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The Essential Spectrum of the Laplacian on Manifolds with Ends

By

Toshiaki Hattori*

Abstract

Let V be a noncompact complete Riemannian manifold. We find a geometric condition which assures that the essential spectrum of the Laplacian on V contains a half-line, by means of fiber bundle structures and the asymptotic behavior of mean curvatures on the ends of V, and give lower bounds of the essential spectrum. Our criteria can be applied to locally symmetric spaces of finite volume and manifolds of infinite volume canonically obtained from manifolds with corners.

Introduction

Let V be an n-dimensional complete Riemannian manifold. When V is noncompact, it is possible that the essential spectrum of the Laplacian on V is nonempty. For example, it is well-known that in the case of n-dimensional complete hyperbolic manifolds of finite volume, the essential spectrum is the half-line $[(n-1)^2/4, \infty)$. In the case of locally symmetric spaces of finite volume, the essential spectrum is known to be a half-line. In this paper we find a geometric condition which assures that the essential spectrum contains a half-line, by means of fiber bundle structures and the asymptotic behavior of mean curvatures on the ends of V, and give lower bounds of the essential spectrum under some additional condition. It is also our hope to understand the case of locally symmetric spaces of finite volume from a different point of view from Langlands' theory of Eisenstein series ([20], [25]), and investigate manifolds of infinite volume canonically obtained from manifolds with corners.

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*Department of Mathematics, Tokyo Institute of Technology, Oh-okayama, Meguro, Tokyo 152-8551, Japan.
e-mail: hattori@math.titech.ac.jp

For studying the essential spectrum of V, there is a known method, referred to as the decomposition principle ([10], [15]): The essential spectrum is stable under compact perturbations. Thus this spectrum does not change when one modifies the manifold in a compact region. Hence, we are motivated by the following question: What kind of geometric or metric structure of the ends produces a half-line in the essential spectrum ?

First we consider a simple example. Let M be a compact manifold with a connected boundary ∂M. Then we can attach a half-cylinder $\partial M \times [0, \infty)$ to the boundary to produce a complete Riemannian manifold V. If the half-cylinder is equipped with the product metric $g + dt^2$, where g is a metric on ∂M and dt^2 is the standard metric on $[0, \infty)$, the essential spectrum equals $[0, \infty)$. When ∂M is the flat torus and the half-cylinder is equipped with a warped product metric $e^{-2t} g + dt^2$, the essential spectrum equals $[(n-1)^2/4, \infty)$. This corresponds to the case of n-dimensional hyperbolic manifolds of finite volume. Therefore we next consider metrics which are not necessarily warped products.

Let $Y_1, \dots, Y_s$ be closed manifolds. Let $\phi_j : Y_j \times [0, \infty) \longrightarrow V$ be an embedding and $\mathcal{E}_j = \phi_j(Y_j \times (0, \infty))$ for each $j = 1, \dots, s$. Suppose that $V - \bigcup_{j=1}^{s} \mathcal{E}_j$ is a compact submanifold with boundary. In particular, V is a manifold with s ends. We suppose that the induced metric on $Y_j \times [0, \infty)$ through ϕ_j is of the form $g_{j,t} + dt^2$, where $g_{j,t}$ is a metric on Y_j depending on $t \geq 0$. Let $\sqrt{g_{j,t}}(y)$ be the square root of the determinant of the metric tensor of $g_{j,t}$ at $y \in Y_j$. We also suppose that

$$\text{the ratio } \alpha_j(t) := \sqrt{g_{j,t}}(y) / \sqrt{g_{j,0}}(y) \text{ does not depend on } y \qquad (\star)$$

on each $Y_j \times [0, \infty)$. In this paper, we call such a manifold V a *Riemannian manifold with boundaries Y_j at infinity.* This class of manifolds contains $\mathbb{R}$-rank 1 or $\mathbb{Q}$-rank 1 locally symmetric spaces of finite volume. However, we do not necessarily assume that V is nonpositively curved or of finite volume. If Y_j has a structure of some special fiber bundle, it might give rise to some additional structure on the essential spectrum. Because of this and the fact that the end of any higher $\mathbb{Q}$-rank locally symmetric space of finite volume is more complicated, we also consider the following situation.

Let B be a manifold (without boundary) and g_B a complete Riemannian metric on B. Let $\pi : Y \longrightarrow B$ be a fiber bundle with compact fibers. We suppose that the dimension of Y is $n - d$ with $d \geq 1$. Let $\mathcal{C} \subset \mathbb{R}^d$ be an open (infinite) cone, that is, a region enclosed by d hyperplanes, and let $d\mathbf{t}^2$ be the standard metric of $\mathbb{R}^d$. We suppose that there exists an open embedding $\phi : Y \times \mathcal{C} \longrightarrow V$ and that the induced metric on $Y \times \mathcal{C}$ is of the form $g_{\mathbf{t}} + d\mathbf{t}^2$.

Here $\{g_{\mathbf{t}}\}_{\mathbf{t}\in\mathcal{C}}$ is a family of complete Riemannian metrics on Y of the form

$$g_{\mathbf{t}}(y) = g_{\mathbf{t}}^{\perp}(y) + (\pi^* g_B)(y), \tag{$\star\star$}$$

where $g_{\mathbf{t}}^{\perp}(y)$ is a metric on the tangent space E_y of the fiber $\pi^{-1}(\pi(y))$ at y depending on $\mathbf{t}$, and $(\pi^* g_B)(y)$ is the metric on the orthogonal complement of E_y in the tangent space $T_y(Y)$ of Y at y. In particular, this means that $\pi : (Y, g_{\mathbf{t}}) \longrightarrow (B, g_B)$ is a Riemannian submersion. Let $l : [0,\infty) \longrightarrow \mathbb{R}^d$ be a ray contained in the interior of $\mathcal{C}$. In the case $d \geq 2$ we suppose that l is not parallel to any of the boundary hyperplanes of $\mathcal{C}$. We denote by $\mathcal{C}_t$ the hyperplane in $\mathcal{C}$ through $l(t)$ orthogonal to l. Let $(t, t_2, \ldots, t_d)$ be a Cartesian coordinate system of $\mathbb{R}^d$ such that the origin $\mathbf{0}$ is $l(0)$ and that the positive part of the t-axis corresponds to the ray l.

Let $\sqrt{g_{\mathbf{t}}}(y)$ be the square root of the determinant of the metric tensor of $g_{\mathbf{t}}$ at $y \in Y$. We consider the following two conditions.

$$\text{The ratio } \sqrt{g_{\mathbf{t}}}(y)/\sqrt{g_{\mathbf{0}}}(y) \text{ depends only on } t. \tag{$*$}$$

$$\text{The projection } \pi \text{ is harmonic with respect to the metric } g_{\mathbf{t}}. \tag{$**$}$$

These two conditions $(*)$, $(**)$ are satisfied in the case of locally symmetric spaces of finite volume. We explain the higher $\mathbb{Q}$-rank case in later sections. In this paper we suppose that the condition $(*)$ is always satisfied unless otherwise mentioned, and denote the ratio by $\alpha(t)$:

$$\alpha(t) = \sqrt{g_{\mathbf{t}}}(y)/\sqrt{g_{\mathbf{0}}}(y).$$

We do not necessarily assume the condition $(**)$.

