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Intersection Homology $\mathcal{D}$-Module and Bernstein Polynomials Associated with a Complete Intersection

By

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Abstract

Let X be a complex analytic manifold. Given a closed subspace $Y \subset X$ of pure codimension $p \geq 1$, we consider the sheaf of local algebraic cohomology $H^p_{[Y]}(\mathcal{O}_X)$, and $\mathcal{L}(Y, X) \subset H^p_{[Y]}(\mathcal{O}_X)$ the intersection homology $\mathcal{D}_X$-Module of Brylinski-Kashiwara. We give here an algebraic characterization of the spaces Y such that $\mathcal{L}(Y, X)$ coincides with $H^p_{[Y]}(\mathcal{O}_X)$, in terms of Bernstein-Sato functional equations.

§1. Introduction

Let X be a complex analytic manifold of dimension $n \geq 2$, $\mathcal{O}_X$ be the sheaf of holomorphic functions on X and $\mathcal{D}_X$ the sheaf of differential operators with holomorphic coefficients. At a point $x \in X$, we identify the stalk $\mathcal{O}_{X,x}$ (resp. $\mathcal{D}_{X,x}$) with the ring $\mathcal{O} = \mathbf{C}\{x_1, \ldots, x_n\}$ (resp. $\mathcal{D} = \mathcal{O}\langle \partial/\partial x_1, \ldots, \partial/\partial x_n \rangle$).

Given a closed subspace $Y \subset X$ of pure codimension $p \geq 1$, we denote by $H^p_{[Y]}(\mathcal{O}_X)$ the sheaf of local algebraic cohomology with support in Y. Let $\mathcal{L}(Y, X) \subset H^p_{[Y]}(\mathcal{O}_X)$ be the intersection homology $\mathcal{D}_X$-Module of Brylinski-Kashiwara ([6]). This is the smallest $\mathcal{D}_X$-submodule of $H^p_{[Y]}(\mathcal{O}_X)$ which coincides with $H^p_{[Y]}(\mathcal{O}_X)$ at the generic points of Y ([6], [3]).

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A natural problem is to characterize the subspaces Y such that $\mathcal{L}(Y,X)$ coincides with $H^p_{[Y]}(\mathcal{O}_X)$. We prove here that it may be done locally using Bernstein functional equations. This supplements a work of D. Massey ([16]), who studies the analogous problem with a topological viewpoint. Indeed, from the Riemann-Hilbert correspondence of Kashiwara-Mebkhout ([12], [18]), the regular holonomic $\mathcal{D}_X$-Module $H^p_{[Y]}(\mathcal{O}_X)$ corresponds to the perverse sheaf $\mathbf{C}_Y[n-p]$ ([8], [10], [17]) where as $\mathcal{L}(Y,X)$ corresponds to the intersection complex $IC^\bullet_Y$ ([6]). By this way, this condition $\mathcal{L}(Y,X) = H^p_{[Y]}(\mathcal{O}_X)$ is equivalent to the following one: *the real link of Y at a point $x \in Y$ is a rational homology sphere*.

First of all, we have an explicit local description of $\mathcal{L}(Y,X)$. This comes from the following result, due to D. Barlet and M. Kashiwara.

Theorem 1.1 ([3]). *The fundamental class $C^Y_X \in H^p_{[Y]}(\mathcal{O}_X) \otimes \Omega^p_X$ of Y in X belongs to $\mathcal{L}(Y,X) \otimes \Omega^p_X$.*

For more details about C^Y_X, see [1]. In particular, if h is an analytic morphism $(h_1,\ldots,h_p) : (X,x) \to (\mathbf{C}^p,0)$ which defines the complete intersection (Y,x) - reduced or not -, then the inclusion $\mathcal{L}(Y,X)_x \subset H^p_{[Y]}(\mathcal{O}_X)_x$ may be identified with:

$$\mathcal{L}_h = \sum_{1\le k_1<\cdots<k_p\le n} \mathcal{D}\cdot \frac{m_{k_1,\ldots,k_p}(h)}{h_1\cdots h_p} \subset \mathcal{R}_h = \frac{\mathcal{O}[1/h_1\cdots h_p]}{\sum_{i=1}^p \mathcal{O}[1/h_1\cdots \check{h}_i \cdots h_p]}$$

where $m_{k_1,\ldots,k_p}(h) \in \mathcal{O}$ is the determinant of the columns $k_1,\ldots,k_p$ of the Jacobian matrix of h. In the following, $\mathcal{J}_h \subset \mathcal{O}$ denotes the ideal generated by the $m_{k_1,\ldots,k_p}(h)$, and $\delta_h \in \mathcal{R}_h$ the section defined by $1/h_1\cdots h_p$.

When Y is a hypersurface, we have the following characterization.

Theorem 1.2. *Let $Y \subset X$ be a hypersurface and $h \in \mathcal{O}_{X,x}$ denote a local equation of Y at a point $x \in Y$. The following conditions are equivalent:*

1. *$\mathcal{L}(Y,X)_x$ coincides with $H^p_{[Y]}(\mathcal{O}_X)_x$.*
2. *The reduced Bernstein polynomial of h has no integral root.*
3. 1 *is not an eigenvalue of the monodromy acting on the reduced cohomology of the fibers of the Milnor fibrations of h around any singular points of Y contained in some open neighborhood of x in Y.*

Let us recall that the *Bernstein polynomial* $b_f(s)$ of a nonzero germ $f \in \mathcal{O}$ is the monic generator of the ideal of the polynomials $b(s) \in \mathbf{C}[s]$ such that:

$$b(s)f^s = P(s) \cdot f^{s+1}$$

in $\mathcal{O}[1/f, s]f^s$, where $P(s) \in \mathcal{D}[s] = \mathcal{D} \otimes \mathbf{C}[s]$. The existence of such a nontrivial equation was proved by M. Kashiwara ([9]). When f is not a unit, it is easy to check that -1 is a root of $b_f(s)$. The quotient of $b_f(s)$ by $(s+1)$ is the so-called *reduced Bernstein polynomial* of f, denoted $\tilde{b}_f(s)$. Let us recall that their roots are rational negative numbers in $]-n, 0[$ (see [21] for the general case, [27] for the isolated singularity case).

