Publ. RIMS, Kyoto Univ.
45 (2009), 451–473

C*-Crossed Products by R, II

By

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Abstract

We propose a condition called *no energy gap* for a flow on a C*-algebra, which is expressed in terms of spectral subspaces, and find an equivalent condition on the primitive ideals of the crossed product which says that any primitive ideal is monotonely increasing or decreasing under the dual flow, when the C*-algebra is simple and the flow is outer. We also discuss some examples involving UHF flows.

§1. Introduction

Let α be a flow on a C*-algebra A, i.e., α is a continuous homomorphism $\mathbf{R}$ into the automorphism group of A (which is equipped with the topology of pointwise convergence). For this system (A, α) we construct the dual system $(A \times_\alpha \mathbf{R}, \hat{\alpha})$ by the method of crossed product, where we know that we keep as much information as on the original system by the Takesaki-Takai duality. Not only that there is a merit; part (or much) of the information on α is now mapped into the C*-algebra $A \times_\alpha \mathbf{R}$. The recent accomplishment on the classification theory of C*-algebras gives us some hope that we might be able to get some insight into α by classifying the crossed products.

In [6] we studied some crossed products by $\mathbf{R}$. A simple case is the crossed product by a uniformly continuous flow. When A is simple we proved that $A \times_\alpha \mathbf{R}$ is isomorphic to $A \otimes C_0(\mathbf{R})$ *without using the fact that α is inner* and then proved this fact due to Sakai.

Another case treated there is when α is outer in a strong sense. More precisely when A is unital and α-simple and α is faithful and has an automorphism γ such that $\gamma\alpha_t = \alpha_t\gamma,\ t \in \mathbf{R}$ and $\|[\gamma^n(x), y]\| \to 0$ for any $x, y \in A$, then

Communicated by M. Kashiwara. Received December 25, 2007. Revised June 2, 2008.
2000 Mathematics Subject Classification(s): 46L55.
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the crossed product $A \times_\alpha \mathbf{R}$ is simple if and only if (A, α) has neither ground states nor ceiling states. Hence, in particular, if $A \times_\alpha \mathbf{R}$ is not simple, there is an ideal I of $A \times_\alpha \mathbf{R}$ such that $t \mapsto \hat{\alpha}_t(I)$ is either decreasing or increasing and $\bigcup_t \hat{\alpha}_t(I)$ is dense in $A \times_\alpha \mathbf{R}$ as well as $\bigcap_t \hat{\alpha}_t(I) = \{0\}$ [10]. Here I need not be primitive.

In this note we continue to study the ideal structures of crossed products by $\mathbf{R}$ along the same lines by introducing the notion of *no energy gap*, which is a condition on the spectral subspaces of the flow on invariant hereditary C*-subalgebras. We will translate this into a condition on the primitive ideals of the crossed product. More precisely, if A is simple and α_t is outer for all $t \neq 0$, then the no energy gap condition is equivalent to saying that all primitive ideals of the crossed product $A \times_\alpha \mathbf{R}$ is monotone under the dual flow $\hat{\alpha}$. In particular we know that the no energy gap condition is stable under cocycle perturbations in this case.

Before closing this section we give some basic facts which will be frequently used later.

Let $f \in L^1(\mathbf{R})$ and define a Fourier transform $\hat{f}$ of f by

$$\hat{f}(\lambda) = \int f(t) e^{-i\lambda t} dt.$$

Let $K^1(\mathbf{R})$ be the ideal of $L^1(\mathbf{R})$ consisting of f with $\text{supp}(\hat{f})$ compact.

Let α be a flow on a C*-algebra A. For $f \in K^1(\mathbf{R})$ and $x \in A$ we define

$$\alpha_f(x) = \int f(t) \alpha_t(x) dt.$$

The α-spectrum $\text{Sp}_\alpha(x)$ of x is defined as the hull of $I \equiv \{f \in K^1(\mathbf{R}) \mid \alpha_f(x) = 0\}$, i.e., the intersection of $\hat{f}^{-1}(0),\ f \in I$. Note that the α-spectrum of $\alpha_f(x)$ is included in $\text{supp}(\hat{f})$. The spectrum $\text{Sp}(\alpha)$ is defined as the hull of $\{f \in K^1(\mathbf{R}) \mid \alpha_f = 0\}$.

Let V be a non-empty open subset of $\mathbf{R}$. We denote by $A^\alpha(V)$ the closure of the set of $x \in A$ with $\text{Sp}_\alpha(x) \subset V$, which is the same as the closure of the set of $\alpha_f(x), x \in A$ and $f \in K^1(\mathbf{R})$ with $\text{supp}(\hat{f}) \subset V$; $A^\alpha(V)$ is an α-invariant closed subspace of A. We define $\tilde{\text{Sp}}(\alpha)$ to be the set of $p \in \mathbf{R}$ such that for any open neighborhood V of p the linear span of $A^\alpha(V)^* A A^\alpha(V)$ is dense in A.

Let $H^\alpha(A)$ be the set of non-zero α-invariant hereditary C*-subalgebras of A. The Connes spectrum $\mathbf{R}(\alpha)$ of α is defined to be the intersection of $\text{Sp}(\alpha|B)$ for all $B \in H^\alpha(A)$, which is a closed subgroup of $\mathbf{R}$. The strong Connes spectrum $\tilde{\mathbf{R}}(\alpha)$ is defined to be the intersection of $\tilde{\text{Sp}}(\alpha|B)$ for all $B \in H^\alpha(A)$, which is a closed subsemigroup of $\mathbf{R}$. See [12, 4, 13, 5] for details.

For $\lambda > 0$ and $B \in H^\alpha(A)$ we denote by B_λ the C*-subalgebra of B generated by $B^\alpha(-\lambda, \lambda)$, where α also denotes the restriction of α to B. Note that $\lambda \mapsto B_\lambda$ is increasing. We are concerned with the condition that $B_\lambda = B$ for all $B \in H^\alpha$ and $\lambda > 0$; which will be called the *no energy gap condition.* This can happen as shown by the following:

Proposition 1.1. *Let α be a flow on a C*-algebra A. If $\tilde{\mathbf{R}}(\alpha) = \mathbf{R}$, then α satisfies the no energy gap condition.*

Proof. Let $\lambda > 0$ and $\mu \geq \lambda$. Let $n \in \mathbf{N}$ and $\nu \in [0, \lambda/2)$ be such that $\mu = n\lambda/2 + \nu$. We set $\epsilon = \lambda/4(n+1)$.

Let $B \in H^\alpha(A)$ and let $x \in B^\alpha(\mu - \lambda/2, \mu + \lambda/2)$. Let $z_i \in B^\alpha(\lambda/2 - \epsilon, \lambda/2 + \epsilon)$ for $i = 1, 2, \ldots, n$ and $z_{n+1} \in B^\alpha(\nu - \epsilon, \nu + \epsilon)$. Then it follows that

$$xz_1^* z_2^* \cdots z_{n+1}^* \in B^\alpha(-\lambda, \lambda) \subset B_\lambda.$$

Since the hereditary C*-subalgebra of B_λ generated by $z_{n+1}^* z_{n+1}$, $z_{n+1} \in B^\alpha(\nu - \epsilon, \nu + \epsilon)$ is B_λ itself, we get that $xz_1^* \cdots z_n^* \in B_\lambda$. Repeating this process we obtain that $x \in B_\lambda$. Since this is true for any $\mu \geq \lambda$, we get that $B = B_\lambda$. □

Certainly this is not the only situation where the no energy gap condition is satisfied. See section 3 for some examples. We also include some observations there on the ideals of crossed products by flows.

§2. No Energy Gap

When α is a flow, we denote by δ_α the infinitesimal generator of α. Let $h \in A_{sa}$ and denote by $\operatorname{ad} ih$ the derivation on A defined by $x \mapsto i[x, h]$. We call the flow generated by $\delta_\alpha + \operatorname{ad} ih$ an inner perturbation of α.

Theorem 2.1. *Let α be a flow on a C*-algebra A. Suppose that for each $t \neq 0$ A is α_t-simple and $\mathbf{T}(\alpha_t) = \mathbf{T}$. Then the following conditions are equivalent*:

1. *α satisfies the no energy gap condition.*
2. *All primitive ideals of $A \times_\alpha \mathbf{R}$ are monotone under $\hat{\alpha}$.*
3. *For any $B \in H^\alpha(A)$ and for any inner perturbation of β of $\alpha|B$, $B_{(0,\lambda)}$ is independent of $\lambda > 0$ and $B_{(-\lambda,0)}$ is independent of $\lambda > 0$, where*

$$B_V = \overline{B^\beta(V)^* B B^\beta(V)}$$

for any open subset V of $\mathbf{R}$.

Moreover if the above conditions are satisfied, then $\mathbf{R}(\alpha) = \mathbf{R}$ (*or* $A \times_\alpha \mathbf{R}$ *is prime*).

That A is α_t-simple means that if I is an α_t-invariant ideal of A, then $I = \{0\}$ or $I = A$. Note that $\mathrm{Sp}(\alpha|B)$ is unbounded for any $B \in H^\alpha(A)$, because otherwise the Connes spectrum $\mathbf{T}(\alpha_t)$ must be zero (for any fixed $t \neq 0$).

Under the situation of the above theorem, if $\mathbf{R}(\alpha) \neq \mathbf{R}$, then the last statement implies that α does not satisfy the no energy gap condition. There are such flows.

Proof of the last statement.

Let I, J be non-zero ideals of $A \times_\alpha \mathbf{R}$ such that $\hat{\alpha}_t(I) \subset I$ and $\hat{\alpha}_t(J) \subset J$ for all $t > 0$. We shall show that $I \cap J \neq \{0\}$.

There are $B, C \in H^\alpha(A)$ and non-zero $f, g \in K^1(\mathbf{R})$ such that $U(f)B \subset I$ and $U(g)C \subset J$, where U is the canonical unitary-multiplier flow of $A \times_\alpha \mathbf{R}$ (see [7]). Let $b = \inf \mathrm{supp}(\hat{f})$; then $U(f')B \subset I$ for any $f' \in K^1(\mathbf{R})$ with $\mathrm{supp}(\hat{f}') \subset [b, \infty)$. Let $x \in BAC$ be a non-zero element such that $\mathrm{Sp}_\alpha(x)$ is compact. It follows then that if $h \in K^1(\mathbf{R})$ satisfies that $\mathrm{Sp}_\alpha(x) + \mathrm{supp}(\hat{h}) \subset (b, \infty)$, then $xU(h) \in I$. Because if $f' \in K^1(\mathbf{R})$ satisfies that $\mathrm{supp}(\hat{f}') \subset [b, \infty)$ and $\hat{f}' = 1$ on an open neighborhood of $\mathrm{Sp}_\alpha(x) + \mathrm{supp}(\hat{h})$, then $xU(h) = U(f')xU(h) \in I$. If C_1 is the α-invariant hereditary C*-subalgebra of A generated by x^*x, this implies that $U(h)^*C_1 \subset I$. Since $C_1 \subset C$, we also have that $U(g)C_1 \subset J$. Let $c = \max\{\inf \mathrm{supp}(\hat{h}), \inf \mathrm{supp}(\hat{g})\}$. For all $k \in K^1(\mathbf{R})$ with $\mathrm{supp}(\hat{k}) \subset [c, \infty)$ we get that $U(k)C_1 \subset I \cap J$. Thus $I \cap J \neq \{0\}$.

The same procedure applies to two non-zero ideals which are monotonely increasing under $\hat{\alpha}$.

Let I, J be non-zero ideals of $A \times_\alpha \mathbf{R}$ such that $\hat{\alpha}_t(I) \subset I$ and $\hat{\alpha}_t(J) \supset J$ for all $t > 0$. We will show that $I \cap J \neq \emptyset$.

