS83b, (34)Lemma]). For an object V of $\mathcal{O}'_P$, we define the character $\Theta_P(V)$ of V by

$$\Theta_P(V) = \sum_{\mu \in \mathfrak{a}^*} \Theta_{MA}(V_\mu),$$

where V_μ is the generalized μ-weight space of V. Let G' be the set of regular elements of G. Set

$$A^- = \{a \in A \mid \alpha(\log a) < 0 \text{ for all } \alpha \in \Sigma^+\},$$

$$(MA)^- = \\ \text{interior of } \left\{ g \in MA \;\middle|\; \prod_{\alpha \in \Delta^+ \setminus \Delta_I^+} (1 - \xi_\alpha(ga)) \geq 0 \text{ for all } a \in A^- \right\} \text{ in } MA.$$

Then the Osborn conjecture says that $\Theta_G(\mathbb{X})$ and $\Theta_P(J(\mathbb{X}))$ coincide on $(MA)^- \cap G'$ (See Hecht and Schmid [HS83b, (42)Lemma]). It is easy to calculate the character of a generalized Verma module. We have

$$\Theta_P(M(\mu))(ta) = \frac{\xi_{\rho+\mu}(a)}{\prod_{\alpha\in\Delta^+\setminus\Delta_I^+}(1-\xi_\alpha(ta))} \quad (t \in T,\ a \in A).$$

Consequently we have

$$\Theta_P(J(I(\lambda))) = \sum_{w\in W} \Theta_P(M(w\lambda)).$$

This implies the following theorem when λ is regular.

Theorem 5.5. *There exists a filtration $0 = V_{r+1} \subset V_r \subset \cdots \subset V_1 = J(I(\lambda))$ of $J(I(\lambda))$ such that V_i/V_{i+1} is isomorphic to $M(w_i\lambda)$ for an arbitrary $\lambda \in \mathfrak{a}^*$. Moreover if $w\lambda - \lambda \notin 2\mathcal{P}$ for all $w \in W \setminus \{e\}$ then $J(I(\lambda)) \simeq \bigoplus_{w\in W} M(w\lambda)$.*

Proof. The first part of the theorem has already been proved. For the second part, consider the exact sequence

$$0 \longrightarrow V_{i+1} \longrightarrow V_i \longrightarrow M(w_i\lambda) \longrightarrow 0.$$

If $w\lambda - \lambda \notin 2\mathcal{P}$ for all $w \in W \setminus \{e\}$, then $v_i \in V_i$ is a lowest weight vector from Proposition 5.1. Hence we have a map $M(w_i\lambda) \to V_i$, which gives a splitting of the above exact sequence. □

§6. Structure of Jacquet Modules (Singular Case)

In this section, we shall prove Theorem 5.5 in the singular case using the translation principle. We keep the notation of Section 5. Let λ' be an element of $\mathfrak{a}^*$ such that the following conditions hold:

- The weight λ' is regular.
- The weight $(\lambda - \lambda')/2$ is integral.
- The real part of λ' belongs to the same Weyl chamber that the real part of λ belongs to.

First we define the translation functor $T_{\lambda'}^{\lambda}$. Let $\mathbb{X}$ be a $U(\mathfrak{g})$-module which has an infinitesimal character λ'. (We regard $\mathfrak{a}^* \subset \mathfrak{h}^*$.) We define $T_{\lambda'}^{\lambda}(\mathbb{X})$ by $T_{\lambda'}^{\lambda}(\mathbb{X}) = P_\lambda(\mathbb{X} \otimes E_{\lambda-\lambda'})$ where:

- $E_{\lambda-\lambda'}$ is the finite-dimensional irreducible representation of $\mathfrak{g}$ with an extreme weight $\lambda-\lambda'$.
- $P_\lambda(V) = \{v \in V \mid \text{for some } n > 0 \text{ and all } z \in Z(\mathfrak{g}),\ (z - \lambda(\widetilde{\chi}(z)))^n v = 0\}$ where $Z(\mathfrak{g})$ is the center of $U(\mathfrak{g})$ and $\widetilde{\chi}\colon Z(\mathfrak{g}) \to U(\mathfrak{h})$ is the Harish-Chandra homomorphism.

Notice that P_λ and $T_{\lambda'}^{\lambda}$ are exact functors. Since the functors J and $T_{\lambda'}^{\lambda}$ commute [Mat90, Proposition 3.2.1], Theorem 5.5 in the singular case follows from the following two identities.

(1) $T_{\lambda'}^{\lambda}(I(\lambda')) = I(\lambda)$.

(2) $T_{\lambda'}^{\lambda}(M(w\lambda')) = M(w\lambda)$.

The following lemma is important to prove these identities.