Example 1.3. Let $f = x_1^2 + \cdots + x_n^2$. It is easy to prove that $b_f(s)$ is equal to $(s+1)(s+n/2)$, by using the identity:

$$\left[\frac{\partial}{\partial x_1}^2 + \cdots + \frac{\partial}{\partial x_n}^2\right] \cdot f^{s+1} = (2s+2)(2s+n)f^s$$

In particular, $\mathcal{R}_f$ coincides with $\mathcal{L}_f$ if and only if n is odd.

These polynomials are famous because of the link of their roots with the monodromy of the Milnor fibration associated with f. This was established by B. Malgrange [15] and M. Kashiwara [11]. More generally, by using the algebraic microlocalization, M. Saito [21] prove that $\{e^{-2i\pi\alpha} \mid \alpha \text{ root of } \tilde{b}_f(s)\}$ coincides with the set of the eigenvalues of the monodromy acting on the Deligne vanishing cycle sheaf $\phi_f \mathbf{C}_{X(x)}$ (where $X(x) \subset X$ is a sufficiently small neighborhood of x). Thus the equivalence $2 \Leftrightarrow 3$ is an easy consequence of this deep fact.

We give a direct proof of $1 \Leftrightarrow 2$ in part 4.

Remark 1.4. (i) In [2], D. Barlet gives a characterization of 3 in terms of the meromorphic continuation of the current $\int_{X(x)} f^\lambda \Box$.
(ii) In [4], M. Blickle and R. Bondu study the same problem in positive characteristic.

Remark 1.5. The equivalence $1 \Leftrightarrow 3$ for the isolated singularity case may be due to J. Milnor [19] using the Wang sequence relating the cohomology of the link with the Milnor cohomology. In general, this equivalence is well-known to specialists. It can be proved by using a formalism of weights and by reducing it to the assertion that the N-primitive part of the middle graded piece of the monodromy weight filtration on the nearby cycle sheaf is the intersection complex (this last assertion is proved in [20] (4.5.8) for instance). It would be quite interesting if one can prove the equivalence between 1 and 3 by using only the theory of $\mathcal{D}$-modules.

In the case of hypersurfaces, it is well known that condition 1 requires a strong kind of irreducibility. This may be refinded in terms of Bernstein polynomial.

Proposition 1.6. *Let $f \in \mathcal{O}$ be a nonzero germ such that $f(0) = 0$. Assume that the origin belongs to the closure of the points where f is locally reducible. Then -1 is a root of the reduced Bernstein polynomial of f.*

Example 1.7. If $f = x_1^2 + x_3 x_2^2$, then $(s+1)^2$ divides $b_f(s)$ because $f^{-1}\{0\} \subset \mathbf{C}^3$ is reducible at any $(0,0,\lambda)$, $\lambda \neq 0$ (in fact, we have: $b_f(s) = (s+1)^2(s+3/2)$).

What may be done in higher codimensions ? If $f \in \mathcal{O}$ is such that (h,f) defines a complete intersection, we can consider the Bernstein polynomial $b_f(\delta_h, s)$ of f associated with $\delta_h \in \mathcal{R}_h$. Indeed, we again have nontrivial functional equations:

$$b(s)\delta_h f^s = P(s) \cdot \delta_h f^{s+1}$$

with $P(s) \in \mathcal{D}[s]$ (see part 2). This polynomial $b_f(\delta_h, s)$ is again a multiple of $(s+1)$, and we can define a reduced Bernstein polynomial $\tilde{b}_f(\delta_h, s)$ as above. Meanwhile, in order to generalize Theorem 1.2, the suitable Bernstein polynomial is neither $\tilde{b}_f(\delta_h, s)$ nor $b_f(\delta_h, s)$, but a third one trapped between these two.

Notation 1.8. *Given a morphism $(h,f) = (h_1, \dots, h_p, f) : (\mathbf{C}^n, 0) \to (\mathbf{C}^{p+1}, 0)$ defining a complete intersection, we denote by $b'_f(h,s)$ the monic generator of the ideal of polynomials $b(s) \in \mathbf{C}[s]$ such that*:

$$b(s)\delta_h f^s \in \mathcal{D}[s](\mathcal{J}_{h,f}, f)\delta_h f^s \ . \tag{1}$$

Lemma 1.9. *The polynomial $b'_f(h,s)$ divides $b_f(\delta_h, s)$, and $b_f(\delta_h, s)$ divides $(s+1)b'_f(h,s)$. In other words, $b'_f(h,s)$ is either $b_f(\delta_h, s)$ or $b_f(\delta_h, s)/(s+1)$.*

The first assertion is clear since $\mathcal{D}[s]\delta_h f^{s+1} \subset \mathcal{D}[s](\mathcal{J}_{h,f}, f)\delta_h f^s$. The second relation uses the identities:

$$(s+1)m_{k_1,\dots,k_{p+1}}(h,f)\delta_h f^s = \underbrace{\left[\sum_{i=1}^{p+1} (-1)^{p+i+1} m_{k_1,\dots,\check{k}_i,\dots,k_{p+1}}(h) \frac{\partial}{\partial x_{k_i}}\right]}_{\Delta^h_{k_1,\dots,k_{p+1}}} \cdot \delta_h f^{s+1}$$

for $1 \le k_1 < \cdots < k_{p+1} \le n$, where the vector field $\Delta^h_{k_1,\ldots,k_{p+1}}$ annihilates δ_h. In particular, we have: $(s+1)b'_f(h,s)\delta_h f^s \in \mathcal{D}[s]\delta_h f^{s+1}$, and the assertion follows.

