Publ. RIMS, Kyoto Univ.
45 (2009), 75–87

Algebraic p-Adic L-Functions in Non-Commutative Iwasawa Theory

By

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Abstract

We define canonical algebraic p-adic L-functions in non-commutative Iwasawa theory and establish some of their basic properties.

§1. Introduction

In this note we use a result of Schneider and Venjakob in [8] to define canonical 'algebraic p-adic L-functions' in non-commutative Iwasawa theory. This construction refines both the notion of 'characteristic element' introduced by Venjakob in [10] and the 'Akashi series' introduced by Coates, Schneider and Sujatha in [5] and also plays a key role in descent theory in non-commutative Iwasawa theory. Indeed, in joint work with Venjakob [3], the results proved here are used to establish a general descent theory that, for example, clarifies the precise connection between main conjectures of non-commutative Iwasawa theory in the spirit of Coates, Fukaya, Kato, Sujatha and Venjakob [4] and the relevant cases of the equivariant Tamagawa number conjecture of Flach and the present author [2].

The main contents of this note is as follows. In §2 we define a natural notion of algebraic p-adic L-function. In §3 we prove some of the basic functorial properties of these elements and show that they refine the 'Akashi series' introduced in [5]. In §4 we prove that algebraic p-adic L-functions are 'characteristic elements' in the sense of [10] (and [4]).

Communicated by A. Tamagawa. Received August 2, 2007. Revised February 6, 2008.
2000 Mathematics Subject Classification(s): 11R23.
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It is a pleasure to thank Otmar Venjakob for interesting discussions and the referee for several useful remarks.

§2. Algebraic p-Adic L-Functions

§2.1. Preliminaries

In the sequel 'module' means 'left module'. For any ring R we write $D(R)$ for the derived category of R-modules. We also write $D^{\mathrm{fg}}(R)$, resp. $D^{\mathrm{p}}(R)$, for the full triangulated subcategory of $D(R)$ comprising complexes that are isomorphic to a bounded complex of finitely generated R-modules, resp. to an object of the category $C^{\mathrm{p}}(R)$ of bounded complexes of finitely generated projective R-modules.

We fix a prime p. For any $\mathbb{Z}_p$-module M we write M_{tor} for its torsion submodule and set $M_{\mathrm{tf}} := M/M_{\mathrm{tor}}$. For any profinite group J we write $\Lambda(J)$ for the 'Iwasawa algebra' $\varprojlim_U \mathbb{Z}_p[J/U]$ where U runs over all open normal subgroups of J and the limit is taken with respect to the natural projection maps $\mathbb{Z}_p[J/U] \to \mathbb{Z}_p[J/U']$ for $U \subseteq U'$. If J is a compact p-adic Lie group, then $\Lambda(J)$ is a noetherian ring and we write $Q(J)$ for its total quotient ring.

§2.2. Canonical Ore sets

We assume to be given a compact p-adic Lie group G and a normal subgroup H of G with the property that the quotient group $\Gamma := G/H$ is isomorphic (topologically) to the additive group of $\mathbb{Z}_p$. We fix a topological generator γ of Γ. We recall from [4, §2-§3] that there are canonical left and right denominator sets $S(G,H)$ and $S(G,H)^*$ of $\Lambda(G)$ where

$$S(G,H) := \{\lambda \in \Lambda(G) : \Lambda(G)/(\Lambda(G)\cdot\lambda) \text{ is a finitely generated } \Lambda(H)\text{-module}\}$$

and

$$S(G,H)^* := \bigcup_{i\geq 0} p^i S(G,H).$$

When G and H are clear from context we abbreviate $S(G,H)$ to S. We also write $\mathfrak{M}_S(G)$ and $\mathfrak{M}_{S^*}(G)$ for the categories of finitely generated $\Lambda(G)$-modules M that satisfy $\Lambda(G)_S \otimes_{\Lambda(G)} M = 0$ and $\Lambda(G)_{S^*} \otimes_{\Lambda(G)} M = 0$ respectively. We further recall from [4, Prop. 2.3] that a finitely generated $\Lambda(G)$-module M belongs to $\mathfrak{M}_S(G)$, resp. to $\mathfrak{M}_{S^*}(G)$, if and only if M, resp. M_{tf}, is a finitely generated $\Lambda(H)$-module (by restriction). Thus $\mathfrak{M}_{S^*}(G)$ coincides with the category $\mathfrak{M}_H(G)$ used in [4].

§2.3. Canonical automorphisms

If J is any profinite group, then we write $M\hat{\otimes}_{\Lambda(J)}N$ for the completed tensor product of compact $\Lambda(J)$-modules M and N (cf. [7, p. 230]). In particular, if M is any compact $\Lambda(G)$-module, then

$$\mathrm{I}_H^G(M) := \Lambda(G)\hat{\otimes}_{\Lambda(H)}\mathrm{Res}_H^G(M)$$

has a natural structure as a (compact) $\Lambda(G)$-module via left multiplication. The functor $M \mapsto \mathrm{I}_H^G(M)$ is exact on the category of compact $\Lambda(G)$-modules and if M belongs to $\mathfrak{M}_S(G)$, then $\mathrm{I}_H^G(M)$ identifies with the usual tensor product $\Lambda(G)\otimes_{\Lambda(H)} M$ (cf. [7, p. 241, Ex. 1]).

We define an endomorphism $\Delta_{G,\gamma,M}$ of $\mathrm{I}_H^G(M)$ by setting

$$\Delta_{G,\gamma,M}(x\otimes_{\Lambda(H)} y) := x\tilde{\gamma}^{-1}\otimes_{\Lambda(H)}\tilde{\gamma}(y)$$

for each $x\in\Lambda(G)$ and $y\in M$, where $\tilde{\gamma}$ is any lift of γ through the projection $G\to\Gamma$. We also set

$$\delta_{G,\gamma,M} := \mathrm{id}_{\mathrm{I}_H^G(M)} - \Delta_{G,\gamma,M}.$$

Then $\Delta_{G,\gamma,M}$, and hence also $\delta_{G,\gamma,M}$, is a well-defined endomorphism of the $\Lambda(G)$-module $\mathrm{I}_H^G(M)$ that is independent of the precise choice of $\tilde{\gamma}$. When G and M are both clear from context we usually abbreviate $\Delta_{G,\gamma,M}$ and $\delta_{G,\gamma,M}$ to Δ_γ and δ_γ respectively.

For any compact $\Lambda(G)$-module M we let M_S, resp. M_{S^*}, denote the induced $\Lambda(G)_S$-module $\Lambda(G)_S\hat{\otimes}_{\Lambda(G)}M$, resp. $\Lambda(G)_{S^*}$-module $\Lambda(G)_{S^*}\hat{\otimes}_{\Lambda(G)}M$.

