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Reflexivity of Spaces of Polynomials on Direct Sums of Banach Spaces

By

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Abstract

Let $P(^nE, F)$ be the space of the continuous n-homogeneous polynomials from E into F and $H_b(E, F)$ be the space of the holomorphic mappings from E into F that are bounded in the bounded subsets of E, both spaces endowed with the topology τ_b of uniform convergence on the bounded subsets of E. The reflexivity of $P(^nE, F)$ is studied in connection with the density of the space of the finite type n-homogeneous polynomials in $P(^nE, F)$ and in connection with the equality $[P(^nE, F), \tau_b]' = [P(^nE, F), \tau_0]'$ in case E is a reflexive countable direct sum of complex Banach spaces and F is a reflexive complex Banach space. The reflexivity of $H_b(E)$ is also considered.

§1. Introduction

If E' is the topological dual of a complex locally convex space E, we will write $E'_w = (E', \sigma(E', E''))$ where $\sigma(E', E'')$ denotes the weak topology on E' and $E'_b = (E', \beta(E', E))$ where $\beta(E', E)$ denotes the topology on E' of uniform convergence on the bounded subsets of E. Given complex locally convex spaces E and F, let $P(^nE, F)$ be the space of continuous n-homogeneous polynomials from E into F. As usual, if $E = \mathbb{C}$ we will write $P(^nE)$.

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We shall consider in $P(^nE,F)$ the standard topologies τ_0 and τ_b where τ_0 denotes the compact open topology and τ_b denotes the topology of uniform convergence on the bounded subsets of E. We refer to [16] for the definition of the topology τ_ω on $P(^nE,F)$. In general we have $\tau_0 \preceq \tau_b \preceq \tau_\omega$. The topologies τ_b and τ_ω coincide in $P(^nE,F)$ whenever E is a countable inductive limit of normed spaces (cf. Example 1.25, p. 25–26 in [16]).

The reflexivity of spaces of continuous scalar valued homogeneous polynomials on reflexive Banach spaces was first studied by R. Ryan in his thesis (see [26] and [3]). After his pioneering work, many authors considered the problem of finding necessary and sufficient condition for the space $P(^nE)$ to be reflexive (see [3], [4], [5], [18] and [13]). Alencar, Aron and Dineen showed in [5] that $P(^nT^*)$ is reflexive for every n, where T^* is the Tsirelson's space (see [27]). This was the first example of an infinite dimensional Banach space E for which the space $P(^nE)$ is reflexive for every n. Aron, Moraes and Ryan proved in [8] that if E is a quotient of T^* then $P(^nE)$ is reflexive. The vector valued case was first considered by Alencar who proved in [4] that if E and F are reflexive Banach spaces with the approximation property, then $P(^nE,F)$ is reflexive if and only if each $P \in P(^nE,F)$ is weakly continuous on bounded sets. Later Jaramillo and Moraes showed in [22] that the above necessary and sufficient conditions for the reflexivity of $P(^nE,F)$ remains valid when only E is assumed to have the approximation property. The reflexivity of the space $H_b(E,F)$ of the holomorphic mappings from E into F that are bounded on the bounded sets was also considered in [22].

We define $P_{wu}(^nE,F)$ as the space of the elements of $P(^nE,F)$ that are uniformly weakly continuous on the bounded subsets of E and $P_f(^nE,F)$ as the space of the finite type n-homogeneous polynomials i.e., $P_f(^nE,F) = span\{\varphi^n \otimes b, \varphi \in E', b \in F\}$ where $\varphi^n \otimes b(z) = \varphi^n(z) \cdot b$. The closure of $P_f(^nE,F)$ in $[P(^nE,F),\tau_b]$ will be denoted by $\overline{P_f(^nE,F)}$. For locally convex spaces E and F, Boyd showed that $\overline{P_f(^nE,F)} = P_{wu}(^nE,F)$ whenever E' has the approximation property (cf. [10], Proposition 7). The space of all $P \in P(^nE,F)$ which maps weakly convergent sequences in E to convergent sequences in F will be denoted by $P_{wsc}(^nE,F)$. It is clear that $P_{wu}(^nE,F) \subset P_{wsc}(^nE,F)$.

