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The Action of the Steenrod Algebra on the Cohomology of p-Compact Groups

By

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Abstract

In the present paper, we determine the action of the Steenrod operations on the cohomology of simply connected p-compact groups with no p-torsion in the integral homologies for an odd prime p. To do so we study simple simply connected p-compact groups since any simply connected p-compact group with no p-torsion in the integral homology is a product of such p-compact groups.

§1. Introduction

A p-compact group is a space X with finite mod p cohomology $H^*(X; \mathbb{F}_p)$ together with a homotopy equivalence $X \to \Omega(BX)$ to the loop space of a path connected p-complete space BX which is called the classifying space of X ([9]). In the present paper, we determine the action of the Steenrod operations on the cohomology of simply connected p-torsion free p-compact groups for an odd prime p. Here, by "p-torsion free" we mean "to have no p-torsion in the integral homology".

Typical examples of p-compact groups are the p-completion of connected compact Lie groups. For simply connected compact Lie groups, it is well known

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how the Steenrod operations act on their cohomologies. In particular, if X is one of the p-completion of the Lie groups $SU(n)$, $Spin(2n+1)$, $Sp(n)$, and p-torsion free exceptional Lie groups for an odd prime p, then the following property holds:

($\mathcal{P}$) X has primitive mod p cohomology generators x_i $(1 \le i \le k)$ with

$$H^*(X;\mathbb{F}_p) \cong \Lambda(x_1,\dots,x_k), \quad \deg x_i = 2n_i - 1 \ (n_1 \le \cdots \le n_k)$$

such that if $n_j - n_i = t(p-1)$ with $t > 0$, then

$$\mathcal{P}^t x_i = \binom{n_i - 1}{t} x_j.$$

On the other hand, ($\mathcal{P}$) is not necessarily true for $X = Spin(2n)^\wedge_p$, where $Y^\wedge_p$ denotes the p-completion of a space Y. However, $Spin(2n)^\wedge_p$ splits as

$$Spin(2n)^\wedge_p \simeq Spin(2n-1)^\wedge_p \times (S^{2n-1})^\wedge_p,$$

and ($\mathcal{P}$) holds for $Spin(2n-1)^\wedge_p$ and for an H-space $(S^{2n-1})^\wedge_p$. Thus, we can say that $Spin(2n)^\wedge_p$ splits as a product of H-spaces satisfying ($\mathcal{P}$).

Our first purpose is to show that the similar result holds for simple simply connected p-torsion free p-compact groups for any odd prime p. It is proved that if p is an odd prime, then a simply connected p-compact group X is simple if and only if its classifying space BX realizes a p-adic pseudoreflection group in the list given by Clark and Ewing ([7], [8]). We will show that ($\mathcal{P}$) holds for almost all simple simply connected p-torsion free p-compact groups. Our proof uses the Clark-Ewing list. Only the exceptions are the loop spaces of generalized Grassmanians.

The generalized Grassmanian $BX(m,q;n)$ is a p-complete space defined for any positive integers m, q and n with $q|m$, $m|p-1$, $m > 1$ and $n > 1$. The space $BX(m,q;n)$, which realizes the p-adic pseudoreflecton group of No. 2a in the Clark-Ewing list, satisfies

$$H^*(BX(m,q;n);\mathbb{F}_p) \cong \mathbb{F}_p[u_1,\dots,u_{n-1},v], \\ \deg u_i = 2im, \quad \deg v = 2nq.$$

Then the p-compact group $X(m,q;n) = \Omega BX(m,q;n)$ is a simple simply connected p-compact group with

$$H^*(X(m,q;n);\mathbb{F}_p) \cong \Lambda(x_1,\dots,x_{n-1},y) \\ \deg x_i = 2im-1, \quad \deg y = 2nq-1.$$

Note that

$$Spin(2n)^\wedge_p \simeq X(2,1;n) \quad \text{and} \quad Spin(2n+1)^\wedge_p \simeq X(2,2;n).$$

Then, we show the following[1]

Theorem 1.1. *If p is an odd prime, then any simple simply connected p-torsion free p-compact group other than $X(m,q;n)$ with $q < m$ satisfies* $(\mathcal{P})$.

A complete list for the Steenrod action on the cohomology of simple simply connected p-torsion free p-compact group is given in section 4. We note that $\mathcal{P}^i$ vanishes for $i \geq p$ except for the ones of Nos. 1 and 2a for dimensional reasons.

Now it is proved by Adams and Wilkerson [1, Corollary 1.3] and Dwyer, Miller and Wilkerson [8, Theorem 1.1] that if p is an odd prime, then any simply connected p-torsion free p-compact group splits as a product of simple p-compact groups. Thus, from the above theorem we have the following

Theorem 1.2. *Let X be a simply connected p-torsion free p-compact group for an odd prime p. Then X splits as*

$$X \simeq X_1 \times \cdots \times X_m,$$

where each X_i is a p-compact group satisfying $(\mathcal{P})$ *or the p-completion of an odd sphere.*

For the case of $p = 2$, Thomas [21] proved a similar fact to $(\mathcal{P})$ for 2-torsion free simply connected H-spaces with primitively generated mod 2 cohomology. Moreover, his result includes a stronger consequence. In fact, he showed that if x is a primitive mod 2 cohomology class with $\deg x = 2n-1$, then for any $t > 0$ with $\binom{n-t-1}{t} \not\equiv 0 \bmod 2$, there exists a primitive class y with $\deg y = 2(n-t)-1$ such that $Sq^{2t}(y) = x$, which is not generally true if p is an odd prime.