Let

$$\beta(t) = \frac{1}{2}\log\alpha(t).$$

As we show in Section 1, the mean curvature of $Y \times \mathcal{C}_t$ in $Y \times \mathcal{C}$ depends not on $y \in Y$ but on t, and is equal to

$$-\frac{1}{n-1}\frac{\alpha'(t)}{\alpha(t)} = -\frac{2}{n-1}\beta'(t).$$

We denote this by $\mathcal{K}(t)$. Let $\overline{\Delta}$ be the unique self-adjoint extension ([8], [13]) of the Laplacian Δ on V to the Hilbert space $L^2(V)$ of square integrable (complex-valued) functions on V, and $\overline{\Delta}_B$ the similar extension of the Laplacian Δ_B on B.

Theorem 1. *Suppose that*

$$\lim_{t\to\infty}\left\{\mathcal{K}(t)^2-\frac{2}{n-1}\mathcal{K}'(t)\right\}=\kappa^2$$

exists.

(1) *If B is compact, then for any $r\geq 0$, there exists a family $\{u_{r,\varepsilon}\}_{\varepsilon>0}$ of compactly supported smooth functions on $\mathcal{E}:=\phi(Y\times\mathcal{C})$ satisfying the following two conditions.*

(0.1) *For any compact subset of $\mathcal{E}$, if we take ε sufficiently small, then the support of $u_{r,\varepsilon}$ lies outside this compact set.*

(0.2) *For some positive constant C_1 independent of ε, we have*

$$\left\|\left(\Delta-\left(\frac{(n-1)^2\kappa^2}{4}+r^2\right)\right)u_{r,\varepsilon}\right\|\leq C_1\varepsilon\|u_{r,\varepsilon}\|,$$

where $\| \ \|$ is the L^2-norm on $L^2(V)$.

In particular, every point of $[(n-1)^2\kappa^2/4,\infty)$ belongs to the spectrum of $\overline{\Delta}$.

(2) *Suppose that the condition $(**)$ is satisfied. If $\overline{\Delta}_B$ has a sequence*

$$c_0<c_1<\cdots<c_m<\cdots$$

of eigenvalues, then the following holds.

For each m and any $r\geq 0$, there exists a family $\{u_{m,r,\varepsilon}\}_{\varepsilon>0}$ of compactly supported smooth functions on $\mathcal{E}=\phi(Y\times\mathcal{C})$ satisfying the following three conditions.

(0.3) *For any compact subset of $\mathcal{E}$, if we take ε sufficiently small, then the support of $u_{m,r,\varepsilon}$ lies outside this compact set.*

(0.4) *For some positive constant C_1 independent of ε, we have*

$$\left\|\left(\Delta-\left(\frac{(n-1)^2\kappa^2}{4}+c_m+r^2\right)\right)u_{m,r,\varepsilon}\right\|\leq C_1\varepsilon\|u_{m,r,\varepsilon}\|.$$

(0.5) *If $m\neq m'$, then*

$$\lim_{\substack{\varepsilon\longrightarrow 0\\ \varepsilon'\longrightarrow 0}}(u_{m,r,\varepsilon},\,u_{m',r',\varepsilon'})=0,$$

where $(\ ,\ )$ is the L^2-inner product on $L^2(V)$.

In particular, every point of $[(n-1)^2\kappa^2/4+c_0,\infty)$ belongs to the spectrum of $\overline{\Delta}$.

Remark. In the case (2), if a non-negative number c belongs to the essential spectrum of $\overline{\Delta}_B$, then we can construct for any $r \geq 0$ a family $\{u_{r,\varepsilon}\}_{\varepsilon>0}$ of compactly supported smooth functions on $\mathcal{E}$ such that the same condition as (0.3) and the inequality

$$\left\| \left(\Delta - \left(\frac{(n-1)^2\kappa^2}{4} + c + r^2 \right) \right) u_{r,\varepsilon} \right\| \leq C_1 \varepsilon \|u_{r,\varepsilon}\|$$

are satisfied (cf. Proposition 1.2).

Remark. Although we followed the decomposition principle to obtain the statement of Theorem 1, we do not need to use it directly in the proof.

In order to give lower bounds of the essential spectrum, we can use the following known result on Rayleigh quotients. Since there seem to be no suitable references, we also include its proof in Section 2 for convenience.

Lemma 1. *Let N be an open subset of a complete Riemannian manifold. Let Z be a C^1-vector field on N such that*
(0.6) *its norm $|Z|$ is bounded from above by some positive constant C: $|Z| \leq C$, and that*
(0.7) *the divergence of Z is bounded away from zero: that is,* $\operatorname{div} Z \geq \varepsilon > 0$ *for some ε.*

Then we have

$$\inf \frac{\int_N |\operatorname{grad} u|^2 \, d\mu_N}{\int_N |u|^2 \, d\mu_N} \geq \left(\frac{\varepsilon}{2C} \right)^2 \qquad \text{on } N,$$

where μ_N is the canonical measure on N induced from the Riemannian metric and u runs through all the compactly supported smooth functions on N.

We first restrict ourselves to the case of Riemannian manifolds with boundaries Y_j at infinity. We denote by $\mathcal{K}_j(t)$ the mean curvature of $Y_j \times \{t\}$ in $Y_j \times (0, \infty)$. Applying Lemma 1 to the vector field $\frac{\partial}{\partial t}$ or $-\frac{\partial}{\partial t}$ on each end, we obtain the following immediately from the decomposition principle.

Proposition 1. *Let V be a Riemannian manifold with boundaries Y_j at infinity. Suppose that $\lim_{t\to\infty} \mathcal{K}_j(t) = \kappa_j$ exists for each j. Let $\kappa^2 = \min_j(\kappa_j)^2$. Then the essential spectrum of $\overline{\Delta}$ is contained in the interval $[(n-1)^2\kappa^2/4, \infty)$.*

We show the following theorem of different type.

Theorem 2. *Let V be a Riemannian manifold with boundaries Y_j at infinity. Suppose that there exists a positive number a such that*

$$\mathcal{K}_j(t)^2 - \frac{2}{n-1}\mathcal{K}_j'(t) \geq a^2 \qquad \text{on } (0, \infty) \tag{0.8}$$

for each j. We also suppose that for each j, there exist $C_2(j)$, $C_3(j)$ such that

$$0 < C_2(j) < C_3(j), \tag{0.9}$$

$$\mathcal{K}_j(t) > 0 \text{ on } [C_2(j), C_3(j)). \tag{0.10}$$

The numbers $C_2(j)$, $C_3(j)$ may depend on j. Then, on the interval $[0, (n-1)^2a^2/4)$ the spectrum of $\overline{\Delta}$ consists of at most a finite number of eigenvalues of finite multiplicity. If $(n-1)^2a^2/4$ is an eigenvalue, then its multiplicity is finite.