As a consequence of this result, $b'_f(h,s)$ coincides with $\tilde{b}_f(\delta_h, s)$ when -1 is not a root of $b'_f(h,s)$; but it is not always true (see part 3). We point out some facts about this polynomial in part 3.

Theorem 1.10. *Let $Y \subset X$ be a closed subspace of pure codimension $p+1 \ge 2$, and $x \in Y$. Let $(h,f) = (h_1,\ldots,h_p,f) : (X,x) \to (\mathbf{C}^{p+1},0)$ be an analytic morphism such that the common zero set of $h_1,\ldots,h_p,f$ is Y in a neighbourhood of x. Up to replace h_i by h_i^m for some non negative integer $m \ge 1$, let us assume that $\mathcal{D}\delta_h = \mathcal{R}_h$. The following conditions are equivalent*:

1. *$\mathcal{L}(Y,X)_x$ coincides with $H^p_{[Y]}(\mathcal{O}_X)_x$.*
2. *The polynomial $b'_f(h,s)$ has no strictly negative integral root.*

Let us observe that the condition $\mathcal{D}\delta_h = \mathcal{R}_h$ is not at all a constraining condition on (Y,x). Moreover, using the boundaries of the roots of the classical Bernstein polynomial, on can take $m = n-1$ (since $1/(h_1\cdots h_p)^{n-1}$ generates the $\mathcal{D}$-module $\mathcal{O}[1/h_1\cdots h_p]$, using Proposition 4.2 below). Finally, one can observe that this technical condition $\mathcal{D}\delta_h = \mathcal{R}_h$ is difficult to verify in practice. Thus, let us give an inductive criterion which is a corollary of Proposition 4.2 below:

Corollary 1.11. *Let $h = (h_1,\ldots,h_p) : (X,x) \to (\mathbf{C}^p,0)$ be an analytic morphism defining a germ of complete intersection of codimension $p \ge 1$. Assume that -1 is the only integral root of the Bernstein polynomial $b_{h_1}(s)$. Moreover, if $p \ge 2$, assume that -1 is the smallest integral root of $b_{h_{i+1}}(\delta_{\tilde{h}_i}, s)$ with $\tilde{h}_i = (h_1,\ldots,h_i) : (X,x) \to (\mathbf{C}^i,0)$, for $1 \le i \le p-1$. Then the left $\mathcal{D}$-module $\mathcal{R}_h$ is generated by δ_h.*

Example 1.12. Let $n=3$, $p=2$, $h_1 = x_1^2 + x_2^3 + x_3^4$ and $h_2 = x_1^2 - x_2^3 + 2x_3^4$. As h_1 defines an isolated singularity and $h = (h_1,h_2)$ defines a weighted-homogeneous complete intersection isolated singularity, we have closed formulas for $b_{h_1}(s)$ and $b_{h_2}(\delta_{h_1}, s)$, see [29], [23]. From the explicit expression of these two polynomials, we see that they have no integral root smaller than -1. Thus δ_h generates $\mathcal{R}_h$.

The proofs of Theorems 1.2 & 1.10 are given in part 4. They are based on a natural generalization of a classical result due to M. Kashiwara which links the roots of $b_f(s)$ to some generators of $\mathcal{O}[1/f]f^\alpha$, $\alpha \in \mathbf{C}$ (Proposition 4.2). The last part is devoted to remarks and comments about Theorem 1.10.

§2. Bernstein Polynomials Associated with a Section of a Holonomic $\mathcal{D}$-Module

In this paragraph, we recall some results about Bernstein polynomials associated with a section of a holonomic $\mathcal{D}_X$-Module.

Given a nonzero germ $f \in \mathcal{O}_{X,x} \cong \mathcal{O}$ and a local section $m \in \mathcal{M}_x$ of a holonomic $\mathcal{D}_X$-Module $\mathcal{M}$ without f-torsion, M. Kashiwara [10] proved that there exists a functional equation:

$$b(s)mf^s = P(s) \cdot mf^{s+1}$$

in $(\mathcal{D}m) \otimes \mathcal{O}[1/f, s]f^s$, where $P(s) \in \mathcal{D}[s] = \mathcal{D} \otimes \mathbf{C}[s]$ and $b(s) \in \mathbf{C}[s]$ are nonzero. The *Bernstein polynomial* of f associated with m, denoted by $b_f(m,s)$, is the monic generator of the ideal of polynomials $b(s) \in \mathbf{C}[s]$ which satisfies such an equation. When f is not a unit, it is easy to check that if $m \in \mathcal{M}_x - f\mathcal{M}_x$, then -1 is a root of $b_f(m,s)$ (see [22] for instance).

Of course, if $\mathcal{M} = \mathcal{O}_X$ and $m = 1$, this is the classical notion recalled in the introduction.

Let us recall that when $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-Module, the roots of the polynomials $b_f(m,s)$ are closely linked to the eigenvalues of the monodromy of the perverse sheaf $\psi_f(Sol(\mathcal{M}))$ around x, the Deligne nearby cycle sheaf, see [13] for example. Here $Sol(\mathcal{M})$ denotes the complex $RHom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ of holomorphic solutions of $\mathcal{M}$, and the relation is similar to the one given in the introduction (since $Sol(\mathcal{O}_X) \cong \mathbf{C}_X$). This comes from the algebraic construction of vanishing cycles, using Malgrange-Kashiwara V-filtration [15], [11].

Now, if $Y \subset X$ is a subspace of pure codimension p, then the regular holonomic $\mathcal{D}_X$-Module $H^p_{[Y]}(\mathcal{O}_X)$ corresponds to $\mathbf{C}_Y\,[n-p]$. Thus the roots of the polynomials $b_f(\delta, s)$, $\delta \in H^p_{[Y]}(\mathcal{O}_X)_x$, are linked to the monodromy associated with $f : (Y, 0) \to (\mathbf{C}, 0)$. For more results about these polynomials, see [22].