Lemma 2.1. *If $\Sigma\in\{S,S^*\}$ and $M\in\mathfrak{M}_\Sigma(G)$, then $\mathrm{I}_H^G(M)_\Sigma$ is a finitely generated $\Lambda(G)_\Sigma$-module and $\delta_{G,\gamma,M}$ induces an automorphism of $\mathrm{I}_H^G(M)_\Sigma$.*

Proof. We assume first that $\Sigma = S$ and fix M in $\mathfrak{M}_S(G)$. Then $\mathrm{I}_H^G(M)$ is a finitely generated $\Lambda(G)$-module and hence $\mathrm{I}_H^G(M)_S$ is a finitely generated $\Lambda(G)_S$-module. There is also a short exact sequence of (finitely generated) $\Lambda(G)$-modules

$$0\to\mathrm{I}_H^G(M)\xrightarrow{\delta_\gamma}\mathrm{I}_H^G(M)\to M\to 0 \qquad (1)$$

in which the third arrow is induced by $x\otimes_{\Lambda(H)} m\mapsto x(m)$ for each $x\in\Lambda(G)$ and $m\in M$. Indeed, the exactness of this sequence has been proved by Schneider and Venjakob [8, Prop. 2.2, Rem. 2.3]. Thus, by applying the (exact) scalar extension functor $\Lambda(G)_S\hat{\otimes}_{\Lambda(G)}-$ to (1) and noting that $M_S = 0$ (by assumption), we may deduce that δ_γ induces an automorphism of $\mathrm{I}_H^G(M)_S$.

From here the analogous results for modules M in $\mathfrak{M}_{S^*}(G)$ follow easily from the fact that M_{tf} belongs to $\mathfrak{M}_S(G)$ and $\mathrm{I}_H^G(M)_{S^*} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{I}_H^G(M_{\mathrm{tf}})_S$. □

§2.4. Algebraic p-adic L-functions

The ring $\Lambda(G)_{S^*}$ is both noetherian and regular [6, Prop. 4.3.4]. Hence there is a natural group isomorphism between the algebraic K-group $K_1(\Lambda(G)_{S^*})$ and the group $G_1(\Lambda(G)_{S^*})$ that is generated (multiplicatively) by symbols of the form $\langle \alpha \mid N\rangle$ where α is an automorphism of a finitely generated $\Lambda(G)_{S^*}$-module N (cf. [9, Th. 16.11]).

Definition 2.2. *Let $A^\bullet$ be a complex in $D^{\mathrm{fg}}(\Lambda(G))$ such that $\Lambda(G)_S\hat{\otimes}_{\Lambda(G)}A^\bullet$ is acyclic. Then (following Lemma 2.1) we define the Algebraic p-adic L-function of $A^\bullet$ (relative to G and γ) to be the element of $K_1(\Lambda(G)_{S^*})$ obtained by setting*

$$\mathcal{L}^{\mathrm{alg}}_{G,\gamma}(A^\bullet) := \prod_{i\in\mathbb{Z}} \langle \delta_{G,\gamma,H^i(A^\bullet)} \mid \mathrm{I}_H^G(H^i(A^\bullet))_{S^*}\rangle^{(-1)^i}.$$

The following result explicates this definition in a classical setting.

Lemma 2.3. *If $G = \Gamma$, then $\Lambda(G)_{S^*}$ is the total quotient ring $Q(\Gamma)$ of $\Lambda(\Gamma)$ and there is a natural isomorphism of groups $\iota : K_1(\Lambda(G)_{S^*}) \cong Q(\Gamma)^\times$. If M is any $\Lambda(\Gamma)$-module that is finitely generated over $\mathbb{Z}_p$, then*

$$\iota(\mathcal{L}^{\mathrm{alg}}_{G,\gamma}(M[0])) = (1+T)^{-\lambda(M)}\mathrm{char}_T(M)$$

where $\lambda(M)$ is the Iwasawa λ-invariant of M and $\mathrm{char}_T(M)$ the characteristic polynomial of M with respect to the variable $T = \gamma - 1$.

Proof. The first sentence is well-known (with the isomorphism ι induced by taking determinants over $Q(\Gamma)$). Further, by the structure theory of finitely generated $\Lambda(\Gamma)$-modules, it suffices to prove the second sentence in the case that $M = \Lambda(\Gamma)/(f)$ where f is a distinguished polynomial. Now $\mathcal{L}^{\mathrm{alg}}_{G,\gamma}(M[0]) = \langle \delta_\gamma \mid Q(\Gamma)\otimes_{\mathbb{Z}_p} M\rangle$ and so, if we consider $N := \mathbb{Z}_p[[T]]\otimes_{\mathbb{Z}_p}(\mathbb{Z}_p[[T]]/(f(T)))$ as a (free) $\mathbb{Z}_p[[T]]$-module via left multiplication, then $\iota(\mathcal{L}^{\mathrm{alg}}_{G,\gamma}(M[0])) = \det_{\mathbb{Z}_p[[T]]}(\alpha)$ where α is the endomorphism of N given by multiplication by

$$\mathrm{id} - (1+T)^{-1}\otimes(1+T) = ((1+T)^{-1}\otimes \mathrm{id})(T\otimes\mathrm{id} - \mathrm{id}\otimes T).$$

But, by using the $\mathbb{Z}_p[[T]]$-basis $\{1 \otimes (T^i \bmod(f(T))) : 0 \le i < \deg(f)\}$ of N, one computes that $\det_{\mathbb{Z}_p[[T]]}((1+T)^{-1} \otimes \mathrm{id} \mid N) = (1+T)^{-\mathrm{rank}_{\mathbb{Z}_p[[T]]}(N)} = (1+T)^{-\deg(f)} = (1+T)^{-\lambda(M)}$ and $\det_{\mathbb{Z}_p[[T]]}(T \otimes \mathrm{id} - \mathrm{id} \otimes T \mid N) = f(T) = \mathrm{char}_T(M)$. The displayed equality is therefore clear. □

Remark 2.4. In joint work with Venjakob [3] we reinterpret Definition 2.2 in terms of the localized K_1-groups introduced by Fukaya and Kato in [6]. If G has no element of order p, then in [3] we also extend Definition 2.2 (and the results proved in §3 and §4 below) to the case of complexes $A^\bullet$ in $D^{\mathrm{fg}}(\Lambda(G))$ for which only $\Lambda(G)_{S^*}\hat{\otimes}_{\Lambda(G)}A^\bullet$ is assumed to be acyclic.

