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The p-Schrödinger Equations on Finite Networks

By

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Abstract

We introduce the discrete p-Schrödinger operator $\mathcal{L}_{p,\omega}$ and solve the following p-Schrödinger equation:

$$\mathcal{L}_{p,\omega}u = -\Delta_{p,\omega}u + q|u|^{p-2}u = f$$

on networks. To show the uniqueness of solutions of the p-Schrödinger equation, we first solve the eigenvalue problem for the p-Schrödinger operator and obtain some properties of the smallest eigenvalue and its corresponding eigenfunction of the p-Schrödinger operator.

§1. Introduction

Many fields of our life can be expressed by using network structures, for example, nervous systems, organizations, global economies, food webs, molecules, internet webs and so on, which phenomena are represented by mathematical equations containing a discrete Laplacian on networks. So studying these phenomena has attracted great attention from many researchers in various fields. Especially, a number of authors ([1], [2], [3], [4] and [6]) have studied the direct problems such as Dirichlet and Neunann boundary value problems.

In the paper [5], the authors introduced another approach on studying the direct problems with a linear operator, called by the Laplacian Δ_ω on

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networks. To prove the solvability of direct problems on networks, they first adapted discrete analogues of some notions on vector calculus such as integration, directional derivative, gradient and Laplacian, and they showed some fundamental properties, for example maximum principle, Green's theorem on graphs.

But most of phenomena on networks are not expressed by linear equations because they usually have various and complicated interconnections governed by their intrinsic characteristics. To make up for these point, in [7], the second and third author defined a nonlinear p-Laplacian $\Delta_{p,\omega}$, which generalizes the Laplacian Δ_ω on networks, and showed the existence of solution of the Poisson equation and the Dirichlet and Neumann boundary value problems containing of the form

$$-\Delta_{p,\omega} u(x) = f(x).$$

Moreover, they introduced the typical eigenvalue problem for the p-Laplacian.

In this paper, we discuss a nonlinear p-Schrödinger operator $\mathcal{L}_{p,\omega}$, which is a generalization of the p-Laplacian defined as follows:

$$\mathcal{L}_{p,\omega} u(x) := -\Delta_{p,\omega} u(x) + q(x)|u(x)|^{p-2}u(x),$$

where x is a vertex of graphs and q is a function on graphs. The main concerns of this paper are to solve the eigenvalue problem and to show the existence of solutions of the following equation

$$\mathcal{L}_{p,\omega} u(x) = f(x) \tag{1.1}$$

for all vertices x in a graph.

We organized this paper as follows: first, we study vector calculus on graphs by recalling the paper [7] and define the p-Schrödinger operator in section 2. In section 3, we deal with the typical eigenvalue problem for the p-Schrödinger operator. Especially, after we find the smallest eigenvalue, we discuss the properties of an eigenfunction corresponding to the smallest eigenvalue. Finally, in section 4, we first prove some properties which are very useful to prove our main results, and then we show the equivalent conditions to make the smallest eigenvalue a positive number. Moreover, it guarantee the existence of solutions of the equation (1.1).

§2. Preliminaries

In this section, we start with the graph theoretic notions frequently used throughout this paper.

By a *graph* $G = G(V, E)$ we mean a finite set $V(G)$ (or simply V) of *vertices* with a set $E(G)$ (or simply E), a subset of $V \times V$ whose elements are called *edges*. By $\{x, y\} \in E$ or $x \sim y$ we mean that two vertices x and y are joined by an edge. Conventionally used, we denote by $x \in V$ or $x \in G$ the fact that x is a vertex in G.

A graph G is said to be *simple* if it has neither multiple edges nor loops, and a graph G is said to be *connected* if for every pair of vertices x and y, there exists a sequence (termed a *path*) of vertices $x = x_0 \sim x_1 \sim \cdots \sim x_n = y$.

Through this paper, all the graphs in our concerns are assumed to be simple and connected graph.

A *weight* on a graph $G(V, E)$ is a function $\omega : V \times V \to [0, \infty)$ satisfying

(i) $\omega(x, x) = 0, \ x \in V$,
(ii) $\omega(x, y) = \omega(y, x)$ if $x \sim y$,
(iii) $\omega(x, y) = 0$ if and only if $\{x, y\} \notin E$.

In particular, a weight function ω satisfying

$$\omega(x, y) = 1, \text{ if } \ x \sim y$$

is called the *standard* weight on G. A graph associated with a weight is said to be a *weighted graph* or *network*. The *degree* of a vertex x, denoted by $d_\omega x$, is defined to be

$$d_\omega x := \sum_{y \in V} \omega(x, y).$$

Throughout this paper, a function on a graph is understood as a function defined just on the set of vertices of the graph. The integration of a function $f : G \to \mathbb{R}$ on a graph G is defined by

$$\int_G f d_\omega \quad \text{(or simply } \int_G f) := \sum_{x \in G} f(x) d_\omega x.$$

For $p > 1$, the p-*directional derivative* of a function $f : G \to \mathbb{R}$ to the direction y is defined by

$$D_{p,\omega,y} f(x) := |f(y) - f(x)|^{p-2}(f(y) - f(x))\sqrt{\frac{\omega(x, y)}{d_\omega x}}$$

for $x \in G$, and the p-*gradient* $\nabla_{p,\omega}$ of a function f is defined to be

$$\nabla_{p,\omega} f(x) := (D_{p,\omega,y} f(x))_{y \in G}.$$

In particular, in the case of $p = 2$, we write simply ∇_ω instead of $\nabla_{2,\omega}$. For $p > 1$, the *p-Laplacian* $\Delta_{p,\omega}$ of a function $f : G \to \mathbb{R}$ on a graph G is defined by

$$\Delta_{p,\omega} f(x) := \sum_{y \in G} |f(y) - f(x)|^{p-2}(f(y) - f(x)) \frac{\omega(x,y)}{d_\omega x},\ x \in G$$

and for a given function $q : G \to \mathbb{R}$, the *p-Schrödinger operators* $\mathcal{L}_{p,\omega}$ of a function $f : G \to \mathbb{R}$ is defined by

$$\mathcal{L}_{p,\omega} f(x) := -\Delta_{p,\omega} f(x) + q(x)|f(x)|^{p-2} f(x),\ x \in G.$$

In what follows, p is always assumed to be a real number bigger than one.

The next theorem proved in the paper [7] by S.-Y. Chung and J.-H. Kim.