§3. The Polynomials $b'_f(h,s)$ and $\tilde{b}_f(\delta_h, s)$

Let us recall that $b'_f(h,s)$ is always equal to one of the two polynomials $b_f(\delta_h, s)$ and $\tilde{b}_f(\delta_h, s)$. In this paragraph, we point out some facts about these Bernstein polynomials associated with an analytic morphism $(h,f) = (h_1,\ldots,h_p,f) : (\mathbf{C}^n, 0) \to (\mathbf{C}^{p+1}, 0)$ defining a complete intersection.

First we have a closed formula for $b'_f(h,s)$ when h and (h,f) define weighted-homogeneous isolated complete intersection singularities.

Proposition 3.1 ([23]). *Let $f, h_1, \dots, h_p \in \mathbf{C}[x_1, \dots, x_n]$, $p < n$, be some weighted-homogeneous of degree $1, \rho_1, \dots, \rho_p \in \mathbf{Q}^{*+}$ for a system of weights $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbf{Q}^{*+})^n$. Assume that the morphisms $h = (h_1, \dots, h_p)$ and (h, f) define two germs of isolated complete intersection singularities. Then the polynomial $b'_f(h,s)$ is equal to*:

$$\prod_{q \in \Pi} (s + |\alpha| - \rho_h + q)$$

where $|\alpha| = \sum_{i=1}^n \alpha_i$, $\rho_h = \sum_{i=1}^p \rho_i$ and $\Pi \subset \mathbf{Q}^+$ is the set of the weights of the elements of a weighted-homogeneous basis of $\mathcal{O}/(f, h_1, \dots, h_p)\mathcal{O} + \mathcal{J}_{h,f}$.

When h is not reduced, the determination of $b'_f(h,s)$ is more difficult, even if (h,f) is a homogeneous morphism; in particular, there does not exist a closed formula as above.

Example 3.2. Let $p = 1$, $f = x_1$ and $h = (x_1^2 + \cdots + x_n^2)^\ell$ with $\ell \geq 1$. By using a formula given in [25], Remark 4.12, the polynomial $\tilde{b}_{x_1}(\delta_h, s)$ is equal to $(s + n - 2\ell)$ for any $\ell \in \mathbf{N}^* = \mathbf{C}^* \cap \mathbf{N}$. For $\ell \geq n/2$, let us determine $b'_{x_1}(h,s)$ with the help of Theorem 1.10. From Example 1.3, we have $\mathcal{R}_h = \mathcal{D}\delta_h$ if $\ell \geq n/2$, and $\mathcal{L}_{h,x_1} = \mathcal{R}_{h,x_1}$ if and only if n is even.

If $n \leq 2\ell$ is odd, $b'_{x_1}(h,s)$ must coincide with $b_{x_1}(\delta_h, s) = (s+1)(s+n-2\ell)$ because of Theorem 1.10 (since $\mathcal{L}_{h,x_1} \neq \mathcal{R}_{h,x_1}$). On the other hand, if $n \leq 2\ell$ is even, we have $b'_{x_1}(h,s) = (s + n - 2\ell) = \tilde{b}_{x_1}(\delta_h, s)$ by the same arguments. Let us refind this last fact by a direct calculus.

As $\tilde{b}_{x_1}(\delta_h, s) = (s + n - 2\ell)$ divides $b'_{x_1}(h,s)$, we just have to check that this polynomial $(s + n - 2\ell)$ provides a functional equation for $\tilde{b}_{x_1}(\delta_h, s)$ when n is even. First, we observe that

$$(s + n - 2\ell)\delta_h x_1^s = \left[\sum_{i=1}^n \frac{\partial}{\partial x_i} x_i\right] \cdot \delta_h x_1^s$$

If $\ell = 1$, we get the result (since $\mathcal{J}_{h,x_1} = (x_2, \dots, x_n)\mathcal{O}$ in that case). Now we assume that $\ell \geq 2$. Let us prove that $x_i \delta_h x_1^s$ belongs to $\mathcal{D}(\mathcal{J}_{h,x_1}, x_1)\delta_h x_1^s$ for $2 \leq i \leq n$. We denote by g the polynomial $x_1^2 + \cdots + x_n^2$ and, for $0 \leq j \leq \ell - 1$, by $\mathcal{N}_j \subset \mathcal{D}(\mathcal{J}_{h,x_1}, x_1)\delta_h x_1^s$ the submodule generated by $x_1 \delta_h x_1^s$, $x_2 g^j \delta_h x_1^s, \dots,$ $x_n g^j \delta_h x_1^s$. In particular, $h = g^\ell$, $\mathcal{N}_{\ell-1} = \mathcal{D}(\mathcal{J}_{h,x_1}, x_1)\delta_h x_1^s$ and $\mathcal{N}_{j+1} \subset \mathcal{N}_j$ for $1 \leq j \leq \ell - 2$. To conclude, we have to check that $\mathcal{N}_0 = \mathcal{N}_{\ell-1}$.

By a direct computation, we obtain the identity:

$$\frac{\partial}{\partial x_i}\left[2(j-\ell)g^{j-1}x_1^2+\sum_{k=2}^{n}\frac{\partial}{\partial x_k}x_k g^j\right]\cdot\delta_h x_1^s = 2(j-\ell)(n+2(j-\ell)-1)x_i g^{j-1}\delta_h x_1^s$$

for $2 \leq i \leq n$, $j > 0$. As n is even, we deduce that $x_i g^{j-1}\delta_h x_1^s$ belongs to $\mathcal{N}_j$ for $2 \leq i \leq n$. In other words, $\mathcal{N}_{j-1} = \mathcal{N}_j$ for $1 \leq j \leq \ell - 1$; thus $\mathcal{N}_0 = \mathcal{N}_{\ell-1}$, as it was expected.