§3. Basic Properties

If Σ denotes either S or S^*, then we write $D^{\mathrm{p}}_{\Sigma}(\Lambda(G))$ for the full triangulated subcategory of $D^{\mathrm{p}}(\Lambda(G))$ comprising complexes $A^\bullet$ in $D^{\mathrm{p}}(\Lambda(G))$ for which $\Lambda(G)_{\Sigma}\hat{\otimes}_{\Lambda(G)}A^\bullet$ is acyclic. For each M in $\mathfrak{M}_{S^*}(G)$ we also set

$$\mathcal{E}_{G,\gamma}(M) := \langle \delta_{G,\gamma,M} \mid \mathrm{I}^G_H(M)_{S^*}\rangle \in K_1(\Lambda(G)_{S^*}).$$

Proposition 3.1. (*Additivity*) *If $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ is an exact triangle in $D^{\mathrm{p}}_S(\Lambda(G))$, then $\mathcal{L}^{\mathrm{alg}}_{G,\gamma}(B^\bullet) = \mathcal{L}^{\mathrm{alg}}_{G,\gamma}(A^\bullet)\mathcal{L}^{\mathrm{alg}}_{G,\gamma}(C^\bullet)$.*

Proof. Applying the exact functor $\mathrm{I}^G_H(-)_{S^*}$ to the long exact cohomology sequence of the given triangle gives a commutative diagram of finitely generated $\Lambda(G)_{S^*}$-modules

$$\begin{array}{ccccccccc}
\to \mathrm{I}^G_H(H^i(A^\bullet))_{S^*} & \to & \mathrm{I}^G_H(H^i(B^\bullet))_{S^*} & \to & \mathrm{I}^G_H(H^i(C^\bullet))_{S^*} & \to & \mathrm{I}^G_H(H^{i+1}(A^\bullet))_{S^*} & \to \\
\downarrow{\scriptstyle \delta_{G,\gamma,H^i(A^\bullet)}} & & \downarrow{\scriptstyle \delta_{G,\gamma,H^i(B^\bullet)}} & & \downarrow{\scriptstyle \delta_{G,\gamma,H^i(C^\bullet)}} & & \downarrow{\scriptstyle \delta_{G,\gamma,H^{i+1}(A^\bullet)}} & \\
\to \mathrm{I}^G_H(H^i(A^\bullet))_{S^*} & \to & \mathrm{I}^G_H(H^i(B^\bullet))_{S^*} & \to & \mathrm{I}^G_H(H^i(C^\bullet))_{S^*} & \to & \mathrm{I}^G_H(H^{i+1}(A^\bullet))_{S^*} & \to
\end{array}$$

in which both rows are exact. Taken in conjunction with the defining relations of $K_1(\Lambda(G)_{S^*})$ this diagram implies the required equality

$$\begin{aligned}
\mathcal{L}^{\mathrm{alg}}_{G,\gamma}(B^\bullet) &= \prod_{i\in\mathbb{Z}} \mathcal{E}_{G,\gamma}(H^i(B^\bullet))^{(-1)^i} \\
&= \prod_{i\in\mathbb{Z}} \mathcal{E}_{G,\gamma}(H^i(A^\bullet))^{(-1)^i} \prod_{i\in\mathbb{Z}} \mathcal{E}_{G,\gamma}(H^i(C^\bullet))^{(-1)^i} = \mathcal{L}^{\mathrm{alg}}_{G,\gamma}(A^\bullet)\mathcal{L}^{\mathrm{alg}}_{G,\gamma}(C^\bullet).
\end{aligned}$$

□

Now let U be a closed subgroup of H that is normal in G and set $\overline{G} := G/U$, $\overline{H} := H/U$ and $\overline{S} := S(\overline{G},\overline{H})$. Then the natural projection $\Lambda(G) \to \Lambda(\overline{G})$ extends to a ring homomorphism $\Lambda(G)_{S^*} \to \Lambda(\overline{G})_{\overline{S}^*}$ and hence induces a homomorphism of groups $\pi_{\overline{G}} : K_1(\Lambda(G)_{S^*}) \to K_1(\Lambda(\overline{G})_{\overline{S}^*})$.

Proposition 3.2. (*Change of group*) *If $A^\bullet$ belongs to $D^{\mathrm{p}}_S(\Lambda(G))$, then $\Lambda(\overline{G})\hat{\otimes}^{\mathbb{L}}_{\Lambda(G)}A^\bullet$ belongs to $D^{\mathrm{p}}_{\overline{S}}(\Lambda(\overline{G}))$ and $\pi_{\overline{G}}(\mathcal{L}^{\mathrm{alg}}_{G,\gamma}(A^\bullet)) = \mathcal{L}^{\mathrm{alg}}_{\overline{G},\gamma}(\Lambda(\overline{G})\hat{\otimes}^{\mathbb{L}}_{\Lambda(G)}A^\bullet)$.*

Proof. The first assertion follows directly from the fact that there is a natural isomorphism in $D^{\mathrm{p}}(\Lambda(\overline{G})_{\overline{S}})$ of the form

$$\Lambda(\overline{G})_{\overline{S}}\hat{\otimes}_{\Lambda(\overline{G})}(\Lambda(\overline{G})\hat{\otimes}^{\mathbb{L}}_{\Lambda(G)}A^\bullet) \cong \Lambda(\overline{G})_{\overline{S}}\hat{\otimes}^{\mathbb{L}}_{\Lambda(G)_S}(\Lambda(G)_S\hat{\otimes}_{\Lambda(G)}A^\bullet).$$

To prove the second assertion we first recall that each term $\mathcal{E}_{G,\gamma}(H^i(A^\bullet))$ can be computed explicitly as follows. One can fix a complex $P_i^\bullet$ in $C^{\mathrm{p}}(\Lambda(G)_{S^*})$, an isomorphism $\psi_i : P_i^\bullet \to \mathrm{I}^G_H(H^i(A^\bullet))_{S^*}[0]$ in $D^{\mathrm{p}}(\Lambda(G)_{S^*})$ and a morphism $\alpha_i : P_i^\bullet \to P_i^\bullet$ in $C^{\mathrm{p}}(\Lambda(G)_{S^*})$ which is bijective in each degree and such that the following diagram commutes in $D^{\mathrm{p}}(\Lambda(G)_{S^*})$