In this paper we are going to study the density of $P_f(^nE,F)$ in $P(^nE,F)$ in connection with the reflexivity of $[P(^nE,F),\tau_b]$ and the reflexivity of $H_b(E,F)$ in the case E is an countable direct sum of Banach spaces and F is a Banach space.

We refer to [15], [16] and [23] for background information on polynomials and holomorphic mappings over locally convex spaces, in particular for the definitions of the topologies τ_δ and τ_ω in $H(E,F)$.

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§2. Reflexivity of Spaces of Polynomials in Direct Sums of Banach Spaces

Given any countable family of complex Banach spaces E_i, let $E = \sum_{i=1}^{\infty} E_i$ be the direct sum of the spaces E_i. For every $m \in \mathbb{N}$ let $F_m = \sum_{i=1}^{m} E_i$. We remark that E is the strict inductive limit of the countable family of complex Banach spaces F_m and consequently E is a (DF)-space which is barrelled and bornological.

Remark 1. By using the polarization formula and Example 1.25 in [16] it is easy to verify that for each $n \in \mathbb{N}$ we have that $P \in P(^nE, F)$ if and only if $P|F_m \in P(^nF_m, F)$ for every $m \in \mathbb{N}$.

Proposition 2.1. *Let $E = \sum_{i=1}^{\infty} E_i$ where E_i is a complex Banach space for every $i \in \mathbb{N}$ and F be a complex Banach space. Then, for each $n \in \mathbb{N}$, we have that $P_{wu}(^nE, F) = P(^nE, F)$ if and only if $P_{wu}(^nF_m, F) = P(^nF_m, F)$ for all $m \in \mathbb{N}$.*

Proof. The proof uses the above remark, the fact that E is a regular inductive limit of the spaces F_m and the Hahn-Banach Theorem. ☐

The following lemma was proved by Dineen in the scalar case (see the proof of Proposition 3.1 in [14] or Proposition 4 in [17]). The same argument works to prove the vectorial case.

Lemma 2.1. *Let $E = \sum_{i=1}^{\infty} E_i$ where E_i is a complex Banach space for every $i \in \mathbb{N}$ and F be a complex Banach space. Let p be a τ_δ-continuous seminorm on $H(E,F)$. Then there exists a positive integer m such that $f \in H(E,F)$ and $f|F_m = 0$ imply $p(f) = 0$.*

We remark that the topology τ_δ in $H(E,F)$ induces the topology τ_ω in $P(^nE, F)$ if E is a locally convex space and F is a Banach space (cf. Proposition 2.41 in [15]) and we recall that the topologies τ_b and τ_ω coincide in $P(^nE, F)$ whenever $E = \sum_{i=1}^{\infty} E_i$ where E_i is a complex Banach space for every $i \in \mathbb{N}$ and F be a complex Banach space.

Proposition 2.2. *Let $E = \sum_{i=1}^{\infty} E_i$ where E_i is a complex Banach space for every $i \in \mathbb{N}$ and F be a complex Banach space. Then for each $n \in \mathbb{N}$ we have that $[P(^nE,F),\tau_b]' = [P(^nE,F),\tau_0]'$ if and only if $[P(^nF_m,F),\tau_b]' = [P(^nF_m,F),\tau_0]'$, for all $m \in \mathbb{N}$.*