As the polynomial $b'_f(h,s)$ plays the rule of $\tilde{b}_f(s)$ in Theorem 1.10, a natural question is to compare these polynomials $b'_f(h,s)$ and $\tilde{b}_f(\delta_h,s)$. Of course, when $(s+1)$ is not a factor of $b'_f(h,s)$, then $b'_f(h,s)$ must coincide with $\tilde{b}_f(\delta_h,s)$; from Theorem 1.10, this sufficient condition is satisfied when $\mathcal{D}\delta_h = \mathcal{R}_h$ and $\mathcal{R}_{h,f} = \mathcal{L}_{h,f}$. But in general, all the cases are possible (see Example 3.2); nevertheless, we do not have found an example with f and h reduced and $b'_f(h,s) = b_f(\delta_h,s)$. Is $b'_f(h,s)$ always equal to $\tilde{b}_f(h,s)$ in this context ? The question is open. Let us study this problem when (h,f) defines an isolated complete intersection singularity. In that case, let us consider the short exact sequence:

$$0 \to \mathcal{K} \hookrightarrow \frac{\mathcal{D}[s]\delta_h f^s}{\mathcal{D}[s](\mathcal{J}_{h,f},f)\delta_h f^s} \twoheadrightarrow (s+1)\frac{\mathcal{D}[s]\delta_h f^s}{\mathcal{D}[s]\delta_h f^{s+1}} \to 0$$

where the three $\mathcal{D}$-modules are supported by the origin. Thus the polynomial $b'_f(h,s)$ is equal to $\mathrm{l.c.m}(s+1,\tilde{b}_f(\delta_h,s))$ if $\mathcal{K} \neq 0$ and it coincides with $\tilde{b}_f(\delta_h,s)$ if not. Remark that $\mathcal{K}$ is not very explicit, since there does not exist a general Bernstein functional equation which defines $\tilde{b}_f(\delta_h,s)$ - contrarily to $\tilde{b}_f(s)$, see part 4. In [24], [25], we have investigated some contexts where such a functional equation may be given. In particular, this may be done when the following condition is satisfied:

A(δ_h)**:** The ideal $\mathrm{Ann}_{\mathcal{D}}\,\delta_h$ of operators annihilating δ_h is generated by $\mathrm{Ann}_{\mathcal{O}}\,\delta_h$ and operators $Q_1,\ldots,Q_w \in \mathcal{D}$ of order 1.

Indeed, because of the relations: $Q_i \cdot \delta_h f^{s+1} = (s+1)[Q_i,f]\delta_h f^s$, $1 \leq i \leq w$, we have the following isomorphism:

$$\frac{\mathcal{D}[s]\delta_h f^s}{\mathcal{D}[s](\widetilde{\mathcal{J}}_{h,f},f)\delta_h f^s} \xrightarrow{\cong} (s+1)\frac{\mathcal{D}[s]\delta_h f^s}{\mathcal{D}[s]\delta_h f^{s+1}}$$

where $\widetilde{\mathcal{J}}_{h,f} \subset \mathcal{O}$ is generated by the commutators $[Q_i,f] \in \mathcal{O}$, $1 \leq i \leq w$. Thus $\tilde{b}_f(\delta_h,s)$ may also be defined using the functional equation:

$$b(s)\delta_h f^s \in \mathcal{D}[s](\widetilde{\mathcal{J}}_{h,f},f)\delta_h f^s$$

and $\mathcal{K} = \mathcal{D}[s](\widetilde{\mathcal{J}}_{h,f}, f)\delta_h f^s / \mathcal{D}[s](\mathcal{J}_{h,f}, f)\delta_h f^s$. For more details about this condition $\mathbf{A(}\delta_h\mathbf{)}$, see [26].

§4. The Proofs

Let us recall that $\tilde{b}_h(s)$ may be defined as the unitary nonzero polynomial $b(s) \in \mathbf{C}[s]$ of smallest degree such that:

$$b(s)h^s = P(s) \cdot h^{s+1} + \sum_{i=1}^{n} P_i(s) \cdot h'_{x_i} h^s \tag{2}$$

where $P(s), P_1(s), \ldots, P_n(s) \in \mathcal{D}[s]$ (see [14]).

Remark 4.1. The equation (2) is equivalent to the following one:

$$b(s)h^s = \sum_{i=1}^{n} Q_i(s) \cdot h'_{x_i} h^s$$

where $Q_i(s) \in \mathcal{D}[s]$ for $1 \leq i \leq n$. Indeed, one can prove that $h^{s+1} \in \mathcal{D}[s](h'_{x_1}, \ldots, h'_{x_n})h^s$ *i.e.* h belongs to the ideal $I = \mathcal{D}[s](h'_{x_1}, \ldots, h'_{x_n}) + \mathrm{Ann}_{\mathcal{D}[s]} h^s$. This requires some computations like in [25] 2.1., using that: $h\partial_{x_i} - sh'_{x_i} \in I$, $1 \leq i \leq n$.

Proof of Proposition 1.6. By semi-continuity of the Bernstein polynomial, it is enough to prove the assertion for a reducible germ f. Let us write $f = f_1 f_2$ where $f_1, f_2 \in \mathcal{O}$ have no common factor. Assume that -1 is not a root of $\tilde{b}_f(s)$. Then, by fixing $s = -1$ in (2), we get:

$$\frac{1}{f} \in \sum_{i=1}^{n} \mathcal{D}\frac{f'_{x_i}}{f} + \mathcal{O} \ \subset \ \mathcal{O}[1/f_1] + \mathcal{O}[1/f_2]$$

since $f'_{x_i}/f = f'_{1,x_i}/f_1 + f'_{2,x_i}/f_2$, $1 \leq i \leq n$. But this is absurd since $1/f_1 f_2$ defines a nonzero element of $\mathcal{O}[1/f_1 f_2]/\mathcal{O}[1/f_1] + \mathcal{O}[1/f_2]$ under our assumption on f_1, f_2. Thus -1 is a root of $\tilde{b}_f(s)$. □

The proofs of the equivalence between 1 and 2 in Theorem 1.2 and of Theorem 1.10 are based on the following result:

Proposition 4.2. *Let $f \in \mathcal{O}$ be a nonzero germ such that $f(0) = 0$. Let m be a section of a holonomic $\mathcal{D}$-module $\mathcal{M}$ without f-torsion, and $\ell \in \mathbf{Z}$. The following conditions are equivalent*:

1. *The smallest integral root of* $b_f(m,s)$ *is strictly greater than* $-\ell-1$.