$$\begin{array}{ccc} P_i^\bullet & \xrightarrow{\psi_i} & \mathrm{I}^G_H(H^i(A^\bullet))_{S^*}[0] \\ {\scriptstyle\alpha_i}\downarrow & & \downarrow{\scriptstyle\delta_{G,\gamma,H^i(A^\bullet)}[0]} \\ P_i^\bullet & \xrightarrow{\psi_i} & \mathrm{I}^G_H(H^i(A^\bullet))_{S^*}[0]; \end{array}$$

one then has $\mathcal{E}_{G,\gamma}(H^i(A^\bullet)) = \prod_{j\in\mathbb{Z}}\langle\alpha_i^j \mid P_i^j\rangle^{(-1)^j}$. Hence

$$\begin{aligned} \pi_{\overline{G}}(\mathcal{L}^{\mathrm{alg}}_{G,\gamma}(A^\bullet)) &= \prod_{i\in\mathbb{Z}}\pi_{\overline{G}}(\mathcal{E}_{G,\gamma}(H^i(A^\bullet))^{(-1)^i} \\ &= \prod_{i\in\mathbb{Z}}\prod_{j\in\mathbb{Z}}\langle \mathrm{id}\otimes\alpha_i^j \mid \Lambda(\overline{G})_{\overline{S}^*}\otimes_{\Lambda(G)_{S^*}} P_i^j\rangle^{(-1)^{i+j}} \\ &= \prod_{i\in\mathbb{Z}}\prod_{j\in\mathbb{Z}}\langle H^j(\mathrm{id}\otimes\alpha_i) \mid H^j(\Lambda(\overline{G})_{\overline{S}^*}\otimes_{\Lambda(G)_{S^*}} P_i^\bullet)\rangle^{(-1)^{i+j}} \qquad (2) \end{aligned}$$

where the last equality follows from the regularity of $\Lambda(\overline{G})_{\overline{S}^*}$ and the defining relations of $K_1(\Lambda(\overline{G})_{\overline{S}^*})$. Now there are natural isomorphisms of $\Lambda(\overline{G})_{\overline{S}^*}$-modules

$$H^j(\Lambda(\overline{G})_{\overline{S}^*}\otimes_{\Lambda(G)_{S^*}} P_i^\bullet) \cong \mathrm{I}^{\overline{G}}_{\overline{H}}(\mathrm{Tor}^{\Lambda(G)}_j(\Lambda(\overline{G}), H^i(A^\bullet)))_{\overline{S}^*}$$

under which $H^j(\mathrm{id}\otimes\alpha_i)$ corresponds to the endomorphism $\delta_{\overline{G},\gamma,\mathrm{Tor}^{\Lambda(G)}_j(\Lambda(\overline{G}),H^i(A^\bullet))}$.

The product expression (2) is therefore equal to

$$\begin{aligned}
&\prod_{i\in\mathbb{Z}}\prod_{j\in\mathbb{Z}}\langle\delta_{\overline{G},\gamma,\mathrm{Tor}_j^{\Lambda(G)}(\Lambda(\overline{G}),H^i(A^\bullet))}\mid \mathrm{I}_{\overline{H}}^{\overline{G}}(\mathrm{Tor}_j^{\Lambda(G)}(\Lambda(\overline{G}),H^i(A^\bullet)))_{\overline{S}^*}\rangle^{(-1)^{i+j}}\\
=&\prod_{i\in\mathbb{Z}}\langle\delta_{\overline{G},\gamma,H^i(\Lambda(\overline{G})\hat{\otimes}^{\mathbb{L}}_{\Lambda(G)}A^\bullet)}\mid \mathrm{I}_{\overline{H}}^{\overline{G}}(H^i(\Lambda(\overline{G})\hat{\otimes}^{\mathbb{L}}_{\Lambda(G)}A^\bullet))_{\overline{S}^*}\rangle^{(-1)^{i}}\\
=&\prod_{i\in\mathbb{Z}}\mathcal{E}_{\overline{G},\gamma}(H^i(\Lambda(\overline{G})\hat{\otimes}^{\mathbb{L}}_{\Lambda(G)}A^\bullet))^{(-1)^i}\\
=&\mathcal{L}^{\mathrm{alg}}_{\overline{G},\gamma}(\Lambda(\overline{G})\hat{\otimes}^{\mathbb{L}}_{\Lambda(G)}A^\bullet)
\end{aligned}$$

where the first displayed equality is a consequence of the spectral sequence $E_2^{r,s}=\mathrm{Tor}_r^{\Lambda(G)}(\Lambda(\overline{G}),H^s(A^\bullet))\implies H^{s-r}(\Lambda(\overline{G})\hat{\otimes}^{\mathbb{L}}_{\Lambda(G)}A^\bullet)$. □

Remark 3.3. (Akashi series) If G has no element of order p, then for each module M in $\mathfrak{M}_S(G)$ the complex $M[0]$ belongs to $D^{\mathrm{p}}_S(\Lambda(G))$. Proposition 3.2 (with $\overline{G}=\Gamma$) thus combines with Lemma 2.3 to imply that the composite homomorphism $K_1(Q(\Gamma))\cong Q(\Gamma)^\times\to Q(\Gamma)^\times/\Lambda(\Gamma)^\times$ sends the element $\pi_\Gamma(\mathcal{L}^{\mathrm{alg}}_{G,\gamma}(M[0]))=\mathcal{L}^{\mathrm{alg}}_{\Gamma,\gamma}(\Lambda(\Gamma)\hat{\otimes}^{\mathbb{L}}_{\Lambda(G)}M[0])$ to the 'Akashi series' f_M of M that is introduced by Coates, Schneider and Sujatha in [5, §4] (and is denoted by $\mathrm{Ak}(M)$ in [4, §3]).

§4. Characteristic Elements

We write $G_0(\mathfrak{M}_{S^*}(G))$ for the Grothendieck group of the category $\mathfrak{M}_{S^*}(G)$ and for each module M in $\mathfrak{M}_{S^*}(G)$ we write $[M]$ for the associated element of $G_0(\mathfrak{M}_{S^*}(G))$. We also write $K_0(\Lambda(G),\Lambda(G)_{S^*})$ for the relative algebraic K_0-group of the natural homomorphism $\Lambda(G)\to\Lambda(G)_{S^*}$ and recall that this group is generated by triples of the form (P,κ,Q) where P and Q are finitely generated projective $\Lambda(G)$-modules and κ is an isomorphism of $\Lambda(G)_{S^*}$-modules $P_{S^*}\cong Q_{S^*}$ (for further details see [9, p. 215]).

