Publ. RIMS, Kyoto Univ.
45 (2009), 383–391

Representation of Spaces of Entire Functions on Banach Spaces

By

José M. Ansemil*, Richard M. Aron** and Socorro Ponte***

Abstract

We prove the impossibility of expressing the space of entire functions on any infinite dimensional complex Banach space with a Schauder basis E as a countable union of spaces of entire functions that are bounded on countable open covers of E.

§1. Notation and Preliminaries

In the study of the space of holomorphic functions on infinite dimensional complex normed spaces $\mathcal{H}(E)$, several natural locally convex topologies appear. Among them, the three most important are the compact open topology τ_0, the Nachbin "ported" topology τ_ω, and the bornological topology τ_δ associated to these two topologies. One useful characterization of the τ_δ topology, introduced independently by Nachbin [6] and Cœuré [2] about 30 years ago, is that it is generated by all seminorms $p : \mathcal{H}(E) \to \mathbb{C}$ which satisfy the following property:

Communicated by H. Okamoto. Received December 18, 2007. Revised June 26, 2008.
2000 Mathematics Subject Classification(s): Primary 46G20; Secondary 46A03, 46A13.
Key words: entire function on a Banach space, inductive limit, Schauder basis.
*Departamento de Análisis Matemático, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, Spain.
e-mail: jm_ansemil@mat.ucm.es
Supported by Project PR41/06-14939 SANTANDER/COMPLUTENSE.
**Department of Mathematical Sciences, Kent State University, Kent, Ohio 44242, USA.
e-mail: aron@math.kent.edu
Supported in part by the MEC and FEDER Project MTM2008-3211.
***Departamento de Análisis Matemático, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, Spain.
e-mail: socorro_ponte@mat.ucm.es
Supported by Project PR41/06-14939 SANTANDER/COMPLUTENSE.

For every increasing countable open cover $\Theta = \{V_j \mid j \in \mathbb{N}\}$ *of* E, *there is a positive constant* C *and a positive natural number* j_0 *such that*

$$p(f) \leq C sup_{x\in V_{j_0}} |f(x)| \text{ for all } f \in \mathcal{H}(E).$$

In fact, it is easy to see that the τ_δ topology on $\mathcal{H}(E)$ can be defined as an inductive limit of the metrizable spaces $\mathcal{H}_\Theta(E) = \{f \in \mathcal{H}(E) \mid f$ is bounded on each $V_j\}$ (with the topology of uniform convergence on the V_j's considered and where Θ ranges over the family of all increasing countable open covers of E). Note that given $f \in \mathcal{H}(E)$, then $f \in \mathcal{H}_\Theta(E)$ where Θ is the increasing open cover of E given by

$$\Theta = \{V_1, V_2, ...\}$$

with $V_j = \{x \in E : |f(x)| < j\}$.

It is natural to wonder whether, in fact, $(\mathcal{H}(E), \tau_\delta)$ is a *countable* inductive limit of such spaces. The aim of this paper is to prove that this inductive limit cannot be such a countable union, at least in the case when E is any infinite dimensional complex Banach space with a Schauder basis.

In our main result, below, we will be discussing sequences $\{\Theta_n\}_{n=0}^{\infty}$ of increasing countable open covers of E. That is, for every n,

$$\Theta_n = \{V_{n,1}, V_{n,2}, ...\},$$

where each $V_{n,j}$ is open, $V_{n,j} \subset V_{n,j+1}$ for all j and $\cup_{j=1}^{\infty} V_{n,j} = E$. As usual, for a function h and a set V, $\|h\|_V = \sup_{x\in V} |h(x)|$.

§2. Countable Unions of Spaces of Entire Functions

Our main aim in this paper is to prove the result mentioned in the abstract:

Theorem. *For any infinite dimensional complex Banach space* E *with a Schauder basis and for any sequence* $\{\Theta_n\}_{n=0}^{\infty}$ *of increasing countable open covers of* E,

$$\mathcal{H}(E) \neq \cup_{n=o}^{\infty} \mathcal{H}_{\Theta_n}(E).$$

In fact, we will show the density of $\mathcal{H}(E) \setminus \cup_{n=o}^{\infty} \mathcal{H}_{\Theta_n}(E)$ in the space of all entire functions $\mathcal{H}(E)$.

The proof of the theorem will use the following two propositions that we believe are also interesting in themselves. The first proposition is an adaptation of an argument of the second author in [1].