2. *The* $\mathcal{D}$*-module* $(\mathcal{D}m)[1/f]$ *is generated by* $mf^{-\ell}$.

3. *The* $\mathcal{D}$*-module* $(\mathcal{D}m)[1/f]/\mathcal{D}m$ *is generated by* $m\dot{f}^{-\ell}$.

4. *The following* $\mathcal{D}$*-linear morphism is an isomorphism*:

$$\begin{aligned}\pi_\ell \ : \ \frac{\mathcal{D}[s]mf^s}{(s+\ell)\mathcal{D}[s]mf^s} &\longrightarrow (\mathcal{D}m)[1/f] \\ P(s)\dot{\cdot}\, mf^s &\mapsto P(-\ell)\cdot mf^{-\ell}\end{aligned}$$

Proof. This is a direct generalization of a well known result due to M. Kashiwara and E. Björk for $m=1\in\mathcal{O}=\mathcal{M}$ ([9] Proposition 6.2, [5] Propositions 6.1.18, 6.3.15 & 6.3.16).

Let us prove $1\Rightarrow 4$. First, we establish that π_ℓ is surjective. It is enough to see that for all $P\in\mathcal{D}$ and $l\in\mathbf{Z}$: $(P\cdot m)f^l\in\mathcal{D}mf^{-\ell}$. By using the following relations:

$$\left(\left[\frac{\partial}{\partial x_i}Q\right]\cdot m\right)f^l=\left[\frac{\partial}{\partial x_i}f-l\frac{\partial f}{\partial x_i}\right]\cdot((Q\cdot m)f^{l-1})$$

where $1\le i\le n$, $Q\in\mathcal{D}$ and $l\in\mathbf{Z}$, we obtain that for all $P\in\mathcal{D}$, $l\in\mathbf{Z}$, there exist $Q\in\mathcal{D}$ and $k\in\mathbf{Z}$ such that $(P\cdot m)f^l=Q\cdot mf^k$. Thus, we just have to prove that: $mf^k\in\mathcal{D}mf^{-\ell}$ for $k<-\ell$.

Let $R\in\mathcal{D}[s]$ be a differential operator such that:

$$b_f(m,s)mf^s=R\cdot mf^{s+1} \tag{3}$$

and let $k\in\mathbf{Z}$ be such that $k<-\ell$. Iterating (3), we get the following identity in $\mathcal{D}m[1/f,s]f^s$:

$$\underbrace{b_f(m,s-\ell-k-1)\cdots b_f(m,s+1)b_f(m,s)}_{c(s)}mf^s=Q(s)\cdot mf^{s-\ell-k} \tag{4}$$

where $Q(s)\in\mathcal{D}[s]$. By assumption on ℓ, we have: $c(k)\neq 0$. Thus, by fixing $s=k$ in (4), we get $mf^k\in\mathcal{D}mf^{-\ell}$ and π_ℓ is surjective.

Let us prove the injectivity of π_ℓ. If we fix $P(s)\in\mathcal{D}[s]$, then we have the following identity in $\mathcal{D}m[1/f,s]f^s$:

$$P(s)\cdot mf^s=(Q(s)\cdot m)f^{s-l}$$

where $Q(s) \in \mathcal{D}[s]$ and l is the degree of P. Assume that $P(s) \cdot mf^s \in \ker \pi_\ell$. Thus there exists a non negative integer $j \in \mathbf{N}$ such that $f^j Q(-\ell)$ annihilates $m \in \mathcal{M}$. In particular: $P(s) \cdot mf^s = (s+\ell)(Q' \cdot m) f^{s-l}$, where $Q' \in \mathcal{D}[s]$ is the quotient of the division of Q by $(s+\ell)$. As in the beginning of the proof, we obtain that $P(s) \cdot mf^s = (s+\ell)\widetilde{Q} \cdot mf^{s-k}$ where $\widetilde{Q} \in \mathcal{D}[s]$ and $k \in \mathbf{N}^*$. From (3), we get:

$$\underbrace{b_f(m, s-1) \cdots b_f(m, s-k+1) b_f(m, s-k)}_{d(s)} P(s) \cdot mf^s = (s+\ell)\widetilde{Q} S \cdot mf^s$$

where $S \in \mathcal{D}[s]$. By division of $d(s)$ by $(s+\ell)$, we obtain the identity :

$$d(-\ell) P(s) \cdot mf^s = (s+\ell)[\widetilde{Q} S + e(s) P(s)] \cdot mf^s$$

where $e(s) \in \mathbf{C}[s]$. Remark that $d(-\ell) \neq 0$ by assumption on ℓ. Thus $P(s) \cdot mf^s \in (s+\ell)\mathcal{D}[s] mf^s$, and π_ℓ is injective. Hence the condition 1 implies that π_ℓ is an isomorphism.