If G has no element of order p, then $\Lambda(G)$ is a noetherian regular ring and the groups $K_0(\Lambda(G),\Lambda(G)_{S^*})$ and $G_0(\mathfrak{M}_{S^*}(G))$ are naturally isomorphic. We normalise this isomorphism in the following way: if $g=s^{-1}h$ with $s\in S^*$ and $h\in\Lambda(G)\cap\Lambda(G)_{S^*}^\times$, then the element $(\Lambda(G),\mathrm{r}_g,\Lambda(G))$ of $K_0(\Lambda(G),\Lambda(G)_{S^*})$ corresponds to the element $[\mathrm{cok}(\mathrm{r}_h)]-[\mathrm{cok}(\mathrm{r}_s)]$ of $G_0(\mathfrak{M}_{S^*}(G))$ where $\mathrm{r}_g,\mathrm{r}_h$ and r_s denote the automorphisms of $\Lambda(G)_{S^*}$ that are induced by right multiplication by g,h and s respectively.

We next note that, irrespective of whether G has an element of order p, each complex $A^\bullet$ in $D^{\mathrm{p}}_{S^*}(\Lambda(G))$ gives rise to a canonical 'euler characteristic' element $\chi(A^\bullet)$ in $K_0(\Lambda(G),\Lambda(G)_{S^*})$. We define this element by identifying $K_0(\Lambda(G),\Lambda(G)_{S^*})$ with π_0 of a natural Picard category that is constructed from the categories of virtual objects $\mathcal{V}(\Lambda(G))$ and $\mathcal{V}(\Lambda(G)_{S^*})$ associated to the categories of finitely generated projective $\Lambda(G)$-modules and finitely generated projective $\Lambda(G)_{S^*}$-modules respectively (for further details see, for example, [1, Lem. 5.1]). Then, with respect to this identification, we let $\chi(A^\bullet)$ denote the *inverse* of the element of $K_0(\Lambda(G),\Lambda(G)_{S^*})$ that corresponds to the pair $([P^\bullet],\iota_{P^\bullet})$ where $[P^\bullet]$ is the object of $\mathcal{V}(\Lambda(G))$ associated to any $P^\bullet$ in $C^{\mathrm{p}}(\Lambda(G))$ that is isomorphic in $D^{\mathrm{p}}(\Lambda(G))$ to $A^\bullet$ and $\iota_{P^\bullet}$ is the morphism in $\mathcal{V}(\Lambda(G)_{S^*})$ associated to the isomorphism $\Lambda(G)_{S^*}\otimes_{\Lambda(G)}P^\bullet\cong\Lambda(G)_{S^*}\hat{\otimes}_{\Lambda(G)}P^\bullet\cong\Lambda(G)_{S^*}\hat{\otimes}_{\Lambda(G)}A^\bullet\cong 0$ in $D^{\mathrm{p}}(\Lambda(G)_{S^*})$. This element $\chi(A^\bullet)$ is the inverse of the element $\chi_{\Lambda(G),\Lambda(G)_{S^*}}(A^\bullet,t)$ that is defined in [1, Def. 5.5] with t equal to the isomorphism $\bigoplus_{i\in\mathbb{Z}}H^{2i}(A^\bullet)_{S^*}\cong 0\cong\bigoplus_{i\in\mathbb{Z}}H^{2i+1}(A^\bullet)_{S^*}$. (We prefer to define $\chi(A^\bullet)$ in terms of the inverse in order to ensure that if G has no element of order p and M belongs to $\mathfrak{M}_{S^*}(G)$, then the isomorphism $K_0(\Lambda(G),\Lambda(G)_{S^*})\cong G_0(\mathfrak{M}_{S^*}(G))$ described above sends $\chi(M[0])$ to $[M]$.)

In the next result we use the natural connecting homomorphisms in K-theory

$$\partial'_G: K_1(\Lambda(G)_{S^*})\cong G_1(\Lambda(G)_{S^*})\to G_0(\mathfrak{M}_{S^*}(G))$$

and

$$\partial_G: K_1(\Lambda(G)_{S^*})\to K_0(\Lambda(G),\Lambda(G)_{S^*}).$$

Theorem 4.1. *Let $A^\bullet$ be a complex in $D^{\mathrm{p}}_S(\Lambda(G))$.*

(i) $\partial'_G(\mathcal{L}^{\mathrm{alg}}_{G,\gamma}(A^\bullet))=\sum_{i\in\mathbb{Z}}(-1)^i[H^i(A^\bullet)]$.

(ii) *If G has rank one (as a p-adic Lie group), then $\partial_G(\mathcal{L}^{\mathrm{alg}}_{G,\gamma}(A^\bullet))=\chi(A^\bullet)$.*

Remark 4.2. (Characteristic elements) In the setting of Lemma 2.3 claim (i) of this result recovers the fact that $\mathrm{char}_T(M)$ generates the characteristic ideal of M. More generally, claim (i) implies that if G has no element of order p, then $\mathcal{L}^{\mathrm{alg}}_{G,\gamma}(A^\bullet)$ is a 'characteristic element of $A^\bullet$' in the sense of [4, (33)]. (If G has rank one, then) the equality of claim (ii) is in general much finer than that of claim (i) and plays a key role in the descent theory formulated in [3].

§4.1. The proof of Theorem 4.1(i)

We write $\partial''_G: G_1(\Lambda(G)_{S^*})\to G_0(\mathfrak{M}_{S^*}(G))$ for the natural connecting homomorphism (that occurs in the above definition of ∂'_G) and recall that if ν

is an automorphism of a finitely generated $\Lambda(G)_{S^*}$-module N, then

$$\partial''_G(\langle\nu, N\rangle) = [\mathcal{N}/(\mathcal{N}\cap\nu(\mathcal{N}))] - [\nu(\mathcal{N})/(\mathcal{N}\cap\nu(\mathcal{N}))] \tag{3}$$

where $\mathcal{N}$ is any finitely generated $\Lambda(G)$-submodule of N with $\mathcal{N}_{S^*} = N$.