Since the above theorem determines the action of the Steenrod algebra on the cohomology of all the simple simply connected p-compact groups, we shall have many applications. For example, we have the following

Corollary 1.3. *Let X be a simply connected p-compact group. Suppose that the action of the Steenrod algebra on $H^*(X;\mathbb{F}_p)$ is trivial. If $p \geq 5$, then X is p-regular, that is, X is p-equivalent to a product of odd spheres; if $p = 3$, then X is 3-equivalent to a product of odd spheres and some copies of the exceptional Lie group G_2.*

[1] After the final submission, the authors found that Theorem 1.1. was independently proved by preprint; D. Davis; Homotopy type and v_1-periodic homotopy groups of p-compact groups; preprint

In the above corollary we need not to assume that X is p-torsion free.

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§2. Simple p-Compact Groups

From now on, we assume that p is an odd prime. Moreover, any p-compact group is assumed to be simply connected and p-torsion free.

Let B be a p-complete space such that the mod p cohomology $H^*(B;\mathbb{F}_p)$ is a finitely generated polynomial algebra:

$$H^*(B;\mathbb{F}_p) \cong \mathbb{F}_p[u_1,\dots,u_k], \quad \deg u_i = 2n_i.$$

Then, by Dwyer, Miller and Wilkerson [8, Theorem 1.1], there exists a p-adic pseudoreflection group $W \subset GL(k,\mathbb{Z}_p^\wedge)$ and a map $f\colon BT \to B$ equivariant up to homotopy with respect to the natural action of W on BT and the trivial action of W on B such that f induces an isomorphism $H^*(B;\mathbb{F}_p) \cong H^*(BT;\mathbb{F}_p)^W$, where $\mathbb{Z}_p^\wedge$ is the ring of p-adic integers, and BT is the classifying space of the p-completion of the k-dimensional torus T^k. The space B is called a realization of W. We notice that the order of W equals to the product $n_1 \dots n_k$. If the order of W is divisible by p, then W and $\mathbb{F}_p[u_1,\dots,u_k]$ are called modular.

All irreducible p-adic pseudoreflection groups are classified by Clark and Ewing [7]. Their result is based on the classification of the irreducible complex reflection groups by Shephard and Todd [20]. A p-compact group X is simple if its classifying space BX realizes one of the irreducible p-adic pseudoreflection groups in the Clark-Ewing list.

Then Theorem 1.2 and Corollary 1.3 are proved from Theorem 1.1.

Proof of Theorem 1.2 *from Theorem* 1.1. Let X be a p-compact group. According to Adams and Wilkerson [1, Corollary 1.3] and Dwyer, Miller and Wilkerson [8, Theorem 1.1], X splits as the product of some simple p-compact groups. Moreover, it is proved by [5, Corollary 1.4] that if $q < m$, then

$$X(m,q;n) \simeq_p X(m,m;n-1) \times (S^{2nq-1})^\wedge_p.$$

Thus, X splits as the product of simple p-compact groups other than $X(m,q;n)$ with $q < m$ and the p-completion of odd spheres. Thus we have the result by Theorem 1.1. □

Proof of Corollary 1.3. First we note that the integral homology of X has no p-torsion. In fact, if it does, then the action of the Bockstein operation on $H^*(X;\mathbb{F}_p)$ is non trivial ([13]). Thus, we have only to show the statement for the case that X is simple by [1, Corollary 1.3] and [8, Theorem 1.1]. Then, by using Theorem 1.1, we see easily that X is either of No. 2b of type $(2,6)$ for $p=3$ or of type $(n_1,\dots,n_k)$ with $n_k-n_1<p-1$. For the former case X is 3-equivalent to G_2, while for the later case X is p-regular by [12]. □

§3. Proof of the Main Theorem

Let X be a p-compact group. Then the mod p cohomology of X is an exterior algebra:

$$H^*(X;\mathbb{F}_p)\cong\Lambda(x_1,\dots,x_k) \tag{3.1}$$

for some primitive classes x_i. The sequence $(n_1,\dots,n_k)$, where $\deg x_i=2n_i-1$, is called the type of X. We note that the type of X is the same as the one of the polynomial algebra $H^*(BX;\mathbb{F}_p)$, where the type of a polynomial algebra $\mathbb{F}_p[u_1,\dots .u_k]$ is a sequence $(n_1,\dots,n_k)$ with $\deg u_i=2n_i$ $(1\le i\le k)$.

Now we show the following

Lemma 3.1. *Let X be a p-compact group of type $(n_1,\dots,n_k)$. Let n be a positive integer with $n\not\equiv 0 \bmod p$ and $n>p$. Suppose that $\mathcal{P}^1\colon PH^{2tn-2(p-1)-1}(X;\mathbb{F}_p)\to PH^{2tn-1}(X;\mathbb{F}_p)$ is a monomorphism for $2\le t\le p$, where $PH^*(X;\mathbb{F}_p)$ is the primitive module of $H^*(X;\mathbb{F}_p)$. Then $\mathcal{P}^1: PH^{2n-2(p-1)-1}(X;\mathbb{F}_p)\to PH^{2n-1}(X;\mathbb{F}_p)$ is an epimorphism.*

Proof. By the assumption, we have $H^*(BX;\mathbb{F}_p)\cong\mathbb{F}_p[u_1,\dots,u_k]$ for some u_i $(1\le i\le k)$ with $\deg u_i=2n_i$. Here, by the same reason as [11, Lemma 4.2] we can choose u_i to satisfy the following condition:

If $\mathcal{P}^1(u_i)$ is not decomposable then $\mathcal{P}^1(u_i)=u_j$ for some u_j; if $\mathcal{P}^1(u_i)=\mathcal{P}^1(u_j)=u_k$ then $u_i=u_j$.