These results give alternative proofs of some of the known results on rank 1 locally symmetric spaces. Let G be a connected semisimple Lie group having finite center and no compact factors. Let K be a maximal compact subgroup of G and $X = G/K$ the associated symmetric space of noncompact type with the canonical left invariant metric. Let Γ be a torsion-free irreducible non-uniform lattice of G and $V = \Gamma\backslash X$. Let us call V an $\mathbb{R}$-rank k locally symmetric space of finite volume if X is a rank k symmetric space. We call V a $\mathbb{Q}$-rank k locally symmetric space of finite volume if G has trivial center and is the identity component of the group of real points of some connected semisimple linear algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$ of $\mathbb{Q}$-rank k and if Γ is an arithmetic subgroup of $\mathbf{G}$. Let ${}_{\mathbb{R}}\rho$ (resp. ρ) be the half sum of the positive roots (resp. $\mathbb{Q}$-roots). Then any $\mathbb{R}$-rank 1 (resp. $\mathbb{Q}$-rank 1) locally symmetric space of finite volume is a Riemannian manifold with boundaries Y_j at infinity with $\mathcal{K}_j(t) = 2|{}_{\mathbb{R}}\rho|/(n-1)$ (resp. $2|\rho|/(n-1)$) ([14], [12], [4], [26], [5], see also [9], [23]). Hence we have

Theorem 3 (cf. [9], [23], [6]). *Let V be an $\mathbb{R}$-rank 1 (resp. a $\mathbb{Q}$-rank 1) locally symmetric space of finite volume. Then the essential spectrum of $\overline{\Delta}$ is the half-line $\big[|{}_{\mathbb{R}}\rho|^2, \infty\big)$ (resp. $\big[|\rho|^2, \infty\big)$). If $|{}_{\mathbb{R}}\rho|^2$ (resp. $|\rho|^2$) is an eigenvalue, then its multiplicity is finite.*

Remark. Let V be a $\mathbb{Q}$-rank 1, $\mathbb{R}$-rank ≥ 2 locally symmetric space of finite volume. Then each boundary Y_j at infinity admits a fiber bundle structure $\pi_j : Y_j \longrightarrow B_j$ satisfying the conditions $(\star\star)$, $(**)$ (see *Remark* before Theorem 4 in Section 3). Let $\overline{\Delta}_{B_j}$ be the unique self-adjoint extension of the Laplacian on the base space B_j of Y_j. Then it follows from Theorem 1 that for each eigenvalue c of $\overline{\Delta}_{B_j}$, there exists a family of compactly supported smooth functions on V which assures that the half-line $[|\rho|^2 + c, \infty)$ is contained in the essential spectrum of $\overline{\Delta}$.

In the case where V is a higher $\mathbb{Q}$-rank locally symmetric spaces of finite volume, there is a compactification $\overline{V}$ of V ([7]) such that $\overline{V}$ is a manifold with corners and its boundary $\partial\overline{V}$ is connected. Each stratum Y_j of $\partial\overline{V}$ admits a fiber bundle structure $\pi_j : V_j \longrightarrow B_j$ satisfying the conditions ($\star\star$), ($**$). The continuous spectrum of V is controlled by the eigenvalues of the base spaces B_j: For each eigenspace of B_j with eigenvalue c, there exists a certain subspace of $L^2(V)$ corresponding to the continuous spectrum. The space $L^2(V)$ is the closure of the union of such subspaces and the eigenspaces corresponding to the point spectrum of V. This follows from Langlands' spectral resolution of the regular representation of G on $L^2(\Gamma\backslash G)$ ([20], [25]). Although the situation is slightly different from the one in Theorem 1, the argument in the proof of Theorem 1 can be also applied to this case and we obtain a similar result (Theorem 4) to Theorem 3 by constructing a vector field as in Lemma 1. Thus we can give alternative proofs of some of the above facts without using Langlands' theory of Eisenstein series. We postpone describing Theorem 4 until Section 3, since we need more notations.

Theorem 1 can be applied to complete manifolds canonically obtained from manifolds with corners as in Figure 7 (see Section 5 for the precise definition).

Corollary 1. *Let W be a manifold with compact corners and V the complete manifold obtained from W by gluing cylinders successively to boundary components. Then the essential spectrum of V is the half-line* $[0,\infty)$.

Our calculations in the proof of Theorems 1 and 2 are based on higher dimensional generalizations of Lax-Phillips' ones ([21, §4]). In order to construct the sequences $\{u_{m,r,\varepsilon}\}_{\varepsilon>0}$ in Theorem 1, under the identification of $\mathcal{E}$ with $Y \times \mathcal{C}$, we first consider the function $f_0(y, \mathbf{t}) = \alpha(t)$ on $\mathcal{E}$ and multiply f_0 by an oscillation to obtain f_r. We take a product of f_r with an eigenfunction φ of B with eigenvalue c_m and control the support of this function by using a suitable cut-off function h (see Figure 1). In particular, in the case of locally symmetric spaces of finite volume the function f_0 is induced from the exponential of a constant multiple of the Busemann function on X with respect to some geodesic $\gamma : [0,\infty) \longrightarrow X$ which is projected on l.

This paper is organized as follows. In Section 1 we first show the explicit relation between the functions $\alpha(t)$, $\beta(t)$, and the mean curvature $\mathcal{K}(t)$. Then we prove Theorem 1. In Section 2 we prove Theorems 2. In Sections 3 and 4 we explain the case of higher $\mathbb{Q}$-rank locally symmetric spaces. In the last section we discuss some consequences of our theorems including Corollary 1.

As usual we denote by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{N}$ the set of the complex numbers, the real numbers, the rational numbers, the natural numbers, respectively.

§1. The Existence of a Half-Line in the Essential Spectrum

In this section we prove Theorem 1. We identify $\mathcal{E}$ with $Y \times \mathcal{C}$ by the diffeomorphism ϕ. First we show that the mean curvature of $Y \times \mathcal{C}_t$ in $Y \times \mathcal{C}$ depends only on t.

Let $\mathbf{t} = (t, t_2, \ldots, t_d)$ be a Cartesian coordinate system of $\mathbb{R}^d$ such that the positive part of the t-axis corresponds to the ray l. We define a unit vector field ξ on $Y \times \mathcal{C}$ by $\xi = \frac{\partial}{\partial t}$, which is orthogonal to $Y \times \mathcal{C}_t$ for each $t \geq 0$. We denote by $T_P(Y \times \mathcal{C}_t)$ the tangent space of $Y \times \mathcal{C}_t$ at $P \in Y \times \mathcal{C}_t$.

Definition 1.1. For each point P of $Y \times \mathcal{C}_t$, let $A_\xi : T_P(Y \times \mathcal{C}_t) \longrightarrow T_P(Y \times \mathcal{C}_t)$ be the shape operator with respect to ξ defined by

$$A_\xi(X) = -\nabla_X \xi \qquad \text{for each } X \in T_P(Y \times \mathcal{C}_t),$$

where ∇ is the covariant derivative on V. We define the mean curvature $\mathcal{K}(P)$ of $Y \times \mathcal{C}_t$ at P (with respect to ξ) by

$$\mathcal{K}(P) = \frac{1}{n-1} \operatorname{trace}(A_\xi).$$

Proposition 1.1. *We have*

$$\mathcal{K}(P) = -\frac{1}{n-1} \frac{\alpha'(t)}{\alpha(t)} = -\frac{2}{n-1} \beta'(t)$$

for all $P \in Y \times \mathcal{C}_t$.