For each integer i we set $M^i := H^i(A^\bullet)$. Then Lemma 4.3(ii) below (with $M = M^i$) implies that the (finitely generated) $\Lambda(G)$-module $\mathrm{I}_H^G(M^i_{\mathrm{tf}})$ is isomorphic to its image in $\mathrm{I}_H^G(M^i)_{S^*}$. Hence, if we set $\delta^i_\gamma := \delta_{G,\gamma,M^i}$, then in $G_0(\mathfrak{M}_{S^*}(G))$ one has

$$\begin{aligned}\partial'_G(\mathcal{L}^{\mathrm{alg}}_{G,\gamma}(A^\bullet)) &= \sum_{i\in\mathbb{Z}}(-1)^i\partial''_G(\langle\delta^i_\gamma \mid \mathrm{I}_H^G(M^i)_{S^*}\rangle) = \sum_{i\in\mathbb{Z}}(-1)^i[\mathrm{I}_H^G(M^i_{\mathrm{tf}})/\delta^i_\gamma(\mathrm{I}_H^G(M^i_{\mathrm{tf}}))]\\ &= \sum_{i\in\mathbb{Z}}(-1)^i[M^i_{\mathrm{tf}}] = \sum_{i\in\mathbb{Z}}(-1)^i([M^i] - [M^i_{\mathrm{tor}}]).\end{aligned}$$

Here the second equality follows from (3) (with $\mathcal{N} = \mathrm{I}_H^G(M^i_{\mathrm{tf}})$ and $\nu = \delta^i_\gamma$) and the fact that $\delta^i_\gamma(\mathrm{I}_H^G(M^i_{\mathrm{tf}})) \subseteq \mathrm{I}_H^G(M^i_{\mathrm{tf}})$ and the third from the isomorphism $\mathrm{I}_H^G(M^i_{\mathrm{tf}})/\delta^i_\gamma(\mathrm{I}_H^G(M^i_{\mathrm{tf}})) \cong M^i_{\mathrm{tf}}$ that is induced by (1) with $M = M^i_{\mathrm{tf}}$. Given the last displayed formula, the proof of Theorem 4.1(i) is therefore completed by applying Lemma 4.3(i) below with $M = M^i$ (for each i).

Lemma 4.3.

(i) *If M belongs to $\mathfrak{M}_S(G)$, then $[M_{\mathrm{tor}}] = 0$ in $G_0(\mathfrak{M}_{S^*}(G))$.*

(ii) *If M belongs to $\mathfrak{M}_{S^*}(G)$, then the natural map $\mathrm{I}_H^G(M_{\mathrm{tf}}) \to \mathrm{I}_H^G(M)_{S^*}$ is injective.*

Proof. Claim (i) follows from the fact that if M is in $\mathfrak{M}_S(G)$, then (1) with $M = M_{\mathrm{tor}}$ is a short exact sequence of objects of $\mathfrak{M}_{S^*}(G)$ and hence implies that $[M_{\mathrm{tor}}] = [\mathrm{I}_H^G(M_{\mathrm{tor}})] - [\mathrm{I}_H^G(M_{\mathrm{tor}})] = 0$ in $G_0(\mathfrak{M}_{S^*}(G))$.

To prove claim (ii) we note that $\mathrm{I}_H^G(M_{\mathrm{tf}})$ is a finitely generated $\Lambda(G)$-module that is $\mathbb{Z}_p$-torsion-free (and hence that a $\Lambda(G)$-submodule of $\mathrm{I}_H^G(M_{\mathrm{tf}})$ belongs to $\mathfrak{M}_{S^*}(G)$ if and only if it belongs to $\mathfrak{M}_S(G)$). It therefore suffices to prove that if M is any $\mathbb{Z}_p$-torsion-free module in $\mathfrak{M}_S(G)$ and N a $\Lambda(G)$-submodule of $\mathrm{I}_H^G(M)$ that is finitely generated over $\Lambda(H)$, then $N = 0$. To do this we fix a pro-p open subgroup J of H that is normal in G. We write $I(J)$ for the kernel of the projection map $\Lambda(G) \to \Lambda(G/J)$ (so $M_J \cong M/I(J)M$) and let $\overline{N}$ denote the image of N under the canonical projection $\mathrm{I}_H^G(M) \to \mathrm{I}_H^G(M_J) \cong \mathrm{I}_{H/J}^{G/J}(M_J)$.

Now if $\tilde{\Gamma}$ denotes the subgroup of G/J that is generated topologically by a choice of pre-image $\tilde{\gamma}$ of γ under the surjection $G/J \to \Gamma$, then $\mathrm{I}_{H/J}^{G/J}(M_J)$ is isomorphic as a $\Lambda(\tilde{\Gamma})$-module to $\Lambda(\tilde{\Gamma}) \otimes_{\mathbb{Z}_p} M_J$ and $\overline{N}$ identifies with a $\Lambda(\tilde{\Gamma})$-submodule of $\Lambda(\tilde{\Gamma}) \otimes_{\mathbb{Z}_p} M_J$ that is finitely generated over $\Lambda(H/J) = \mathbb{Z}_p[H/J]$ and hence also over $\mathbb{Z}_p$. We set $t := \tilde{\gamma} - 1 \in \Lambda(\tilde{\Gamma})$. Now $\overline{N}_{\rm tor}$ is a finite $\Lambda(\tilde{\Gamma})$-submodule of $\Lambda(\tilde{\Gamma}) \otimes_{\mathbb{Z}_p} M_{J,\rm tor} \subseteq \Lambda(\tilde{\Gamma}) \otimes_{\mathbb{Z}_p} M_J$ and so if $x \in \overline{N}_{\rm tor}$, then there exist integers $r > s > 0$ such that $t^r x = t^s x$. But every element of $\Lambda(\tilde{\Gamma}) \otimes_{\mathbb{Z}_p} M_{J,\rm tor}$ can be written uniquely in the form $\sum_{a \geq 0} t^a \otimes y_a$ with $y_a \in M_{J,\rm tor}$ and so $t^r x = t^s x$ implies that $x = 0$. It follows that $\overline{N}_{\rm tor} = 0$ and hence we can regard $\overline{N}$ as a $\Lambda(\tilde{\Gamma})$-submodule of $\Lambda(\tilde{\Gamma}) \otimes_{\mathbb{Z}_p} M_{J,\rm tf}$ which is itself finitely generated over $\mathbb{Z}_p$. But $M_{J,\rm tf}$ is a free $\mathbb{Z}_p$-module so $\Lambda(\tilde{\Gamma}) \otimes_{\mathbb{Z}_p} M_{J,\rm tf}$ is a free $\Lambda(\tilde{\Gamma})$-module and hence cannot contain any non-zero $\Lambda(\tilde{\Gamma})$-submodule which is finitely generated over $\mathbb{Z}_p$. Hence $\overline{N} = 0$. From the exact sequence $0 \to {\rm I}_H^G(I(J)M) \to {\rm I}_H^G(M) \to {\rm I}_H^G(M_J) \to 0$ we therefore deduce that $N \subseteq {\rm I}_H^G(I(J)M)$.