Proposition 2.1. *Let E be an infinite dimensional Banach space with a normalized Schauder basis $\{e_n\}$ with basis constant M. Then given $s > 0$, there is an entire function h on E such that $||h||_{B(0,\frac{1}{3M})} < \infty$ and such that for every $p \in \mathbb{N}$ there is $q \in \mathbb{N}$ such that $||h||_{B(e_q,\frac{s}{2^p})} = \infty$.*

Proof. Let us fix a natural number γ_1 such that $\frac{1}{2^{\gamma_1}} < s$ and consider the sequence $\{\gamma_j\}_{j=2}^{\infty}$ where $\gamma_j = \gamma_1 + j$. For each $n \in \mathbb{N}$, let $\phi_n : E \to \mathbb{C}$ be given by $\phi_n(x) = 2x_n$, where $x = \sum_{n=1}^{\infty} x_n e_n \in E$. Note that as we assume that the projections $x \mapsto x_n$ have norm less than or equal to M, then our ϕ_n have norm less or equal than $2M$. Write $\mathbb{N} = \cup_j \mathbb{N}_j$ where each $\mathbb{N}_j$ is an increasing infinite sequence and $\mathbb{N}_j \cap \mathbb{N}_k = \phi$ for $j \neq k$. Let $\mathbb{N}_j = \{n_1^j, n_2^j, ...\}$, where we assume that $n_1^1 = 1$, and let h be the function defined on E by

$$h(x) = \sum_{j=1}^{\infty}\sum_{k=2}^{\infty} \left[(\phi_{n_1^j}(x))^{\gamma_j}\phi_{n_k^j}(x)\right]^{jk}.$$

The function h is holomorphic because the series converges uniformly on compact subsets of E. Moreover, h is bounded in $B(0, \frac{1}{3M})$. Indeed for x in $B(0, \frac{1}{3M})$ we have

$$|h(x)| \leq \sum_{j=1}^{\infty}\sum_{k=2}^{\infty} \left|(\phi_{n_1^j}(x))^{\gamma_j}\phi_{n_k^j}(x)\right|^{jk} \leq$$

$$\sum_{j=1}^{\infty}\sum_{k=2}^{\infty}[(2M||x||)^{\gamma_j}2M||x||]^{jk} \leq$$

$$\sum_{j=1}^{\infty}\sum_{k=2}^{\infty}\left[\left(\frac{2}{3}\right)^{\gamma_j}\frac{2}{3}\right]^{jk} \leq \sum_{j=1}^{\infty}\sum_{k=2}^{\infty}\left(\frac{4}{9}\right)^{kj} < \infty.$$

Given $p \in \mathbb{N}$, take another natural number $t \geq 2$ and consider

$$w_{p,t} := e_{n_1^p} + \frac{s}{2^{p+1}}e_{n_t^p}$$

Then we have

$$h(w_{p,t}) = \sum_{j=1, j\neq p}^{\infty}\sum_{k=2}^{\infty}\left[(\phi_{n_1^j}(w_{p,t}))^{\gamma_j}\phi_{n_k^j}(w_{p,t})\right]^{jk} + \sum_{k=2}^{\infty}\left[(\phi_{n_1^p}(w_{p,t}))^{\gamma_p}\phi_{n_k^p}(w_{p,t})\right]^{pk}$$

The first sum is 0 since $j \neq p$, and so

$$h(w_{p,t}) = \sum_{k=2}^{\infty}\left[(\phi_{n_1^p}(w_{p,t}))^{\gamma_p}\phi_{n_k^p}(w_{p,t})\right]^{pk} = \left[(2\cdot 1)^{\gamma_p}\frac{s}{2^{p+1}}\cdot 2)\right]^{pt}$$

which goes to ∞ with t because $2^{\gamma_p}\frac{s}{2^p} > 1$. Since $w_{p,t} \in B(e_{n_1^p}, \frac{s}{2^p})$ for all $t \in \mathbb{N}$, we have that $||h||_{B(e_{n_1^p},\frac{s}{2^p})} = \infty$, this proves the result if we take $q = n_1^p$. □