There are $B, C \in H^\alpha(A)$ and $f, g \in K^1(\mathbf{R})$ such that $U(f)B \subset I$ and $U(g)C \subset J$. Let $b = \inf \mathrm{supp}(\hat{f})$ and $c = \sup \mathrm{supp}(\hat{g})$. Let H be the closed linear span of BAC, which is α-invariant. It follows that $\mathrm{Sp}(\alpha|H)$ is not bounded (because $\mathrm{Sp}(\alpha|B)$ is not bounded). Hence there is $x \in H$ such that $\mathrm{Sp}_\alpha(x) \subset (b - c + \lambda, b - c + \lambda + 1)$ for some $\lambda > 1$. If $h \in K^1(\mathbf{R})$ satisfies that $\mathrm{supp}(\hat{h}) \subset (c - \lambda, c - \lambda + 1)$, we obtain that $xU(h) \in I \cap J$. This implies that $I \cap J \neq \{0\}$.

Let I, J be non-zero ideals of $A \times_\alpha \mathbf{R}$. Then I (resp. J) are given as $I_1 \cap I_2$ (resp. $J_1 \cap J_2$), where I_1 and J_1 are monotonely decreasing and I_2 and J_2 are monotonely increasing under $\hat{\alpha}$. Then $I \cap J = (I_1 \cap J_1) \cap (I_2 \cap J_2)$, which is non-empty by the above arguments. This completes the proof that $A \times_\alpha \mathbf{R}$ is prime, which implies $\mathbf{R}(\alpha)=\mathbf{R}$ together with α-simplicity of A (see [12]).

Proof of (1)⇒(2).

Let P be a non-zero primitive ideal of $A\times_\alpha\mathbf{R}$. Since A is α-simple, P is not invariant under $\hat{\alpha}$.

Suppose that $s\mapsto\hat{\alpha}_s(P)$ is periodic and let s_0 be the smallest positive s with $\hat{\alpha}_s(P)=P$. Since P contains a non-zero element of the form $U(f)x$ with $x\in A$ and $f\in L^1(\mathbf{R})$, it contains all $U(e^{ins_0t}f)x$, $n\in\mathbf{Z}$, where U is the canonical unitary-multiplier flow implementing α and $U(f)=\int f(t)U_tdt$. Let B be the α-invariant hereditary C*-subalgebra generated by xx^*. Then $\{f\in L^1(\mathbf{R})\mid U(f)B\subset P\}$ is a non-zero proper ideal of $L^1(\mathbf{R})$ invariant under the multiplication by $t\mapsto e^{is_0t}$. Let ρ be an irreducible representation of $A\times_\alpha\mathbf{R}$ such that the kernel of ρ is equal to P. Then it follows that the spectrum of $U_B\equiv\overline{\rho}(U)|[\overline{\rho}(B)H_\rho]$ is a proper closed subset invariant under the translation by s_0, where $\overline{\rho}$ is the unique extension of ρ to a representation of the multiplier algebra of $A\times_\alpha\mathbf{R}$. If $\lambda>0$ is smaller than a gap of the spectrum, then $\overline{\rho}(B_\lambda)$ leaves invariant the spectral subspace of U_B corresponding to a closed interval bounded by gaps bigger than λ. Since this is invariant under U_B, it follows that $B_\lambda\neq B$. This contradiction shows that P cannot be periodic under $\hat{\alpha}$.

Now P satisfies that $\hat{\alpha}_s(P)\neq P$ for any $s\neq 0$. If ρ is an irreducible representation of $A\times_\alpha\mathbf{R}$ on a Hilbert space $\mathcal{H}$ such that its kernel is P, then it follows that $\pi=\overline{\rho}|A$ is irreducible. This follows by showing that the center of the weak closure of the image of

$$\int_{\mathrm{R}}^{\oplus}\overline{\rho}\hat{\alpha}_t dt$$

is $1\otimes L^\infty(\mathbf{R})$ (see [8]). Let $V=\overline{\rho}(U)$ and $B\in H^\alpha(A)$. Then (1) implies that $\mathrm{Sp}(V|[\pi(B)\mathcal{H}])$ is connected for the reasoning as in the preceding paragraph. Note that $\mathrm{Sp}(V|[\pi(B)\mathcal{H}])$ is unbounded because otherwise $\alpha|B$ is uniformly continuous, which implies that $\mathbf{T}(\alpha_t)=\{0\}$ for all t, a contradiction. Hence $\mathrm{Sp}(V|[\pi(B)\mathcal{H}])$ is either $\mathbf{R}$, $(-\infty,\mu]$, or $[\mu,\infty)$ for some μ.

If $\mathrm{Sp}(V|[\pi(B)\mathcal{H}])=\mathbf{R}$ for all $B\in H^\alpha(A)$, then $P=\{0\}$, contradicting the choice of P.

Suppose that $\mathrm{Sp}(V|[\pi(B)\mathcal{H}])$ is bounded below for some $B\in H^\alpha(A)$. Let $D\in H^\alpha(A)$. We will show that $\mathrm{Sp}(V|\pi(D)\mathcal{H}])$ cannot be bounded above. Let $a\in DAB$ be a non-zero element with compact α-spectrum. Let D_1 be the hereditary C*-subalgebra generated by $\alpha_s(a)B\alpha_t(a)^*$, $s,t\in\mathbf{R}$. Then $D_1\in H^\alpha(A)$ and $D_1\subset D$. For $x\in B$, $s,t\in\mathbf{R}$, and $\xi\in\mathcal{H}$, we have that

$$V_t\pi(\alpha_s(a)x)\xi=\pi(\alpha_{s+t}(a))V_t\pi(x)\xi.$$

From this we can conclude that $\mathrm{Sp}(V|\pi(D_1)\mathcal{H}])$ is bounded below. Hence $\mathrm{Sp}(V|\pi(D)\mathcal{H}])$ is either $\mathbf{R}$ or bounded below. Since P is a closed linear span of elements of the form $U(f)x$, this shows that the kernel of $\rho\hat{\alpha}_s$ increases in s, i.e., $\hat{\alpha}_s(P) \subset P$ for $s > 0$.

In the same way we prove the conclusion in the other case.

Proof of (2)⇒(3).

Suppose that (3) does not hold. We may suppose there is a $B \in H^\alpha(A)$, $h \in B_{sa}$, and $\mu > \lambda > 0$ such that

$$B_{(0,\lambda)} \subsetneqq B_{(0,\mu)},$$

where $B_{(0,\lambda)} = \overline{B^\beta(0,\lambda)^* B B^\beta(0,\lambda)}$ and β denotes the flow generated by $\delta_\alpha + \operatorname{ad} ih$. Since an approximate identity for $B_{(0,\lambda)}$ (resp. $B_{(0,\mu)}$) can be chosen from $B^\beta(-\lambda,\lambda)$ it follows that $B_{(0,\lambda)} \cap B_\lambda \subsetneqq B_{(0,\mu)} \cap B_\lambda$, where B_λ is the C*-subalgebra generated by $B^\beta(-\lambda,\lambda)$. Let ϕ be a pure state of $B_{(0,\mu)} \cap B_\lambda$ such that $\phi|B_{(0,\lambda)} \cap B_\lambda = 0$. We also denote by ϕ the unique extension of ϕ to a pure state of B_λ. We extend ϕ to a pure state of B and uniquely to a pure state of A. Then it follows by the lemma below that ϕ is β-invariant, where we use the assumption that A is α_t-simple and $\mathbf{T}(\alpha_t) = \mathbf{T}$ for all $t \neq 0$ (which implies that A is β_t-simple and $\mathbf{T}(\beta_t) = \mathbf{T}$).

In the GNS representation π_ϕ we define a unitary flow V by

$$V_t \pi_\phi(x)\Omega_\phi = \pi_\phi \beta_t(x)\Omega_\phi, \quad x \in A.$$

Let $\mathcal{K} = [\pi_\phi(B)\Omega_\phi] = [\pi_\phi(B)\mathcal{H}_\phi]$. Then it follows that $\mathrm{Sp}(V|\mathcal{K}) \ni 0$ and

$$\mathrm{Sp}(V|\mathcal{K}) \cap (0,\lambda) = \emptyset, \quad \mathrm{Sp}(V|\mathcal{K}) \cap [\lambda,\mu) \neq \emptyset.$$

Thus the spectrum of $V|\mathcal{K}$ is not connected and hence the kernel of the irreducible representation $\pi_\phi \times V$ of $A \times_\beta \mathbf{R}$ is not monotone under $\hat{\beta}$. Since $(A \times_\alpha \mathbf{R}, \hat{\alpha})$ is isomorphic to $(A \times_\beta \mathbf{R}, \hat{\beta})$ by the Takesaki-Takai duality, (2) does not hold.

Lemma 2.2. *Suppose that $A_{(0,\lambda)} \subsetneqq A$ for some $\lambda > 0$, where $A_{(0,\lambda)}$ is the hereditary C*-subalgebra of A generated by x^*x, $x \in A^\alpha(0,\lambda)$. Let ϕ be a pure state of A_λ such that $\phi|A_{(0,\lambda)} \cap A_\lambda = 0$, where A_λ is the C*-subalgebra generated by $A^\alpha(-\lambda,\lambda)$. Then ϕ is a ceiling state on A_λ, i.e., $\phi(x^*x) = 0$ for any $x \in A_\lambda$ with $\mathrm{Sp}_\alpha(x) \subset (0,\infty)$ and uniquely extends to an α-invariant pure state of A.*

Proof. Note that A_λ is α-invariant and $A_{(0,\lambda)} \cap A_\lambda \subsetneqq A_\lambda$. We will show that $\phi|A_\lambda$ is a ceiling state.

We consider elements of the form $x_1x_2\cdots x_n$, where the α-spectrum of each x_i is a subset of $(-\lambda,0)$, $(-\lambda/2,\lambda/2)$, or $(0,\lambda)$. We write $S(x_i)=-1,0,+1$ depending on which of $(-\lambda,0)$, $(-\lambda/2,\lambda/2)$, or $(0,\lambda)$ we regard $\mathrm{Sp}_\alpha(x_i)$ as a subset of. Note that the linear span of those elements are dense in A_λ.

By the following arguments these monomials $x_1\cdots x_n$ is expressed as a linear combination of $y_1y_2\cdots y_m$ where $S(y_1)=\cdots=S(y_m)=1,-1$ with $m\le n$ or $S(y_1)=0$ with $m=1$. If $S(x_{i-1})=1$ and $S(x_i)=-1$ or $S(x_{i-1})=S(x_i)=0$, then $x_{i-1}x_i$ can be expressed as $z_1+z_2+z_3$ with $S(z_i)=i-2,\ i=1,2,3$ because $\mathrm{Sp}_\alpha(x_{i-1}x_i)\subset(-\lambda,\lambda)$. If $S(x_{i-1})=1$ and $S(x_i)=0$, then $x_{i-1}x_i$ can be expressed as $x_{i-1}z_1+z_2+z_3+z_4$ with $S(z_1)=1$ and $S(z_i)=i-3,\ i=2,3,4$ (by expressing x_i as z_1+y with $\mathrm{Sp}_\alpha(z_1)\subset(0,\lambda/2)$ and $\mathrm{Sp}_\alpha(y)\subset(-\lambda/2,\epsilon')$ for a sufficiently small $\epsilon'>0$ and then expressing $x_{i-1}y$ as $z_2+z_3+z_4$). By applying this kind of replacements to the products $x_{i-1}x_i$ with $S(x_{i-1})\ne S(x_i)$ or $S(x_{i-1})=0=S(x_i)$ finitely many times we reach the desired conclusion.