Then, the assumption that $\mathcal{P}^1: PH^{2tn-2(p-1)-1}(X;\mathbb{F}_p)\to PH^{2tn-1}(X;\mathbb{F}_p)$ is a monomorphism is equivalent to say that if $\deg u_s=2tn-2(p-1)$ then $\mathcal{P}^1u_s=u_m$ for some m. Then we show that if $\deg u_j=2n$ then $\mathcal{P}^1(u_i)=u_j$ for some u_i, which implies the result clearly.

Let J be the ideal of $H^*(BX;\mathbb{F}_p)$ generated by all the generators u_s with $s\ne j$ so that $H^*(BX;\mathbb{F}_p)/J\cong\mathbb{F}_p[u_j]$. Since $n\not\equiv 0 \bmod p$, we have

$$u_j^p=\mathcal{P}^nu_j=\mathcal{P}^1(n^{-1}\mathcal{P}^{n-1}u_j).$$

Thus we have $\mathcal{P}^1(H^*(BX;\mathbb{F}_p))\not\subset J$, and so there is a generator u_i with $\mathcal{P}^1u_i\notin J$. For dimensional reasons, we have $\deg u_i=2tn-2(p-1)$ for some t with

$1 \le t \le p$. Then, by the assumption, t must be 1 and $\mathcal{P}^1 u_i = u_j$ since $\mathcal{P}^1 u_i$ is not decomposable. □

Next we consider some special cases.

Proposition 3.2. *Let X be a p-compact group of type $(n_1, \ldots, n_{p-1})$ with*

$$n_i = \begin{cases} i(p-1), & 1 \le i \le p-2 \\ p(p-1), & i = p-1 \end{cases}$$

Then, X satisfies $(\mathcal{P})$.

Proof. We show that we can choose primitive generators x_i $(1 \le i \le p-1)$ of $H^*(X; \mathbb{F}_p)$ such that

$$x_i = \binom{p-2}{i-1}^{-1} \mathcal{P}^{i-1} x_1 \quad \text{for } 2 \le i \le p-2. \tag{3.2}$$

Then $(\mathcal{P})$ is satisfied. In fact, if $1 \le i < j \le p-2$, then $n_j - n_i = (j-i)(p-1)$, and we have

$$\begin{aligned} \mathcal{P}^{j-i} x_i &= \binom{p-2}{i-1}^{-1} \mathcal{P}^{j-i} \mathcal{P}^{i-1} x_1 \\ &= \binom{p-2}{i-1}^{-1} \binom{j-1}{i-1} \mathcal{P}^{j-1} x_1 \\ &= \binom{p-i-1}{j-i} \binom{p-2}{j-1}^{-1} \mathcal{P}^{j-1} x_1 \\ &= \binom{n_i - 1}{j-i} x_j \end{aligned}$$

since $n_i - 1 = (i-1)p + (p-i-1)$ and $j - i < p$. On the other hand, $n_{p-1} - n_i = (p-i)(p-1)$ for $i \le p-2$, and

$$\mathcal{P}^{p-i} x_i = (p-i)^{-1} \mathcal{P}^1 \mathcal{P}^{p-i-1} x_i = 0 = \binom{n_i - 1}{p-i} x_{p-1}.$$

Now we show (3.2). By the assumption we have

$$H^*(BX; \mathbb{F}_p) \cong \mathbb{F}_p[u_1, u_2, \ldots, u_{p-1}]$$

with

$$\deg u_i = \begin{cases} 2i(p-1), & 1 \le i \le p-2 \\ 2p(p-1), & i = p-1. \end{cases}$$

First, since $\mathcal{P}^{i-1}u_{p-i} \in D^2H^*(BX;\mathbb{F}_p)$ for dimensional reasons, we have

$$\mathcal{P}^i u_{p-i} = i^{-1}\mathcal{P}^1\mathcal{P}^{i-1}u_{p-i} \in D^2H^*(BX;\mathbb{F}_p), \tag{3.3}$$

where D^tH^* is the module of t-fold decomposables of an algebra H^*.

Let J be an ideal of $H^*(BX;\mathbb{F}_p)$ generated by $u_1,\dots,u_{p-2}$. Then, we show that

$$\mathcal{P}^p u_{p-1} \notin J. \tag{3.4}$$

In fact, if $\mathcal{P}^p u_{p-1} \in J$, then for dimensional reasons, $\mathcal{P}^p u_i \in J$ for $1 \le i \le p-1$. Thus

$$\mathcal{P}^1(H^*(BX;\mathbb{F}_p)) \subset J \quad \text{and} \quad \mathcal{P}^p(H^*(BX;\mathbb{F}_p)) \subset J.$$

Then

$$u_{p-1}^p = \mathcal{P}^{p(p-1)}u_{p-1} = -\mathcal{P}^1\mathcal{P}^{p(p-1)-2}\mathcal{P}^1u_{p-1} - \mathcal{P}^p\mathcal{P}^{p(p-1)}u_{p-1} \in J,$$

and we have a contradiction.

Now we consider the BP cohomology of BX at the prime p (cf. [10, Lemma 8.1]):

$$BP^*(BX) \cong BP^*[[U_1,\dots,U_{p-1}]],$$

where $T(U_i) = u_i$ for the natural map $T\colon BP^*(BX) \to H^*(BX;\mathbb{F}_p)$. We notice that the kernel of T is the ideal $(p,v_1,v_2,\dots)$ generated by $p,v_1,v_2,\dots$.