Theorem 2.1. *Let G be a network. Then for any pair of functions $f : G \to \mathbb{R}$ and $h : G \to \mathbb{R}$, we have*

$$2 \int_G h(-\Delta_{p,\omega} f) = \int_G \nabla_\omega h \cdot \nabla_{p,\omega} f.$$

§3. Eigenvalue Problems for p-Schrödinger Operators

In this section, we deal with a real number λ such that the following equation

$$-\Delta_{p,\omega} u(x) + q(x)|u(x)|^{p-2} u(x) = \lambda |u(x)|^{p-2} u(x),\ x \in G \tag{3.1}$$

has a non-zero solution where $q : G \to \mathbb{R}$ is a function on a graph G. We call it the typical eigenvalue problem for the p-Schrödinger operator because if we have a non-zero solution u and a real number λ satisfying the equation (3.1), then for any $\alpha \in \mathbb{R}$, αu and λ also satisfy the equation (3.1).

The following lemma is very useful result to prove the eigenvalue problem for the p-Schrödinger operators.

Lemma 3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function defined by*

$$f(\mathbf{x}) := \sum_{i,j=1}^{n} a_{ij} |x_i - x_j|^p + \sum_{i=1}^{n} b_i |x_i|^p$$

where $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $a_{ij} \geq 0$ and some of a_{ij} are not zero and $b_i \in \mathbb{R}$ for all i and let for any $\alpha \in \mathbb{R}$, B_α be a set defined by

$$B_\alpha := \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) = \alpha\}.$$

Then we have the followings.
(i) *There exists* $\mathbf{x} \in \mathbb{R}^n$ *such that* $f(\mathbf{x}) \neq 0$.
(ii) *If there exists* $\mathbf{x}' \in \mathbb{R}^n$ *such that* $f(\mathbf{x}') > 0$, *then for any* $\alpha > 0$ *and* $c_i > 0$, $i = 1, 2, \cdots, n$, *the set* B_α *is non-empty and the function*

$$g(\mathbf{x}) := \sum_{i=1}^{n} c_i |x_i|^p$$

has a minimum at a point in B_α.
(iii) *If there exists* $\mathbf{x}' \in \mathbb{R}^n$ *such that* $f(\mathbf{x}') < 0$, *then for any* $\alpha < 0$ *and* $c_i > 0$, $i = 1, 2, \cdots, n$, *the set* B_α *is non-empty and the function*

$$g(\mathbf{x}) := \sum_{i=1}^{n} c_i |x_i|^p$$

has a minimum at a point in B_α.

Proof. (i) Suppose that there exist a_{ij}, b_i such that $f(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^n$. For the unit vector $\mathbf{e}_k$ whose k-th element is 1 and the others are zero,

$$f(\mathbf{e}_k) = \sum_{j=1}^{n} a_{kj} + \sum_{i=1}^{n} a_{ik} + b_k = 0.$$

Thus

$$b_k = -\sum_{j=1}^{n} a_{kj} - \sum_{i=1}^{n} a_{ik}.$$

But for the vector $\mathbf{1} := (1, 1, \cdots, 1)$,

$$f(\mathbf{1}) = \sum_{k=1}^{n} b_k = -\sum_{k=1}^{n}\sum_{j=1}^{n} a_{kj} - \sum_{k=1}^{n}\sum_{i=1}^{n} a_{ik}.$$

Since $a_{ij} \geq 0$ and some of a_{ij} is not non-zero, $f(\mathbf{1}) \neq 0$. This is in contradiction with the assumption.

(ii) Let $\mathbf{x}' = (x_1', x_2', \cdots, x_n')$ such that $f(\mathbf{x}') > 0$. For any $\alpha > 0$,

$$\begin{aligned} f(\alpha^{\frac{1}{p}} f^{-\frac{1}{p}}(\mathbf{x}')\mathbf{x}') &= \sum_{i,j=1}^{n} a_{ij} |\alpha^{\frac{1}{p}} f^{-\frac{1}{p}}(\mathbf{x}')(x_i' - x_j')|^p + \sum_{i=1}^{n} b_i |\alpha^{\frac{1}{p}} f^{-\frac{1}{p}}(\mathbf{x}') x_i'|^p \\ &= (\alpha^{\frac{1}{p}} f^{-\frac{1}{p}}(\mathbf{x}'))^p f(\mathbf{x}') = \alpha. \end{aligned}$$

Thus B_α is non-empty. We now show that the function

$$g(\mathbf{x}) := \sum_{i=1}^{n} c_i |x_i|^p$$

has a minimum at a point in B_α. Since $f(t\mathbf{x}') = |t|^p f(\mathbf{x}') \to \infty$ as $t \to \infty$, for $\alpha > 0$, there exists $t_0 \in \mathbb{R}$ such that $f(t_0\mathbf{x}') > \alpha$. Let $\mathbf{x}_0 = t_0\mathbf{x}'$. Since $f(\mathbb{O}) = 0$ where $\mathbb{O}$ is the origin, there exists $\mathbf{y}$ on the line connecting $\mathbb{O}$ and $\mathbf{x}_0$ such that $f(\mathbf{y}) = \alpha$. Thus $\mathbf{y} \in B_\alpha$. Now we define a set $A_{\mathbf{x}_0}$ as follows

$$A_{\mathbf{x}_0} := \{\mathbf{x} \in \mathbb{R}^n | g(\mathbf{x}) \le g(\mathbf{x}_0)\}.$$

Since g is strictly convex and g has a minimum at $\mathbb{O}$,

$$g(\mathbf{y}) \le g(\mathbf{x}_0).$$

Thus $\mathbf{y} \in A_{\mathbf{x}_0} \cap B_\alpha$. Since $A_{\mathbf{x}_0}$ is compact and B_α is close, $A_{\mathbf{x}_0} \cap B_\alpha$ is compact. Thus g has a minimum on $A_{\mathbf{x}_0} \cap B_\alpha$. Since $g(\mathbf{x}_0) < g(\mathbf{x})$ for any $\mathbf{x} \in A^c_{\mathbf{x}_0}$, g has a minimum on B_α. (iii) can be done in the similar way as (ii). □

We now state and prove the eigenvalue problem for the p-Schrödinger operator.