Proposition 2.2. *Let E be an infinite dimensional Banach space with a normalized Schauder basis $\{e_n\}$ with basis constant M. For every $R > 0$ and $r > 0$, there exists an entire function f such that*

$$||f||_{B(0,R)} < \infty \text{ and } ||f||_{B(3MRe_1,r)} = \infty.$$

Proof. Given R and r let us define a function f on E by

$$f(x) = h\left(\frac{x}{3MR}\right)$$

being h the function associated with $s = \frac{r}{3MR}$ in the above proposition. Then $||f||_{B(0,R)} = ||h||_{B(0,\frac{1}{3M})} < \infty$. On the other hand, if we take $p = 1$, the proof of that proposition gives $q = n_1^1 = 1$, which implies that $||h||_{B(e_1,\frac{s}{2})} = \infty$. Therefore since for $x \in B(e_1, s)$ we have $3MRx \in B(3MRe_1, r)$, we get

$$||f||_{B(3MRe_1,r)} \geq ||h||_{B(e_1,s)} = \infty.$$

□

Remark. It is not known if for any pair of disjoint balls B_1 and B_2 in an infinite dimensional normed space E there is an entire function f such that

$$\|f\|_{B_1} < \infty \text{ and } \|f\|_{B_2} = \infty.$$

This is somewhat surprising, in light of results such as [1] that prove the existence of a dense set (for all the topologies we are considering) of entire functions f such that for every $\varepsilon > 0$, there is $x_\varepsilon \in E$, $\|x_\varepsilon\| \leq 1$, satisfying $||f||_{B(x_\varepsilon,\epsilon)} = \infty$. For the normed spaces for which this occurs, we have the following.

Lemma 2.1. *Given $R > 0$ and $0 < r < R$, let $f \in \mathcal{H}(E)$ be chosen to satisfy the conclusion of the above proposition. Then for every $n \in \mathbb{N}$, there is $g_n \in \mathcal{H}(E)$ such that*

$$\|g_n f\|_{B(0,R)} < \frac{1}{2^n} \text{ and } \|g_n f\|_{B(3MRe_1,r)} = \infty.$$

Proof. Let $x_1 = 3MRe_1$. By the Hahn Banach theorem there are $u \in (E)^*_{\mathbb{R}}$ and $\alpha > 0$ such that

$$u(a) < \alpha \leq u(b) \text{ for all } a \in B(0,R) \text{ and } b \in B(x_1, r).$$

Note that $u(0) = 0$. Let us consider the functional $\upsilon \in E^*$ given by $\upsilon(x) = u(x) - iu(ix)$ and

$$g_n = \frac{\upsilon}{2^n \|f\|_{B(0,R)} \cdot \|\upsilon\|_{B(0,R)}}.$$

Then $g_n \in E^* \subset \mathcal{H}(E)$,

$$\|g_n f\|_{B(0,R)} = \left\| \frac{\upsilon f}{2^n \|f\|_{B(0,R)} \cdot \|\upsilon\|_{B(0,R)}} \right\|_{B(0,R)} \leq \frac{1}{2^n}$$

and

$$\|g_n f\|_{B(x_1,r)} = \left\| \frac{\upsilon f}{2^n ||f||_{B(0,R)} \cdot \|\upsilon\|_{B(0,R)}} \right|_{B(x_1,r)} \geq$$

$$\frac{\alpha}{2^n \|f\|_{B(0,R)} \cdot \|\upsilon\|_{B(0,R)}} \|f\|_{B(x_1,r)} = \infty.$$

□

Proof of the Theorem. There is no loss of generality if we assume that the Schauder basis is normalized. Given a countable collection of increasing open covers $\{\Theta_n\}_{n=0}^{\infty}$, where each $\Theta_n = \{V_{n,j} \mid j \in \mathbb{N}\}$, we will prove the existence of an entire function f on E such that for every n there is j_n such that $\|f\|_{V_{n,j_n}} = \infty$. As a consequence, f will not belong to $\mathcal{H}_{\Theta_n}(E)$ for any n.

Since $\cup_{j=1}^{\infty} V_{0,j} = E$, there is $j_0 \in \mathbb{N}$ such that $0 \in V_{0,j_0}$ and so there is $R_0 > 0$ such that $B(0, R_0) \subset V_{0,j_0}$. It is well-known (see [3], pp. 157-158) that there is an entire function f_0 in E such that $\|f_0\|_{B(0,R_0)} = \infty$.