Let r_t be the Landweber-Novikov operation so that

$$\chi(\mathcal{P}^t)(T(z)) = T(r_t(z))$$

for any $z \in BP^*(BX)$, where χ is the canonical anti-automorphism on the mod p Steenrod algebra. In particular, since $\chi(\mathcal{P}^t) = (-1)^t\mathcal{P}^t$ for $t \le p$ we have

$$(-1)^t\mathcal{P}^t(T(z)) = T(r_t(z)) \quad \text{for } t \le p.$$

Since $\mathcal{P}^p u_{p-1} \notin J$ by (3.4), we have

$$r_pU_{p-1} \equiv a\,U_{p-1}^2 \mod D^3BP^*(BX) + (p,v_1,v_2,\dots) \tag{3.5}$$

for some $a \in \mathbb{Z}_{(p)}$ with $a \not\equiv 0 \bmod p$, where $\mathbb{Z}_{(p)}$ is the ring of p-local integers.

Now we can put

$$\begin{aligned} r_1U_{p-1} \equiv{} & b_1\,U_1U_{p-1} + \sum_{k=3}^{(p+1)/2} b_k\,U_kU_{p+1-k} \\ & + v_1\left(c_2\,U_2U_{p-1} + \sum_{k=4}^{(p+1)/2} c_k\,U_kU_{p+2-k}\right) \\ & \mod D^3BP^*(BX) + (p,v_1,v_2,\dots)^2 \end{aligned}$$

with b_k, $c_k \in \mathbb{Z}_{(p)}$. Here, $r_{p-1}r_1 \equiv p\,r_p \bmod (v_1, v_2, \dots)$, and $r_i U_{p-i} \in D^2BP^*(BX) + (p, v_1, \dots)$ by (3.3). Thus

$$\begin{aligned} p\,r_p U_{p-1} &\equiv r_{p-1}r_1 U_{p-1} \\ &\equiv b_1\,(r_{p-1}U_1)U_{p-1} + \sum_{k=3}^{(p+1/2)} b_k\,(r_{p-k}U_k)(r_{k-1}U_{p+1-k}) \\ &\quad + p\left(c_2\,(r_{p-2}U_2)U_{p-1} + \sum_{k=4}^{(p+1)/2} c_k\,(r_{p-k}U_k)(r_{k-2}U_{p+2-k})\right) \\ &\equiv pb_1 d\,U_{p-1}^2 \\ &\qquad\qquad \bmod D^3BP^*(BX) + (p^2, v_1, v_2, \dots), \end{aligned}$$

where $r_{p-1}U_1 \equiv pd\,U_{p-1} \bmod D^2BP^*(BX) + (p^2, v_1, v_2, \dots)$ with $d \in \mathbb{Z}_{(p)}$. Thus we have $b_1 d \equiv a \not\equiv 0 \bmod p$ by (3.5). In particular, we have

$$r_{p-1}U_1 \not\equiv 0 \quad \bmod D^2BP^*(BX) + (p^2, v_1, v_2, \dots).$$

Suppose that $\mathcal{P}^{p-3}u_1 \in D^2H^*(BX; \mathbb{F}_p)$. Then

$$r_{p-3}U_1 \equiv pe\,U_{p-2} \quad \bmod D^2BP^*(BX) + (p^2, v_1, v_2, \dots)$$

for some $e \in \mathbb{Z}_{(p)}$. Thus, we have

$$\begin{aligned} r_{p-1}U_1 \equiv \binom{p-1}{2}^{-1} r_2 r_{p-3}U_1 \equiv pe\,r_2 U_{p-2} \equiv 0 \\ \bmod D^2BP^*(BX) + (p^2, v_1, v_2, \dots), \end{aligned}$$

which is a contradiction. Therefore, we have

$$\mathcal{P}^{p-3}u_1 \notin D^2H^*(BX; \mathbb{F}_p),$$

and we can choose generators u_i for $2 \le i \le p-2$ such that

$$u_i = \binom{p-2}{i-1}^{-1} \mathcal{P}^{i-1}u_1.$$

Thus by putting $x_i = s(u_i)$, where s is the cohomology suspension, we have (3.2), which completes the proof. □

Next we consider the p-compact group $X(m, m; n)$.

Proposition 3.3. *For any positive integers m and n with $m|p-1$, the p-compact group $X(m, m; n)$ satisfies ($\mathcal{P}$).*

Proof. According to Castellana [6, p.2817] there is a map $Bc\colon BX(q,1;n) \to BU(nq)^\wedge_p$ such that if we put

$$u_j = Bc^*(c_{jq}) \quad (1 \le j \le n)$$

then

$$H^*(BX(m,m;n);\mathbb{F}_p) = \mathbb{F}_p[u_1,\dots,u_n],$$

where c_j is the j-th Chern class so that

$$H^*(BU(nq)^\wedge_p;\mathbb{F}_p) = \mathbb{F}_p[c_1,\dots,c_{nq}].$$

Then by putting $x_i = s(u_i)$ for the cohomology suspension s we have ($\mathcal{P}$) for $X(m,m;n)$ from the one for $U(nq)^\wedge_p$ ([4, Corollaire 11.4], [16]).

□

Now we show Theorem 1.1.

Proof of Theorem 1.1. Let X be a simple p-compact group of type $(n_1,\dots,n_k)$ other than $X(m,q;n)$ with $q<m$.

If $n_k - n_1 < p-1$, then ($\mathcal{P}$) clearly holds since X is p-regular and there are no i and j with $n_j - n_i = t(p-1)$ for any $t>0$.

Suppose that $p-1 \le n_k - n_1 < 2(p-1)$. Then, by checking the Clark Ewing list, we can see that $n_i \ne n_j$ for $i \ne j$.