Proof. Let $P = (y^*, (t^*, {t_2}^*, \ldots, {t_d}^*)) \in Y \times \mathcal{C}_{t^*}$. Take an arbitrary coordinate neighborhood $(U, (y_1, \ldots, y_{n-d}))$ of y^* in Y. We put $y_{n-d+j} = t_{j+1}$ for $j = 1, \ldots, d-1$, and $y_n = t$ to obtain a local coordinate system $(y_1, \ldots, y_n)$ around P in $Y \times \mathcal{C}_t$. Let g_{jk} be the components of the metric tensor with respect to this local coordinate system, $(g^{jk}) = (g_{jk})^{-1}$, and Γ^m_{jk} the Christoffel symbols. We have

$$\begin{aligned}(n-1)\mathcal{K}(P) &= -\sum_{j=1}^{n-1} \Gamma^j_{jn} = -\frac{1}{2} \sum_{j=1}^{n-1} \sum_{m=1}^{n} g^{jm} \left(\frac{\partial g_{nm}}{\partial y_j} + \frac{\partial g_{jm}}{\partial y_n} - \frac{\partial g_{jn}}{\partial y_m} \right) \\ &= -\frac{1}{2} \sum_{j=1}^{n-1} \sum_{m=1}^{n-1} g^{jm} \left(\frac{\partial g_{jm}}{\partial t} \right).\end{aligned}$$

Let $A = (g_{jk})_{1 \leq j,k \leq n-1}$. Then we have

$$(n-1)\mathcal{K}(P) = -\frac{1}{2} \operatorname{trace}\left(A^{-1} \frac{\partial A}{\partial t} \right) = -\frac{1}{2} \frac{\frac{\partial}{\partial t}(\det A)}{\det A} = -\frac{1}{2} \frac{\partial}{\partial t}(\log(\det A)).$$

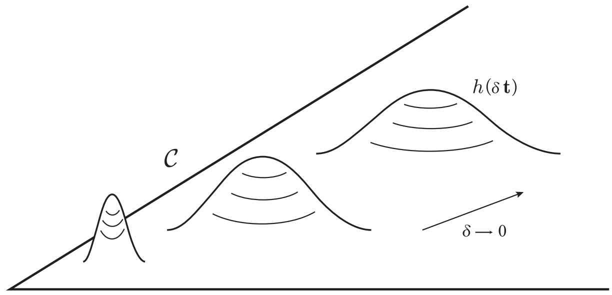


Figure 1.

Since the determinant of A depends only on t and is equal to $(\sqrt{g_0})^2\alpha(t)^2$, we obtain

$$(n-1)\mathcal{K}(P) = -(\log\alpha)'(t^*) = -2\beta'(t^*). \qquad \square$$

The mean curvature $\mathcal{K}(P)$ is constant on each $Y \times \mathcal{C}_t$, which we denote by $\mathcal{K}(t)$.

Let $C_0^\infty(Y \times \mathcal{C})$ be the set of compactly supported smooth functions on $Y \times \mathcal{C}$. We define a map $T : C_0^\infty(Y \times \mathcal{C}) \longrightarrow C_0^\infty(Y \times \mathcal{C})$ by

$$(T(f))(y, \mathbf{t}) = f(y, \mathbf{t})/\sqrt{\alpha(t)}$$

for $f \in C_0^\infty(Y \times \mathcal{C})$. Then we have

$$T^{-1} \circ \Delta \circ T = -\frac{\partial^2}{\partial t^2} + \frac{(n-1)^2}{4}\left\{\mathcal{K}(t)^2 - \frac{2}{n-1}\mathcal{K}'(t)\right\} - \sum_{j=2}^{d}\frac{\partial^2}{\partial t_j^2} + \Delta_{(Y,g_{\mathbf{t}})} \quad (1.1)$$

on $C_0^\infty(Y \times \mathcal{C})$, where $\Delta_{(Y,g_{\mathbf{t}})}$ is the Laplacian on $(Y, g_{\mathbf{t}})$. Let $(\tau_{\mathbf{t}})_y$ be the trace of the second fundamental form of the submanifold $\pi^{-1}(\pi(y))$ of $(Y, g_{\mathbf{t}})$ at $y \in Y$, and $C_0^\infty(B)$ the set of compactly supported smooth functions on B. From the formula on Riemannian submersion in Theorem 4.4 of [19, XIV, §4], we have

$$\Delta_{(Y,g_{\mathbf{t}})}(\varphi \circ \pi)(y) = (\Delta_B\varphi)(\pi(y)) + (\tau_{\mathbf{t}})_y \cdot (\varphi \circ \pi) \quad \text{for all } \varphi \in C_0^\infty(B). \quad (1.2)$$

Let f_0 be the function on $Y \times \mathcal{C}$ defined by

$$f_0(y, \mathbf{t}) = e^{-\beta(t)} = \frac{1}{\sqrt{\alpha(t)}}.$$

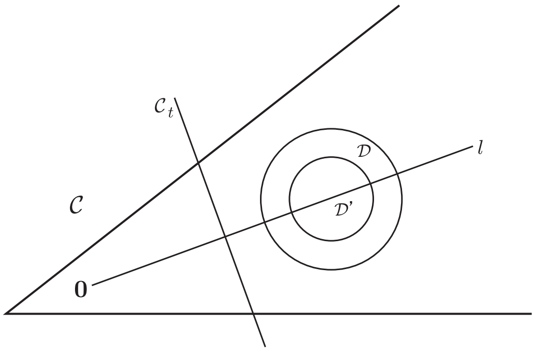


Figure 2.

For each $r \geq 0$, we define a function f_r by

$$f_r(y, \mathbf{t}) = e^{-\beta(t)+\sqrt{-1}\cdot rt}.$$

For $\varphi \in C_0^\infty(B)$, we put

$$f_{\varphi,r}(y, \mathbf{t}) = e^{-\beta(t)+\sqrt{-1}\cdot rt}\varphi(\pi(y)).$$

Let $\mathcal{D}$ and $\mathcal{D}'$ be open balls in $\mathbb{R}^d$ with the same center on the ray l such that $\mathcal{D}' \subset \mathcal{D} \subset \mathcal{C}$ and that $\mathbf{0} \notin \mathcal{D}$, where $\mathbf{0}$ is the origin of $\mathbb{R}^d$. We choose $\mathcal{D}$ such that every ray emanating from the origin which is tangent to the sphere bounding $\mathcal{D}$ is entirely contained in $\mathcal{C}$. We also suppose that

$$\left(\bigcup_{0 \leq t < C_4} \mathcal{C}_t \right) \cap \mathcal{D} = \phi$$

for a positive number C_4 (see Figure 2). Let $h : \mathbb{R}^d \longrightarrow [0,1]$ be a smooth function such that $h > 0$ on $\mathcal{D}$, $h \equiv 0$ outside $\mathcal{D}$, and $h \equiv 1$ on the closure of $\mathcal{D}'$.

For any positive number $\delta \leq 1$, we define a smooth function $f_{\varphi,r,\delta}$ on $Y \times \mathcal{C}$ by

$$f_{\varphi,r,\delta}(y, \mathbf{t}) = h(\delta \mathbf{t}) e^{-\beta(t)+\sqrt{-1}\cdot rt}\varphi(\pi(y)). \tag{1.3}$$

Let $L^2(B)$ be the Hilbert space of square integrable functions on B with L^2-norm $\| \ \|_B$.

Proposition 1.2. *Suppose that* $(\tau_{\mathbf{t}})_y \cdot (\varphi \circ \pi) = 0$ *for any* $y \in Y$, $\mathbf{t} = (t, t_2, \ldots, t_d) \in \mathcal{C}$ *with* $t \geq 0$, *and that* $\lim_{t\to\infty} \left\{ \mathcal{K}(t)^2 - 2\mathcal{K}'(t)/(n-1) \right\} = \kappa^2$.

Then for any $\varepsilon > 0$, there exists a positive number $\delta(\varepsilon)$ such that the following holds.