Proof. Suppose that $[P(^nE,F),\tau_b]' = [P(^nE,F),\tau_0]'$. Fix $m \in \mathbb{N}$. From $\tau_b \succeq \tau_0$ it is clear that $[P(^nF_m,F),\tau_0]' \subset [P(^nF_m,F),\tau_b]'$. Now, given $\phi \in [P(^nF_m,F),\tau_b]'$, we define $\psi : P(^nE,F) \longrightarrow \mathbb{C}$ by letting $\psi(P) = \phi(P|F_m)$. The linearity of ψ is clear and the τ_b-continuity of ψ follows from the τ_b-continuity of ϕ and from the fact that τ_b-convergence in $P(^nE,F)$ implies τ_b-convergence in $P(^nF_m,F)$. Now we use the hypothesis to get $\psi \in [P(^nE,F),\tau_0]'$. Finally, we claim that $\phi \in [P(^nF_m,F),\tau_0]'$. Indeed, given a net $(Q_\lambda)_{\lambda\in\Lambda}$ in $P(^nF_m,F)$ such that $Q_\lambda \to Q$ in $[P(^nF_m,F),\tau_0]$, let $\tilde{Q}_\lambda := Q_\lambda \circ \pi_m$ and $\tilde{Q} := Q \circ \pi_m$ where π_m is the projection of E onto F_m. It is clear that $\tilde{Q}_\lambda \to \tilde{Q}$ in $[P(^nE,F),\tau_0]$, since for all compact $K \subset E$ we have $\|\tilde{Q}_\lambda - \tilde{Q}\|_K = \|Q_\lambda - Q\|_{\pi_m(K)}$ with $\pi_m(K)$ compact. Then $\psi(\tilde{Q}_\lambda) \to \psi(\tilde{Q})$ and consequently $\phi(Q_\lambda) \to \phi(Q)$. Conversely take $\psi \in [P(^nE,F),\tau_b]'$. Since the topology induced in $P(^nE,F)$ by the topology τ_δ in $H(E,F)$ coincides with τ_b, by the Hahn-Banach Theorem there exists $\tilde{\psi} \in [H(E),\tau_\delta]'$ such that $\tilde{\psi}|P(^nE,F) = \psi$. So $p(f) = |\tilde{\psi}(f)|$ is a τ_δ-continuous seminorm on $H(E,F)$ and therefore, by Lemma 2.1, there exists m_0 such that $p(f) = 0$ whenever $f \in H(E,F)$ satisfies $f|F_{m_0} = 0$.

Define $\mu : [P(^nF_{m_0},F),\tau_b] \longrightarrow \mathbb{C}$ by letting $\mu(Q) = \psi(\tilde{Q})$ where $\tilde{Q} = Q \circ \pi_{m_0}$. By hypothesis μ is τ_b-continuous if and only if μ is τ_0-continuous. Moreover if $Q_\lambda \to Q$ in $[P(^nF_{m_0},F),\tau_b]$, then $\tilde{Q}_\lambda \to \tilde{Q}$ in $[P(^nE,F),\tau_b]$, as $\|\tilde{Q}_\lambda - \tilde{Q}\|_B = \|Q_\lambda - Q\|_{\pi_{m_0}(B)}$ and $\pi_{m_0}(B)$ is bounded in F_{m_0} whenever B is a bounded subset of E. Therefore $\mu(Q_\lambda) \to \mu(Q)$, and so μ is τ_b-continuous and consequently τ_0-continuous. Now if $P_\alpha \to P$ in $[P(^nE,F),\tau_0]$, then $P_\alpha|F_{m_0} \to P|F_{m_0}$ in $[P(^nF_{m_0},F),\tau_0]$ and then $\mu(P_\alpha|F_{m_0}) \to \mu(P|F_{m_0})$. But $\mu(P_\alpha|F_{m_0}) = \psi(P_\alpha|F_{m_0}) = \psi(P_\alpha)$ and $\mu(P|F_{m_0}) = \psi(P|F_{m_0}) = \psi(P)$, as $(P_\alpha|F_{m_0} - P_\alpha)|F_{m_0} = 0$ and $(P|F_{m_0} - P)|F_{m_0} = 0$. So, $\psi(P_\alpha) \to \psi(P)$ and we have $[P(^nE,F),\tau_b]' \subset [P(^nE,F),\tau_0]'$. Finally, $\tau_b \succeq \tau_0$ gives the converse inclusion. $\square$

Proposition 2.3. *Let $E = \sum_{i=1}^{\infty} E_i$ where E_i is a complex Banach space for every $i \in \mathbb{N}$ and F be a complex Banach space. Then for each $n \in \mathbb{N}$, $[P(^nE,F),\tau_b]$ is quasi complete for its weak topology if and only if $[P(^nF_m,F),\tau_b]$ is quasi complete for its weak topology for all $m \in \mathbb{N}$.*