Let $n_j - n_i = t(p-1)$ for some $t>0$. Then it is clear that $t=1$. Moreover, by checking the Clark-Ewing list, we can see that $n_j \not\equiv 0 \bmod p$ and $PH^{2sn_j-2(p-1)-1}(X;\mathbb{F}_p) = 0$ for $2 \le s \le p$. Thus by Lemma 3.1, $\mathcal{P}^1\colon PH^{2n_i-1}(X;\mathbb{F}_p) \to PH^{2n_j-1}(X;\mathbb{F}_p)$ is an epimorphism. Since $n_i - 1 \equiv n_j \not\equiv 0 \bmod p$, we can choose generators x_i and x_j with

$$\mathcal{P}^1 x_i = \binom{n_i-1}{1} x_j$$

Thus ($\mathcal{P}$) holds.

Now we consider the remaining case $n_k - n_1 \ge 2(p-1)$. The Clark-Ewing list consists of thirty seven families of irreducible p-adic pseudoreflection groups, with two subfamilies 2a and 2b in No. 2. It is proved that all non-modular irreducible p-adic pseudoreflection groups have realizations ([7]). On the other hand, not all modular irreducible p-adic pseudoreflection groups have realizations. The non-realizable ones for an odd prime p are of Nos. 28, 35, 36 for $p=3$, and 37 for $p=3$, 5 ([22], [2], [18], [19]). The following is the table of all realizable irreducible p-adic pseudoreflection groups with $n_k - n_1 \geq 2(p-1)$.

No.	type	prime
1	$(2, 3, \dots, n+1)$	$p \leq (n+1)/2$
2a	$(m, 2m, \dots, nm)$	$p \leq (n-1)m/2+1$
2b	$(2, 6)$	3
26	$(6, 12, 18)$	7
28	$(2, 6, 8, 12)$	5
29	$(4, 8, 12, 20)$	5
30	$(2, 12, 20, 30)$	11
31	$(8, 12, 20, 24)$	5
32	$(12, 18, 24, 30)$	7
33	$(4, 6, 10, 12, 18)$	7
34	$(6, 12, 18, 24, 30, 42)$	7, 13, 19
35	$(2, 5, 6, 8, 9, 12)$	5
36	$(2, 6, 8, 10, 12, 14, 18)$	5, 7
37	$(2, 8, 12, 14, 18, 20, 24, 30)$	7, 11, 13

Now, the p-adic pseudoreflection groups of Nos. 1, 2b, 28, 35, 36 and 37 are the Weyl groups of the Lie groups $SU(n+1)$, G_2, F_4, E_6, E_7 and E_8, respectively. If BX realizes the Weyl group of a Lie group G, then $H^*(X; \mathbb{F}_p)$ is isomorphic to $H^*(G; \mathbb{F}_p)$ as an algebra over the Steenrod algebra. Thus, for these cases, we have only to show ($\mathcal{P}$) for corresponding Lie groups, which are proved in [4, Corollaire 11.4] (see also [16]), [3], [12] and [15].

For No. 2a, we have already done in Propositions 3.3.

For Nos. 29 for $p = 5$ and 34 for $p = 7$, ($\mathcal{P}$) holds by Proposition 3.2. For the Nos. 30 for $p = 11$ and 31 for $p = 5$, we can apply Lemma 3.1 to show ($\mathcal{P}$). Note that for No. 31, we have

$$\mathcal{P}^2 x_2 = 2^{-1}\mathcal{P}^1\mathcal{P}^1 x_2 = 0 = \binom{12-1}{2} x_3.$$

For No. 34 for $p = 19$, ($\mathcal{P}$) holds by Nishinobu [17, Theorem 1.2].

Now we consider No. 32 for $p = 7$. Then by Lemma 3.1 generators x_i are chosen to satisfy

$$\mathcal{P}^1 x_2 = \binom{18-1}{1} x_3, \quad \mathcal{P}^1 x_3 = \binom{24-1}{1} x_4.$$

Thus, we have only to show that

$$\mathcal{P}^1 x_1 \neq 0.$$

Then by modifying x_1 we have

$$\mathcal{P}^1 x_1 = \binom{12-1}{1} x_2.$$

Suppose contrarily that $\mathcal{P}^1 x_1 = 0$. Then, in $H^*(BX;\mathbb{F}_7) = \mathbb{F}_7[u_1, u_2, u_3, u_4]$ the generators u_i can be chosen to satisfy

$$\mathcal{P}^1 u_1 = 0, \quad \mathcal{P}^1 u_2 = \binom{18-1}{1} u_3, \quad \mathcal{P}^1 u_3 = \binom{24-1}{1} u_4.$$

We show that $\mathcal{P}^1 u_4$ is not in the ideal (u_1, u_3, u_4) generated by u_1, u_3, u_4. In fact, if $\mathcal{P}^1 u_4 \in (u_1, u_3, u_4)$, then $\mathcal{P}^1(H^*(BX;\mathbb{F}_7)) \subset (u_1, u_3, u_4)$ since $\mathcal{P}^1 u_i \in (u_1, u_3, u_4)$ for any $i = 1, 2, 3, 4$. On the other hand, we have

$$\mathcal{P}^1(\mathcal{P}^{17} u_2) = 18\, \mathcal{P}^{18} u_2 = 18\, u_2^7 \notin (u_1, u_3, u_4).$$

This is a contradiction, and so we have

$$\mathcal{P}^1 u_4 \notin (u_1, u_3, u_4).$$

Now, consider the quotient algebra $H^*(BX;\mathbb{F}_7)/(u_1) \cong \mathbb{F}_7[\bar{u}_2, \bar{u}_3, \bar{u}_4]$, where $\bar{u}_i$ is the coset of u_i. Note that this algebra does not necessarily admit the Steenrod action, but has the $\mathcal{P}^1$ action. Then, by the above argument we can put

$$\mathcal{P}^1 \bar{u}_4 = a \bar{u}_2^2$$

with $a \neq 0$. Then by iterating $\mathcal{P}^1$ we have

$$(\mathcal{P}^1)^7 \bar{u}_4 = 3a^3 \bar{u}_2^4 + a^2 \bar{u}_2 \bar{u}_3 \bar{u}_4 + 3a^2 \bar{u}_3^3 \neq 0.$$

This is a contradiction since $(\mathcal{P}^1)^7 = 0$. Thus $\mathcal{P}^1 u_1 \neq 0$ and we have $\mathcal{P}^1 x_1 \neq 0$.