Observe that $4 \Rightarrow 2$ and $2 \Leftrightarrow 3$ are clear. Thus let us prove $2 \Rightarrow 1$. Let $k \in \mathbf{Z}$ denote the smallest integral root of $b_f(m, s)$. Assume that $-\ell > k$. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathcal{D}[s] mf^{s+1} & \hookrightarrow & \mathcal{D}[s] mf^s & \twoheadrightarrow & \mathcal{D}[s] mf^s / \mathcal{D}[s] mf^{s+1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \upsilon & & \\
0 & \to & \mathcal{D}[s] mf^{s+1} & \hookrightarrow & \mathcal{D}[s] mf^s & \twoheadrightarrow & \mathcal{D}[s] mf^s / \mathcal{D}[s] mf^{s+1} & \to & 0 \\
 & & \downarrow u & & \downarrow & & & & \\
 & & \mathcal{D} mf^{k+1} & \overset{i}{\hookrightarrow} & \mathcal{D} mf^k & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

where υ is the left-multiplication by $(s-k)$. Remark that the second column is exact (since $1 \Rightarrow 4$), that u is surjective, and that i is an isomorphism (since $mf^{-\ell} \in \mathcal{D} mf^{k+1}$ generates $\mathcal{D} m[1/f]$ by assumption).

After a diagram chasing, one can check that υ is surjective. Thus the $\mathcal{D}$-module $\mathcal{D}[s] mf^s / \mathcal{D}[s] mf^{s+1}$ is Artinian, as the stalk of a holonomic $\mathcal{D}$-Module [indeed, it is the quotient of two sub-holonomic $\mathcal{D}$-Modules which are isomorphic (see [9], [10])]. As a surjective endomorphism of an Artinian module is also injective, υ is injective. But this is absurd since k is a root of $b_f(m, s)$. Hence, $-\ell$ is less or equal to the smallest integral root of f. $\square$

Remark 4.3. (i) When $m \in \mathcal{M} - f\mathcal{M}$, one can check easily that $s+1$ is a factor of $b_f(m,s)$ [compare with the classical case: $\mathcal{M} = \mathcal{O}$, $m = 1$]. In that case, it is necessary[1] to take $\ell \in \mathbf{N} \cap \mathbf{C}^*$ in order to have the four equivalent conditions satisfied.
(ii) Up to notational changes, the proposition is still true for any $\ell \in \mathbf{C}$ [consider the roots of the form $-\ell - k$ with $k \in \mathbf{N}$ in condition 1, and the $\mathcal{D}$-module $(\mathcal{D}m)[1/f]f^{-\ell}$ in the other conditions].

Obviously, Corollary 1.11 is easily obtained by iterating the equivalence between 1 and 3. Let us prove Theorem 1.10.

Proof of Theorem 1.10. If $b'_f(h,s)$ has no strictly negative integral root, then $\mathcal{D}\delta_{h,f} = \mathcal{R}_{h,f}$ (Lemma 1.9 and Proposition 4.2, using that $\mathcal{D}\delta_h = \mathcal{R}_h$), and we just have to remark that $\delta_{h,f}$ belongs to $\mathcal{L}_{h,f}$ when -1 is not a root of $b'_f(h,s)$. Indeed, by fixing $s=-1$ in the defining equation of $b'_f(h,s)$:

$$b'_f(h,s)\delta_h f^s \in \mathcal{D}[s](\mathcal{J}_{h,f}, f)\delta_h f^s$$

we get:

$$\delta_h f^{-1} \in \sum_{1 \le k_1 < \cdots < k_p \le n}^{n} \mathcal{D}m_{k_1,\ldots,k_p}(h,f)\delta_h f^{-1} + \mathcal{D}\delta_h \subset \mathcal{R}_h[1/f].$$

Thus $\delta_{h,f} \in \mathcal{R}_{h,f} \cong \mathcal{R}_h[1/f]/\mathcal{R}_h$ belongs to $\mathcal{L}_{h,f}$.

Now let us assume that $\mathcal{L}_{h,f} = \mathcal{R}_{h,f}$. As $\mathcal{L}_{h,f} \subset \mathcal{D}\delta_{h,f}$, we also have $\mathcal{D}\delta_{h,f} = \mathcal{R}_{h,f}$ *i.e.* -1 is the smallest integral root of $b_f(\delta_h, s)$ (Proposition 4.2, using the asumption $\mathcal{D}\delta_h = \mathcal{R}_h$). So let us prove that -1 is not a root of $b'_f(h,s)$, following the formulation of [28] Lemma 1.3. Since $\delta_{h,f} \in \mathcal{L}_{h,f} = \sum_{1 \le k_1 < \cdots < k_p \le n} \mathcal{D}m_{k_1,\ldots,k_p}(h,f)\delta_{h,f}$, we have: $1 \in \mathcal{D}\mathcal{J}_{h,f} + \operatorname{Ann}_{\mathcal{D}} \delta_{h,f}$, or equivalently: $1 \in \mathcal{D}(\mathcal{J}_{h,f}, f) + \operatorname{Ann}_{\mathcal{D}} \delta_h f^{-1}$ (using that $\mathcal{D}f(\delta_h f^{-1}) = \mathcal{R}_h$). Moreover, as -1 is the smallest integral root of $b_f(\delta_h, s)$, an operator P belongs to $\operatorname{Ann}_{\mathcal{D}} \delta_h \otimes 1/f$ if and only if there exists $Q(s) \in \mathcal{D}[s]$ such that $P - (s+1)Q(s) \in \operatorname{Ann}_{\mathcal{D}[s]} \delta_h f^s$ (Proposition 4.2). Thus we have:

$$\mathcal{D}[s] = \mathcal{D}[s](s+1, \mathcal{J}_{h,f}, f) + \operatorname{Ann}_{\mathcal{D}[s]} \delta_h f^s.$$

In particular, if $(s+1)$ was a factor of $b'_f(h,s)$, we would have:

$$\frac{b'_f(h,s)}{s+1} \in \mathcal{D}[s](b'_f(h,s), \mathcal{J}_{h,f}, f) + \operatorname{Ann}_{\mathcal{D}[s]} \delta_h f^s$$

[1] Of course, this is not enough in general.