Theorem 3.1. *Let $q : G \to \mathbb{R}$ be a function. Then there exists a nonzero solution u such that*

$$-\Delta_{p,\omega} u(x) + q(x)|u(x)|^{p-2}u(x) = \lambda |u(x)|^{p-2}u(x),\ x \in G$$

for some $\lambda \in \mathbb{R}$.

Proof. It follows from Lemma 3.1 (i) that there exists $v_0 : G \to \mathbb{R}$ such that

$$\begin{aligned}0 \ne& \int_G (-\Delta_{p,\omega} v_0 + q|v_0|^{p-2}v_0)v_0 \\ =& \frac{1}{2}\int_G \nabla_\omega v_0 \cdot \nabla_{p,\omega} v_0 + \int_G q|v_0|^p \\ =& \frac{1}{2}\sum_{x,y\in G} |v_0(y) - v_0(x)|^p \omega(x,y) + \sum_{x\in G} q(x)|v_0(x)|^p d_\omega x.\end{aligned}$$

We first assume that $\int_G \frac{1}{2}\nabla_\omega v_0 \cdot \nabla_{p,\omega} v_0 + q|v_0|^p > 0$. For any $\alpha > 0$, define a set

$$M_\alpha := \{h : G \to \mathbb{R} | \frac{1}{2p}\int_G \nabla_\omega h \cdot \nabla_{p,\omega} h + \frac{1}{p}\int_G q|h|^p = \alpha\},$$

two functionals

$$I_\alpha[h] := \frac{1}{2\alpha p}\int_G \nabla_\omega h \cdot \nabla_{p,\omega} h + \frac{1}{\alpha p}\int_G q|h|^p,$$

and

$$E[h] := \frac{1}{p}\int_G |h|^p$$

for all $h : G \to \mathbb{R}$. Then Lemma 3.1 (ii) yields that $M_\alpha \neq \emptyset$, $I_\alpha[h] = 1$ for any $h \in M_\alpha$, and there exists $g_0 \in M_\alpha$ such that $E[g_0] = \min_{g \in M_\alpha} E[g]$. Moreover, for any function $u : G \to \mathbb{R}$ and $v : G \to \mathbb{R}$, $I_\alpha[u + tv]$ and $E[u + tv]$ are continuously differentiable with respect to $t \in \mathbb{R}$,

$$\begin{aligned}\frac{d}{dt} I_\alpha[u+tv]|_{t=0} =& -\frac{1}{\alpha}\sum_{x,y\in G}|u(y)-u(x)|^{p-2}(u(y)-u(x))v(x)\omega(x,y)\\ &+\frac{1}{\alpha}\sum_{x\in G} q(x)|u(x)|^{p-2}u(x)v(x)d_\omega x,\end{aligned}$$

and

$$\frac{d}{dt}E[u+tv]|_{t=0} = \sum_{x\in G}|u(x)|^{p-2}u(x)v(x)d_\omega x.$$

Since $I_\alpha[g_0] = 1$, for any $h : G \to \mathbb{R}$, there exists $\delta > 0$ such that $I_\alpha[g_0 + th] > 0$ and $I_\alpha^{-\frac{1}{p}}[g_0 + th](g_0 + th) \in M_\alpha$ for $|t| < \delta$. Therefore

$$\begin{aligned}&\frac{d}{dt}E[I_\alpha^{-\frac{1}{p}}[g_0+th](g_0+th)]|_{t=0}\\ =&\frac{d}{dt}I_\alpha^{-1}[g_0+th]E[g_0+th]|_{t=0}\\ =&\left[-I_\alpha^{-2}[g_0+th]E[g_0+th]\frac{d}{dt}I_\alpha[g_0+th]+I_\alpha^{-1}[g_0+th]\frac{d}{dt}E[g_0+th]\right]_{t=0}\\ =&-E[g_0]\int_G\frac{1}{\alpha I_\alpha^2[g_0]}\left(-\Delta_{p,w}g_0+q|g_0|^{p-2}g_0\right)h+\int_G|g_0|^{p-2}g_0h\\ =&-\int_G\frac{E[g_0]}{\alpha}\left(-\Delta_{p,\omega}g_0(x)+q(x)|g_0|^{p-2}g_0\right)h+\int_G|g_0(x)|^{p-2}g_0h.\end{aligned}$$

Put

$$h_0(x) = \frac{E[g_0]}{\alpha}\left(-\Delta_{p,\omega}g_0(x)+q(x)|g_0(x)|^{p-2}g_0(x)\right)+|g_0(x)|^{p-2}g_0(x)$$

for $x \in G$. Then $I_\alpha^{-\frac{1}{p}}[g_0 + th_0](g_0 + th_0) \in M_\alpha$. Moreover, $I_\alpha^{-\frac{1}{p}}[g_0 + th_0](g_0 + th_0) \to g_0$ as $t \to 0$. Since g_0 is a minimizer of E on M_α,

$$\begin{aligned}0 &= \frac{d}{dt}E[I_\alpha^{-\frac{1}{p}}[g_0+th_0](g_0+th_0)]|_{t=0}\\ &=\int_G h_0\cdot h_0 = \sum_{x\in G}h_0^2(x)d_\omega x\end{aligned}$$

Hence $h_0(x) = 0$ for all x in G. Therefore

$$-\Delta_{p,\omega} g_0(x) + q(x)|g_0(x)|^{p-2} g_0(x) = \frac{\alpha}{E[g_0]} |g_0(x)|^{p-2} g_0(x),\ x \in G.$$

On the other hand, if $\int_G \frac{1}{2}\nabla_\omega v_0 \cdot \nabla_{p,\omega} v_0 + q|v_0|^p < 0$ then it is proved by the similar way as the above so that the proof is done. □

From now on, a real number λ is called an *eigenvalue* and a non-zero function ϕ is called an *eigenfunction* corresponding to λ for the p-Schrödinger operator if a real number λ and a non-zero function ϕ satisfy the eigenvalue problem for the p-Schrödinger operator.

We obtain the following result that gives the existence of the smallest eigenvalue for the p-Schrödinger operators. The case $p = 2$ is classical.

Lemma 3.2. *Let $q : G \to \mathbb{R}$ be a function and let λ_0 be defined by*

$$\lambda_0 := \inf_{\phi \neq 0} \frac{\int_G \frac{1}{2}\nabla_\omega \phi \cdot \nabla_{p,\omega}\phi + q|\phi|^p}{\int_G |\phi|^p}.$$

Then there exists a non-zero function $\phi_0 : G \to \mathbb{R}$ such that

$$\frac{\int_G \frac{1}{2}\nabla_\omega \phi_0 \cdot \nabla_{p,\omega}\phi_0 + q|\phi_0|^p}{\int_G |\phi_0|^p} = \lambda_0.$$

Moreover, ϕ_0 is an eigenfunction associated with the eigenvalue λ_0.