The remaining cases are of Nos. 26 and 33 for $p = 7$, and 34 for $p = 13$. The idea of the proof for these cases are essentially the same. First we consider No. 26.

By Lemma 3.1 we can choose generators x_i such that

$$\mathcal{P}^1 x_2 = \binom{12-1}{1} x_3.$$

Thus, we prove

$$\mathcal{P}^1 x_1 \neq 0$$

to show

$$\mathcal{P}^1 x_1 = \binom{6-1}{1} x_2$$

for a suitable x_1.

Suppose contrarily that $\mathcal{P}^1 x_1 = 0$. Then we can choose generators u_i $(i = 1, 2, 3)$ of $H^*(BX; \mathbb{F}_p)$ such that the ideal (u_1) of $H^*(BX; \mathbb{F}_p)$ generated by u_1 is closed under the action of the Steenrod algebra for dimensional reasons. Thus $H^*(BX; \mathbb{F}_p)/(u_1)$ is a polynomial algebra of the type $(12, 18)$ on which the Steenrod algebra acts unstably. According to Adams and Wilkerson [1, Theorem 1.2], the type of any non-modular polynomial algebra on which the Steenrod algebra acts unstably must be a union of types in the Clark-Ewing list, which is not the case. This is a contradiction, and $\mathcal{P}^1 u_1$ is indecomposable. Thus we have $\mathcal{P}^1 x_1 \neq 0$.

Next consider No. 33.

By Lemma 3.1 we can choose generators x_i such that

$$\mathcal{P}^1 x_1 = \binom{4-1}{1} x_3, \quad \mathcal{P}^1 x_4 = \binom{12-1}{1} x_5.$$

Thus we have only to show that

$$\mathcal{P}^1 x_2 \neq 0$$

to show

$$\mathcal{P}^1 x_2 = \binom{6-1}{1} x_4$$

for a suitable x_2.

Suppose contrarily that $\mathcal{P}^1 x_2 = 0$. Then we can choose generators u_i of $H^*(BX; \mathbb{F}_p)$ such that the ideal (u_1, u_2, u_3) of $H^*(BX; \mathbb{F}_p)$ generated by u_1, u_2, u_3 is closed under the action of the operation $\mathcal{P}^1$. Moreover, for dimensional reasons, we have $\mathcal{P}^7 u_3 \in (u_1, u_2, u_3)$. Thus (u_1, u_2, u_3) is closed under the action of the Steenrod algebra, and we have a polynomial algebra $H^*(BX; \mathbb{F}_p)/(u_1, u_2, u_3)$ of the type $(12, 18)$ on which the Steenrod algebra acts unstably. Thus, by the same reason as the case of No. 26, we have a contradiction, and we have $\mathcal{P}^1 x_2 \neq 0$.

Finally, we consider No. 34.

By Lemma 3.1, we can choose generators x_i such that

$$\mathcal{P}^1 x_2 = \binom{12-1}{1} x_4, \quad \mathcal{P}^1 x_3 = \binom{18-1}{1} x_5, \quad \mathcal{P}^1 x_5 = \binom{30-1}{1} x_6.$$

Thus, we have only to show that

$$\mathcal{P}^1 x_1 \neq 0$$

to show

$$\mathcal{P}^1 x_1 = \binom{6-1}{1} x_3$$

for a suitable x_1.

Suppose contrarily that $\mathcal{P}^1 x_1 = 0$. Then we can choose generators u_i of $H^*(BX;\mathbb{F}_p)$ such that the ideal (u_1) of $H^*(BX;\mathbb{F}_p)$ generated by u_1 is closed under the action of the Steenrod algebra for dimensional reasons. Thus we have a polynomial algebra $H^*(BX;\mathbb{F}_p)/(u_1)$ of type $(12, 18, 24, 30, 42)$ on which the Steenrod algebra acts unstably. Thus, by the same reason as the case of No. 26, we have a contradiction, and we have $\mathcal{P}^1 x_1 \neq 0$. This completes the proof of the theorem. □

§4. Steenrod Connection

In this section we list the complete Steenrod actions on the cohomology of simple simply connected p-torsion free p-compact groups for any odd prime p. In No. 2a we only consider $X(m, m; n)$. For $X(m, q; n)$ with $q < m$, the Steenrod actions can be given from the following isomorphism of algebras over the Steenrod algebra:

$$H^*(X(m, q; n);\mathbb{F}_p) \cong H^*(X(m, m; n-1);\mathbb{F}_p) \otimes H^*(S^{2nq-1};\mathbb{F}_p).$$

In the following, we only list non trivial actions, where PH^n denotes the n-dimensional primitive module of $H^*(X;\mathbb{F}_p)$.

(No.1 of type: $(2, 3, \ldots, n+1)$) For any odd prime p with $p \leq n$:

$$\mathcal{P}^k(PH^{2t-1}) = PH^{2(t+k(p-1))-1} \quad \text{for } \binom{t-1}{k} \not\equiv 0 \bmod p,$$

where $t \geq 2$ and $t + k(p-1) \leq n+1$.