Proof. Suppose that $[P(^nF_m,F),\tau_b]$ is quasi complete for its weak topology for every $m \in \mathbb{N}$. Let $(P_\alpha)_\alpha \subset P(^nE, F)$ be a bounded Cauchy net for the weak topology of $[P(^nE, F), \tau_b]$. We claim that $(P_\alpha|F_m)_\alpha$ is a bounded Cauchy net for the weak topology of $[P(^nF_m, F), \tau_b]$, for all $m \in \mathbb{N}$. Indeed, $(P_\alpha|F_m)_\alpha$ is bounded since the unit ball of F_m is a bounded subset of E and $(P_\alpha)_\alpha$ is a bounded net in $[P(^nE, F), \tau_b]$. As in the last proposition, given $\phi \in [P(^nF_m, F), \tau_b]'$, the mapping $\psi : [P(^nE, F), \tau_b] \longrightarrow \mathbb{C}$ defined by $\psi(Q) = \phi(Q|F_m)$ belongs to $[P(^nE, F), \tau_b]'$. Therefore $(\psi(P_\alpha))_\alpha = (\phi(P_\alpha|F_m))_\alpha$ is a Cauchy net in $\mathbb{C}$. By hypothesis, for each m there exists $P_m \in P(^nF_m, F)$ such that $P_\alpha|F_m \to P_m$ in the weak topology of $[P(^nF_m, F), \tau_b]$. In particular, $P_\alpha|F_m(x) \to P_m(x)$, for all $x \in F_m$. Let $P : E \longrightarrow F$ given by $P(x) = P_m(x)$ where m is such that $x \in F_m$. It is clear that P is well defined and $P|F_m = P_m$, for all $m \in \mathbb{N}$. Moreover $P \in P(^nE, F)$ by Remark 1 above. Finally we are going to show that $P_\alpha \to P$ in the weak topology of $[P(^nE, F), \tau_b]$. Let $\psi \in [P(^nE, F), \tau_b]'$. We saw already that there exists $m \in \mathbb{N}$ such that the mapping $\mu : [P(^nF_m, F), \tau_b] \longrightarrow \mathbb{C}$ defined by $\mu(Q) = \psi(\tilde{Q})$, where $\tilde{Q} = Q \circ \pi_m$, is continuous. Therefore $\mu(P_\alpha|F_m) \to \mu(P_m)$. We also saw that $\mu(P_\alpha|F_m) = \psi(P_\alpha)$ and $\mu(P_m) = \psi(P)$. Consequently $\psi(P_\alpha) \to \psi(P)$, and this completes the proof.

Conversely assume that $[P(^nE, F), \tau_b]$ is quasi complete for its weak topology. Fix any $m \in \mathbb{N}$. Suppose that $(P_\alpha)_\alpha$ is a bounded Cauchy net for the weak topology of $[P(^nF_m, F), \tau_b]$. Let $\tilde{P}_\alpha = P_\alpha \circ \pi_m$. We show as in last proposition that for every $\psi \in [P(^nE, F), \tau_b]'$ the mapping $\mu : [P(^nF_m, F), \tau_b] \longrightarrow \mathbb{C}$ defined by $\mu(Q) = \psi(\tilde{Q})$ is continuous. Therefore $(\mu(P_\alpha))_\alpha$ is a Cauchy net in $\mathbb{C}$, and this means that $(\psi(\tilde{P}_\alpha))_\alpha$ is a Cauchy net in $\mathbb{C}$. By hypothesis there exists $\tilde{P} \in P(^nE, F)$ such that $\tilde{P}_\alpha \to \tilde{P}$ in the weak topology of $[P(^nE, F), \tau_b]$. Finally, given $\phi \in [P(^nF_m, F), \tau_b]'$, the mapping $\psi : P(^nE, F) \longrightarrow \mathbb{C}$ defined by $\psi(Q) = \phi(Q|F_m)$ is linear and τ_b-continuous. Therefore, $\psi(\tilde{P}_\alpha) \to \psi(\tilde{P})$ and so $\phi(\tilde{P}_\alpha|F_m) \to \phi(\tilde{P}|F_m)$. The proof follows since $\tilde{P}_\alpha|F_m = P_\alpha$ and $\tilde{P}|F_m \in P(^nF_m, F)$. □

Theorem 2.1. *Let $E = \sum_{i=1}^{\infty} E_i$ where E_i is a reflexive complex Banach space for every $i \in \mathbb{N}$. Then for every reflexive complex Banach space F and for each $n \in \mathbb{N}$ we have that $[P(^nE, F), \tau_0]' = [P(^nE, F), \tau_b]'$ if and only if $[P(^nE, F), \tau_b]$ is reflexive.*