Let $x\in A_\lambda$ be such that $\mathrm{Sp}_\alpha(x)$ is a compact subset of $(0,\infty)$. Then there is an $f\in K^1(\mathbf{R})$ such that $\mathrm{supp}(f)\subset(0,\infty)$ and $\alpha_f(x)=x$. We approximate x by a linear combination of elements of the form $y_1y_2\cdots y_n$ with $S(y_1)=\cdots=S(y_n)=\pm1$ and z with $S(z)=0$. Then applying α_f we conclude that x can be approximated by a linear combination of elements of the form $\alpha_f(y_1y_2\cdots y_n)$ with $S(y_1)=\cdots=S(y_n)=1$ and $\alpha_f(z)$. Since $\phi(a\alpha_t(y_1\cdots y_n))=0,\ t\in\mathbf{R}$ and $\phi(a\alpha_f(z))=0$ for any $a\in A_\lambda$, this implies that $\phi(x^*x)=0$. Namely ϕ is a ceiling state on A_λ.

We will show that ϕ has a unique extension to a pure state of A, which is necessarily α-invariant.

Let p be the support projection of ϕ in A_λ^{**}, where we regard A_λ^{**} as a subalgebra of A^{**}. Let D be the C*-algebra generated by pAp; p is the identity of D. Since $\alpha_t^{**}(p)=p$, α^{**} restricts to a flow γ on D. From the property $pA^\alpha(-\lambda,\lambda)p=\mathbf{C}p$, we will obtain that $D^\gamma(-\lambda,\lambda)=\mathbf{C}p$.

To prove this we consider an element $y\in D$ obtained as a linear combination of elements of the form $px_1px_2p\cdots px_mp$ with $m\le n$ such that $\mathrm{Sp}_\alpha(y)\subset(-\lambda+\epsilon,\lambda-\epsilon)$ for some $\epsilon>0$. Note that there is an $f\in K^1(\mathbf{R})$ such that $\alpha_f(y)=y$ and $\mathrm{supp}(\hat f)\subset(-\lambda+2\epsilon/3,\lambda-2\epsilon/3)$. Then by replacing each x_i by a linear combination, we may assume that $\mathrm{Sp}_\alpha(x_i)\subset(\lambda_i-\epsilon/3n,\lambda_i+\epsilon/3n)$ for some λ_i. (Then y is a linear combination of even more elements of the form $px_1px_2p\cdots px_mp$ with $m\le n$.) If $|\sum_i\lambda_i|<\lambda-\epsilon/3$, then $px_1px_2p\cdots px_mp\in\mathbf{C}p$, which follows by approximating p's in the middle by an element $e\in A$ with $\mathrm{Sp}_\alpha(e)\subset(-\delta,\delta)$ for a sufficiently small $\delta>0$. Otherwise since we have

$\alpha_f(px_1p\cdots px_mp)=0$, we can disregard such a monomial. Thus it follows that $y=pyp\in\mathbf{C}p$.

In this way we conclude that $D^\gamma(-\lambda,\lambda)=\mathbf{C}p$. Thus we obtain that $\mathrm{Sp}(\gamma)\cap(-\lambda,\lambda)=\{0\}$ and γ is ergodic. Then it follows that $\mathrm{Sp}(\gamma)=\{0\}$ or $\mu\mathbf{Z}$ for some $\mu\geq\lambda$. If $\mathrm{Sp}(\gamma)=\{0\}$, then it follows that $D=\mathbf{C}p$, i.e., p is minimal in A^{**} and so ϕ is α-invariant.

Suppose that $\mathrm{Sp}(\gamma)=\mu\mathbf{Z}$. Then there is a unitary $u\in D$ such that $\gamma_t(u)=e^{i\mu t}u$ and D is generated by u. Since D is abelian and pAp is dense in $pA^{**}p$, we obtain that $pA^{**}p$ is abelian (generated by u in the weak* topology). Let $c(p)$ be the central support of p in A^{**}. Then the multiplication by p gives an isomorphism from $Z(A^{**})c(p)$ onto $pA^{**}p$, where $Z(A^{**})$ is the center of A^{**}. (For example, if $q\in pA^{**}p$ is a projection, then $c(q)p=q$.) Let $U\in Z(A^{**})c(p)$ be a unitary such that $Up=u$. Note that $\alpha_t^{**}(U)=e^{i\mu t}U$.

Let ψ be a pure state extension of $\phi|A_\lambda$. Let $\beta=\alpha_{2\pi/\mu}$. Since $\beta(U)=U$, ψ is β-invariant. Since ψ must be a character on $Z(A^{**})c(p)$, we may suppose that $\psi(U)=1$.

Let $x\in A^\alpha(n\mu,n\mu+\lambda)$ for $n\in\mathbf{Z}$. Since xU^{-n} can be approximated by a net in $A^\alpha(0,\lambda)$, it follows that $\psi(x^*x)=\psi(U^nx^*xU^{-n})=0$. Let $\Omega=\{e^{2\pi it/\mu}\mid t\in(0,\lambda)\}$, an open subset of $\mathbf{T}$. Since the linear span of $A^\alpha(n\mu,n\mu+\lambda)$, $n\in\mathbf{Z}$ is dense in $A^\beta(\Omega)$, it follows that $\psi(x^*x)=0$ for $x\in A^\beta(\Omega)$. Define a unitary W on the GNS representation space associated with ψ by $W\pi_\psi(x)\Omega_\psi=\pi_\psi(\beta(x))\Omega_\psi,\ x\in A$. Then W implements β and $\mathrm{Sp}(W)\cap\Omega=\emptyset$, or $A\times_\beta\mathbf{Z}$ is not simple, which implies that $\mathbf{T}(\beta)=\tilde{\mathbf{T}}(\beta)$ is not equal to $\mathbf{T}$ since A is β-simple [5]. This contradiction shows that $\mathrm{Sp}(\gamma)$ must be trivial, or $pAp=\mathbf{C}p$. This concludes the proof. □

Before going to the proof of (3)⇒(1), we prepare a few lemmas.

Lemma 2.3. *Let $\epsilon\in(0,\lambda)$ and $\delta>0$. Let D be the hereditary C*-subalgebra of A generated by y^*y with $y\in A_\lambda^\alpha(-\lambda-\epsilon,-\epsilon)$. Suppose that $x^*x\in D$ for any $x\in A^\alpha(-\lambda-\delta,-\epsilon)$. Then it follows that $A^\alpha(-\lambda-\delta,-\epsilon)\subset A_\lambda$ (and hence $A^\alpha(-\lambda-\delta,\lambda+\delta)\subset A_\lambda$.)*

Proof. Let $\epsilon'=\min(\delta,\epsilon)$. Let $x\in A^\alpha(-\lambda-\epsilon',-\epsilon)$ and $a\in A_\lambda$ with $\mathrm{Sp}_\alpha(a)\subset(-\lambda-\epsilon,-\epsilon)$; from the latter follows that $\mathrm{Sp}_\alpha(a^*)\subset(\epsilon,\lambda+\epsilon)$. Since $\mathrm{Sp}_\alpha(xa^*)\subset(-\lambda,\lambda)$, we obtain that $xa^*\in A_\lambda$; it then follows that $xa^*ba'\in A_\lambda$ for any $a,a'\in A_\lambda^\alpha(-\lambda-\epsilon,-\epsilon)$ and any $b\in A_\lambda$. Since D is α-invariant, D contains an approximate identity in $D^\alpha(-\lambda,\lambda)\subset D\cap A_\lambda$; the latter intersection is a hereditary C*-subalgebra of A_λ, which is the hereditary C*-subalgebra of A_λ obtained as the closed linear span of elements of the form a^*ba' described above or as the hereditary C*-subalgebra of A_λ generated by y^*y with $y \in A_\lambda(-\lambda-\epsilon,-\epsilon)$. (The fact that D has an approximate identity in $D\cap A_\lambda$, which is what we will use just below, also follows from how D has been defined in the first place. The above fact on $D\cap A_\lambda$ will be occasionally used.)

Since x is in the closed left ideal generated by D, if (e_n) is an approximate identity for D then $\|x-xe_n\|\to 0$. By choosing (e_n) from $D\cap A_\lambda$ we conclude that $x\in A_\lambda$. If $\delta\le\epsilon$, then this completes the proof.

If $\delta>\epsilon$, then set $\epsilon'=\min(\delta-\epsilon,\epsilon)$. We repeat this process: If $x\in A^\alpha(-\lambda-\epsilon-\epsilon',-\epsilon)$ and $a\in A_\lambda$ with $\mathrm{Sp}_\alpha(a)\subset(-\lambda-\epsilon,-\epsilon)$, then $\mathrm{Sp}_\alpha(xa^*)\subset(-\lambda-\epsilon,\lambda)$, which implies that $xa^*\in A_\lambda$ by the first step. Then by the reasoning as above, we conclude that $x\in A_\lambda$. If $\delta-\epsilon\le\epsilon$, then this completes the proof.

In general there is an $n\in\mathbf{N}$ such that $n\epsilon<\delta\le(n+1)\epsilon$. We repeat this process n times with $\epsilon'=\epsilon$ and once more with $\epsilon'=\delta-n\epsilon$, to conclude that $A^\alpha(-\lambda-\delta,-\epsilon)\subset A_\lambda$. □

Lemma 2.4. *Let $0<\epsilon<\lambda$. The following two hereditary C*-subalgebras of A_λ are equal: The one D_1 generated by y^*y, $y\in A_\lambda^\alpha(-\infty,-\epsilon)$ and the other one D generated by y^*y, $y\in A_\lambda^\alpha(-\lambda-\epsilon,-\epsilon)$.*

Proof. Obviously $D_1\supset D$.

Let L be the left ideal of A_λ generated by $A_\lambda^\alpha(-\lambda-\epsilon,-\epsilon)$. It suffices to show that $A_\lambda^\alpha(-\infty,-\epsilon)\subset L$.

Let $x\in A_\lambda^\alpha(-\mu,-\epsilon)$ for some $\mu>\epsilon$. Then x can be approximated by a linear combination of elements of the form $x_1x_2\cdots x_n$, where $S(x_i)$ is well-defined as in the proof of 2.2, i.e., $\mathrm{Sp}_\alpha(x_i)$ is contained in $(-\lambda,0)$, $(-\lambda/2,\lambda/2)$, or $(0,\lambda)$ depending on whether $S(x_i)=-1$, 0, or 1. Then as before we express $x_1\cdots x_n$ as the sum of $y_1\cdots y_m$ with $m\le n$ and $S(y_1)=\cdots=S(y_m)=\pm1$ and an element z with $S(z)=0$. By adding those $y_1\cdots y_n$ with $S(y_1)=\cdots=S(y_n)=1$ to z, we conclude that x can be approximated by a sum of elements of the form $y_1\cdots y_n$ with $S(y_i)=-1$ and an element z with $\mathrm{Sp}_\alpha(z)\subset(-\lambda/2,\infty)$. We will denote this sum by x'.

We will express x' as the sum $d+z$, where $d\in L$ and $z\in A_\lambda$ with $\mathrm{Sp}_\alpha(z)\subset(-\lambda-\epsilon,\infty)$, as follows.

We express each term $y_1\cdots y_n$ with $S(y_i)=-1$ as the sum $d+z'$, where $d\in L$ and $z'\in A_\lambda$ with $\mathrm{Sp}(z')\subset(-\lambda-\epsilon,0)$. If $n=1$, there is nothing to prove. Suppose $n>1$. We express y_n as the sum z_1+z_2, where $\mathrm{Sp}_\alpha(z_1)\subset(-\lambda,-\epsilon)$ and $\mathrm{Sp}_\alpha(z_2)\subset(-\epsilon-\epsilon',0)$. Here $\epsilon'>0$ is chosen so that $\mathrm{Sp}(y_{n-1}z_2)\subset(-\lambda-\epsilon,0)$. Note that $y_1\cdots y_{n-1}z_1\in L$. If $n=2$ we are finished by setting $z'=y_1z_2$.