Let $x_1 = 3MR_0 e_1 \in E$. Since $\cup_{j=1}^{\infty} V_{1,j} = E$, there is $j_1 \in \mathbb{N}$ such that $x_1 \in V_{1,j_1}$ and then, since V_{1,j_1} is open and f_0 is continuous, there is $r_1 > 0$ such that $r_1 < R_0$, $B(x_1, r_1) \subset V_{1,j_1}$ and $\|f_0\|_{B(x_1,r_1)} < \infty$.

By Proposition 2.2 we get a function $f_1 \in \mathcal{H}(E)$ such that

$$\|f_1\|_{B(0,R_0)} < \infty \text{ and } \|f_1\|_{B(x_1,r_1)} = \infty.$$

In this way we obtain sequences: $\{R_n\} \subset \mathbb{R}^+$ increasing to ∞, $\{x_n\} \subset E, \{j_n\} \subset \mathbb{N}, \{r_n\} \subset \mathbb{R}^+$, and $\{f_n\} \subset \mathcal{H}(E)$ having the following properties: For every $n \geq 1$, $R_n = (3M+1)R_{n-1}, x_n = 3MR_{n-1}e_1,\ r_n < R_0,\ B(x_n, r_n) \subset V_{n,j_n},\ ||f_j||_{B(x_n,r_n)} < \infty$ for all $j = 0, ..., n-1,\ ||f_n||_{B(0,R_{n-1})} < \infty$, and $||f_n||_{B(x_n,r_n)} = \infty$.

By an application of Lemma 2.1 we get, for every $n \in \mathbb{N}$, a function $g_n \in \mathcal{H}(E)$ such that

$$||g_n f_n||_{B(0,R_{n-1})} < \frac{1}{2^n} \text{ and } ||g_n f_n||_{B(x_n,r_n)} = \infty.$$

To conclude the proof, let us verify that the function

$$f = f_0 + g_1 f_1 + g_2 f_2 + \cdots$$

satisfies our requirements.

Given a ball B in E, choose R_k in the above sequence such that $B \subset B(0, R_k)$. Then

$$\sum_{j=k+1}^{\infty} \|g_j f_j\|_B \leq \sum_{j=k+1}^{\infty} \|g_j f_j\|_{B(0,R_k)} \leq \sum_{j=k+1}^{\infty} \frac{1}{2^j}$$

which implies that the series that defines f is uniformly convergent on balls, and so f is an entire function on E.

On the other hand, since

$$||f_j||_{B(x_n,r_n)} < \infty$$

for every $n \in \mathbb{N}$ and $j = 0, ..., n-1$, we have that $||f_0||_{B(x_n,r_n)} < \infty$ and $||g_j f_j||_{B(x_n,r_n)} < \infty$ for all $j \leq n-1$. Moreover for $j = n+1, n+2, ...$ we have that $B(x_n, r_n) \subset B(0, R_{j-1})$ and so

$$||g_j f_j||_{B(x_n,r_n)} \leq ||g_j f_j||_{B(0,R,_{j-1})} < \frac{1}{2^j}.$$

This implies that

$$\|f\|_{B(x_n,r_n)} = \left\| f_0 + \sum_{j=1}^{n-1} g_j f_j + g_n f_n + \sum_{j=n+1}^{\infty} g_j f_j \right\|_{B(x_n,r_n)} \geq$$

$$\|g_n f_n\|_{B(x_n,r_n)} - \left(\|f_0\|_{B(x_n,r_n)} + \sum_{j=1}^{n-1} \|g_j f_j\|_{B(x_n,r_n)} + \sum_{j=n+1}^{\infty} \|g_j f_j\|_{B(x_n,r_n)} \right) = \infty$$

because $\|g_n f_n\|_{B(x_n,r_n)} = \infty$. Since $B(x_n, r_n)$ is contained in V_{n,j_n} we have that

$$\|f\|_{V_{n,j_n}} = \infty$$

and also

$$\|f\|_{V_{0,j_0}} = \infty$$

because $\|f_0\|_{V_{0,j_0}} = \infty$ and $\sum_{j=1}^{\infty} \|g_j f_j\|_{B(0,R_0)} < \infty$. Therefore

$$f \notin \mathcal{H}_{\Theta_n}(E) \text{ for all } n = 0, 1, 2, ...$$

□

In the next proposition we are going to see that not only is

$$\mathcal{H}(E) \setminus \cup_{n=0}^{\infty} \mathcal{H}_{\Theta_n}(E)$$

non-void, but it is even dense in $\mathcal{H}(E)$ for the strongest topology we are considering on $\mathcal{H}(E)$, namely τ_δ.