Proof. Note that

$$\begin{aligned}
&\inf_{\phi \neq 0} \frac{\int_G \frac{1}{2}\nabla_\omega \phi \cdot \nabla_{p,\omega}\phi + q|\phi|^p}{\int_G |\phi|^p} \\
&= \inf_{\phi\neq 0}\left(\frac{1}{2}\sum_{x,y\in G}\left|\frac{\phi(y)}{(\int_G|\phi|^p)^{\frac{1}{p}}} - \frac{\phi(x)}{(\int_G|\phi|^p)^{\frac{1}{p}}}\right|^p \omega(x,y) + \sum_{x\in G} q(x)\left|\frac{\phi(x)}{(\int_G|\phi|^p)^{\frac{1}{p}}}\right|^p d_\omega x\right) \\
&= \inf_{\int|\phi|^p=1}\int_G \frac{1}{2}\nabla_\omega\phi\cdot\nabla_{p,\omega}\phi + q|\phi|^p.
\end{aligned}$$

Since the set $\{\phi : G \to \mathbb{R} | \int_G |\phi|^p = 1\}$ is compact, there exists $\phi_0 : G \to \mathbb{R}$ such that

$$\int_G \frac{1}{2}\nabla_\omega \phi_0 \cdot \nabla_{p,\omega}\phi_0 + q|\phi_0|^p = \min_{\int|\phi|^p=1}\int_G \frac{1}{2}\nabla_\omega\phi\cdot\nabla_{p,\omega}\phi + q|\phi|^p$$

and

$$\int_G |\phi_0|^p = 1.$$

Define for any $x \in G$, a function $\delta_x : G \to \mathbb{R}$ as following

$$\delta_x(y) = \begin{cases} 1, x = y \\ 0, \text{otherwise.} \end{cases}$$

Take any $x_0 \in G$, then $\int_G |\phi_0 + t\delta_{x_0}|^p \neq 0$ for all $t \in (-1, 1)$ and

$$\lambda_0 \leq \frac{\int_G \frac{1}{2}\nabla_\omega(\phi_0 + t\delta_{x_0}) \cdot \nabla_{p,\omega}(\phi_0 + t\delta_{x_0}) + q|(\phi_0 + t\delta_{x_0})|^p}{\int_G |(\phi_0 + t\delta_{x_0})|^p}, \ t \in (-1, 1).$$

Thus

$$0 \leq \int_G \frac{1}{2}\nabla_\omega(\phi_0 + t\delta_{x_0}) \cdot \nabla_{p,\omega}(\phi_0 + t\delta_{x_0}) + q|(\phi_0 + t\delta_{x_0})|^p - \lambda_0 \int_G |(\phi_0 + t\delta_{x_0})|^p$$

for all $t \in (-1, 1)$. The right-hand side is continuously differentiable with respect to t and equal to zero at $t = 0$. Thus

$$\begin{aligned} 0 =& \frac{d}{dt}\left[\int_G \frac{1}{2}\nabla_\omega(\phi_0 + t\delta_{x_0}) \cdot \nabla_{p,\omega}(\phi_0 + t\delta_{x_0}) + q|(\phi_0 + t\delta_{x_0})|^p \right. \\ & \left. -\lambda_0 \int_G |(\phi_0 + t\delta_{x_0})|^p \right]_{t=0} \\ =& -p \sum_{x,y \in G} |\phi_0(y) - \phi_0(x)|^{p-2}(\phi_0(y) - \phi_0(x))\delta_{x_0}(x)\omega(x, y) \\ & + p \sum_{x \in G} q(x)|\phi_0(x)|^{p-2}\phi_0(x)\delta_{x_0}(x)d_\omega x - \lambda_0 p \sum_{x \in G} |\phi_0(x)|^{p-2}\phi_0(x)\delta_{x_0}(x)d_\omega x \\ =& p(-\Delta_{p,\omega}\phi_0(x_0) + q(x_0)|\phi_0(x_0)|^{p-2}\phi_0(x_0) - \lambda_0|\phi_0(x_0)|^{p-2}\phi_0(x_0))d_\omega x_0. \end{aligned}$$

Since the above equations hold for arbitrary $x_0 \in G$, we have

$$-\Delta_{p,\omega}\phi_0(x) + q(x)|\phi_0(x)|^{p-2}\phi_0(x) = \lambda_0|\phi_0(x)|^{p-2}\phi_0(x), \ x \in G.$$

□

By Lemma 3.2, we know that there exists the smallest eigenvalue for the p-Schrödinger operator. We denote the smallest eigenvalue λ_0 and it's eigenfunction ϕ_0.

The following two results guarantee the existence of an eigenfunction ϕ_0 satisfying

$$\phi_0(x) > 0, x \in G.$$

Theorem 3.2. *Let $q : G \to \mathbb{R}$ be a function. There exists a non-zero solution u satisfying*

$$-\Delta_{p,w}u(x) + q(x)|u(x)|^{p-2}u(x) = \lambda_0|u(x)|^{p-2}u(x) \tag{3.2}$$

and

$$u(x) \geq 0, \;\; x \in G.$$

Proof. It follows from Lemma 3.2, there exists a non-zero solution u satisfying (3.2). Let $u^+(x) := |u(x)|$ for all x in G. Since $\int_G |u^+|^p = \int_G |u|^p$ and

$$\begin{aligned}\int_G \frac{1}{2}\nabla_\omega u \cdot \nabla_{p,\omega} u + q|u|^p =& \frac{1}{2}\sum_{x,y\in G} |u(y)-u(x)|^p w(x,y) + \sum_{x\in G} q(x)|u(x)|^p d_w x \\ \geq& \frac{1}{2}\sum_{x,y\in G} |u^+(y)-u^+(x)|^p w(x,y) \\ &+ \sum_{x\in G} q(x)|u^+(x)|^p d_w x \\ =& \int_G \frac{1}{2}\nabla_\omega u^+ \cdot \nabla_{p,\omega} u^+ + q|u^+|^p,\end{aligned}$$

we have

$$\lambda_0 = \frac{\int_G \frac{1}{2}\nabla_\omega u \cdot \nabla_{p,\omega} u + q|u|^p}{\int_G |u|^p} \geq \frac{\int_G \frac{1}{2}\nabla_\omega u^+ \cdot \nabla_{p,\omega} u^+ + q|u^+|^p}{\int_G |u^+|^p}. \tag{3.3}$$