If $n>2$, we express $y_{n-1}z_2$ as the sum $z_1'+z_2'$, where $\mathrm{Sp}_\alpha(z_1')\subset(-\lambda-\epsilon,-\epsilon)$

and $\mathrm{Sp}_\alpha(z_2') \subset (-\epsilon - \epsilon', 0)$. Here $\epsilon' > 0$ is chosen so that $\mathrm{Sp}_\alpha(y_{n-2}z_2') \subset (-\lambda - \epsilon, 0)$. Note that $y_1 \cdots y_{n-2}z_1' \in L$. If $n = 3$ then we are finished, otherwise we apply the same procedure to $y_{n-2}z_2'$ in $y_1 \cdots y_{n-3}y_{n-2}z_2'$. This way we can conclude that $y_1 \cdots y_n$ is the sum $d + z'$, where $d \in L$ and $z' \in A_\lambda$ with $\mathrm{Sp}(z') \subset (-\lambda - \epsilon, 0)$.

By adding all those z' to z and by all those $d' \in L$ into d, we express x' as $d + z$, where $d \in L$ and $z \in A_\lambda$ with $\mathrm{Sp}_\alpha(z) \subset (-\lambda - \epsilon, \infty)$.

Note that we have shown that $\|x - d - z\| \approx 0$. Since $\mathrm{Sp}_\alpha(x) \subset (-\mu, -\epsilon)$, there is $f \in K^1(\mathbf{R})$ such that $\alpha_f(x) = x$ and $\mathrm{supp}(\hat{f}) \subset (-\mu, -\epsilon)$. We may suppose that $\|f\|_1\|x - d - z\|$ is still sufficiently small. Then x can be approximated by the sum $\alpha_f(d) + \alpha_f(z)$, where $\alpha_f(d) \in L$ (because α_f maps L into L) and $\alpha_f(z) \in L$ (because $\mathrm{Sp}_\alpha(\alpha_f(z)) \subset \mathrm{supp}\hat{f} \cap \mathrm{Sp}_\alpha(z)$). This concludes the proof that $A_\lambda^\alpha(-\infty, -\epsilon) \subset L$. □

Lemma 2.5. *Suppose that* $A_\lambda \subsetneqq A$ *for* $\lambda > 0$. *Let* $\epsilon \in (0, \lambda/5)$ *and let* ϕ *be a pure state of* A_λ *such that* $\phi(x^*x) = 0$ *for all* $x \in A_\lambda^\alpha(-\infty, -\epsilon)$. *Then* ϕ *uniquely extends to a pure state of* A *which has* α*-spectrum in* $[-\epsilon, \epsilon]$ (*and is* α*-covariant*).

Proof. By Lemmas 2.3 and 2.4 it follows from $A_\lambda \subsetneqq A$ that the hereditary C*-subalgebra of A_λ generated by x^*x, $x \in A_\lambda^\alpha(-\infty, -\epsilon)$ is not equal to A_λ. (Otherwise the D defined as in 2.3 would be equal to A in view of 2.4, which would imply that $A^\alpha(-\infty, -\epsilon) \subset A_\lambda$, contradicting $A_\lambda \subsetneqq A$; see the beginning of the proof of (3)⇒(1) below.) This guarantees there is such a pure state ϕ as in the statement.

Since $t \mapsto \phi\alpha_t$ is norm-continuous, ϕ is α-covariant, i.e., there is a unitary flow V on H_ϕ such that $V_t\pi_\phi(x)V_t^* = \pi_\phi\alpha_t(x)$, $x \in A_\lambda$. Since ϕ vanishes on $A_\lambda^\alpha(-\infty, -\epsilon)$, we have that $\mathrm{Sp}_V(\Omega_\phi)$ is contained in a closed interval of length ϵ and we may suppose that $\epsilon \in \mathrm{Sp}_V(\Omega_\phi) \subset [0, \epsilon]$. Then it follows that

$$\mathrm{Sp}(V) \subset [0, \infty).$$

To show this suppose that there is a negative μ in $\mathrm{Sp}(V)$. Then for any $\delta > 0$ there is a unit vector $\xi \in \mathcal{H}_\phi$ such that $\mathrm{Sp}_V(\xi) \subset (\mu - \delta, \mu + \delta)$. Since $\mathrm{Sp}_V(\Omega_\phi) \ni \epsilon$, $\eta = E[\epsilon - \delta, \epsilon]\Omega_\phi \neq 0$, where E is the spectral measure for the generator of V. There is a sequence (e_n) in A_λ such that $\pi_\phi(e_n) \to E[\epsilon - \delta, \epsilon]$ strongly, $0 \leq e_n \leq 1$, and $\mathrm{Sp}_\alpha(e_n) \subset (-\delta, \delta)$. There is a $b \in A_\lambda$ such that $\pi_\phi(b)\eta = \xi$. We may suppose that $\mathrm{Sp}_\alpha(b) \subset (\mu - \epsilon - 2\delta, \mu - \epsilon + 3\delta)$. Since $\pi_\phi(be_n)\Omega_\phi \to \xi$ and $\pi_\phi(A_\lambda^\alpha(-\infty, -\epsilon))\Omega_\phi = 0$, we must have that $\mu - \epsilon + 4\delta \geq -\epsilon$, which is a contradiction for a small $\delta > 0$.

Let p be the support projection of ϕ in $A_\lambda^{**} \subset A^{**}$, which is minimal in A_λ^{**}. We have to show that p is also minimal in A^{**}.

Since $\overline{\pi}_\phi(p)$ is the projection onto the subspace $\mathbf{C}\Omega_\phi$, we have that $\langle \overline{\pi}_\phi \overline{\alpha}_t(p)\xi, \eta\rangle = \langle \xi, V_t\Omega_\phi\rangle\langle V_t\Omega_\phi, \eta\rangle$ for $\xi, \eta \in H_\phi$, where $\overline{\pi}_\phi$ (resp. $\overline{\alpha}_t$) denotes the extension of π_ϕ (resp. α_t) to a representation (resp. an automorphism) of A_λ^{**}. Hence we obtain that $t \mapsto \overline{\alpha}_t(p)$ is norm-continuous and $\mathrm{Sp}_{\overline{\alpha}}(p) \subset [-\epsilon, \epsilon]$. Note that there is a net (p_ι) in A_λ such that p_ι converges to p in the weak* topology and all $\mathrm{Sp}_\alpha(p_\iota)$'s are included in a small neighborhood of $[-\epsilon, \epsilon]$.

Let $\epsilon' > 0$ be such that $5\epsilon + 6\epsilon' < \lambda$. Let $\mu \in \mathbf{R}$ and let $x \in A$ be such that $\mathrm{Sp}_\alpha(x) \subset (\mu - \epsilon', \mu + \epsilon')$. Then x^*px has $\overline{\alpha}$-spectrum in $(-\epsilon - 2\epsilon', \epsilon + 2\epsilon') \subset (-\lambda, \lambda)$. Since $pA^\alpha(-\lambda, \lambda)p = \mathbf{C}p$, one can conclude that $px^*pxp \in \mathbf{C}p$. Since the same is true for xpx^*, if $pxp \neq 0$, then pxp is a constant multiple of a unitary and we fix an $x = x_\mu$ such that $px_\mu p$ is a unitary.

Let S be the set of $\mu \in \mathbf{R}$ for which we have a unitary $px_\mu p$ as above. Let $\mu \in S$ and let $\nu \in \mathbf{R}$. If $x \in A$ satisfies that $\mathrm{Sp}_\alpha(x) \subset (\nu - \epsilon', \nu + \epsilon')$, then $\mathrm{Sp}_\alpha(x_\mu^* px) \subset (\nu - \mu - \epsilon - 2\epsilon', \nu - \mu + \epsilon + 2\epsilon')$. If $|\nu - \mu| \in (\epsilon + \epsilon + 2\epsilon', \lambda - \epsilon - 2\epsilon')$, then $px_\mu^* pxp = 0$, or $\nu \notin S$ because $pA^\alpha(\epsilon, \lambda)p \subset pA_\lambda^\alpha(\epsilon, \infty)p = \{0\}$. If $|\nu - \mu| \leq 2\epsilon + 2\epsilon'$ and $\nu \in S$, then the unitary $px_\nu p$ obtained for ν is a constant multiple of $px_\mu p$.

For $\mu, \nu \in S$ we write $\mu \sim \nu$ if $px_\mu p \in \mathbf{C}px_\nu p$ and $|\mu - \nu| < \lambda - \epsilon - 2\epsilon'$, which induces an equivalence relation on S. We assert that each equivalence class is contained in an interval of length at most $2(\epsilon + \epsilon')$ and that two different equivalence classes are separated at least by a length of $\lambda - \epsilon - 2\epsilon'$. The latter follows immediately from the preceding paragraph.

Let E be an equivalence class and let $\mu, \nu \in E$ with $\mu < \nu$. Then there is a finite sequence $\mu_0 = \mu, \mu_1, \mu_2, \ldots, \mu_n = \nu$ such that $\mu_{i-1} \sim \mu_i$. Then $|\mu_i - \mu_{i-1}| < \lambda - \epsilon - 2\epsilon'$, which implies that $|\mu_{i-1} - \mu_i| \leq 2(\epsilon + \epsilon')$. If $|\mu - \nu| > 2(\epsilon + \epsilon')$, i.e., $|\mu - \nu| \geq \lambda - \epsilon - 2\epsilon'$, there must be i such that $|\mu - \mu_i| \in (2(\epsilon + \epsilon'), \lambda - \epsilon - 2\epsilon')$ because $\lambda - \epsilon - 2\epsilon' > 4(\epsilon + \epsilon')$. This contradiction shows that $|\mu - \nu| \leq 2(\epsilon + \epsilon')$.

Let E and F be equivalence classes in S. For $\mu \in E$ and $\nu \in F$ there is a $\tau \in S$ in $I = (\mu + \nu - \epsilon - \epsilon', \mu + \nu + \epsilon + \epsilon')$ such that $px_\tau p \in \mathbf{C}px_\mu px_\nu p$, which is obtained by covering $\mathrm{Sp}_\alpha(x_\mu px_\nu)$ by a finite number of $(\tau - \epsilon', \tau + \epsilon')$, $\tau \in I$ as $\mathrm{Sp}_\alpha(x_\mu px_\nu) \subset (\mu + \nu - \epsilon - 2\epsilon', \mu + \nu + \epsilon + 2\epsilon')$. If there is another τ' satisfying the same condition, then $\tau \sim \tau'$. Hence we can define a product among the equivalence classes. Note that $0 \in S$ and px_0p is a constant multiple of p. Since $\mu \in S$ implies that $-\mu \in S$ and the product of equivalence classes $[\mu]$ and $[-\mu]$ is $[0]$, we can conclude that $S/\sim$ is a group. If it is non-zero, it must be isomorphic to the integers; if $\mu \in S$ is the smallest positive number among those not equivalent to μ in $\{\nu \in S \mid \nu > 0, \nu \not\sim 0\}$, then it follows that the equivalence class $[\mu]$ generates $S/\sim$ and pAp is generated by $px_\mu p$, i.e., pAp is commutative. Hence $pA^{**}p$ is commutative.