Proposition 2.3. *Let E be an infinite dimensional Banach space with a Schauder basis. For any sequence $\{\Theta_n\}_{n=0}^{\infty}$ of increasing countable open covers of E, the set*

$$\mathcal{H}(E) \setminus \cup_{n=0}^{\infty} \mathcal{H}_{\Theta_n}(E)$$

is dense in $(\mathcal{H}(E), \tau_\delta)$.

Proof. Without loss of generality, we may assume that the sequence $\{\Theta_n\}_{n=0}^{\infty}$ is decreasing in n. That is, for every n and j we assume $V_{n,j} \supset V_{n+1,j}$. Indeed, if this is not the case it is enough to consider a new sequence $\{\Phi_n\}_{n=0}^{\infty}$ where $\Phi_n = \{W_{n,j}\}_{j=1}^{\infty}$ and $W_{n,j} = \cap_{k=0}^{n} V_{k,j}$ for all j. Note that each Φ_n is an increasing open covering of E. (Given $n \in \mathbb{N}, x \in E$ and $k = 0, 1, ..., n$, let j_k be chosen so that x belongs to V_{k,j_k}; then x belongs to $W_{n,max\{j_k : k=0,..,n\}}$.) Note also that $\mathcal{H}(E) \setminus \cup_{n=0}^{\infty} \mathcal{H}_{\Theta_n}(E) \supset \mathcal{H}(E) \setminus \cup_{n=0}^{\infty} \mathcal{H}_{\Phi_n}(E)$. This assumption allows us to ignore any finite number of terms in the expression of the function f in the above theorem and this will be used later.

Let us prove first that for every continuous seminorm p in $(\mathcal{H}(E), \tau_\delta)$ and for every $\varepsilon > 0$ there exists $g_{p,\varepsilon} \in \mathcal{H}(E) \setminus \cup_{n=o}^{\infty} \mathcal{H}_{\Theta_n}(E)$ such that $p(g_{p,\varepsilon}) < \varepsilon$. To do this, we will make use of the functions f_n, g_n and real numbers R_m that appear in the proof of the theorem.

With the notation of Lemma 2.1 let $\Theta = \{\mathring{A}_j\}_{j=1}^{\infty}$, where $A_j = \{x \in E : |f_k(x) g_k(x)| < \frac{1}{2^k}$ for all $k \geq j\}$. We claim that the collection Θ is an increasing open cover of E. Indeed, every $x \in E$ is in one of the balls $B(0, R_{j-1})$, and then for every $k \geq j$ we have,

$$||f_k g_k||_{B(0,R_{j-1})} \leq ||f_k g_k||_{B(0,R_{k-1})} < \frac{1}{2^k}. \tag{1}$$

This implies that $B(0, R_{j-1}) \subset A_j$ and so $x \in \mathring{A}_j$. By the definition of the τ_δ topology, given p there is a positive constant C and a natural number j_0 such that

$$p(g) \leq C ||g||_{\mathring{A}_{j_0}} \quad \text{for all } g \in \mathcal{H}(E). \tag{2}$$

Given ε assume that the above j_0 satisfies $\sum_{j=j_0}^{\infty} \frac{1}{2^j} < \frac{\varepsilon}{C}$. We observe that the function $g_{p,\varepsilon}$ in $\mathcal{H}(E) \setminus \cup_{n=0}^{\infty} \mathcal{H}_{\Theta_j}(E)$, given by

$$g_{p,\varepsilon} = g_{j_0} f_{j_0} + g_{j_0+1} f_{j_0+1} + \cdots,$$

satisfies $p(g_{p,\varepsilon}) < \varepsilon$. Indeed, from (1) and (2) we get that

$$p(g_{p,\varepsilon}) \leq C \sum_{j=j_0}^{\infty} ||g_j f_j||_{A_{j_0}} \leq C \sum_{j=j_0}^{\infty} \frac{1}{2^j} < \varepsilon.$$