Moreover by the definition of λ_0,

$$\lambda_0 \leq \frac{\int_G \frac{1}{2}\nabla_\omega u^+ \cdot \nabla_{p,\omega} u^+ + q|u^+|^p}{\int_G |u^+|^p}. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\lambda_0 = \frac{\int_G \frac{1}{2}\nabla_\omega u^+ \cdot \nabla_{p,\omega} u^+ + q|u^+|^p}{\int_G |u^+|^p}.$$

It follows from Lemma 3.2 that

$$-\Delta_{p,\omega} u^+(x) + q(x)|u^+(x)|^{p-2}u^+(x) = \lambda_0 |u^+(x)|^{p-2}u^+(x), \;\; x \in G.$$

□

Theorem 3.3. *Let $q : G \to \mathbb{R}$ be a function. There exists a positive solution u satisfying the equation* (3.2).

Proof. Theorem 3.2 guarantees that there exists a nonnegative solution u satisfying (3.2). It is enough to show that if there exists x_0 in G such that

$u(x_0) = 0$, then $u \equiv 0$. Let $m = |\min_{x \in G} q(x)|$. It follows from the definition of λ_0 that

$$\begin{aligned}\lambda_0 &= \inf_{\phi \neq 0} \frac{\int_G \frac{1}{2} \nabla_\omega \phi \cdot \nabla_{p,\omega} \phi + q|\phi|^p}{\int |\phi|^p} \\ &= \inf_{\int_G |\phi|^p = 1} \int_G \frac{1}{2} \nabla_\omega \phi \cdot \nabla_{p,\omega} \phi + q|\phi|^p \\ &\geq \inf_{\int_G |\phi|^p = 1} \int_G q|\phi|^p \\ &\geq \inf_{\int_G |\phi|^p = 1} \min_{x \in G} q(x) \int_G |\phi|^p \\ &= \min_{x \in G} q(x)\end{aligned}$$

which implies $\lambda_0 + m \geq 0$. Thus

$$-\Delta_{p,\omega} u(x) + (q(x) + m)|u(x)|^{p-2} u(x) = (\lambda_0 + m)|u(x)|^{p-2} u(x) \geq 0, \ x \in G.$$

The assumption $u(x_0) = 0$ implies

$$\begin{aligned}0 &\leq -\Delta_{p,\omega} u(x_0) + (q(x_0) + m)|u(x_0)|^{p-2} u(x_0) \\ &= -\sum_{y \in G} |u(y) - u(x_0)|^{p-2} (u(y) - u(x_0)) \frac{\omega(x_0, y)}{d_\omega x_0} \\ &= -\sum_{y \in G} |u(y)|^{p-2} u(y) \frac{\omega(x_0, y)}{d_\omega x_0}.\end{aligned}$$

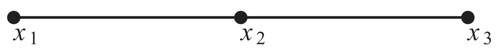
Hence $u(y) = 0$ for all $y \sim x_0$. Take any $y \sim x_0$. By repeating the above process, we conclude $u(z) = 0$ for each $z \sim y$. Since G is a connected graph, $u(x) = 0$ for all $x \in G$. □

§4. A Characterization of Positive Solutions

For given functions $q : G \to \mathbb{R}$ and $f : G \to \mathbb{R}$, the following equation

$$(4.1) \qquad -\Delta_{p,\omega} u(x) + q(x)|u(x)|^{p-2} u(x) = f(x), \ x \in G$$

is said to be the p-Schrödinger equation. The existence of the solution of the p-Schrödinger equation is guaranteed by the function q. For example, consider a network G whose vertices are $\{x_1, x_2, x_3\}$ with the weights $\omega(x_1, x_2) = 1$, $\omega(x_2, x_3) = 1$ and $\omega(x_1, x_3) = 0$ as follows:



If we assume $q \equiv 0$ and $f \equiv -1$ on G then it is easily seen that there is no solution satisfying the p-Schrödinger equation. Accordingly, the main goal in this section is to find equivalent conditions which guarantee the existence of the solution of the p-Schrödinger equation.

The following results are very useful to prove our main theorem.

Lemma 4.1. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function defined by*

$$f(x, y, z) = |x - y|^p + |x|^p|z - 1|^{p-2}(z - 1) + |y|^p \left|\frac{1}{z} - 1\right|^{p-2} \left(\frac{1}{z} - 1\right).$$

Then the function $f \geq 0$ on $B = \{(x, y, z) \in \mathbb{R}^3 : x, y \geq 0,\ z > 0\}$. Moreover, f equals 0 if and only if $y = xz$.

Proof. We show that for any $z > 0$, f is non-negative on the plane $\{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0\}$. For any $t > 0$,

$$f(tx, ty, z) = t^p f(x, y, z).$$

Hence, for any $z > 0$, the function f has the same sign on the set $\{(tx, ty, z) : t > 0\}$. Therefore it is enough to show that for any $z > 0$, f is non-negative on the two line segments $L_z = \{(x, y) \in \mathbb{R}^2 : x \in [0, 1],\ y = 1\}$ and $M_z = \{(x, y) \in \mathbb{R}^2 : x = 1,\ y \in [0, 1]\}$.

First, if $z = 1$, then we have

$$f(x, y, 1) = |x - y|^p \geq 0, \tag{4.2}$$

where the equality holds if and only if $x = y$.

We now assume that $z > 1$. Then for any $(x, y) \in L_z$, we have

$$\begin{aligned}\frac{d}{dx} f(x, y, z) &= \frac{d}{dx}\left((1 - x)^p + x^p(z - 1)^{p-1} - \left(1 - \frac{1}{z}\right)^{p-1}\right)\\ &= p\left(x^{p-1}(z - 1)^{p-1} - (1 - x)^{p-1}\right).\end{aligned}$$

It is easy to see that $\frac{d}{dx} f(x, y, z) > 0$ for all $(x, y) \in L_z$ with $x > \frac{1}{z}$ and $\frac{d}{dx} f(x, y, z) < 0$ for all $(x, y) \in L_z$ with $x < \frac{1}{z}$. Hence f has the minimum at $(\frac{1}{z}, 1) \in L_z$ and we have

$$f\left(\frac{1}{z}, 1, z\right) = 0.$$

Hence $f \geq 0$ on L_z and the equality holds when $x = \frac{1}{z}$.