Proof. Suppose $[P(^nE, F), \tau_0]' = [P(^nE, F), \tau_b]'$. Then by Proposition 2.1 above and Theorem 3.1 of [24] we have that $[P(^nF_m, F), \tau_b]$ is reflexive for all $m \in \mathbb{N}$. So, by Proposition 3 in [21], p. 228, $[P(^nF_m, F), \tau_b]$ is quasi complete for its weak topology for all $m \in \mathbb{N}$ and, by Proposition 2.3 above, $[P(^nE, F), \tau_b]$

is quasi complete for its weak topology. Now, using again Proposition 3 in [21], p. 228, and the fact that $[P(^nE,F),\tau_b]$ is a barrelled space whenever E is a (DF)-space we get the reflexivity of $[P(^nE,F),\tau_b]$.

Conversely if $[P(^nE,F),\tau_b]$ is reflexive, we have that $[P(^nE,F),\tau_b]$ is quasi complete for its weak topology and, by Proposition 2.3, $[P(^nF_m,F),\tau_b]$ is quasi complete for its weak topology for all $m \in \mathbb{N}$. So, $[P(^nF_m,F),\tau_b]$ is reflexive (recall that it is a barrelled space) and, by Theorem 3.1 of [24] we have that $[P(^nF_m,F),\tau_0]' = [P(^nF_m,F),\tau_b]'$ for all $m \in \mathbb{N}$. By Proposition 2.2 this means that $[P(^nE,F),\tau_0]' = [P(^nE,F),\tau_b]'$. □

Remark 2. From the fact that $P(^nE,F) = P(^nE,F_w)$ when E and F are Banach spaces, it is easy to show that this equality remains true for $E = \sum_{i=1}^{\infty} E_i$ (E_i is a complex Banach space for every $i \in \mathbb{N}$) and F a complex Banach space. If, in addition, F is reflexive, following ideas of C. Boyd in [11] we can introduce in $P(^nE,F)$ a topology τ_γ by $[P(^nE,F),\tau_\gamma] = [P(^nE,F_\gamma),\tau_0]$ where F_γ is the space F endowed with the topology of uniform convergence on compact subsets of F'. A net $(\tilde{Q}_\lambda)$ converges to $\tilde{Q}$ in $[P(^nE,F),\tau_\gamma]$ if $\sup_{x\in K}\ p\circ(\tilde{Q}_\lambda - \tilde{Q})(x) \to 0$ for every compact subset K of E and for every continuous seminorm p on F_γ and it is known that τ_γ is weaker then τ_0. Now the proof of Proposition 2.2 can be easily adapted to prove that for $E = \sum_{i=1}^{\infty} E_i$ (where E_i is a complex Banach space for every $i \in \mathbb{N}$), F a reflexive complex Banach space and for each $n \in \mathbb{N}$ we have that $[P(^nE,F),\tau_b]' = [P(^nE,F),\tau_\gamma]'$ if and only if $[P(^nF_m,F),\tau_b]' = [P(^nF_m,F),\tau_\gamma]'$, for all $m \in \mathbb{N}$.

Theorem 2.2. *Let $E = \sum_{i=1}^{\infty} E_i$ where E_i is a reflexive complex Banach space for every $i \in \mathbb{N}$. Then for every reflexive complex Banach space F and for each $n \in \mathbb{N}$ we have that $[P(^nE,F),\tau_\gamma]' = [P(^nE,F),\tau_b]'$ if and only if $[P(^nE,F),\tau_b]$ is reflexive.*

Proof. By the remarks following Theorem 8 of [11] we obtain that $[P(^nF_m,F),\tau_b]$ is reflexive if and only if $[P(^nF_m,F),\tau_\gamma]' = [P(^nF_m,F),\tau_b]'$. (Theorem 8 of [11] is actually true for holomorphic functions but, as the author remarks, the analogous result is also true for spaces of homogeneous polynomials.) By using this equivalence and Remark 2 we get that $[P(^nE,F),\tau_\gamma]' = [P(^nE,F),\tau_b]'$ if and only if $[P(^nF_m,F),\tau_b]$ is reflexive for every $m \in \mathbf{N}$. Finally our theorem follows by using Theorem 3.1 of [24], Proposition 2.2 and Theorem 2.1. □