By successively repeating the above argument with M replaced by $I(J)M$, then $I(J)^2 M$ etc., we deduce that N is contained in $\bigcap_{k \geq 0} {\rm I}_H^G(I(J)^k M)$. Now $I(J)^{k'} M \subseteq I(J)^k M$ and hence also ${\rm I}_H^G(I(J)^{k'} M) \subseteq {\rm I}_H^G(I(J)^k M)$ for each $k' \geq k$ and so the intersections $\bigcap_{k \geq 0} I(J)^k M$ and $\bigcap_{k \geq 0} {\rm I}_H^G(I(J)^k M)$ can both be computed as inverse limits. Since completed tensor products commute with inverse limits it follows that $\bigcap_{k \geq 0} {\rm I}_H^G(I(J)^k M) = {\rm I}_H^G(\bigcap_{k \geq 0} I(J)^k M)$. But $I(J)$ is contained in the radical of $\Lambda(G)$ (since J is pro-p) and so $\bigcap_{k \geq 0} I(J)^k M = 0$ (cf. [7, Prop. (5.2.17)]). Hence $N = 0$, as required. □

§4.2. The proof of Theorem 4.1(ii)

In this subsection we assume that G has rank one as a p-adic Lie group and hence that $\Lambda(G)_{S^*}$ is equal to the semisimple artinian ring $Q(G)$. We note first that since $A^\bullet$ belongs to $D_S^{\rm p}(\Lambda(G))$ the complex ${\rm I}_H^G(A^\bullet)$ belongs to $D^{\rm p}(\Lambda(G))$. We may thus choose a complex $P^\bullet$ in $C^{\rm p}(\Lambda(G))$ for which there exists an isomorphism $\psi : P^\bullet \xrightarrow{\sim} {\rm I}_H^G(A^\bullet)$ in $D^{\rm p}(\Lambda(G))$ and a morphism $\alpha : P^\bullet \to P^\bullet$ in $C^{\rm p}(\Lambda(G))$ such that the following diagram commutes in $D^{\rm p}(\Lambda(G))$

$$\begin{array}{ccc} P^\bullet & \xrightarrow{\alpha} & P^\bullet \\ \psi \downarrow & & \downarrow \psi \\ {\rm I}_H^G(A^\bullet) & \xrightarrow{\delta_\gamma^\bullet} & {\rm I}_H^G(A^\bullet), \end{array} \tag{4}$$

where $\delta_\gamma^\bullet$ is the morphism with $\delta_\gamma^i = \delta_{G,\gamma,A^i}$ in each degree i. Now $H^i({\rm I}_H^G(A^\bullet)) = {\rm I}_H^G(H^i(A^\bullet))$ and so (1) with $M = H^i(A^\bullet)$ implies that $H^i(\delta_\gamma^\bullet)$ (and therefore also $H^i(\alpha)$) is injective in each degree i. Hence, by using Lemma 4.4 below, we may change α by a homotopy in order to assume that α^i is itself injective in each degree i. Thus there exists a short exact sequence of (bounded) complexes of finitely generated $\Lambda(G)$-modules

$$0 \to P^\bullet \xrightarrow{\alpha} P^\bullet \to \mathrm{cok}(\alpha)^\bullet \to 0 \tag{5}$$

where $\mathrm{cok}(\alpha)^i = \mathrm{cok}(\alpha^i)$ in each degree i and the differentials of $\mathrm{cok}(\alpha)^\bullet$ are induced by those of $P^\bullet$. This sequence implies that for each i the complex $\mathrm{cok}(\alpha)^i[-i]$ is naturally quasi-isomorphic to $P^i \xrightarrow{\alpha^i} P^i$, where the first term occurs in degree $i-1$, and hence both belongs to $D^{\mathrm{p}}_{S^*}(\Lambda(G))$ and also satisfies

$$\chi(\mathrm{cok}(\alpha)^i[-i]) = (-1)^i(P^i, \alpha^i, P^i) = \partial_G(\langle \alpha^i | P^i_{S^*}\rangle^{(-1)^i}), \tag{6}$$

where the first equality is a consequence of our chosen normalisation of $\chi(-)$ and the second a consequence of the definition of ∂_G.

Next we combine the exact sequence (5) with the commutativity of (4) to deduce that $\mathrm{cok}(\alpha)^\bullet$ is isomorphic in $D^{\mathrm{p}}(\Lambda(G))$ to the mapping cone $\mathrm{cone}(\delta^\bullet_\gamma)$ of $\delta^\bullet_\gamma$. On the other hand, by using the exact sequences (1) with $M = H^i(A^\bullet)$ for each i, it is straightforward to show that the morphism $\mathrm{cone}(\delta^\bullet_\gamma) \to A^\bullet$ which, in each degree i, sends $(x^i, x^{i+1}) \in \mathrm{I}^G_H(A^i) \oplus \mathrm{I}^G_H(A^{i+1}) = \mathrm{cone}(\delta^\bullet_\gamma)^i$ to the image of x^i under the natural map $\mathrm{I}^G_H(A^i) \to A^i$ is a quasi-isomorphism. It follows that $\mathrm{cok}(\alpha)^\bullet$ is isomorphic in $D^{\mathrm{p}}(\Lambda(G))$ to $A^\bullet$ and hence that $\chi(A^\bullet) = \chi(\mathrm{cok}(\alpha)^\bullet)$ by [1, Prop. 5.6]. Now in each degree i there is an exact sequence of complexes $0 \to \mathrm{cok}(\alpha)^i[-i] \to \tau_{\leq i}(\mathrm{cok}(\alpha)^\bullet) \to \tau_{\leq i-1}(\mathrm{cok}(\alpha)^\bullet) \to 0$ where $\tau_{\leq d}$ denotes naive truncation in degree d. By applying [1, Th. 5.7] to each of these exact sequences we obtain an equality $\chi(\mathrm{cok}(\alpha)^\bullet) = \sum_{i\in\mathbb{Z}} \chi(\mathrm{cok}(\alpha)^i[-i])$ and hence

$$\begin{aligned}
\partial_G(\mathcal{L}^{\mathrm{alg}}_{G,\gamma}(A^\bullet)) &= \partial_G\left(\prod_{i\in\mathbb{Z}} \langle \delta_{G,\gamma,H^i(A^\bullet)} | \mathrm{I}^G_H(H^i(A^\bullet))_{S^*}\rangle^{(-1)^i}\right) \\
&= \partial_G\left(\prod_{i\in\mathbb{Z}} \langle H^i(\alpha) | H^i(P^\bullet)_{S^*}\rangle^{(-1)^i}\right) \\
&= \partial_G\left(\prod_{i\in\mathbb{Z}} \langle \alpha^i | P^i_{S^*}\rangle^{(-1)^i}\right) \\
&= \sum_{i\in\mathbb{Z}} \chi(\mathrm{cok}(\alpha)^i[-i]) \\
&= \chi(A^\bullet)
\end{aligned}$$

where the second equality follows from the commutativity of (4), the third from the defining relations of $K_1(\Lambda(G)_{S^*})$ and the fourth from (6). This completes our proof of Theorem 4.1(ii).