On the other hand, for any $(x, y) \in M_z$, we obtain

$$\frac{d}{dy} f(x, y, z) = -p\left((1-y)^{p-1} + y^{p-1}\left(1 - \frac{1}{z}\right)\right) < 0.$$

It follows that for any $(x, y) \in M_z$, we have

$$\begin{aligned} f(x, y, z) &\geq f(1, 1, z) \\ &= (z-1)^{p-1} - \left(1 - \frac{1}{z}\right)^{p-1} \\ &= |z-1|^{p-2}(z-1)\left(1 - \frac{1}{z^{p-1}}\right) \\ &> 0. \end{aligned}$$

Therefore it follows from (4.2) that we have $f \geq 0$ on $\{(x, y, z) : x, y \geq 0 \textit{ and } z > 1\}$ and the equality holds if and only if $x = \frac{y}{z}$.

We now assume $z < 1$. Since $f(x, y, z) = f(y, x, \frac{1}{z})$, we have a similar result of the case $z > 1$, that is, we obtain $f(x, y, z) \geq 0$ for $x, y \geq 0$ and $0 < z < 1$. Moreover the equality holds if and only if $y = xz$. □

Theorem 4.1. *Let u be a nonnegative function and v be a positive function on a graph G. Then*

$$\nabla_\omega u \cdot \nabla_{p,\omega} u - \nabla_\omega\left(\frac{u^p}{v^{p-1}}\right) \cdot \nabla_{p,\omega} v \geq 0 \text{ on } G.$$

Moreover, the equality holds if and only if $u \equiv tv$ for some constant $t > 0$.

Proof. For any $x \in G$,

$$\begin{aligned} &(\nabla_\omega u \cdot \nabla_{p,\omega} u)(x) - \left(\nabla_\omega\left(\frac{u^p}{v^{p-1}}\right) \cdot \nabla_{p,\omega} v\right)(x) \\ =& \sum_{y\in G}\left[|u(y)-u(x)|^p - \left(\frac{u^p(y)}{v^{p-1}(y)} - \frac{u^p(x)}{v^{p-1}(x)}\right)\right. \\ &\left.\times |v(y)-v(x)|^{p-2}(v(y)-v(x))\right]\frac{\omega(x,y)}{d_\omega x} \\ =& \sum_{y\in G}\left[|u(y)-u(x)|^p + u^p(y)\left|\frac{v(x)}{v(y)} - 1\right|^{p-2}\left(\frac{v(x)}{v(y)} - 1\right)\right. \\ &\left.+ u^p(x)\left|\frac{v(y)}{v(x)} - 1\right|^{p-2}\left(\frac{v(y)}{v(x)} - 1\right)\right]\frac{\omega(x,y)}{d_\omega x}. \end{aligned}$$

It follows from Lemma 4.1 that each term of the above equation is nonnegative. Thus

$$(\nabla_\omega u \cdot \nabla_{p,\omega} u)(x) - \left(\nabla_\omega \left(\frac{u^p}{v^{p-1}}\right) \cdot \nabla_{p,\omega} v\right)(x) \geq 0$$

for all x in G.

We now assume that the equality holds. Then by Lemma 4.1, for any x, y in G satisfying $x \sim y$,

$$\begin{aligned} |u(y) - u(x)|^p + u^p(y) \left|\frac{v(x)}{v(y)} - 1\right|^{p-2} \left(\frac{v(x)}{v(y)} - 1\right) \\ + u^p(x) \left|\frac{v(y)}{v(x)} - 1\right|^{p-2} \left(\frac{v(y)}{v(x)} - 1\right) = 0 \end{aligned}$$

which implies $u(x) = u(y)\frac{v(x)}{v(y)}$. Since G is a connected graph, $u(x) = u(y)\frac{v(x)}{v(y)}$ for all x, y in G. Thus $u = cv$ for some $c > 0$.

We now assume that $u = cv$ for some $c > 0$. Then by a simple calculation, we prove that the equality holds □

Corollary 4.1. *Let u and v be positive functions on a graph G. Then*

$$I[u,v] := \int_G -\Delta_{p,\omega} u \left(\frac{u^p - v^p}{u^{p-1}}\right) - \Delta_{p,\omega} v \left(\frac{v^p - u^p}{v^{p-1}}\right) \geq 0.$$

Moreover, $I[u,v] = 0$ if and only if $u \equiv tv$ for some $t > 0$.

Proof. For given functions u and v,

$$\begin{aligned} I[u,v] &= \int_G u(-\Delta_{p,\omega} u) + \Delta_{p,\omega} v \left(\frac{u^p}{v^{p-1}}\right) + \Delta_{p,\omega} u \left(\frac{v^p}{u^{p-1}}\right) + v(-\Delta_{p,\omega} v) \\ &= \frac{1}{2} \int_G \nabla_\omega u \cdot \nabla_{p,\omega} u - \nabla_\omega \left(\frac{u^p}{v^{p-1}}\right) \cdot \nabla_{p,\omega} v \\ &\quad + \frac{1}{2} \int_G \nabla_\omega v \cdot \nabla_{p,\omega} v - \nabla_\omega \left(\frac{v^p}{u^{p-1}}\right) \cdot \nabla_{p,\omega} u. \end{aligned}$$

It follows from Theorem 4.1 that

$$\nabla_\omega u \cdot \nabla_{p,\omega} u - \nabla_\omega \left(\frac{u^p}{v^{p-1}}\right) \cdot \nabla_{p,\omega} v \geq 0$$

and

$$\nabla_\omega v \cdot \nabla_{p,\omega} v - \nabla_\omega \left(\frac{v^p}{u^{p-1}}\right) \cdot \nabla_{p,\omega} u \geq 0$$

on G. Thus

$$I[u,v] \geq 0.$$

Moveover, $I[u,v]=0$ if and only if

$$\nabla_\omega u \cdot \nabla_{p,\omega} u - \nabla_\omega \left(\frac{u^p}{v^{p-1}}\right) \cdot \nabla_{p,\omega} v = 0$$

and

$$\nabla_\omega v \cdot \nabla_{p,\omega} v - \nabla_\omega \left(\frac{v^p}{u^{p-1}}\right) \cdot \nabla_{p,\omega} u = 0$$

on G. Thus by Theorem 4.1, $u = tv$ for some $t > 0$. □

Finally, we are in a position to state and prove our main result of this paper.