Theorem 2.3. *Let $E = \sum_{i=1}^{\infty} E_i$ where E_i is a reflexive complex Banach space with the approximation property for every $i \in \mathbb{N}$ and let F be a*

reflexive complex Banach space. For every $n \in \mathbb{N}$, the following conditions are equivalent:
(1) $[P(^nE,F),\tau_0]' = [P(^nE,F),\tau_b]'$.
(2) $[P(^nE,F),\tau_b]$ *is reflexive.*
(3) $[P(^nE,F),\tau_\gamma]' = [P(^nE,F),\tau_b]'$.
(4) $P_{wu}(^nE,F) = P(^nE,F)$.
(5) $\overline{P_f(^nE,F)} = P(^nE,F)$.

Proof. The equivalence between (1) and (2) and between (2) and (3) are established by Theorem 2.1 and Theorem 2.2, respectively. The equivalence between (1) and (5) follows by Theorem 3.1 of [24], Proposition 7 of [10], Proposition 2.2 and Proposition 2.1. The equivalence between (1) and (4) follows by Theorem 3.1 of [24], Proposition 2.2 and Proposition 2.1. □

Remark 3. We remark that the implications (5) ⇒ (4) ⇒ (3) ⇔ (2) ⇔ (1) are true even if the spaces E_i don't have the approximation property and the implication (2) ⇒ (4) remains true if the spaces E_i have the compact approximation property instead of the approximation property.

We also remark that $E = \sum_{i=1}^\infty E_i$ has the approximation property if and only if E_i has the approximation property for every $i \in \mathbb{N}$.

Proposition 2.4. *Let $E = \mathbb{C}^{(\mathbb{N})}$ and let F be an arbitrary complex Banach space. Then $\overline{P_f(^nE,F)} = P(^nE,F)$ for every $n \in \mathbb{N}$.*

Proof. By Lemma 1 of [9] every bounded subset of $\mathbb{C}^{(\mathbb{N})}$ is finite dimensional and consequently every homogeneous polynomial $P : \mathbb{C}^{(\mathbb{N})} \to F$ is weakly continuous on bounded sets. Moreover, as the strong dual of $\mathbb{C}^{(\mathbb{N})}$ has the approximation property, by Proposition 7 of [10] we have $\overline{P_f(^n\mathbb{C}^{(\mathbb{N})},F)} = P_{wu}(^n\mathbb{C}^{(\mathbb{N})},F)$ and the statement follows . □

Proposition 2.5. *Let $E = \sum_{i=1}^\infty E_i$ where E_i is a complex Banach space for every $i \in \mathbb{N}$ and let F be an arbitrary complex Banach space. If each F_m does not contain a copy of ℓ_1, then $P_{wsc}(^nE,F) = P_{wu}(^nE,F)$ for all $n \in \mathbb{N}$.*

Proof. If $(x_n)_n$ is a sequence in E such that $x_n \to x$ as $n \to \infty$ in the topology $\sigma(E,E')$, then $B = \{x_n : n \in \mathbb{N}\} \cup \{x\}$ is a weakly bounded subset of E and consequently it is bounded. It follows that $P_{wu}(^nE,F) \subset P_{wsc}(^nE,F)$. Conversely, take $P \in P_{wsc}(^nE,F)$. For each $m \in \mathbb{N}$ we have that if (x_n) is an arbitrary sequence in F_m which converges to $x \in F_m$ in the weak topology of F_m then (x_n) converges to x in the weak topology of E. Consequently $P|F_m \in P_{wsc}(^nF_m, F)$ for every $m \in \mathbb{N}$. Moreover, as F_m is a Banach space that does not contain a copy of ℓ_1, by Proposition 2.12 of [7] we have that $P_{wsc}(^nF_m, F) = P_{wu}(^nF_m, F)$. So, $P|F_m \in P_{wu}(^nF_m, F)$ for all $m \in \mathbb{N}$ and by using the Hahn-Banach Theorem and the fact that E is a regular inductive limit we get that $P \in P_{wu}(^nE, F)$. □

We remark that the above result remains true for more general cases of regular inductive limits. For instance, it is true in case E is a weakly compact regular inductive limit of Banach spaces $(E_\alpha)_{\alpha \in A}$ such that $\ell_1 \not\subseteq E_\alpha$ for every $\alpha \in A$ and in case $E = \underset{n \to \infty}{ind}\ E_n$ is a regular countable inductive limit with the property M_0 of Banach spaces E_n such that $\ell_1 \not\subseteq E_n$ for every $n \in \mathbb{N}$ (see Theorems 2.5 and 2.6 in [1]).