Let δ be a positive number smaller than $\delta(\varepsilon)$, $c \geq 0$, and $\varphi \in C_0^\infty(B)$ with

$$\|(\Delta_B - c)\varphi\|_B < \varepsilon\|\varphi\|_B.$$

Then we have

$$\left\| \left(\Delta - \left(\frac{(n-1)^2\kappa^2}{4} + c + r^2 \right) \right) f_{\varphi,r,\delta} \right\| < C_5 \varepsilon \|f_{\varphi,r,\delta}\|, \tag{1.4}$$

where C_5 is a constant depending only on h, r, and φ.

We denote by μ, μ_Y, μ_B, $\mu_{\mathbf{t}}$ the Riemannian measure of V, $(Y, g_{\mathbf{0}})$, (B, g_B), $\mathcal{C}$, respectively, and $\mu_{Y,\mathbf{t}} = e^{2\beta(t)}\mu_Y$ the Riemannian measure of $(Y, g_{\mathbf{t}})$.

Proof. Let

$$\begin{aligned} A_1(y,\mathbf{t}) &= \left\{ (\beta'(t)^2 + \beta''(t)) - \frac{(n-1)^2\kappa^2}{4} \right\} f_{\varphi,r,\delta}(y,\mathbf{t}) \\ &\quad - \delta \left\{ 2\sqrt{-1}r \frac{\partial h}{\partial t}(\delta\mathbf{t}) + \delta \left(\frac{\partial^2 h}{\partial t^2} + \sum_{j=2}^{d} \frac{\partial^2 h}{\partial t_j^2} \right)(\delta\mathbf{t}) \right\} \varphi(\pi(y)) e^{-\beta(t)+\sqrt{-1}\cdot rt} \end{aligned}$$

and

$$A_2(y,\mathbf{t}) = (\Delta_B - c)\varphi(\pi(y)) h(\delta\mathbf{t}) e^{-\beta(t)+\sqrt{-1}\cdot rt}.$$

Then, it follows from (1.1), (1.2) that

$$\left(\Delta - \left(\frac{(n-1)^2\kappa^2}{4} + c + r^2 \right) \right) f_{\varphi,r,\delta}(y,\mathbf{t}) = A_1(y,\mathbf{t}) + A_2(y,\mathbf{t}). \tag{1.5}$$

We have

$$\begin{aligned} |A_1(y,\mathbf{t})| &\leq \left| \frac{(n-1)^2\kappa^2}{4} - \left\{ (\beta')^2 + \beta'' \right\} \right| |f_{\varphi,r,\delta}| \\ &\quad + \delta \left| 2\sqrt{-1}r \frac{\partial h}{\partial t}(\delta\mathbf{t}) + \delta \left(\frac{\partial^2 h}{\partial t^2} + \sum_{j=2}^{d} \frac{\partial^2 h}{\partial t_j^2} \right)(\delta\mathbf{t}) \right| |\varphi| e^{-\beta(t)}. \end{aligned}$$

Since the support of h is compact, there exists a positive number C_6 (which depends only on h and r) such that

$$\left| 2\sqrt{-1}r \frac{\partial h}{\partial t}(\delta\mathbf{t}) + \delta \left(\frac{\partial^2 h}{\partial t^2} + \sum_{j=2}^{d} \frac{\partial^2 h}{\partial t_j^2} \right)(\delta\mathbf{t}) \right| \leq C_6.$$

We also have

$$|f_{\varphi,r,\delta}| = |h||\varphi|e^{-\beta(t)} \le |\varphi|e^{-\beta(t)}.$$

Since the support of $h(\delta\mathbf{t})$ is contained in the closure of the ball $\frac{1}{\delta}\mathcal{D}$ and $\lim_{\delta\to 0}\frac{C_4}{\delta} = \infty$, the following holds.

For any $\varepsilon > 0,$ there exists a positive number $\delta(\varepsilon)$ smaller than ε such that

$$\left|\frac{(n-1)^2\kappa^2}{4} - \left\{(\beta')^2 + \beta''\right\}\right| < \varepsilon$$

on the support of $h(\delta\mathbf{t})$ for each $\delta < \delta(\varepsilon)$.

We have, for such δ,

$$|A_1(y,\mathbf{t})| \le (\delta C_6 + \varepsilon)|\varphi|e^{-\beta(t)}$$

and

$$\begin{aligned}\|A_1\|^2 &= \int_{Y\times\mathcal{C}} |A_1(y,\mathbf{t})|^2 d\mu = \int_{\mathcal{C}}\int_Y |A_1(y,\mathbf{t})|^2 d\mu_{Y,\mathbf{t}} d\mu_{\mathbf{t}} \\ &\le (\delta C_6+\varepsilon)^2 \int_{\mathcal{D}/\delta}\int_Y |\varphi|^2 e^{-2\beta(t)} d\mu_{Y,\mathbf{t}} d\mu_{\mathbf{t}} \\ &= (\delta C_6+\varepsilon)^2 \int_{\mathcal{D}/\delta}\int_Y |\varphi|^2 d\mu_Y d\mu_{\mathbf{t}} \le \varepsilon^2 (C_6+1)^2 \cdot \frac{1}{\delta^d}\mathrm{vol}(\mathcal{D}) \int_Y |\varphi|^2 d\mu_Y.\end{aligned}$$

Since the support of φ is compact, we can take positive numbers C_7, C_8 such that

$$C_7 \le \mathrm{vol}\ (\pi^{-1}(z)) \le C_8$$

for all z in the support of φ. Consequently, we obtain

$$\|A_1\|^2 \le C_8 \cdot \varepsilon^2 (C_6+1)^2 \frac{1}{\delta^d}\mathrm{vol}(\mathcal{D})\|\varphi\|_B^2. \tag{1.6}$$

Similarly, we have

$$|A_2(y,\mathbf{t})| \le |(\Delta_B - c)\varphi|e^{-\beta(t)}.$$

It follows that

$$\begin{aligned}\|A_2\|^2 &\le \int_{\mathcal{C}}\int_Y |(\Delta_B - c)\varphi|^2 e^{-2\beta(t)} d\mu_{Y,\mathbf{t}} d\mu_{\mathbf{t}} \\ &= \int_{\mathcal{D}/\delta}\int_Y |(\Delta_B - c)\varphi|^2 d\mu_Y d\mu_{\mathbf{t}} \le C_8 \cdot \frac{1}{\delta^d}\mathrm{vol}(\mathcal{D})\|(\Delta_B - c)\varphi\|_B^2.\end{aligned}$$

Thus we obtain

$$\|A_2\|^2 < C_8 \cdot \varepsilon^2 \cdot \frac{1}{\delta^d} \mathrm{vol}(\mathcal{D}) \|\varphi\|_B^2. \tag{1.7}$$

From (1.5), (1.6) and (1.7), we have

$$\left\| \left(\Delta - \left(\frac{(n-1)^2\kappa^2}{4} + c + r^2 \right) \right) f_{\varphi,r,\delta} \right\|^2 < C_8 \varepsilon^2 (C_6+2)^2 \frac{1}{\delta^d} \mathrm{vol}(\mathcal{D}) \|\varphi\|_B^2. \tag{1.8}$$