(No.2a of type: $(m, 2m, \ldots, nm)$) For any odd prime p with $p \leq n$:

$$\mathcal{P}^k(PH^{2tm-1}) = PH^{2(tm+k(p-1))-1} \quad \text{for } \binom{tm-1}{k} \not\equiv 0 \bmod p,$$

where $t \geq 1$ and $tm + k(p-1) \leq nm$.

(No.2b of type: $(2, p+1)$) For any odd prime p:

$$\mathcal{P}^1(PH^3) = PH^{2p+1}.$$

(No.5 of type: $(6,12)$) For $p=7$:

$$\mathcal{P}^1(PH^{11})=PH^{23}.$$

(No.8 of type: $(8,12)$) For $p=5$:

$$\mathcal{P}^1(PH^{15})=PH^{23}.$$

(No.9 of type: $(8,24)$) For $p=17$:

$$\mathcal{P}^1(PH^{15})=PH^{47}.$$

(No.10 of type: $(12,24)$) For $p=13$:

$$\mathcal{P}^1(PH^{23})=PH^{47}.$$

(No.12 of type: $(6,8)$) For $p=3$:

$$\mathcal{P}^1(PH^{11})=PH^{15}.$$

(No.14 of type: $(6,24)$) For $p=19$:

$$\mathcal{P}^1(PH^{11})=PH^{47}.$$

(No.16 of type: $(20,30)$) For $p=11$:

$$\mathcal{P}^1(PH^{39})=PH^{59}.$$

(No.17 of type: $(20,60)$) For $p=41$:

$$\mathcal{P}^1(PH^{39})=PH^{119}.$$

(No.18 of type: $(30,60)$) For $p=31$:

$$\mathcal{P}^1(PH^{59})=PH^{119}.$$

(No.20 of type: $(12,30)$) For $p=19$:

$$\mathcal{P}^1(PH^{23})=PH^{59}.$$

(No.24 of type: $(4,6,14)$) For $p=11$:

$$\mathcal{P}^1(PH^{7})=PH^{27}.$$

(No.25 of type: $(6,9,12)$) For $p=7$:

$$\mathcal{P}^1(PH^{11})=PH^{23}.$$

(No.26 of type: $(6,12,18)$) For $p=7$:

$$\mathcal{P}^1(PH^{11}) = PH^{23},$$
$$\mathcal{P}^2(PH^{11}) = \mathcal{P}^1(PH^{23}) = PH^{35}.$$

For $p=13$:

$$\mathcal{P}^1(PH^{11}) = PH^{35}.$$

(No.27 of type: $(6,12,30)$) For $p=19$:

$$\mathcal{P}^1(PH^{23}) = PH^{59}.$$

(No.28 of type: $(2,6,8,12)$) For $p=5$:

$$\mathcal{P}^1(PH^{3}) = PH^{11},$$
$$\mathcal{P}^1(PH^{15}) = PH^{23}.$$

For $p=7$:

$$\mathcal{P}^1(PH^{3}) = PH^{15},$$
$$\mathcal{P}^1(PH^{11}) = PH^{23}.$$

For $p=11$:

$$\mathcal{P}^1(PH^{3}) = PH^{23}.$$

(No.29 of type: $(4,8,12,20)$) For $p=5$:

$$\mathcal{P}^1(PH^{7}) = PH^{15},$$
$$\mathcal{P}^2(PH^{7}) = \mathcal{P}^1(PH^{15}) = PH^{23}.$$

For $p=13$:

$$\mathcal{P}^1(PH^{15}) = PH^{39}.$$

For $p=17$:

$$\mathcal{P}^1(PH^{7}) = PH^{39}.$$

(No.30 of type: $(2,12,20,30)$) For $p=11$:

$$\mathcal{P}^1(PH^{3}) = PH^{23},$$
$$\mathcal{P}^1(PH^{39}) = PH^{59}.$$

For $p=19$:

$$\mathcal{P}^1(PH^{3}) = PH^{39},$$
$$\mathcal{P}^1(PH^{23}) = PH^{59}.$$

For $p = 29$:

$$\mathcal{P}^1(PH^3) = PH^{59}.$$

(No.31 of type: $(8, 12, 20, 24)$) For $p = 5$:

$$\mathcal{P}^1(PH^{15}) = PH^{23},$$
$$\mathcal{P}^1(PH^{39}) = PH^{47}.$$

For $p = 13$:

$$\mathcal{P}^1(PH^{15}) = PH^{39},$$
$$\mathcal{P}^1(PH^{23}) = PH^{47}.$$

For $p = 17$:

$$\mathcal{P}^1(PH^{15}) = PH^{47}.$$

(No.32 of type: $(12, 18, 24, 30)$) For $p = 7$:

$$\mathcal{P}^1(PH^{23}) = PH^{35},$$
$$\mathcal{P}^2(PH^{23}) = \mathcal{P}^1(PH^{35}) = PH^{47},$$
$$\mathcal{P}^3(PH^{23}) = \mathcal{P}^2(PH^{35}) = \mathcal{P}^1(PH^{47}) = PH^{59}.$$

For $p = 13$:

$$\mathcal{P}^1(PH^{23}) = PH^{47},$$
$$\mathcal{P}^1(PH^{35}) = PH^{59}.$$

For $p = 19$:

$$\mathcal{P}^1(PH^{23}) = PH^{59}.$$

(No.33 of type: $(4, 6, 10, 12, 18)$) For $p = 7$:

$$\mathcal{P}^1(PH^7) = PH^{19},$$
$$\mathcal{P}^1(PH^{11}) = PH^{23},$$
$$\mathcal{P}^2(PH^{11}) = \mathcal{P}^1(PH^{23}) = PH^{35}.$$