Aron and Dineen showed in [6] that the Tsirelson-James space T_J^* satisfies the equation $\overline{P_f(^nT_J^*)} = P(^nT_J^*)$ for every $n \in \mathbb{N}$. For vector valued polynomials it is known that $\overline{P_f(^nT_J^*, F)} = P(^nT_J^*, F)$ for all $n \in \mathbb{N}$ whenever the Banach space F has positive rank (see Example 2.3 in [20]). We recall that a Banach space has positive rank if there exists $\alpha \in (0,1)$ such that every sequence (x_n) satisfying $\|\sum_{n \in B} x_n\| \leq c|B|^\alpha$ for $c \geq 0$ and for all finite $B \subset \mathbb{N}$ (where $|B|$ is the number of elements of B) converges with respect to the norm.

Proposition 2.6. *Let $E = \sum_{i=1}^\infty E_i$ where $E_i = T_J^*$ for every $i \in \mathbb{N}$. Then $\overline{P_f(^nE, F)} = P(^nE, F)$ for all $n \in \mathbb{N}$ whenever F is a complex Banach space which is Schur.*

Proof. It is known (see Example 3 of [12] or the proof of Proposition 5.5 in [28], p.87) that $\overline{P_f(^nE)} = P(^nE)$, for all $n \in \mathbb{N}$. By Proposition 2.1, for every $n \in \mathbb{N}$, we have that $P_{wu}(^nF_m, F) = P(^nF_m, F)$ for all $m \in \mathbb{N}$. Consequently each F_m does not contain a copy of ℓ_1 and by Proposition 2.5 we get $P_{wsc}(^nE, F) = P_{wu}(^nE, F)$ for all $n \in \mathbb{N}$ and for all complex Banach space F. Now, all we have to show is $P_{wsc}(^nE, F) = P(^nE, F)$ whenever F is a complex Banach space which is Schur. Indeed, given any $P \in P(^nE, F)$, for each $\varphi \in F'$ we have that $\varphi \circ P \in P(^nE) = P_{wsc}(^nE)$. So, if (x_n) converges weakly to x in E, $(\varphi \circ P(x_n))$ converges to $\varphi \circ P(x)$ in $\mathbb{C}$ and so the sequence $(P(x_n))$ converges weakly to $P(x)$ in F. This completes the proof since F is Schur. □

Remark 4. We recall that the Tsirelson-James space is a quasi-reflexive Banach space that is not reflexive and so if $E = \sum_{i=1}^\infty E_i$ where $E_i = T_J^*$ for every $i \in \mathbb{N}$ then E is not a reflexive space. Since $\overline{P_f(^nT_J^*)} = P(^nT_J^*)$ for all

$n \in \mathbb{N}$, this space E provides an example showing that the reflexivity of E is necessary in Theorem 2.3. For $E = \sum_{i=1}^{\infty} T_J^*$, $[P(^1E), \tau_0]' = [E', \tau_0]' \cong E \neq E'' \cong (E'_b)' = [P(^1E), \tau_b]'$, as T_J^* is not reflexive.

The same comment applies to $E = \sum_{i=1}^{\infty} c_0$.

Let $H_b(E, F)$ be the space of holomorphic mappings from E into F that are bounded on the bounded subsets of E and let $H_{wu}(E, F)$ be the space of the holomorphic mappings from E into F that are uniformly weakly continuous on the bounded subsets of E, both endowed with the topology τ_b of uniform convergence on the bounded subsets of E.

Theorem 2.4. *Let $E = \sum_{i=1}^{\infty} E_i$ where E_i is a reflexive complex Banach space with the approximation property for every $i \in \mathbb{N}$ and let F be a reflexive complex Banach space. The following conditions are equivalent*:

(1) $[H_b(E, F), \tau_b]$ *is reflexive.*

(2) $H_b(E, F) = H_{wu}(E, F)$.