Let $c(p)$ be the central support of p in A^{**}. Since $t \mapsto \overline{\alpha}_t(p)$ is (infinitely often) differentiable, we can define $h \in (A_\lambda)^{**}$ by $ih = \overline{\delta}_\alpha(p)p - p\overline{\delta}_\alpha(p)$, where $\overline{\delta}_\alpha$ is the generator of $\overline{\alpha}$ on the norm-continuous part of A^{**}. Since h is in the domain of $\overline{\delta}_\alpha$, one can define an $\overline{\alpha}$-cocycle u by $du_t/dt = -u_t i\overline{\alpha}_t(h)$. Then $\mathrm{Ad}u_t\overline{\alpha}_t(p) = p$ and hence $\overline{\alpha}_t(c(p)) = \mathrm{Ad}\, u_t\overline{\alpha}_t(c(p)) = c(p)$.

Since $pA^{**}p$ is commutative, we obtain $Z(A^{**})c(p) \cong pA^{**}p$, which implies that there is a unitary $U \in Z(A^{**})c(p)$ such that $pU = px_\mu p$. Since $p\overline{\alpha}_t(U) = \mathrm{Ad}\, u_t\overline{\alpha}_t(pU) = p\mathrm{Ad}\, u_t\alpha_t(x_\mu)p$, it follows that $t \mapsto \overline{\alpha}_t(U)$ is norm-continuous. Let C be the $\overline{\alpha}$-invariant C*-algebra generated by U. Then $Cp \supset pAp$.

If $Q \in C$ satisfies that $\mathrm{Sp}_{\overline{\alpha}}(Q) \subset (-\lambda, \lambda)$ then $Qp \in \mathbf{C}p$ from the reasoning as above (i.e., $pA^\alpha(-\lambda, \lambda)p = \mathbf{C}p$), which implies that $\overline{\alpha}|C$ is ergodic and $\mathrm{Sp}(\overline{\alpha}|C) = \nu\mathbf{Z}$ for some $\nu \geq \lambda$. We find a unitary V in C such that $\overline{\alpha}_t(V) = e^{i\nu t}V,\ t \in \mathbf{R}$.

Let $\hat{\phi}$ be a pure state extension of ϕ. We may suppose that $\hat{\phi}(V) = 1$, i.e., $\hat{\phi}(xV) = \hat{\phi}(x)$ for any $x \in A^{**}$. Let $x \in A$ be such that $\mathrm{Sp}_\alpha(x) \subset (k\nu - \lambda, k\nu - \epsilon)$. Then xV^{-k} is in the weak* closure of $A^\alpha(-\lambda, -\epsilon)$. Hence we have that $\hat{\phi}(V^{-k}x^*xV^k) = 0$, which implies that $\hat{\phi}(x^*x) = 0$. Let $\Omega = \{e^{2\pi it/\nu} \mid t \in (-\lambda, -\epsilon)\}$ and let $\beta = \alpha_{2\pi/\nu}$. Then $\hat{\phi}$ vanishes on $A^\beta(\Omega)^*A^\beta(\Omega)$. This implies that the hereditary C*-subalgebra generated by $x^*x,\ x \in A^\beta(\Omega)$ does not equal A, which in turn implies that $\tilde{\mathbf{T}}(\beta) \neq \mathbf{T}$. Since this contradicts the assumption that A is β-simple and $\mathbf{T}(\beta) = \mathbf{T}$, p must be minimal in A^{**}, i.e., ϕ has a unique pure state extension.

If $x \in A$ has α-spectrum in $(\lambda - \epsilon, \infty)$, then pxp has α-spectrum in $(\lambda - 3\epsilon, \infty) \subset (\epsilon, \infty)$ and hence $pxp = 0$ (because $pxp \in \mathbf{C}p$). Hence, since $\mathrm{Sp}_\alpha(\phi|A_\lambda) \subset [-\epsilon, \epsilon]$, we can conclude that $\mathrm{Sp}_\alpha(\phi) \subset [-\epsilon, \epsilon]$. □

Lemma 2.6. *Let $\epsilon > 0$ and $\delta > 0$ and let D denote the hereditary C*-subalgebra of A generated by $y^*y,\ y \in A^\alpha_\lambda(-\lambda - \epsilon, -\epsilon)$. Suppose that there is an $x \in A$ such that $\mathrm{Sp}_\alpha(x) \subset (-\mu, -\mu + \epsilon)$ for some $\mu > \lambda$ and $x^*x \notin D$. Then for any $\epsilon' > 0$ there is an $x \in A^\alpha(-\mu, -\mu + \epsilon)$ such that $\|x\| = 1$ and $\inf\{\|x^*x - d\| \mid d \in D\} > 1 - \epsilon'$.*

Proof. If L is the left ideal of A generated by D, we have that $d_0 = \inf\{\|x - z\| \mid z \in L\} > 0$. Let (e_n) be an approximate identity for D such that $\mathrm{Sp}_\alpha(e_n) \downarrow \{0\}$; more precisely we assume $\mathrm{Sp}(xe_n) \subset (-\mu, -\mu + \epsilon)$. Since $\|x(1 - e_n)\| \geq d_0$ and $\|x - z\| \geq \|x(1 - e_n) - z(1 - e_n)\| \geq \lim_n \|x(1 - e_n)\|$

for $z \in L$, it follows that $d_0 = \lim \|x(1-e_n)\|$. We choose an n so that $d_0/\|x(1-e_n)\| > (1-\epsilon')^{1/2}$ and set $y = x(1-e_n)/\|x(1-e_n)\|$, which is a norm-one element with α-spectrum in $(-\mu, -\mu+\epsilon)$.

Since $d_1 = \inf\{\|y^*y - d\| \mid d \in D\}$ satisfies that $d_1 \geq \lim_m \|y^*y(1-e_m)\| \geq \lim_m \|y(1-e_m)\|^2$, we conclude that $d_1 \geq (d_0/\|x(1-e_n)\|)^2 > 1-\epsilon'$. □

Lemma 2.7. *Let D be a hereditary C^*-subalgebra of A and $z \in A$ such that $z \geq 0$, $\|z\| = 1$, and $\inf\{\|z-d\| \mid d \in D\} > 1-\epsilon'$ for some $\epsilon' > 0$. Then there is a pure state ϕ of A such that $\phi|D = 0$ and $\phi(z) > 1-\epsilon'-2\sqrt{\epsilon'}$.*

Proof. By the Hahn-Banach theorem there is an $f \in A^*$ such that $\|f\| = 1$, $f^* = f$, and $f|D = 0$, and $f(z) > 1-\epsilon'$. Let $f = f_+ - f_-$ be the orthogonal decomposition: $f_\pm \geq 0$, $\|f_+\| + \|f_-\| = 1$. Since $f(z) = f_+(z) - f_-(z)$, we get that $f_+(z) > 1-\epsilon'$, which implies that $\|f_+\| > 1-\epsilon'$ and $\|f_-\| < \epsilon'$. If E is the open projection for D, then we have that $f_+(E) = f_-(E) < \epsilon'$. Let $\phi' = f_+((1-E)\cdot(1-E))$. Then $\phi' \geq 0$, $\phi'|D = 0$, and $\phi'(z) > 1-\epsilon'-2\sqrt{\epsilon'}$.

Thus if S denotes the closed convex set of positive functionals ϕ of A such that $\phi|D = 0$ and $\|\phi\| \leq 1$, then $\sup\{\phi(z) \mid \phi \in S\}$ is greater than $1-\epsilon-2\sqrt{\epsilon'}$. An extreme point of S is either 0 or a pure state. Hence we may find a required pure state as an extreme point of S. □

Lemma 2.8. *Let $\lambda > 0$ and $\epsilon > 0$. Let $x \in A$ be such that $\|x\| = 1$, $\mathrm{Sp}_\alpha(x) \subset (-\infty, -\lambda)$ and let $h = h^* \in A$ with $\|h\| < \epsilon\lambda/2$. Let ϕ be a ground state of A for $\alpha^{(h)}$. Then it follows that $\phi(x^*x) < \epsilon$.*

Proof. In the GNS representation associated with ϕ, let U be the canonical unitary flow defined by $U_t\pi_\phi(x)\Omega_\phi = \pi_\phi\alpha_t^{(h)}(x)\Omega_\phi$, $x \in A$. Let H be the generator of U: $U_t = e^{itH}$. By the assumption we have that $H \geq 0$ and $H\Omega_\phi = 0$. Let $H_1 = H - \pi_\phi(h) - E_1$, where $E_1 = \inf \mathrm{Sp}(H - \pi_\phi(h))$. Note that $|E_1| \leq \|h\|$. Let F be the spectral measure for H_1. Then

$$\lambda\langle F[\lambda,\infty)\Omega_\phi, \Omega_\phi\rangle \leq \langle H_1\Omega_\phi, \Omega_\phi\rangle = -\langle(\pi_\phi(h)+E_1)\Omega_\phi, \Omega\phi\rangle \leq 2\|h\|.$$

Since $\mathrm{Sp}_\alpha(x) \subset (-\infty, -\lambda)$ and $\mathrm{Ad}\, e^{itH_1}$ implements α, it follows that

$$\pi_\phi(x)F[0,\lambda]\Omega_\phi = 0,$$

which implies that

$$\pi_\phi(x)\Omega_\phi = \pi_\phi(x)F(\lambda,\infty)\Omega_\phi.$$

Hence it follows that $\|\pi_\phi(x)\Omega_\phi\|^2 \leq \|F(\lambda,\infty)\Omega_\phi\|^2 < \epsilon$. This completes the proof. □

Proof of (3)⇒(1).

Suppose that (1) does not hold, i.e., $B_\lambda \neq B$ for some $B \in H^\alpha(A)$ and $\lambda > 0$. We denote B by A below.

Let $\epsilon \in (0, \lambda/10)$ and let ϵ' be a sufficiently small positive number. Let D be the hereditary C*-subalgebra of A generated by $y^*y,\ y \in A_\lambda^\alpha(-\lambda-\epsilon, -\epsilon)$. Since $A \neq A_\lambda$, there is a $\delta > 0$ and $x \in A$ such that $\mathrm{Sp}_\alpha(x) \subset (-\lambda-\delta, -\epsilon)$ and $x^*x \notin D$ (by 2.3). We may suppose that $\mathrm{Sp}_\alpha(x) \subset (-\mu, -\mu+\epsilon)$ for some $\mu > \lambda$ and by 2.6 that $\mathrm{dist}(x^*x, D) > 1-\epsilon'$. Note that $x^*x \in A_\lambda$ since $\mathrm{Sp}_\alpha(x^*x) \subset (-\epsilon, \epsilon)$. By 2.7 there is a pure state ϕ of A_λ such that $\phi|D \cap A_\lambda = 0$ and $\phi(x^*x) > 1-\epsilon'-2\sqrt{\epsilon'}$. We now assume that $\phi(x^*x) > 1-\epsilon'$ by making the starting ϵ' even smaller. We also denote by ϕ the unique extension of ϕ to a state of A (see 2.5). Note that $\mathrm{Sp}_\alpha(\phi) \subset [-\epsilon, \epsilon]$.