For $p = 13$:

$$\mathcal{P}^1(PH^{11}) = PH^{35}.$$

(No.34 of type: $(6, 12, 18, 24, 30, 42)$) For $p = 7$:

$$\mathcal{P}^1(PH^{11}) = PH^{23},$$
$$\mathcal{P}^2(PH^{11}) = \mathcal{P}^1(PH^{23}) = PH^{35},$$
$$\mathcal{P}^3(PH^{11}) = \mathcal{P}^2(PH^{23}) = \mathcal{P}^1(PH^{35}) = PH^{47},$$
$$\mathcal{P}^4(PH^{11}) = \mathcal{P}^3(PH^{23}) = \mathcal{P}^2(PH^{35}) = \mathcal{P}^1(PH^{47}) = PH^{59}.$$

For $p = 13$:

$$\mathcal{P}^1(PH^{11}) = PH^{35},$$
$$\mathcal{P}^1(PH^{23}) = PH^{47},$$
$$\mathcal{P}^2(PH^{11}) = \mathcal{P}^1(PH^{35}) = PH^{59},$$
$$\mathcal{P}^3(PH^{11}) = \mathcal{P}^2(PH^{35}) = \mathcal{P}^1(PH^{59}) = PH^{83}.$$

For $p = 19$:

$$\mathcal{P}^1(PH^{11}) = PH^{47},$$
$$\mathcal{P}^1(PH^{23}) = PH^{59},$$
$$\mathcal{P}^2(PH^{11}) = \mathcal{P}^1(PH^{47}) = PH^{83}.$$

For $p = 31$:

$$\mathcal{P}^1(PH^{23}) = PH^{83}.$$

For $p = 37$:

$$\mathcal{P}^1(PH^{11}) = PH^{83}.$$

(No.35 of type: $(2, 5, 6, 8, 9, 12)$) For $p = 5$:

$$\mathcal{P}^1(PH^{3}) = PH^{11},$$
$$\mathcal{P}^1(PH^{9}) = PH^{17},$$
$$\mathcal{P}^1(PH^{15}) = PH^{23}.$$

For $p = 7$:

$$\mathcal{P}^1(PH^{3}) = PH^{15},$$
$$\mathcal{P}^1(PH^{11}) = PH^{23}.$$

For $p = 11$:

$$\mathcal{P}^1(PH^{3}) = PH^{23}.$$

(No.36 of type: $(2, 6, 8, 10, 12, 14, 18)$) For $p = 5$:

$$\mathcal{P}^1(PH^{3}) = PH^{11},$$
$$\mathcal{P}^1(PH^{15}) = PH^{23},$$
$$\mathcal{P}^1(PH^{19}) = PH^{27},$$
$$\mathcal{P}^2(PH^{19}) = \mathcal{P}^1(PH^{27}) = PH^{35}.$$

For $p = 7$:

$$\mathcal{P}^1(PH^3) = PH^{15},$$
$$\mathcal{P}^1(PH^{11}) = PH^{23},$$
$$\mathcal{P}^2(PH^{11}) = \mathcal{P}^1(PH^{23}) = PH^{35}.$$

For $p = 11$:

$$\mathcal{P}^1(PH^3) = PH^{23},$$
$$\mathcal{P}^1(PH^{15}) = PH^{35}.$$

For $p = 13$:

$$\mathcal{P}^1(PH^3) = PH^{27},$$
$$\mathcal{P}^1(PH^{11}) = PH^{35}.$$

For $p = 17$:

$$\mathcal{P}^1(PH^3) = PH^{35}.$$

(No.37 of type: $(2, 8, 12, 14, 18, 20, 24, 30)$) For $p = 7$:

$$\mathcal{P}^1(PH^3) = PH^{15},$$
$$\mathcal{P}^1(PH^{23}) = PH^{35},$$
$$\mathcal{P}^1(PH^{27}) = PH^{39},$$
$$\mathcal{P}^2(PH^{23}) = \mathcal{P}^1(PH^{35}) = PH^{47},$$
$$\mathcal{P}^3(PH^{23}) = \mathcal{P}^2(PH^{35}) = \mathcal{P}^1(PH^{47}) = PH^{59}.$$

For $p = 11$:

$$\mathcal{P}^1(PH^3) = PH^{23},$$
$$\mathcal{P}^1(PH^{15}) = PH^{35},$$
$$\mathcal{P}^1(PH^{27}) = PH^{47},$$
$$\mathcal{P}^1(PH^{39}) = PH^{59}.$$

For $p = 13$:

$$\mathcal{P}^1(PH^3) = PH^{27},$$
$$\mathcal{P}^1(PH^{15}) = PH^{39},$$
$$\mathcal{P}^1(PH^{23}) = PH^{47},$$
$$\mathcal{P}^1(PH^{35}) = PH^{59}.$$

For $p = 17$:

$$\mathcal{P}^1(PH^3) = PH^{35},$$
$$\mathcal{P}^1(PH^{15}) = PH^{47},$$
$$\mathcal{P}^1(PH^{27}) = PH^{59}.$$

For $p = 19$:

$$\mathcal{P}^1(PH^3) = PH^{39},$$
$$\mathcal{P}^1(PH^{23}) = PH^{59}.$$

For $p = 23$:

$$\mathcal{P}^1(PH^3) = PH^{47},$$
$$\mathcal{P}^1(PH^{15}) = PH^{59}.$$

For $p = 29$:

$$\mathcal{P}^1(PH^3) = PH^{59}.$$

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