But from the identity (1), we have:

$$b'_f(h,s) \in \mathcal{D}[s](\mathcal{J}_{h,f}, f) + \mathrm{Ann}_{\mathcal{D}[s]}\,\delta_h f^s$$

and this is a defining equation of $b'_f(h,s)$. Thus:

$$\frac{b'_f(h,s)}{s+1} \in \mathcal{D}[s](\mathcal{J}_{h,f}, f) + \mathrm{Ann}_{\mathcal{D}[s]}\,\delta_h f^s$$

In particular, $b'_f(h,s)$ divides $b'_f(h,s)/(s+1)$, which is absurd. Therefore -1 is not a root of $b'_f(h,s)$, and this ends the proof. □

Proof of the equivalence between 1 *and* 2 *in Theorem* 1.2. Up to notational changes, the proof is the very same than the previous one. Assume that $\tilde{b}_h(s)$ has no integral root. On one hand, $\mathcal{D}\delta_h$ coincides with $\mathcal{R}_h = \mathcal{O}[1/h]/\mathcal{O}$ by Proposition 4.2 [take $m = 1$ and $\mathcal{M} = \mathcal{O}$]. On the other hand, by fixing $s = -1$ in (2), we get

$$\frac{1}{h} \in \sum_{i=1}^{n} \mathcal{D}\frac{h'_{x_i}}{h} + \mathcal{O} \subset \mathcal{O}[1/h]$$

and $\delta_h \in \mathcal{L}_h$. Hence $\mathcal{L}_h = \mathcal{R}_h$.

Now let us assume that $\mathcal{L}_h = \mathcal{R}_h$. As $\mathcal{L}_h \subset \mathcal{D}\delta_h \subset \mathcal{R}_h$, δ_h generates $\mathcal{R}_h$. In particular, -1 is the only integral root of $b_h(s)$ by using Proposition 4.2 (since the roots of $b_h(s)$ are strictly negative). By the same arguments as in the proof of Theorem 1.10, one can prove that -1 is not a root of $\tilde{b}_h(s)$. Thus $\tilde{b}_h(s)$ has no integral root, as it was expected □

Remark 4.4. Under the assumption $\mathcal{D}\delta_{h,f} = \mathcal{R}_{h,f}$, we show in the proof of Theorem 1.10 that if $\delta_{h,f}$ belongs to $\mathcal{L}_{h,f}$ then -1 is not a root of $b'_f(h,s)$. As the reverse relation is obvious, a natural question is to know if this assumption is necessary. In terms of reduced Bernstein polynomial, does the condition: -1 *is not a root of* $\tilde{b}_h(s)$ characterize the membership of δ_h in $\mathcal{L}_h$?

§5. Some Remarks

Let us point out some facts about Theorem 1.10:

- The assumption $\mathcal{D}\delta_h = \mathcal{R}_h$ is necessary. This appears clearly in the following examples.

Example 5.1. Let $p = 1$ and $h = x_1^2 + \cdots + x_4^2$. As $b_h(s) = (s+1)(s+2)$, we have $\mathcal{D}\delta_h \neq \mathcal{R}_h$ (Proposition 4.2). If $f_1 = x_1$, then $b'_{f_1}(h,s) = (s+2)$

by using Proposition 3.1 where as $\mathcal{L}_{h,f_1} = \mathcal{R}_{h,f_1}$ (Example 1.3, or because $\mathcal{D}\delta_{f_1} = \mathcal{R}_{f_1}$ and $b'_{f_1}(h,s) = (s+3/2)$).

Now if we take $f_2 = x_5$, we have $\mathcal{L}_{h,f_2} \neq \mathcal{R}_{h,f_2}$ and $b'_{f_2}(h,s) = 1$ since:

$$\frac{\dot{2}}{x_1^2 + \cdots + x_4^2} x_5^s = \left[\sum_{i=1}^{4} \frac{\partial}{\partial x_i} x_i \right] \cdot \frac{\dot{1}}{x_1^2 + \cdots + x_4^2} x_5^s \; .$$

- If $p = 1$, this condition $\mathcal{D}\delta_h = \mathcal{R}_h$ just means that the only integral root of $b_h(s)$ is -1 (Proposition 4.2).

- The condition $\mathcal{L}_h = \mathcal{R}_h$ clearly implies $\mathcal{D}\delta_h = \mathcal{R}_h$, but it is not necessary; see Example 1.7 for instance. An other example with $p = 1$ is given by $h = x_1x_2(x_1+x_2)(x_1+x_2x_3)$ since $b_h(s) = (s+5/4)(s+1/2)(s+3/4)(s+1)^3$.

- Contrarly to the classical Bernstein polynomial, it may happen that an integral root of $b'_f(h,s)$ is positive or zero (see Example 3.2, with $f = x_1$, $h = (x_1^2 + \cdots + x_4^2)^\ell$ and $\ell \geq 2$). In particular, 1 is an eigenvalue of the monodromy acting on $\phi_f \mathbf{C}_{h^{-1}\{0\}}$. For that reason, we do not have here the analogue of condition 3, Theorem 1.2.

- In [7], the authors introduce a notion of Bernstein polynomial for an arbitrary variety Z. In the case of hypersufaces, this polynomial $b(s)$ coincides with the classical Bernstein-Sato polynomial. But it does not seem to us that its integral roots are linked to the condition $\mathcal{L}_{h,f} = \mathcal{R}_{h,f}$. For instance, if $h = x_1^2 + x_2^2 + x_3^2$ and $f = x_4^2 + x_5^2 + x_6^2$ then one can check that $b'_f(h,s) = \tilde{b}(f^s,s) = (s+3/2)$; in particular $\mathcal{L}_{h,f} = \mathcal{R}_{h,f}$. Meanwhile, by using [7], Theorem 5, we get $b(s) = (s+3)(s+5/2)(s+2)$ if $Z = V(h,f) \subset \mathbf{C}^6$.

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