On the other hand, since

$$\begin{aligned} \|f_{\varphi,r,\delta}\|^2 &= \int_{Y\times\mathcal{C}} |f_{\varphi,r,\delta}|^2 d\mu = \int_{\mathcal{C}} \int_Y |f_{\varphi,r,\delta}|^2 d\mu_{Y,\mathbf{t}} d\mu_{\mathbf{t}} \\ &\geq \int_{\mathcal{D}'/\delta} \int_Y |f_{\varphi,r,\delta}|^2 d\mu_{Y,\mathbf{t}} d\mu_{\mathbf{t}} \\ &= \int_{\mathcal{D}'/\delta} \int_Y |\varphi|^2 e^{-2\beta(t)} d\mu_{Y,\mathbf{t}} d\mu_{\mathbf{t}} = \int_{\mathcal{D}'/\delta} \int_Y |\varphi|^2 d\mu_Y d\mu_{\mathbf{t}}, \end{aligned}$$

we have

$$\|f_{\varphi,r,\delta}\|^2 \geq C_7 \cdot \frac{1}{\delta^d} \mathrm{vol}(\mathcal{D}') \|\varphi\|_B^2. \tag{1.9}$$

From (1.8) and (1.9), we have

$$\left\| \left(\Delta - \left(\frac{(n-1)^2\kappa^2}{4} + c + r^2 \right) \right) f_{\varphi,r,\delta} \right\|^2 < (C_6+2)^2 \cdot \frac{C_7}{C_8} \cdot \frac{\mathrm{vol}(\mathcal{D})}{\mathrm{vol}(\mathcal{D}')} \cdot \varepsilon^2 \|f_{\varphi,r,\delta}\|^2.$$

Let

$$C_5 = (C_6+2)\sqrt{\frac{C_7 \,\mathrm{vol}(\mathcal{D})}{C_8 \,\mathrm{vol}(\mathcal{D}')}}.$$

Then we obtain the inequality (1.4). □

Proof of Theorem 1. We first prove (2). Let φ_* be an eigenfunction belonging to the eigenvalue c_m. From the definition of $\overline{\Delta}_B$, there exists a sequence $\{\varphi_i\}$ in $C_0^\infty(B)$ such that $\lim_{i\to\infty} \varphi_i = \varphi_*$ and $\lim_{i\to\infty} \Delta_B\varphi_i = \overline{\Delta}_B\varphi_*$ in $L^2(B)$. Then, by the triangle inequality, we can find for each $\varepsilon > 0$ a function $\varphi \in C_0^\infty(B)$ such that

$$\|(\Delta_B - c_m)\varphi\|_B < \varepsilon\|\varphi\|_B.$$

Since the condition $(*)$ is satisfied, the estimate in Proposition 1.2 is valid for any positive number $\delta < \delta(\varepsilon)$. We take one of such δ and put

$$u_{m,r,\varepsilon} = f_{\varphi,r,\delta}.$$

Choose the functions φ so that $\|\varphi_* - \varphi\|_B$ are sufficiently small. Then the resultant family $\{u_{m,r,\varepsilon}\}_{\varepsilon>0}$ of compactly supported smooth functions on $\mathcal{E}$ satisfies the conditions (0.3)–(0.5).

To prove (1), we take a constant function $\varphi \equiv 1$ and consider $f_{\varphi,r,\delta}$ in (1.3). Since $(\tau_{\mathbf{t}})_y \cdot (\varphi \circ \pi) = 0$ and $\Delta_B \varphi = 0$, the estimate in Proposition 1.2 is valid for $c = 0$. By repeating the same argument as above, the conclusion follows. □

Remark. In the case (2), if there is another eigenfunction $\overline{\varphi}_*$ belonging to the eigenvalue c_m such that φ_* and $\overline{\varphi}_*$ are mutually orthogonal with respect to the L^2-inner product on $L^2(B)$, then we can construct for any $r \geq 0$ a family $\{\overline{u}_{m,r,\varepsilon}\}_{\varepsilon>0}$ of compactly supported smooth functions on $\mathcal{E}$ satisfying the similar conditions as (0.2)–(0.5) and

$$\lim_{\substack{\varepsilon \longrightarrow 0 \\ \varepsilon' \longrightarrow 0}} (u_{m,r,\varepsilon}, \overline{u}_{m,r',\varepsilon'}) = 0.$$

Remark. If B is compact, we can take an eigenfunction of $\overline{\Delta}_B$ as φ in the above proof of (2).

§2. Lower Bounds of the Essential Spectrum

In this section we prove Theorem 2. It suffices to show the following.

There exists a finite dimensional subspace $\mathcal{V}$ of $L^2(V)$ such that

$$\left(\left(\overline{\Delta} - \frac{(n-1)^2 a^2}{4}\right) u,\, u\right) > 0 \text{ for all } u \in \mathcal{V}^\perp \cap \operatorname{dom}(\overline{\Delta}) - \{0\}\,, \tag{2.1}$$

where $\mathcal{V}^\perp$ is the orthogonal complement of $\mathcal{V}$ in $L^2(V)$ and $\operatorname{dom}(\overline{\Delta})$ is the domain of definition of $\overline{\Delta}$.

Let

$$\mathcal{E}_j^\wedge = \phi_j(Y_j \times (C_3(j), \infty)), \qquad \partial_j W = \phi_j(Y_j \times \{C_3(j)\})$$

for each $j = 1, \ldots, s$ and let

$$W = V - \bigcup_{j=1}^{s} \mathcal{E}_j^\wedge.$$

The boundary of the compact manifold W decomposes as

$$\partial W = \bigcup_{j=1}^{s} \partial_j W.$$

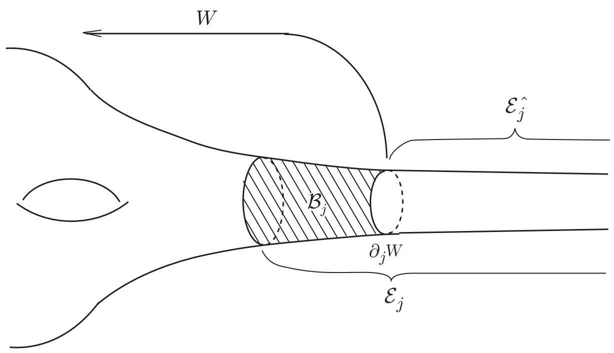


Figure 3.

We also put

$$\mathcal{B}_j = \phi_j(Y_j \times (C_2(j), C_3(j)]).$$

Then $\mathcal{B}_j$ is a collar neighborhood of $\partial_j W$ (see Figure 3). We relate the integral in (2.1) on the noncompact manifold V to some integrals on the compact manifold W.

For any vector field Z on V we denote by $|Z|(x)$ the norm of Z_x with respect to the Riemannian metric of V at the point $x \in V$. Let $L^2(W)$ be the space of all square integrable functions on W. For any $w_1, w_2 \in L^2(W)$, let

$$(w_1, w_2)_W = \int_W w_1 \cdot \overline{w}_2 d\mu$$

be the L^2-inner product. We denote by $\| \ \|_W$ the L^2-norm on $L^2(W)$:

$$\|w\|_W = \sqrt{\int_W |w|^2 d\mu}\ .$$

Let $C^{1,2}(W)$ be the space of smooth functions w on W such that $|\text{grad } w| \in L^2(W)$, where grad w is the gradient vector field of w. We denote by $\|\text{grad } w\|$ the L^2-norm of $|\text{grad } w|$. Let $L^{1,2}(W)$ be the completion of $C^{1,2}(V)$ with respect to the $L^{1,2}$-norm,

$$\|w\|_{1,2} = \sqrt{\|w\|_W^2 + \|\text{grad } w\|_W^2}\ .$$

For $u \in \text{dom}(\overline{\Delta})$, we only write $\|u\|_{1,2}$ to denote the $L^{1,2}$-norm of the restriction $u|_W$ of u to W: $\|u\|_{1,2} = \|u|_W\|_{1,2}$.