Lemma 6.1. *Let ν be a weight of $E_{\lambda-\lambda'}$ and $w \in W$. Assume $\nu = w\lambda - \lambda'$. Then $\nu = \lambda - \lambda'$.*

Proof. See Vogan [Vog81, Lemma 7.2.18]. □

Proof of $T_{\lambda'}^{\lambda}(I(\lambda')) = I(\lambda)$. We may assume that λ' is dominant. Notice that we have $I(\lambda') \simeq \operatorname{Ind}_P^G(1 \otimes \lambda')_K$. Let $0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E_{\lambda-\lambda'}$ be a P-stable filtration with the trivial induced action of N on E_i/E_{i-1}. We may assume that E_i/E_{i-1} is irreducible. Let ν_i be the highest weight of E_i/E_{i-1}. Notice that $\operatorname{Ind}_P^G(1 \otimes \lambda') \otimes E_{\lambda-\lambda'} = \operatorname{Ind}_P^G((1 \otimes \lambda') \otimes E_{\lambda-\lambda'})$. Then $\operatorname{Ind}_P^G(1\otimes\lambda')\otimes E_{\lambda-\lambda'}$ has a filtration $\{M_i\}$ such that $M_i/M_{i-1} \simeq \operatorname{Ind}_P^G((1\otimes\lambda')\otimes(E_i/E_{i-1}))$. Since $\operatorname{Ind}_P^G((1 \otimes \lambda') \otimes (E_i/E_{i-1}))$ has an infinitesimal character $\lambda + \nu_i$, $P_\lambda(M_i/M_{i-1}) = 0$ if $\nu_i \neq w\lambda - \lambda'$ for all $w \in W$. By Lemma 6.1 we have $T_{\lambda'}^{\lambda}(\operatorname{Ind}_P^G(1\otimes\lambda')) = \operatorname{Ind}_P^G((1\otimes\lambda')\otimes(E_i/E_{i-1}))$ where $\nu_i = \lambda - \lambda'$. By the conditions of λ', the action of M on the $(\lambda-\lambda')$-weight space of $E_{\lambda-\lambda'}$ is trivial. Consequently $T_{\lambda'}^{\lambda}(\operatorname{Ind}_P^G(1\otimes\lambda')) = \operatorname{Ind}_P^G((1\otimes\lambda')\otimes(\lambda-\lambda')) = \operatorname{Ind}_P^G(1\otimes\lambda)$. □

Proof of $T_{\lambda'}^{\lambda}(M(w\lambda')) = M(w\lambda)$. We may assume $w = e \in W$. Since $M(\lambda') \otimes E_{\lambda-\lambda'} = U(\mathfrak{g}) \otimes (\mathbb{C}_{\lambda'} \otimes E_{\lambda-\lambda'})$, the equation follows by the same argument of the proof of $T_{\lambda'}^{\lambda}(I(\lambda')) = I(\lambda)$. □

Acknowledgments

The author is grateful to his advisor Hisayosi Matumoto for his advice and support. He would also like to thank Professor Toshio Oshima for his comments.

References

[Cas80] W. Casselman, Jacquet modules for real reductive groups, in *Proceedings of the International Congress of Mathematicians (Helsinki, 1978)*, 557–563, Acad. Sci. Fennica, Helsinki.

[CO78] W. Casselman and M. S. Osborne, The restriction of admissible representations to $\mathfrak{n}$, Math. Ann. **233** (1978), no. 3, 193–198.

[Col85] D. H. Collingwood, *Representations of rank one Lie groups*, Pitman, Boston, MA, 1985.

[Col91] ———, Jacquet modules for semisimple Lie groups having Verma module filtrations, J. Algebra **136** (1991), no. 2, 353–375.

[GW80] R. Goodman and N. R. Wallach, Whittaker vectors and conical vectors, J. Funct. Anal. **39** (1980), no. 2, 199–279.

[Hel62] S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.

[HS83a] H. Hecht and W. Schmid, Characters, asymptotics and $\mathfrak{n}$-homology of Harish-Chandra modules, Acta Math. **151** (1983), no. 1-2, 49–151.

[HS83b] ———, On the asymptotics of Harish-Chandra modules, J. Reine Angew. Math. **343** (1983), 169–183.

[KKM+78] M. Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima and M. Tanaka, Eigenfunctions of invariant differential operators on a symmetric space, Ann. of Math. (2) **107** (1978), no. 1, 1–39.

[Kna01] A. W. Knapp, *Representation theory of semisimple groups*, Reprint of the 1986 original, Princeton Univ. Press, Princeton, NJ, 2001.

[Kna02] ———, *Lie groups beyond an introduction*, Second edition, Birkhäuser Boston, Boston, MA, 2002.

[Kos75] B. Kostant, On the existence and irreducibility of certain series of representations, in *Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971)*, 231–329, Halsted, New York.

[Kos78] ———, On Whittaker vectors and representation theory, Invent. Math. **48** (1978), no. 2, 101–184.

[Lep75] J. Lepowsky, Existence of conical vectors in induced modules, Ann. of Math. (2) **102** (1975), no. 1, 17–40.

[Mat90] H. Matumoto, Whittaker modules associated with highest weight modules, Duke Math. J. **60** (1990), no. 1, 59–113.

[Osh84] T. Ōshima, Boundary value problems for systems of linear partial differential equations with regular singularities, in *Group representations and systems of differential equations (Tokyo, 1982)*, 391–432, Adv. Stud. Pure Math., 4, North-Holland, Amsterdam.

[Osh88] ———, A realization of semisimple symmetric spaces and construction of boundary value maps, in *Representations of Lie groups, Kyoto, Hiroshima, 1986*, 603–650, Adv. Stud. Pure Math., 14, Academic Press, Boston, MA.

[SW82] J. T. Stafford and N. R. Wallach, The restriction of admissible modules to parabolic subalgebras, Trans. Amer. Math. Soc. **272** (1982), no. 1, 333–350.

[Vog81] D. A. Vogan, Jr., *Representations of real reductive Lie groups*, Progress in Mathematics, vol. 15, Birkhäuser Boston, Mass., 1981.

[Wal88] N. R. Wallach, *Real reductive groups. I*, Academic Press, Boston, MA, 1988.