(3) $P_{wu}(^nE, F) = P(^nE, F)$ *for all* $n \in \mathbb{N}$.

(4) $[P(^nE, F), \tau_b]$ *is reflexive for all* $n \in \mathbb{N}$.

Proof. The reflexivity of $[H_b(E, F), \tau_b]$ implies the reflexivity of $[P(^nE, F), \tau_b]$ for every $n \in \mathbb{N}$ since $[P(^nE, F), \tau_b]$ is a closed subspace of $[H_b(E, F), \tau_b]$ and, by Theorem 4 in [19], $[H_b(E, F), \tau_b]$ is a Fréchet space. Conversely, by Propositions 3.1 and 2.3 in Chapter 4 of [28] we have that the reflexivity of $[P(^nE, F), \tau_b]$ for every $n \in \mathbb{N}$ implies the reflexivity of $[H_b(E, F), \tau_b]$. The equivalence between (2) and (3) follows, for instance, as a particular case of Proposition 2.1 in [2]. Finally, (3) and (4) are equivalent by Theorem 2.3. □

We recall that given a Fréchet algebra $\mathcal{A}$, a multiplicative linear function $T : \mathcal{A} \to \mathbb{C}$ is called an homomorphism. For every $x \in E$, we will denote by δ_x the homomorphism $f \in H_b(E) \to f(x) \in \mathbb{C}$. We say that δ_x $(x \in E)$ is an evaluation.

It is well known that $[H_b(E), \tau_b]$ is a Fréchet algebra. As a particular case of Theorem 3.9 in [2] we have the following:

Theorem 2.5. *Let $E = \sum_{i=1}^{\infty} E_i$ where E_i is a complex Banach space with the approximation property for every $i \in \mathbb{N}$. If E is reflexive then each τ_b-continuous homomorphism $T : H_b(E) \to \mathbb{C}$ is an evaluation, if and only if $P(^nE) = P_{wu}(^nE)$ for every $n \in \mathbb{N}$.*

The next theorem provides a converse for this result.

Theorem 2.6. *Let $E = \sum_{i=1}^{\infty} E_i$ where E_i is a complex Banach space with the approximation property for every $i \in \mathbb{N}$. If each τ_b-continuous complex valued homomorphism on $H_b(E)$ is an evaluation, then E is reflexive and $P(^nE) = P_{wu}(^nE)$ for every $n \in \mathbb{N}$.*

Proof. If E is not reflexive, then $F_m = \sum_{i=1}^{m} E_i$ is not reflexive for some $m \in \mathbb{N}$ and, by Theorem 1.2 in [25], there exists an homomorphism $T : H_b(F_m) \longrightarrow \mathbb{C}$ such that $T \neq \delta_x$ for any $x \in F_m$. Define $\overline{T} : H_b(E) \longrightarrow \mathbb{C}$ by $\overline{T}(g) = T(g|F_m)$ for every $g \in H_b(E)$. It is easy to see that $\overline{T}$ is a τ_b-continuous homomorphism on $H_b(E)$ and by hypothesis we have $\overline{T} = \delta_z$ for some $z \in E$. Now, to each $f \in H_b(F_m)$ we can associate $\overline{f} = f \circ \pi_m \in H_b(E)$ and so we get

$$T(f) = T(\overline{f}|F_m) = \overline{T}(\overline{f}) = \delta_z(\overline{f}) = \overline{f}(z) = f(\pi_m(z)) = \delta_{\pi_m(z)}(f)$$

and this gives a contradiction.

So if every continuous complex valued homomorphism on $H_b(E)$ is an evaluation we have that E is reflexive and by [2] Theorem 3.9 we have $P(^nE) = P_{wu}(^nE)$ for every $n \in \mathbb{N}$. □

Now, the next result is a immediate consequence of Theorems 2.4, 2.5 and 2.6.

Theorem 2.7. *Let $E = \sum_{i=1}^{\infty} E_i$ where E_i is a complex Banach space with the approximation property for every $i \in \mathbb{N}$. Then every τ_b-continuous homomorphism $T : H_b(E) \longrightarrow \mathbb{C}$ is an evaluation if and only if $H_b(E)$ is reflexive.*

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