We show the following.

There exist positive constants C_9, C_{10} such that

$$C_{10}\|u\|_W^2 + \left(\left(\overline{\Delta} - \frac{(n-1)^2a^2}{4}\right)u,\, u\right) \geq C_9\|u\|_{1,2}^2 \text{ for all } u \in \mathrm{dom}(\overline{\Delta}). \tag{2.2}$$

Then the assertion (2.1) follows from (2.2): Let $\mathcal{H}$ be the image of the restriction map $\mathrm{dom}(\overline{\Delta}) \ni u \longmapsto u|_W \in L^2(W)$. It follows from Rellich-Kondrachov lemma (cf. [1, Théorème 10]) that for any positive number $\varepsilon < \sqrt{C_9/(2C_{10})}$ there exists a finite dimensional subspace $\mathcal{L}$ of $\mathcal{H}$ such that $\varepsilon\|w\|_{1,2} > \|w\|_W$ for any $w \neq 0$ in the orthogonal complement of $\mathcal{L}$ in $\mathcal{H}$ with respect to the inner product $(\ ,\ )_W$ on $L^2(W)$. It suffices to take the image of the natural inclusion $\mathcal{L} \hookrightarrow L^2(W)$ as $\mathcal{V}$.

In order to prove (2.2) it suffices to show the inequality for compactly supported smooth functions u on V. We remark that

$$\begin{aligned}\left(\left(\Delta - \frac{(n-1)^2a^2}{4}\right)u,\, u\right) &= \int_V \left(|\mathrm{grad}\ u|^2 - \frac{(n-1)^2a^2}{4}|u|^2\right)d\mu \\ = \int_W \left(|\mathrm{grad}\ u|^2 - \frac{(n-1)^2a^2}{4}|u|^2\right)d\mu & \\ + \sum_{j=1}^{s}\int_{\mathcal{E}_j^\wedge} &\left(|\mathrm{grad}\ u|^2 - \frac{(n-1)^2a^2}{4}|u|^2\right)d\mu\end{aligned} \tag{2.3}$$

for such u. Hence, we estimate the integrals in the last term of (2.3) as follows.

Lemma 2.1. *For any compactly supported smooth function u on V, we have*

$$\begin{aligned}&\int_{\mathcal{E}_j^\wedge}\left(|\mathrm{grad}\ u|^2 - \frac{(n-1)^2a^2}{4}|u|^2\right)d\mu \\ &\geq \beta_j'(C_3(j))\left(\frac{1}{C_3(j)-C_2(j)} - 4\beta_j'(C_3(j))\right)\int_{\mathcal{B}_j}|u|^2 d\mu - \frac{1}{4}\int_{\mathcal{B}_j}|\mathrm{grad}\ u|^2 d\mu.\end{aligned}$$

Proof. From now on to (2.13), we drop the index j for simplicity. We write just $\mathcal{E}^\wedge$, Y, ∂W, $\mathcal{B}$, $\beta(t)$, C_2, C_3, ϕ, $\mathcal{K}(t)$ instead of $\mathcal{E}_j^\wedge$, Y_j, $\partial_j W$, $\mathcal{B}_j$, $\beta_j(t)$, $C_2(j)$, $C_3(j)$, ϕ_j, $\mathcal{K}_j(t)$, etc.

On $\mathcal{E}^\wedge$ we use the same local coordinate system $(y_1, \ldots, y_{n-1}, t)$ as in the proof of Proposition 1.1. In this case $d = 1$ and we can write $u = u(y,t)$. We can decompose the gradient vector field of u as

$$\mathrm{grad}\ u = Z + \frac{\partial u}{\partial t}\frac{\partial}{\partial t}, \quad Z \perp \frac{\partial}{\partial t}.$$

Then we have

$$|\mathrm{grad}\ u|^2 = |Z|^2 + \left|\frac{\partial u}{\partial t}\right|^2 . \tag{2.4}$$

We remark that

$$e^{-2\beta(t)}\left|\frac{\partial}{\partial t}\left(ue^{\beta(t)}\right)\right|^2 = \left|\frac{\partial u}{\partial t}\right|^2 + (\beta'(t))^2|u|^2 + 2\beta'(t)\mathrm{Re}\left(u\frac{\partial \overline{u}}{\partial t}\right), \tag{2.5}$$

where Re( ) means the real part. Since u has a compact support, there exists a positive number $C_{11} = C_{11}(j)$ such that the support of u is contained in $V - \phi(Y \times [C_{11}, \infty))$. Then we have

$$\begin{aligned}\int_{\mathcal{E}^\wedge} 2\beta'(t)\mathrm{Re}\left(u\frac{\partial \overline{u}}{\partial t}\right) d\mu &= \lim_{t\to\infty}\int_{C_3}^{t}\left\{\int_Y 2\beta'(t)\mathrm{Re}\left(u\frac{\partial \overline{u}}{\partial t}\right)e^{2\beta(t)}d\mu_Y\right\}dt\\ &= \int_{C_3}^{C_{11}}\left\{\int_Y 2\beta'(t)\mathrm{Re}\left(u\frac{\partial \overline{u}}{\partial t}\right)e^{2\beta(t)}d\mu_Y\right\}dt,\end{aligned}$$

where μ_Y is the canonical measure of Y. Since

$$\frac{\partial}{\partial t}\left[\beta'(t)|u|^2e^{2\beta(t)}\right] = \left\{2\beta'(t)\mathrm{Re}\left(u\frac{\partial \overline{u}}{\partial t}\right) + \beta''(t)|u|^2 + 2(\beta'(t))^2|u|^2\right\}e^{2\beta(t)},$$

we have

$$\begin{aligned}&\int_{\mathcal{E}^\wedge} 2\beta'(t)\mathrm{Re}\left(u\frac{\partial \overline{u}}{\partial t}\right) d\mu\\ &= \int_{C_3}^{C_{11}}\int_Y\left\{\frac{\partial}{\partial t}\left[\beta'(t)|u|^2e^{2\beta(t)}\right] - |u|^2\beta''e^{2\beta(t)} - 2(\beta')^2|u|^2e^{2\beta(t)}\right\}d\mu_Y dt\\ &= -\int_Y \beta'(C_3)|u|^2e^{2\beta(C_3)}d\mu_Y - \int_{\mathcal{E}^\wedge}|u|^2(\beta'' + 2(\beta')^2)d\mu\\ &= -\int_{\partial W}\beta'|u|^2d\nu - \int_{\mathcal{E}^\wedge}|u|^2(\beta'' + 2(\beta')^2)d\mu,\end{aligned} \tag{2.6}$$

where ν is the canonical measure on ∂W induced from the Riemannian metric of V. From (2.5) and (2.6), we have

$$\begin{aligned}&\int_{\mathcal{E}^\wedge} e^{-2\beta(t)}\left|\frac{\partial}{\partial t}\left(ue^{\beta}\right)\right|^2 d\mu\\ &= \int_{\mathcal{E}^\wedge}\left(\left|\frac{\partial u}{\partial t}\right|^2 - |u|^2(\beta'' + (\beta')^2)\right)d\mu - \int_{\partial W}\beta'|u|^2d\nu.\end{aligned} \tag{2.7}$$