Theorem 4.2. *Let $q : G \to \mathbb{R}$ be a function. Then the following are equivalent.*
(i) *If a function u on a graph G satisfies*

$$-\Delta_{p,\omega} u(x) + q(x)|u(x)|^{p-2}u(x) \geq 0, \ \ x \in G,$$

then $u(x) \geq 0$ for all $x \in G$.
(ii) *If a non-zero function u on a graph G satisfies*

$$-\Delta_{p,\omega} u(x) + q(x)|u(x)|^{p-2}u(x) \geq 0, \ \ x \in G,$$

then $u(x) > 0$ for all $x \in G$.
(iii) *The smallest eigenvalue λ_0 is positive.*
(iv) *For a nonnegative function f on a graph G satisfying $f \not\equiv 0$, there exists a positive function u on a graph G such that*

$$-\Delta_{p,\omega} u(x) + q(x)|u(x)|^{p-2}u(x) \geq f(x), \ \ x \in G.$$

(v) *For a nonnegative function f on a graph G, there exists a unique function u on a graph G such that*

$$-\Delta_{p,\omega} u(x) + q(x)|u(x)|^{p-2}u(x) = f(x), \ \ x \in G.$$

Moreover, $u(x) \geq 0$ for all $x \in G$.

Proof. (i)⇒ (ii) By arguing as in the proof of Theorem 3.3, it is shown that $u(x) > 0$ for all x in G.

(ii)⇒ (iii) Assume $\lambda_0 \leq 0$. Since by Theorem 3.3, there exists an eigenfunction ϕ_0 satisfying $\phi_0(x) > 0$ for all x in G,

$$-\Delta_{p,\omega}\phi_0(x) + q(x)|\phi_0(x)|^{p-2}\phi_0(x) = \lambda_0|\phi_0(x)|^{p-2}\phi_0(x) \leq 0, \ \ x \in G.$$

Put $\psi_0(x) = -\phi_0(x)$ for all x in G. Then $-\Delta_{p,\omega}\psi_0(x) + q(x)|\psi_0(x)|^{p-2}\psi_0(x) \geq 0,\ x \in G$ but $\psi_0(x) < 0$. This is contradiction to the assumption (ii).

(iii) $\Rightarrow$ (i) Suppose that $u(x) < 0$ for some x in G. Let we define a function $v(x) := \min\{u(x), 0\}$ for all x in G. Then $v(x) \leq 0$. Since $-\Delta_{p,\omega}u(x) + q(x)|u(x)|^{p-2}u(x) \geq 0$ for all $x \in G$,

$$\{-\Delta_{p,\omega}u(x) + q(x)|u(x)|^{p-2}u(x)\}v(x) \leq 0 \tag{4.3}$$

for all x in G. It follows from the definition of a function v that $q(x)|u(x)|^{p-2}u(x)v(x) = q(x)|v(x)|^p$ for $x \in G$ satisfying $u(x) < 0$ and $q(x)|u(x)|^{p-2}u(x)v(x) = 0$ for $x \in G$ satisfying $u(x) \geq 0$. Thus

$$\sum_{x\in G} q(x)|u(x)|^{p-2}u(x)v(x) = \sum_{x\in G} q(x)|v(x)|^p. \tag{4.4}$$

It is easy to see that

$$|u(y) - u(x)|^{p-2}(u(y) - u(x))v(x) \leq |v(y) - v(x)|^{p-2}(v(y) - v(x))v(x)$$

for all x, y in G. It implies that

$$\int_G (-\Delta_{p,\omega}u)v \geq \int_G (-\Delta_{p,\omega}v)v \tag{4.5}$$

for all x in G. By (4.3), (4.4) and (4.5),

$$\int_G (-\Delta_{p,\omega}v)v + q|v|^p \leq 0.$$

Thus

$$\frac{\int_G (-\Delta_{p,\omega}v)v + q|v|^p}{\int_G |v|^p} \leq 0.$$

This is in contradiction with the assumption $\lambda_0 > 0$.

(iii) $\Rightarrow$ (iv) Let λ_0 and ϕ_0 satisfy

$$-\Delta_{p,\omega}\phi_0(x) + q(x)|\phi_0(x)|^{p-2}\phi_0(x) = \lambda_0|\phi_0(x)|^{p-2}\phi_0(x),\ x \in G,$$

and $\phi_0(x) > 0$ for all x in G. Then for any $\alpha \in \mathbb{R}$,

$$-\Delta_{p,\omega}(\alpha\phi_0(x)) + q(x)|(\alpha\phi_0(x))|^{p-2}(\alpha\phi_0(x)) = \lambda_0|(\alpha\phi_0(x))|^{p-2}(\alpha\phi_0(x)),\ x \in G$$

holds. Thus there exists a sufficiently large α such that

$$-\Delta_{p,\omega}(\alpha\phi_0(x)) + q(x)|(\alpha\phi_0(x))|^{p-2}(\alpha\phi_0(x)) \geq f(x),\ x \in G.$$

(iv) $\Rightarrow$ (iii) Let ϕ_0 be an eigenfunction satisfying $\phi_0(x) > 0$ for all $x \in G$. Choose $C > 0$ such that $C \geq \max_{x\in G}\left[\frac{u(x)}{\phi_0(x)}\right]$. Put $\psi(x) = C\phi_0(x) > 0$ for all $x \in G$. Assume $\lambda_0 \leq 0$. Since $-\Delta_{p,\omega}u(x) + q(x)u(x)^{p-1} \geq 0$ for all $x \in G$,

$$\int_G -\Delta_{p,\omega}u\left(\frac{\psi^p - u^p}{u^{p-1}}\right) \geq \int_G -q\left(\psi^p - u^p\right).$$