Let V be a unitary flow on the GNS representation on $\mathcal{H}_\phi$ such that $V_t\pi_\phi(x)V_t^* = \pi_\phi(\alpha_t(x))$ for $x \in A$ and $\epsilon \in \mathrm{Sp}_V(\Omega_\phi) \subset [0, \epsilon]$. Since $V_t\Omega_\phi \in [\pi_\phi(A_\lambda)\Omega_\phi]$ (as $\mathrm{Sp}_V(V_t\Omega_\phi) \subset [0,\epsilon]$), V leaves $[\pi_\phi(A_\lambda)\Omega_\phi]$ invariant. Since $\pi_\phi(A_\lambda^\alpha(-\lambda-\epsilon, -\epsilon))\Omega_\phi = 0$, it follows by 2.4 that $\pi_\phi(A_\lambda^\alpha(-\infty, -\epsilon))\Omega_\phi = 0$. Then it follows as in the proof of Lemma 2.5 that

$$\mathrm{Sp}(V|[\pi_\phi(A_\lambda)\Omega_\phi]) \subset [0, \infty).$$

Since $\phi(a^*a) = 0$ for $a \in A^\alpha(-\lambda, -\epsilon) \subset A_\lambda^\alpha(-\lambda-\epsilon, -\epsilon)$, it also follows that $\mathrm{Sp}(V) \cap (-\lambda+\epsilon, 0) = \emptyset$. Let H be the generator of V and let

$$H_1 = H - 2\epsilon\pi_\phi(x^*x) - E_1$$

where E_1 is the infimum of the spectrum of $H - 2\epsilon\pi_\phi(x^*x)$ on $[\pi_\phi(A_\lambda)\Omega_\phi]$. Here we should note that $H - 2\epsilon\pi_\phi(x^*x)$ leaves $[\pi_\phi(A_\lambda)\Omega_\phi]$ invariant since $x^*x \in A_\lambda$. Since $\langle (H - 2\epsilon\pi_\phi(x^*x))\Omega_\phi, \Omega_\phi\rangle \leq \epsilon - 2\epsilon\phi(x^*x)$ and $H - 2\epsilon\pi_\phi(x^*x) \geq H - 2\epsilon$, we have that $-2\epsilon \leq E_1 \leq -\epsilon(1-2\epsilon')$. If (ξ_n) is a sequence in the domain $D(H^{1/2})$ in $[\pi_\phi(A_\lambda)\Omega_\phi]$ such that $\|H_1^{1/2}\xi_n\| \to 0$, then it follows that

$$\langle \pi_\phi(x^*x)\xi_n, \xi_n\rangle > \phi(x^*x) - 1/2$$

for all large n since

$$\begin{aligned} -2\epsilon\langle \pi_\phi(x^*x)\xi_n, \xi_n\rangle &\leq \langle (H - 2\epsilon\pi_\phi(x^*x))\xi_n, \xi_n\rangle \\ &\leq \langle (H - 2\epsilon\pi_\phi(x^*x))\Omega_\phi, \Omega_\phi\rangle < \epsilon - 2\epsilon\phi(x^*x). \end{aligned}$$

Thus a weak* limit point φ of the sequence of states $a \mapsto \langle \pi_\phi(a)\xi_n, \xi_n\rangle$ satisfies that $\varphi(x^*x) > 1/2 - \epsilon'$. We replace φ by $\varphi/\|\varphi\|$ to make φ a state, which still satisfies that $\varphi(x^*x) > 1/2 - \epsilon'$.

Let $\mathcal{H}_-$ be the spectral subspace of H corresponding to $(-\infty, 0)$ (or equivalently $(-\infty, -\lambda+\epsilon]$). The supremum of the spectrum of H_1 on $\mathcal{H}_-$ is less than or equal to $-\lambda+\epsilon-E_1 \leq -\lambda+3\epsilon$. Hence $\mathrm{Sp}(H_1)\cap(-\lambda+3\epsilon, 0)=\emptyset$. Let β be the flow generated by $\delta_\alpha - 2\epsilon\,\mathrm{ad}\, ix^*x$. Since e^{itH_1} implements β and $\|e^{itH_1}\xi_n-\xi_n\|\rightarrow 0$, we have that φ is β-invariant. Moreover if $y\in A$ satisfies that $\mathrm{Sp}_\beta(y)\subset(-\lambda+3\epsilon, 0)$, it follows that $\|\pi_\phi(y)\xi_n\|\rightarrow 0$, which implies that $\varphi(y^*y)=0$.

Let $A_{\lambda-3\epsilon,\beta}$ be the C*-subalgebra generated by $A^\beta(-\lambda+3\epsilon, \lambda-3\epsilon)$. Then $\varphi|A_{\lambda-3\epsilon,\beta}$ is a ground state for β. We claim that φ is not a ground state on A for β. If it were a ground state on A, then by the previous lemma, we would get that

$$\varphi(x^*x) < 4\epsilon/(\lambda-\epsilon)$$

because $\alpha=\beta^{(2\epsilon x^*x)}$ and $\mathrm{Sp}_\alpha(x)\subset(-\mu, -\mu+\epsilon)$ with $\mu>\lambda$. This is a contradiction for $\epsilon'<1/18$ because $\varphi(x^*x)>1/2-\epsilon'$. This implies that (3) of the theorem is not satisfied for this perturbation β. This concludes the proof of the last implication (3)⇒(1).

§3. UHF Flows

When (μ_n) is a sequence in **R**, we define a flow α on a UHF algebra A of 2^∞ type by

$$\alpha_t = \bigotimes_{n=1}^{\infty} \mathrm{Ad}\begin{pmatrix} e^{i\mu_n t} & 0 \\ 0 & 1 \end{pmatrix}$$

This is what we call a (special type of) UHF flow [9]. (More generally β is a UHF flow if it is defined as $\beta_t=\bigotimes_{i=1}^\infty \beta_t^{(i)}$ where $\beta^{(i)}$ is a flow on $M_{2^{n_i}}$ with (n_i) is a sequence of integers greater than 0.) If $\lim_n \mu_n = 0$ and $\sum_n \mu_n^2=\infty$, then the corresponding α is a universal UHF flow in the sense that if β is another UHF flow on A then $\alpha\otimes\beta$ on $A\otimes A$ is cocycle conjugate to α [9].

We can show that a universal UHF flow satisfies the no energy gap condition.

We shall prove the following more general result:

Proposition 3.1. *Let (A,α) be a universal UHF flow as above and let β be a flow on a C*-algebra B. Then the tensor product system $(A\otimes B, \alpha\otimes\beta)$ satisfies the no energy gap condition.*

Proof. Let $\lambda>0$ and $\mu\geq\lambda$. Let $n\in\mathbf{N}$ and $\nu\in[0,\lambda/2)$ be such that $\mu=n\lambda/2+\nu$. We set $\epsilon=\lambda/4(n+1)$.

Let $D \in H^{\alpha\otimes\beta}(A \otimes B)$ and let $x \in D^{\alpha}(\mu - \lambda/2, \mu + \lambda/2)$. There is a central sequence (u_k) in $A^{\alpha}(\lambda/2 - \epsilon, \lambda/2 + \epsilon)$ such that $u_k^* u_k + u_k u_k^* \to 1$. There is another central sequence (v_k) in $A^{\alpha}(\nu - \epsilon, \nu + \epsilon)$ such that $v_k^* v_k + v_k v_k^* \to 1$ (because α is universal in the above sense). Let (e_k) be an approximate identity for D in $D^{\beta}(-\epsilon/2, \epsilon/2)$.

Let $x \in D^{\alpha\otimes\beta}(\mu - \lambda/2, \mu + \lambda/2)$. Then it follows from the computation of $\alpha \otimes \beta$-spectrum that

$$y \equiv x(e_{k_1} u_{m_1}^* e_{k_1}) \cdots (e_{k_n} u_{m_n}^* e_{k_n})(e_{k_{n+1}} v_{m_{n+1}}^* e_{k_{n+1}}) \in D_\lambda,$$

where $u_{m_i}^*$ denotes $u_{m_i}^* \otimes 1$ etc. Then by taking the limit of $y(e_{k_{n+1}} v_{m_{n+1}} e_{k_{n+1}}) \in D_\lambda$ and $(e_{k_{n+1}} v_{m_{n+1}} e_{k_{n+1}})y \in D_\lambda$ as $m_{n+1} \to \infty$ and then $k_{n+1} \to \infty$, we obtain that

$$x(e_{k_1} u_{m_1}^* e_{k_1}) \cdots (e_{k_n} u_{m_n}^* e_{k_n}) \in D_\lambda.$$

We repeat this process to conclude that $x \in D_\lambda$. Thus we obtain that $D^{\alpha}(\mu - \lambda/2, \mu + \lambda/2) \subset D_\lambda$, which implies that $D = D_\lambda$. □

Corollary 3.2. *Let (A, α) be a universal UHF flow as above and let β be a flow on a unital simple C^*-algebra B. Then $(A \otimes B) \times_{\alpha\otimes\beta} \mathbf{R}$ is simple if and only if (B, β) has no ground states nor ceiling states.*

Proof. Since (A, α) has a ground state (or a ceiling state), if (B, β) has one, then so does $(A \otimes B, \alpha \otimes \beta)$, which gives a non-trivial ideal of the crossed product. This proves the 'only if' part.

Suppose that (B, β) has no ground states nor ceiling states. Then $(A \otimes B, \alpha \otimes \beta)$ has no ground states nor ceiling states. Since $\alpha \otimes \beta$ satisfies the no energy gap condition by the above proposition, all the primitive ideals of $(A \otimes B) \times_{\alpha\otimes\beta} \mathbf{R}$ are monotone under the dual flow by Theorem 2.1. Since $A \otimes B$ is simple and unital, such a non-trivial primitive ideal should give a ground state or a ceiling state for $(A \otimes B, \alpha \otimes \beta)$. Thus one concludes that $(A \otimes B) \times_{\alpha\otimes\beta} \mathbf{R}$ has no non-trivial primitive ideals, i.e., is simple. □

Probably we would have a similar result for (μ_n) satisfying a weaker condition.

Proposition 3.3. *Let (μ_n) be a (strictly) decreasing sequence of positive numbers such that $\lim_n \mu_n = 0$ and define a flow α on the UHF algebra A as above. Then the following conditions are equivalent*:

1. *α satisfies the no energy gap condition.*

2. $\sum_{n=1}^{\infty} \mu_n = \infty$.

Proof. Suppose that (2) does not hold. Then we can define $h = h^* \in A$ by

$$h = \sum_n \mu_n e_{1,1}^{(n)},$$

where $e_{i,j}^{(n)}$ is a matrix unit for the n'th factor. The spectrum of h contains 0 and $\mu = \sum_n \mu_n > 0$ and is the closure of the set of all finite sums of (μ_n). Let f be a continuous function on $\mathbf{R}$ such that $f = 0$ on $[\mu/3, 2\mu/3]$ and $f = 1$ at 0 and μ and $0 \le f \le 1$. Since $\alpha_t = \operatorname{Ad} e^{ith}$, the closure B of $f(h)Af(h)$ is an α-invariant hereditary C*-subalgebra of A. It follows that $B_{\mu/3} \ne B$. This shows that (1)⇒(2).

Suppose that (2) holds. The dual system $(A \times_\alpha \mathbf{R}, \hat{\alpha})$ is isomorphic to the inductive limit of (D_n, γ), where $D_n = M_{2^n} \otimes C_0(\mathbf{R}) \cong M_{2^n} \times_\alpha \mathbf{R} \subset A \times_\alpha \mathbf{R}$, γ is induced by the translation on $\mathbf{R}$, and the map $\phi_n : D_{n-1} \to D_n = M_2 \otimes D_{n-1}$ is given by

$$f \mapsto f(\cdot + \mu_n) \oplus f.$$

Note that the center of D_n is generated by the flow $U_t \exp\{-it \sum_{i=1}^n \mu_i e_{1,1}^{(i)}\}$. Since ϕ_n is injective, we regard $A \times_\alpha \mathbf{R}$ as the closure of the union $\bigcup_n D_n$ (see [2] for more general results).