Lemma 4.4. *Let $P^\bullet$ in $C^{\mathrm{p}}(\Lambda(G))$ be as in* (4) *and assume that G has rank one. If $\alpha : P^\bullet \to P^\bullet$ is any morphism of complexes for which $H^i(\alpha)$ is injective in each degree i, then there is a morphism of complexes $\hat{\alpha} : P^\bullet \to P^\bullet$ that is homotopic to α and injective in each degree.*

Proof. In each degree i there are tautological exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z^i & \longrightarrow & P^i & \xrightarrow{d^i} & B^{i+1} & \longrightarrow & 0 \\ 0 & \longrightarrow & B^i & \longrightarrow & Z^i & \longrightarrow & H^i & \longrightarrow & 0 \end{array} \tag{7}$$

where B^i, Z^i and H^i are the coboundaries, cocycles and cohomology of $P^\bullet$.

Now every finitely generated $\Lambda(G)[\frac{1}{p}]$-module is of projective dimension at most one. (Indeed, by [7, Prop. (5.3.19)(i)], this claim is true if we replace G by any (open) subgroup that is topologically isomorphic to $\mathbb{Z}_p$ and then the result for G itself follows by the argument of [6, Prop. 4.3.4].) Thus, the (image under $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} -$ of the) exact sequences (7) allow one to prove, by descending induction on i, that each $\Lambda(G)[1/p]$-module $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} B^i$ is projective. The upper sequence of (7) therefore splits after applying $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} -$ and so in each degree i we can choose a $\Lambda(G)$-submodule $\hat{B}^{i+1}$ of P^i such that $\hat{B}^{i+1} \cap Z^i = 0$ and $B^{i+1}/d^i(\hat{B}^{i+1})$ is p-torsion.

We next construct a homomorphism of $\Lambda(G)$-modules $k^i : P^i \to \hat{B}^i$ with

$$k^i(\hat{B}^{i+1}) = 0 \tag{8}$$

and such that the quotient module

$$B^i/(d^{i-1} \circ (\alpha^{i-1} + k^i \circ d^{i-1})(\hat{B}^i)) \tag{9}$$

is p-torsion. To do this we claim first that the cokernel of the map

$$\eta^i : \mathrm{Hom}_{\Lambda(G)}(P^i/\hat{B}^{i+1}, \hat{B}^i) \to \mathrm{Hom}_{\Lambda(G)}(B^i, \hat{B}^i)$$

that is induced by the composite $B^i \subset P^i \to P^i/\hat{B}^{i+1}$ is p-torsion. Indeed, the natural map $Z^i \to P^i/\hat{B}^{i+1}$ is injective with p-torsion cokernel and so there is a natural complex $\mathrm{Hom}_{\Lambda(G)}(P^i/\hat{B}^{i+1}, \hat{B}^i) \to \mathrm{Hom}_{\Lambda(G)}(B^i, \hat{B}^i) \to \mathrm{Ext}^1_{\Lambda(G)}(H^i(P^\bullet), \hat{B}^i)$ which has p-torsion cohomology in the central degree. Thus one has $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{cok}(\eta^i) = 0$ if $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{Ext}^1_{\Lambda(G)}(H^i(P^\bullet), \hat{B}^i) = 0$. But $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{B}^i$ is a projective $\Lambda(G)[1/p]$-module and so $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{Ext}^1_{\Lambda(G)}(H^i(P^\bullet), \hat{B}^i) = 0$ if $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{Ext}^1_{\Lambda(G)}(H^i(P^\bullet), \Lambda(G)) = 0$. Moreover, the module $H^i(P^\bullet)$ is isomorphic to $\Lambda(G) \otimes_{\Lambda(H)} H^i(A^\bullet)$ and so $\mathrm{Ext}^1_{\Lambda(G)}(H^i(P^\bullet), \Lambda(G))$ is isomorphic to $\mathrm{Ext}^1_{\Lambda(H)}(H^i(A^\bullet), \Lambda(H)) \otimes_{\Lambda(H)} \Lambda(G)$ (cf. the discussion following [8, Prop. 3.1]). To prove that $\mathrm{cok}(\eta^i)$ is p-torsion it is thus enough to note that $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{Ext}^1_{\Lambda(H)}(H^i(A^\bullet), \Lambda(H)) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{Ext}^1_{\mathbb{Z}_p[H]}(H^i(A^\bullet), \mathbb{Z}_p[H])$ vanishes because H is finite.

Next we note that the vanishing of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} (B^{i+1}/d^i(\hat{B}^{i+1}))$ implies that the kernel and cokernel of the homomorphism $\mathrm{Hom}_{\Lambda(G)}(B^i, \hat{B}^i) \to \mathrm{Hom}_{\Lambda(G)}(\hat{B}^i, B^i)$ that sends k to $d^{i-1} \circ k \circ d^{i-1}$ are both p-torsion. Since $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{cok}(\eta^a) = 0$ in each degree a this allows us to choose k^i in $\mathrm{Hom}_{\Lambda(G)}(P^i/\hat{B}^{i+1}, \hat{B}^i)$ so that $d^{i-1} \circ \alpha^i + d^{i-1} \circ k^i \circ d^{i-1} \in \mathrm{Hom}_{\Lambda(G)}(\hat{B}^i, B^i)$ has p-torsion kernel and p-torsion cokernel. Then these homomorphisms k^i satisfy both (8) and (9) and we set $\hat{\alpha}^i := \alpha^i - (d^{i-1} \circ k^i + k^{i+1} \circ d^i)$. We thereby obtain a morphism of complexes $\hat{\alpha}$ that is homotopic to α via the homotopy $\{-k^i\}_{i\in\mathbb{Z}}$. Also (8) and (9) combine to imply that $B^{i+1}/d^i(\hat{\alpha}^i(\hat{B}^{i+1}))$ is p-torsion in each degree i. But each map $d^i : \hat{B}^{i+1} \to B^i$ is injective with p-torsion cokernel and hence $\hat{\alpha}^{i+1} : B^{i+1} \to B^{i+1}$ is also injective with p-torsion cokernel in each degree $i+1$. In addition $H^i(\hat{\alpha}) = H^i(\alpha)$ is assumed to be injective in each degree i and hence, by an easy exercise involving the exact sequences (7), we may deduce that each homomorphism $\hat{\alpha}^i$ is itself injective, as required. □

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