It follows that

$$\begin{aligned}
I[\psi, u] &= \int_G -\Delta_{p,\omega}\psi\left(\frac{\psi^p - u^p}{\psi^{p-1}}\right) - \int_G -\Delta_{p,\omega}u\left(\frac{\psi^p - u^p}{u^{p-1}}\right)\\
&\leq \int_G -\Delta_{p,\omega}\psi\left(\frac{\psi^p - u^p}{\psi^{p-1}}\right) - \int_G -q\left(\psi^p - u^p\right)\\
&= \int_G \left(-\Delta_{p,\omega}\psi + q\psi^{p-1}\right)\left(\frac{\psi^p - u^p}{\psi^{p-1}}\right)\\
&= \int_G \left(\lambda_0\psi^{p-1}\right)\left(\frac{\psi^p - u^p}{\psi^{p-1}}\right)\\
&= \int_G \lambda_0\left(C^p\phi_0^p - u^p\right)\\
&= \int_G \lambda_0\phi_0^p\left(C^p - \frac{u^p}{\phi_0^p}\right) \leq 0.
\end{aligned}$$

It follows from Corollary 4.1 that $I[\psi, u] = 0$ and then $\psi = \gamma u$ for some $\gamma > 0$. However,

$$0 \geq C^{p-1}\lambda_0|\phi_0|^{p-2}\phi_0 = -\Delta_{p,\omega}\psi + q|\psi|^{p-2}\psi \geq \gamma^{p-1}f.$$

Since $f \geq 0$ and $f \neq 0$, this is in contradiction with the assumption $\lambda_0 \leq 0$. Thus $\lambda_0 > 0$.

This completes the equivalence of assertions (i) to (iv). Since (v) implies (iv), now we show that (iii) implies (v).

(iii) $\Rightarrow$ (v) If $f \equiv 0$ then $u \equiv 0$ is a solution. Now, we assume $f \not\equiv 0$. we define for a function $v : G \to \mathbb{R}$,

$$E_p[v] := \int_G \frac{1}{2}\nabla_\omega v \cdot \nabla_{p,\omega}v + q|v|^p - pfv$$

and for any $r \in [0, \infty)$,

$$S_r := \left\{v : G \to \mathbb{R} \mid \int_G |v|^p = r^p\right\}.$$

Since $\lambda_0 \le \int_G -\Delta_{p,\omega} v \cdot v + q|v|^p$ for all $v \in S_1$,

$$E_p[\varepsilon v] = \varepsilon^p \left(\int_G \frac{1}{2} \nabla_\omega v \cdot \nabla_{p,\omega} v + q|v|^p \right) - \varepsilon \int_G pfv$$
$$\ge \varepsilon^p \lambda_0 - \varepsilon \int_G pfv.$$

Since $\lambda_0 > 0$, $E_p[\varepsilon v] \to \infty$ as $\varepsilon \to \infty$ for all $v \in S_1$. Hence there exists $r > 0$ such that $E_p[v] > 0$ for $v \in S_r$. Now we define for any $r \in [0, \infty)$,

$$B_r := \left\{ v : G \to \mathbb{R} | \int_G |v|^p \le r^p \right\}.$$

Since $E_p[\mathbb{O}] = 0$ and B_r is compact, there exists w_0 in the interior of B_r such that

$$E_p[w_0] = \min_{v \in B_r} E_p[v].$$

Thus for any $v : G \to \mathbb{R}$,

$$\begin{aligned} 0 =& \frac{d}{dt} E_p[w_0 + tv]|_{t=0} \\ =& - p \sum_{x,y \in G} |w_0(y) - w_0(x)|^{p-2} (w_0(y) - w_0(x)) v(x) \omega(x,y) \\ &+ p \sum_{x \in G} (q(x)|w_0(x)|^{p-2} w_0(x) - f(x)) v(x) d_\omega x. \end{aligned}$$

Define for any $x \in G$, a function $\delta_x : G \to \mathbb{R}$ as following

$$\delta_x(y) = \begin{cases} 1, x = y \\ 0, \text{otherwise.} \end{cases}$$

Take any $x \in G$ and put $v \equiv \delta_x$ then we have

$$-\sum_{y \in G} |w_0(y) - w_0(x)|^{p-2} (w_0(y) - w_0(x)) \omega(x,y) + q(x)|w_0(x)|^{p-2} w_0(x) d_\omega x = f(x)$$

for all $x \in G$. Now we show the uniqueness. The equivalence of assertions (i) to (ii) implies that for a given nonnegative function f, a solution u is either a positive function or zero function. Let u_i satisfy $-\Delta_{p,\omega} u_i + q|u_i|^{p-2} u_i = f$, $i = 1, 2$ and $u_1 \not\equiv u_2$. If $u_1 \equiv 0$ or $u_2 \equiv 0$, then $f \equiv 0$. It implies that $\lambda_0 \le 0$. This contradicts the assumption $\lambda_0 > 0$. Now, we assume that $u_1 > 0$ and

$u_2 > 0$. Then by Corollary 4.1

$$\begin{aligned} 0 &\le I[u_1, u_2] \\ &= \int_G -\Delta_{p,\omega} u_1 \left(\frac{u_1^p - u_2^p}{u_1^{p-1}} \right) - \int_G -\Delta_{p,\omega} u_2 \left(\frac{u_1^p - u_2^p}{u_2^{p-1}} \right) \\ &= \int_G (f - q|u_1|^{p-2} u_1) \left(\frac{u_1^p - u_2^p}{u_1^{p-1}} \right) - \int_G (f - q|u_2|^{p-2} u_2) \left(\frac{u_1^p - u_2^p}{u_2^{p-1}} \right) \\ &= \int_G f \frac{(u_1^p - u_2^p)(u_2^{p-1} - u_1^{p-1})}{u_1^{p-1} u_2^{p-1}} \le 0. \end{aligned}$$

Thus by Corollary 4.1, $u_1 \equiv c u_2$ for some $c > 0$. It follows that

$$\begin{aligned} f &= -\Delta_{p,\omega} u_1 + q|u_1|^{p-2} u_1 \\ &= c^{p-1}(-\Delta_{p,\omega} u_2 + q|u_2|^{p-2} u_2) \\ &= c^{p-1} f. \end{aligned}$$

Thus $c = 1$. □

In Theorem 4.2, we don't use the condition that a function f is nonnegative when we prove the existence of a solution of the p-Schrödinger equation. Accordingly, if we remove the condition that the function f is nonnegative in (v), then we get the following result by the same proof.

Corollary 4.2. *Let f be a given function on G. If a function q satisfies that $\lambda_0 > 0$ then there exists a solution u of the following equation*

$$-\Delta_{p,\omega} u(x) + q(x)|u(x)|^{p-2} u(x) = f(x), \ \ x \in G.$$

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