Let J be an ideal of $A \times_\alpha \mathbf{R}$. Let $J_n = J \cap D_n$. Then J_n is determined by an open subset V_n of $\mathbf{R}$ by $J_n \cong M_{2^n} \otimes C_0(V_n) \subset D_n$. Note that J is determined by the sequence (J_n). We show that they satisfy that $V_n \supset V_{n-1} \cup (V_{n-1} - \mu_n)$ and $V_n \cap (V_n + \mu_n) \subset V_{n-1}$.

For any $t \in V_{n-1}$ there is an $f \in J_{n-1}$ such that $f(t) \ne 0$. Then, since $\phi_n(f)(t) \ne 0$, it follows that $t \in V_n$. For any $t \in V_{n-1} - \mu_n$ there is an $f \in J_{n-1}$ such that $f(t + \mu_n) \ne 0$. Then $\phi_n(f)(t) \ne 0$, which implies that $t \in V_n$. Thus one can conclude that $V_n \supset V_{n-1} \cup (V_{n-1} - \mu_n)$.

Let $t \in V_n \cap (V_n + \mu_n)$. Then there is a $f \in C_0(\mathbf{R})$ such that $f(t) \ne 0$ and $\operatorname{supp} f \subset V_n \cap (V_n + \mu_n)$. Since $\operatorname{supp} f$ and $\operatorname{supp} f(\cdot + \mu_n)$ are contained in V_n, we have that $\phi_n(f) \in J_n$ which means that $f \in J_{n-1}$. Since $f(t) \ne 0$ it follows that $t \in V_{n-1}$. One can conclude that $V_n \cap (V_n + \mu_n) \subset V_{n-1}$.

Let I_1, I_2 be two open intervals in V_0. More specifically let $I_i = (a_i, b_i)$ with $a_1 < b_1 < a_2 < b_2$. We choose $n \in \mathbf{N}$ such that $\mu_n < \delta \equiv \min(b_1 - a_1, b_2 - a_2)$ and $m > n$ such that $\sum_{k=n}^m \mu_k > \max(\delta, \mu_1, a_2 - b_1)$. Since $V_m \supset I_i - \sum_{k=n}^{\ell} \mu_k$ for any $n \le \ell \le m$, we have that $V_m \supset (a_1 - \sum_{k=n}^m \mu_k, b_2)$. Then by using the same condition for $\mu_k, k < n$, we conclude that $V_m \supset (a_1 - \sum_{k=1}^m \mu_k, b_2)$. It then follows from $V_\ell \cap (V_\ell + \mu_\ell) \subset V_{\ell-1}$ for $\ell \le m$ that $V_0 \supset (a_1, b_2)$. Hence we can conclude that V_0 is connected. In this way we conclude all V_n's are connected. If $V_0 = (a, \infty)$ or $(-\infty, b)$, then it follows that J is monotone under $\hat{\alpha}$. Let $V_0 = (a, b)$. Then we denote by P_1 the ideal corresponding to the constant sequence $(-\infty, b)$ of open intervals of $\mathbf{R}$ and by P_2 the ideal corresponding $(a - \sum_{i=1}^{n} \mu_n, \infty)_{n=0}^{\infty}$. Both P_1, P_2 are monotone under $\hat{\alpha}$ and we can show that $J = P_1 \cap P_2$ or J is not primitive. Thus we can conclude that all the primitive ideals are monotone under $\hat{\alpha}$. □

Remark 3.4. From the proof of the above proposition we see that there are primitive ideals P_+, P_- of $A \times_\alpha \mathbf{R}$ such that $\hat{\alpha}_t(P_+) \supset P_+$ and $\hat{\alpha}_t(P_-) \subset P_-$ for all $t > 0$ and all other primitive ideals are in the orbits of these two ideals under $\hat{\alpha}$. It follows that the primitive ideal space of $A \times_\alpha \mathbf{R}$ is identified with $\mathbf{R} \sqcup \mathbf{R} \sqcup \{0\}$; $\hat{\alpha}$ acts as translations on $\mathbf{R}$ and the closure of a point t is $(-\infty, t]$ in the first copy of $\mathbf{R}$ and $[t, \infty)$ in the second copy. Thus if $\sum_n \mu_n = \infty$, the primitive ideal space is no use to distinguish $A \times_\alpha \mathbf{R}$.

Remark 3.5. There is a unital simple C*-algebra A with a flow α such that the primitive ideal space of $A \times_\alpha \mathbf{R}$ is identified with $\mathbf{R} \sqcup \{0\}$. For example we take a closed subset F of the interval $[0, 1]$ such that $F \ni 0$ and $F \not\ni 1$ and construct a unital simple C*-algebra B and a flow γ with period 1 as in 3.2 of [3]. Note that (B, γ) has a unique ground state but not a ceiling state (the exact property of (B, γ) we require will be given below). We take (A, α) as in Proposition 3.1 and its tensor product with (B, γ). Then $\beta = \alpha \otimes \gamma$ has a unique ground state but not a ceiling state and furthermore satisfies the no energy gap condition. Thus we can conclude that all non-zero primitive ideals of $A \otimes B \times_\beta \mathbf{R}$ is monotonly increasing under $\hat{\beta}$ and on just one orbit.

To prove the assertion of no energy gap in the above remark note that $B \times_\gamma \mathbf{R}$ is identified with the mapping torus M of $\hat{\gamma}_1$, where we regard γ as a faithful homomorphism from $\mathbf{T}$ into the automorphism group of B and the mapping torus M is the C*-algebra of bounded continuous functions x of $C_b(\mathbf{R}, B \times_\gamma \mathbf{T})$ with $x(t+1) = \hat{\gamma}_1(x(t))$ for $t \in \mathbf{R}$ [1]. Note that $B \times_\gamma \mathbf{T}$ has a non-zero primitive ideal I such that $n \in \mathbf{Z} \mapsto I_n = \hat{\gamma}_n(I)$ is increasing and exhausts all proper non-zero ideals. Set $I_{-\infty} = \{0\}$ and $I_\infty = B \times_\gamma \mathbf{T}$. Then $(A \otimes B) \times_\beta \mathbf{R}$ is isomorphic to the inductive limit of $D_n = M_{2^n} \otimes M = M_2 \otimes D_{n-1}$, where the map $\phi_n : D_{n-1} \to D_n$ is given by $\phi_n(x)(t) = x(t + \mu_n) \oplus x(t)$.

Let J be a proper non-zero ideal of $(A \otimes B) \times_\beta \mathbf{R}$ and set $J_n = J \cap D_n$. Let $J_n(t)$ denote the ideal $\{x(t) \mid x \in J_n\}$ of $M_{2^n} \otimes B \times_\gamma \mathbf{T}$ and define $\psi_n : \mathbf{R} \to \mathbf{Z} \cup \{-\infty, \infty\}$ by $M_{2^n} \otimes I_{\psi_n(t)} = J_n(t)$. Note that $t \mapsto \psi_n(t)$ is lower semi-continuous and $\psi_n(t + 1) = \psi_n(t) + 1$. We assert that $\psi_n(t) \geq$

$\max\{\psi_{n-1}(t), \psi_{n-1}(t+\mu_n)\}$ and $\psi_{n-1}(t) \geq \min\{\psi_n(t), \psi_n(t-\mu_n)\}$.

Fix $t \in \mathbf{R}$. If $m = \psi_{n-1}(t)$, then there is an $x \in J_{n-1}$ such that $x(t) \in M_{2^{n-1}} \otimes I_m \setminus M_{2^{n-1}} \otimes I_{m-1}$. Then $\phi_n(x) \in J_n$ and $\phi_n(x)(t) = x(t+\mu_n) \oplus x(t)$, which implies that $\psi_n(t) \geq m$. If $m = \psi_{n-1}(t+\mu_n)$, then there is an $x \in J_{n-1}$ such that $x(t+\mu_n) \in M_{2^{n-1}} \otimes I_m \setminus M_{2^{n-1}} \otimes I_{m-1}$. Then $\phi_n(x) \in J_n$ and $\phi_n(x)(t) = x(t+\mu_n) \oplus x(t)$, which implies that $\psi_n(t) \geq m$. This proves the first inequality. Let $m = \min\{\psi_n(t), \psi_n(t-\mu_n)\}$. Let $x \in D_{n-1}$ be such that $\mathrm{supp}(x)$ is concentrated around t and $x(t) \in M_{2^{n-1}} \otimes I_m \setminus M_{2^{n-1}} \otimes I_{m-1}$. Then $\phi_n(x)$ is concentrated around $t-\mu_n$ and t and satisfies that $\phi_n(x)(t), \phi_n(x)(t-\mu_n) \in M_{2^n} \otimes I_m \setminus M_{2^n} \otimes I_{m-1}$. For a suitable choice of x (with small support) we get that $\phi_n(x) \in J_n$ and hence $x \in J_{n-1}$. Since $x(t) \in M_{2^{n-1}} \otimes I_m \setminus M_{2^{n-1}} \otimes I_{m-1}$, we conclude that $\psi_{n-1}(t) \geq m$. This proves the second inequality.

Hence it follows that for any $m > n$, $\psi_m(t) \geq \max_K\{\psi_{n-1}(t+\sum_{k\in K}\mu_k)\}$ and $\psi_{n-1}(t) \geq \min_K\{\psi_m(t-\sum_{k\in K}\mu_k)\}$, where K runs over all the subsets of $\{n, \ldots, m\}$. If $\psi_{n-1}(t_i) \geq N$ for $i = 1, 2$ with $t_1 < t_2$, then the same is true on a small neighborhood of t_i. Hence it follows that for a sufficiently large m, $\psi_m(s) \geq N$ on the interval $[t_1 - \sum_{k=n}^m \mu_k, t_2]$ and then $\psi_{n-1}(s) \geq N$ on the interval $[t_1, t_2]$. Thus we can conclude that $t \mapsto \psi_n(t)$ is non-decreasing (taking values in $\mathbf{Z}$) and $\{t \mid \psi(t) = N\}$ is given as $(a_N, a_N+1]$ for all $N \in \mathbf{N}$. Since the ideal J is determined by the sequence (J_n) (which is described by (ψ_n)), we have reached the conclusion.

Remark 3.6. In general the primitive ideal space of $A \times_\alpha \mathbf{R}$ would be complicated. We will give some description of it which may be useful when the C*-algebra is unital.

Let α be a flow on a unital A satisfying the no energy gap condition as in Theorem 2.1. Then A has a ground state for α if and only if $A \times_\alpha \mathbf{R}$ has a primitive ideal which increases up to the whole algebra under $\hat{\alpha}$.

For $\mu \in \mathbf{R}$ we denote by $A^{\mu+}$ the hereditary C*-subalgebra of A generated by xx^*, $x \in A^\alpha(\mu, \infty)$ and set $A^\mu = \bigcap_{\nu<\mu} A^{\nu+}$. Note that A^μ is an α-invariant hereditary C*-subalgebra of A and $A^\mu = A$ for $\mu \leq 0$. We assume that $A^{0+} \neq A$, which is to say A has a ground state for α. Define a subset $\mathcal{F}$ of $A \times_\alpha \mathbf{R}$ by

$$\mathcal{F} = \{U(f)a \mid a \in A, f \in K^1(\mathbf{R}), \mathrm{supp}\hat{f} \subset (-\infty, 0)\}.$$

Then the ideal I generated by $\mathcal{F}$ equals the closed linear span of

$$I_0 = \bigcup_{\mu\in\mathbf{R}} \{U(f)xa \mid f \in K^1(\mathbf{R}),\ \mathrm{supp}\hat{f} \subset (-\infty, \mu),\ x \in A^\mu,\ a \in A\}.$$

(If there is only one orbit under $\hat{\alpha}$ as in the previous remark, I should be primitive.) If (π, V) is a representation of (A, α) such that π is irreducible and $\mathrm{Sp}(V) = [0, \infty)$, where we used the property $\mathrm{Sp}(V)$ is connected, then we obtain that $I \subset \ker \pi \times V$. The kernel of $\pi \times V$ is described as follows: Let E be the spectral measure for V and let

$$D_\pi^\mu = \{x \in A \mid \pi(x) = E[\mu, \infty)\pi(x)E[\mu, \infty)\},$$

where we should note the fact V depends only on π. Then D_π^μ is a hereditary C*-subalgebra of A containing A^μ and that $\ker \pi \times V$ is the closed linear span of I_0 with $x \in A^\mu$ replaced by $x \in D_\pi^\mu$. It also follows that $A^\mu = \bigcap_\pi D_\pi^\mu$ and that the ideal I is the intersection of all those primitive ideals obtained as $\ker \pi \times V$, whose orbits under $\hat{\alpha}$ exhausts all primitive ideals which monotonely increase under $\hat{\alpha}$.