Finally, let $f \in \mathcal{H}(E)$, a τ_δ seminorm p, and $\epsilon > 0$ be given, and take $j_0 \in \mathbb{N}$ as in the above paragraph. Let us assume that $f \notin \mathcal{H}(E) \backslash \cup_{n=0}^{\infty} \mathcal{H}_{\Theta_n}(E)$. There is $n_0 \in \mathbb{N}$, that we may assume to be greater than j_0, such that $f \in \mathcal{H}_{\Theta_{n_0}}(E)$. By our assumption on the open covers, it follows that $f \in \mathcal{H}_{\Theta_n}(E)$, and so $||f||_{V_{n,j}} < \infty$ for all $n \geq n_0$ and $j \in \mathbb{N}$. Given p and ε, we have that

$$f + g_{p,\varepsilon} \in \mathcal{H}(E) \backslash \cup_{n=0}^{\infty} \mathcal{H}_{\Theta_n}(E).$$

To see this, if $n \geq n_0$ there is j_n as in the proof of the theorem such that

$$||f + g_{p,\varepsilon}||_{V_{n,j_n}} \geq ||g_{p,\varepsilon}||_{V_{n,j_n}} - ||f||_{V_{n,j_n}} = \infty.$$

Also, $V_{1,j_{n_0}} \supset V_{2,j_{n_0}} \supset \cdots \supset V_{n_0,j_{n_0}}$ and therefore $||f + g_{p,\varepsilon}||_{V_{n,j_{n_0}}} = \infty$ for $n = 0, ..., n_0 - 1$, which implies that $f + g_{p,\varepsilon} \in \mathcal{H}(E) \backslash \cup_{n=0}^{\infty} \mathcal{H}_{\Theta_n}(E)$. Thus, $p(f + g_{p,\varepsilon} - f) = p(g_{p,\varepsilon}) < \varepsilon$. □

Remarks.

1. The above arguments can also be applied to normed spaces with a Schauder basis such that the projections are uniformly bounded. For instance, this happens for c_{00}, the space of all eventually null sequences with any of the p norms, and for every normed space which is dense in a Banach space with a Schauder basis, since by the Perturbation of Basis Theorem these spaces also have a Schauder basis with uniformly bounded projections (see, e.g., [4, Prop. 1.a.9.]). In particular this applies to $C[0,1]$ with any p norm, $p \in [1, \infty)$, since they are normed spaces which are dense in $L^p[0,1]$ which has a Schauder basis.

2. As we have shown in the proof of Proposition 2.3, from a given sequence $\{\Theta_n\}$ of increasing open coverings of E, we can construct another sequence $\{\Phi_n\}$ of increasing open coverings of E such that for all n,

$$\mathcal{H}_{\Phi_n}(E) \subset \mathcal{H}_{\Phi_{n+1}}(E)$$

with continuous inclusions and $\cup_{n=0}^{\infty} \mathcal{H}_{\Theta_n}(E) \subset \cup_{n=0}^{\infty} \mathcal{H}_{\Phi_n}(E)$. Then our Theorem proves that the (LF)-space $\underrightarrow{lim}\ \mathcal{H}_{\Phi_n}(E)$ does not agree with $(H(E), \tau_\delta)$. (LF)-spaces have been studied and applied to holomorphy by several authors, particularly Mujica in [5].

Acknowledgements

This work was started when the second author was a visitor to the Departamento de Análisis Matemático of the Universidad Complutense de Madrid (April/June 2006), and concluded when the first and third authors visited the

Mathematics Department of Kent State University (September 2007). They acknowledge the hospitality of both institutions.

References

[1] R. M. Aron, Entire functions of unbounded type on a Banach space, Boll. Un. Mat. Ital. (4) **9** (1974), 28–31.

[2] G. Coeuré, Fonctions plurisousharmoniques sur les espaces vectoriels topologiques et applications àe l'étude des fonctions analytiques, Ann. Inst. Fourier (Grenoble) **20** (1970), fasc. 1, 361–432.

[3] S. Dineen, *Complex analysis on infinite-dimensional spaces*, Springer Monographs in Mathematics, Springer, London, 1999.

[4] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. I and II*, Classics in Mathematics, Springer, Berlin, 1977.

[5] J. Mujica, Spaces of germs of holomorphic functions, in *Studies in analysis*, Adv. in Math. Suppl. Stud. **4**, 1–41, Academic Press, New York, 1979.

[6] L. Nachbin, *Topology on spaces of holomorphic mappings*, Ergebnisse der Mathematik und ihrer Grenzgebiete **47**, Springer-Verlag New York Inc., New York, 1969.