To prove that the closed linear span $\mathcal{I}$ of I_0 is an ideal, we should note that $I_0 A \subset I_0$, $I_0 U_t \subset I_0$, and $U_t I_0 \subset I_0$. We shall show the remaining property that AI_0 is contained in $\mathcal{I}$. First note that

$$\mathcal{S} = \bigcup_{\mu \in \mathbf{R}} \{U(f)xa \mid f \in K^1(\mathbf{R}),\ \mathrm{supp}\hat{f} \subset (-\infty, \mu),\ x, a \in A,\ \mathrm{Sp}_\alpha(x) \subset (\mu, \infty)\}$$

is dense in I_0. Let $y \in A$ be an element with compact α-spectrum. Let $U(f)xa$ be an element as in the above set. Then there is a $\delta > 0$ such that $\mathrm{supp}\hat{f} \subset (-\infty, \mu - \delta)$ and $\mathrm{Sp}_\alpha(x) \subset (\mu + \delta, \infty)$. We express y as a finite sum $\sum_i y_i$ with $\mathrm{Sp}_\alpha(y_i) \subset (\mu_i - \delta, \mu_i + \delta)$ for some $\mu_i \in \mathbf{R}$. Since $K = \mathrm{Sp}_\alpha(y_i) + \mathrm{supp}\hat{f} \subset (-\infty, \mu + \mu_i)$, there is a $g \in K^1(\mathbf{R})$ such that $\hat{g} = 1$ on the compact set K and $\mathrm{supp}\,\hat{g} \subset (-\infty, \mu + \mu_i)$. Then $y_i U(f)xa = U(g)y_i U(f)xa$ equals

$$\int U(g)U_t\alpha_{-t}(y_i)xaf(t)dt,$$

where the integrand is norm-continuous. Since $\mathrm{Sp}_\alpha(\alpha_{-t}(y_i)x) \subset (\mu + \mu_i, \infty)$, we can conclude that each $y_i U(f)xa$ is in the closed linear span $\mathcal{I}$ of I_0, which shows that $yU(f)xa \in \mathcal{I}$. This concludes the proof that $\mathcal{I}$ is an ideal.

To prove that $\mathcal{I}$ is generated by $\mathcal{F}$ we shall show that I_0 is contained in the closed linear span of $A\mathcal{F}$. Let $U(f)xa$ be as in $\mathcal{S}$ with $\mathrm{supp}\hat{f} \subset (-\infty, \mu)$ and $\mathrm{Sp}_\alpha(x) \subset (\mu, \infty)$. Then there is a $g \in K^1(\mathbf{R})$ such that $\hat{g} = 1$ on $-\mathrm{Sp}_\alpha(x) + \mathrm{supp}\hat{f}$ and $\mathrm{supp}\,\hat{g} \subset (-\infty, 0)$. Since $U(f)xa = U(f)xU(g)a$ equals

$$\int \alpha_t(x)U_tU(g)af(t)dt$$

and $U_tU(g)a \in \mathcal{F}$, we can conclude that $U(f)xa$ is in the closed linear span of $A\mathcal{F}$.

It is obvious that D^μ_π is a hereditary C*-subalgebra. To show that $A^\mu \subset D^\mu_\pi$ let $x \in A^\mu$. Then $\pi(x)E[0,\nu]\mathcal{H}_\pi = 0$ for any $\nu < \mu$ because $x \in A^{\nu+}$. Thus we obtain that $\pi(x)E[0,\mu) = 0$, which implies $A^\mu \subset D^\mu_\pi$. We also have that $\ker\pi \times V$ contains the set defined as I_0 with $x \in A^\mu$ replaced by $x \in D^\mu_\pi$, which we will denote by I'_0. We will show the converse: $\ker\pi \times V \subset I'_0$. Let $U(f)x \in \ker\pi \times V$ with $f \in K^1(\mathbf{R})$ and $x \in A$. Let $\mu = \sup\operatorname{supp}\hat{f}$. Since the primitive ideal $\ker\pi \times V$ satisfies that $\hat{\alpha}_t(\ker\pi \times V) \supset \ker\pi \times V$ for $t > 0$, we obtain that $E(-\infty,\mu)\pi(x) = 0$. Thus it follows that $E[\mu,\infty)\pi(x) = \pi(x)$ and so $xx^* \in D^\mu$ and $U(f)x \in I'_0$. Since $\ker\pi \times V$ is the closed linear span of such $U(f)x$, we get the conclusion.

To show the last statement let D^μ be the intersection of all those D^μ_π. We claim that $D^\mu = A^\mu$ for all μ. Suppose that $A^\mu \subsetneqq D^\mu$, which implies that there is a $\nu < \mu$ such that $D^\mu \not\subset A^{\nu+}$. Then there is an $x \in D^\mu$ such that $x \geq 0$ and $x \notin A^{\nu+}$. Thus we obtain a pure state ϕ of A such that $\phi|A^{\nu+} = 0$ and $\phi(x) > 0$. Then π_ϕ is α-covariant and we can find a unitary flow V implementing α such that $\mathrm{Sp}(V) = [0,\infty)$. Since $\mathrm{Sp}_V(\Omega_\phi) \subset [0,\nu]$ and $\pi_\phi(x)\Omega_\phi \neq 0$, we reach the contradiction that $\pi(x)\Omega_\phi = \pi(x)E[\mu,\infty)E[0,\nu]\Omega\phi = 0$. It is obvious that I is the intersection of all those $\ker\pi \times V$.

The following proposition shows the crossed products by UHF flows (with $\lim_n \mu_n = 0$) are not isomorphic between in the case $\sum_n \mu_n^2 = \infty$ and in the case $\sum_n \mu_n^2 < \infty$.

Proposition 3.7. *Under the same situation as in Proposition* 3.3*, the following conditions are equivalent*:

1. $A \times_\alpha \mathbf{R}$ *has a tracial state.*
2. $\sum_n \mu_n^2 < \infty$.

Proof. Suppose that (2) holds, i.e., $\sigma^2 = \sum_i \mu_i^2 < \infty$. Define a random variable X_n by $\mathrm{Prob}(X_n = \pm\mu_n/2) = 1/2$ and suppose that all X_n's are independent. Since $\sum_n \mu_n^2 < \infty$, the sum $X = \sum_n X_n$ converges almost surely ([11], page 248). Let ν be the probability distribution of X.

Let τ be the normalized trace on M_{2^n} and let $t_n = -(1/2)\sum_{i=1}^n \mu_i$. We define a tracial state $\tau_{n,t}$ on $D_n = M_{2^n} \otimes C_0(\mathbf{R})$ by $\tau_{n,t}(f) = \tau(f(t))$ and extend it to a state of $A \times_\alpha \mathbf{R}$, denoted by the same symbol.

Let ϕ be a weak* limit point of τ_{n,t_n}. Since the map of D_{n-1} into D_n is given by $f \mapsto f(\cdot + \mu_n) \oplus f$, we have that $\tau_{n,t_n}(1 \otimes f) = \mathcal{E}f(\sum_{i=1}^n X_i)$, where

$\mathcal{E}$ denotes the expectation of a random variable. Thus $\phi|D_0$ satisfies that

$$\phi(f) = \int f(t)d\nu(t).$$

Thus ϕ is actually a state. It is obvious that ϕ is a trace.

Suppose that (1) holds and let ϕ be a tracial state of $A\times_\alpha\mathbf{R}$. Let us denote by Φ_n the embedding of D_n into $A\times_\alpha\mathbf{R}$ and let ν_n be the probability measure on $\mathbf{R}$ such that $\phi(\Phi_n(1\otimes f)) = \int f(t)d\nu_n(t)$. Note that $\nu_{n-1} = 1/2(\nu_n(\cdot - \mu_n) + \nu_n)$. Let T_n be a random variable such that T_n is independent of $X_1, \dots, X_n$ and the probability distribution of T_n is given by $\nu_n(\cdot + 1/2\sum_{i=1}^n \mu_i)$; then ν_0 is the probability distribution of $X_1+\cdots+X_n+T_n$ for any n: $\mathcal{E}f(T_0) = \mathcal{E}f(X_1+T_1) = \mathcal{E}f(X_1+X_2+T_2) = \cdots$. If f_n denotes the characteristic function of X_n, then it follows that $\Pi_{n=1}^\infty |f_n| \neq 0$ on the set of positive Lebesgue measure; otherwise it would imply that $\int e^{ist}d\nu_0(t) = 0$ for almost all s, a contradiction. Then the series $\sum_{n=1}^\infty X_n$ is essentially convergent ([11], page 263); thus $\sum_{n=1}^\infty X_n$ converges, or $\sum_{n=1}^\infty \mu_n^2 < \infty$ ([11], page 248). □

The no energy gap condition is not satisfied if α is a non-trivial inner flow or periodic flow. Moreover we have:

Example 3.8. Let (μ_n) be a (strictly) decreasing sequence of positive numbers such that $\lim_n \mu_n = \mu > 0$ and $\sum_n(\mu_n - \mu) = \infty$. We define a flow α on the UHF algebra A by using (μ_n) in the same way as in the beginning of this section. Then α does not satisfy the no energy gap condition and has $\mathbf{R}(\alpha) = \mathbf{R}$. Moreover $\lambda > 0 \mapsto A_\lambda$ increases at infinitely many points.

That $\mathbf{R}(\alpha) = \mathbf{R}$ follows by constructing, for any $\lambda > 0$, a central sequence (x_n) in A such that $\|x_n\| = 1$, $\mathrm{Sp}_\alpha(x_n) = \{\lambda_n\}$ and $\lambda_n \to \lambda$.

The restriction of α on the tensor product $A^{(n)}$ of the first n factors is determined by the set S_n of eigenvalues (with multiplicity) given by $\{e_n(I) \mid I \subset \{1, 2, \dots, n\}\}$, where $e_n(I) = \sum_{i\in I, i\leq n} \mu_i$. Let $\lambda > 0$. We divide S_n into clusters such that two points belong to the same cluster if they have a series of points which are closer than λ and do not otherwise. We assign to each cluster the matrix subalgebra of $A^{(n)}$ containing all the corresponding eigenprojections. Then $A_\lambda^{(n)}$ is given as the direct sum of these matrix algebras and A_λ is obtained as the inductive limit of $A_\lambda^{(n)}$. Suppose that $\mu_1 < 2\mu$. Then for any $i \in \mathbf{N}$ there is an $\epsilon > 0$ such that the ϵ-neighborhood of $e_n(\{i\})$ does not contain any other points in S_n for any n. As a matter of fact we may set $\epsilon = \min\{\mu_i - \mu_{i+1}, \mu_{i-1} - \mu_i\}$ with $\mu_0 = 2\mu$. Hence if $\lambda < \epsilon$ then i forms a cluster by itself in S_n for any $n > i$. This implies the claim made in the example.

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