From (2.4), (2.7), we have

$$\begin{aligned}
&\int_{\mathcal{E}^\wedge} (|\text{grad } u|^2 - |u|^2(\beta'' + (\beta')^2))d\mu \\
&= \int_{\mathcal{E}^\wedge} \left\{ |Z|^2 + e^{-2\beta(t)} \left| \frac{\partial}{\partial t}(ue^{\beta(t)}) \right|^2 \right\} d\mu + \int_{\partial W} \beta'|u|^2 d\nu \qquad (2.8) \\
&\geq \int_{\partial W} \beta'|u|^2 d\nu.
\end{aligned}$$

Let us estimate the integral $\int_{\partial W} \beta'|u|^2 d\nu$ from below. Put $\psi(t) = (t - C_2)/(C_3 - C_2)$. Then we have

$$\begin{aligned}
|u(y, C_3)|^2 \, e^{2\beta(C_3)} &= e^{2\beta(C_3)} \int_{C_2}^{C_3} \frac{\partial}{\partial t} \left\{ \psi(t)|u|^2 \right\} dt \\
&= e^{2\beta(C_3)} \int_{C_2}^{C_3} \left\{ \frac{|u|^2}{C_3 - C_2} + 2\psi \text{Re} \left(u \frac{\partial \overline{u}}{\partial t} \right) \right\} dt.
\end{aligned}$$

We remark that $\beta(C_3) \leq \beta(t)$ on $[C_2, C_3]$. It follows that

$$|u(y, C_3)|^2 \, e^{2\beta(C_3)} \leq \int_{C_2}^{C_3} \frac{|u|^2 e^{2\beta(t)}}{C_3 - C_2} dt + 2 \int_{C_2}^{C_3} \left| u \frac{\partial \overline{u}}{\partial t} \right| e^{2\beta(t)} dt. \qquad (2.9)$$

Since

$$\left| u \frac{\partial \overline{u}}{\partial t} \right| e^{2\beta(t)} = |u| \cdot 2\sqrt{-\beta'(C_3)} \, e^{\beta(t)} \times \left| \frac{\partial \overline{u}}{\partial t} \right| \frac{1}{2\sqrt{-\beta'(C_3)}} \, e^{\beta(t)},$$

we have, from the arithmetic geometric mean inequality,

$$\begin{aligned}
&2 \int_{C_2}^{C_3} \left| u \frac{\partial \overline{u}}{\partial t} \right| e^{2\beta(t)} dt \\
&\leq 2 \sqrt{\int_{C_2}^{C_3} (-4)\beta'(C_3)|u|^2 e^{2\beta(t)} dt} \sqrt{\int_{C_2}^{C_3} \frac{1}{-4\beta'(C_3)} \left| \frac{\partial \overline{u}}{\partial t} \right|^2 e^{2\beta(t)} dt} \\
&\leq -4\beta'(C_3) \int_{C_2}^{C_3} |u|^2 e^{2\beta(t)} dt - \frac{1}{4\beta'(C_3)} \int_{C_2}^{C_3} \left| \frac{\partial \overline{u}}{\partial t} \right|^2 e^{2\beta(t)} dt.
\end{aligned}$$

We obtain

$$\begin{aligned}
&- \beta'(C_3)|u(y, C_3)|^2 e^{2\beta(C_3)} \\
&\leq -\beta'(C_3) \left(\frac{1}{C_3 - C_2} - 4\beta'(C_3) \right) \int_{C_2}^{C_3} |u|^2 e^{2\beta(t)} dt \\
&\qquad + \frac{1}{4} \int_{C_2}^{C_3} |\text{grad } u|^2 e^{2\beta(t)} dt.
\end{aligned} \qquad (2.10)$$

Hence

$$
\begin{aligned}
&-\int_{\partial W} \beta' |u|^2 d\nu = -\beta'(C_3) \int_Y |u|^2 e^{2\beta(C_3)} d\mu_Y \\
&\le -\beta'(C_3) \left(\frac{1}{C_3 - C_2} - 4\beta'(C_3) \right) \int_Y \int_{C_2}^{C_3} |u|^2 e^{2\beta(t)} dt d\mu_Y \\
&\qquad + \frac{1}{4} \int_Y \int_{C_2}^{C_3} |\text{grad } u|^2 e^{2\beta(t)} dt d\mu_Y \\
&= -\beta'(C_3) \left(\frac{1}{C_3 - C_2} - 4\beta'(C_3) \right) \int_{\mathcal{B}} |u|^2 d\mu + \frac{1}{4} \int_{\mathcal{B}} |\text{grad } u|^2 d\mu,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\partial W} \beta' |u|^2 d\nu \\
&\ge \beta'(C_3) \left(\frac{1}{C_3 - C_2} - 4\beta'(C_3) \right) \int_{\mathcal{B}} |u|^2 d\mu - \frac{1}{4} \int_{\mathcal{B}} |\text{grad } u|^2 d\mu.
\end{aligned}
\tag{2.11}
$$

Combining (2.8) with (2.11), we obtain

$$
\begin{aligned}
&\int_{\mathcal{E}^\wedge} (|\text{grad } u|^2 - |u|^2 (\beta'' + (\beta')^2) d\mu \\
&\ge \beta'(C_3) \left(\frac{1}{C_3 - C_2} - 4\beta'(C_3) \right) \int_{\mathcal{B}} |u|^2 d\mu - \frac{1}{4} \int_{\mathcal{B}} |\text{grad } u|^2 d\mu.
\end{aligned}
\tag{2.12}
$$

From Proposition 1.1 and the assumption, we have

$$
\beta'' + (\beta')^2 = \frac{(n-1)^2}{4} \left\{ \mathcal{K}(t) - \frac{2}{n-1} \mathcal{K}'(t) \right\} \ge \frac{(n-1)^2 a^2}{4}. \tag{2.13}
$$

The desired inequality now follows from (2.12) and (2.13). □

Since $\beta_j'(C_3(j)) \le 0$ and $\bigcup_{j=1}^s \mathcal{B}_j \subset W$, we have, from (2.3) and Lemma

2.1,

$$\begin{aligned}
&\left(\left(\Delta-\frac{(n-1)^2a^2}{4}\right)u,\,u\right)-\int_W\left(|\text{grad }u|^2-\frac{(n-1)^2a^2}{4}|u|^2\right)d\mu\\
&\geq\sum_{j=1}^{s}\beta_j'(C_3(j))\left(\frac{1}{C_3(j)-C_2(j)}-4\beta_j'(C_3(j))\right)\int_{\mathcal{B}_j}|u|^2d\mu\\
&\qquad\qquad\qquad\qquad\qquad-\frac{1}{4}\sum_{j=1}^{s}\int_{\mathcal{B}_j}|\text{grad }u|^2d\mu\\
&\geq\left\{\sum_{j=1}^{s}\beta_j'(C_3(j))\left(\frac{1}{C_3(j)-C_2(j)}-4\beta_j'(C_3(j))\right)\right\}\int_W|u|^2d\mu\\
&\qquad\qquad\qquad\qquad\qquad-\frac{1}{4}\int_W|\text{grad }u|^2d\mu.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left(\left(\Delta-\frac{(n-1)^2a^2}{4}\right)u,\,u\right)\\
&+\left\{\frac{(n-1)^2a^2}{4}+\sum_{j=1}^{s}\left(-\beta_j'(C_3(j))\left(\frac{1}{C_3(j)-C_2(j)}-4\beta_j'(C_3(j))\right)\right)+\frac{3}{4}\right\}\\
&\qquad\qquad\qquad\times\int_W|u|^2d\mu\;\geq\;\frac{3}{4}\int_W\left\{|u|^2+|\text{grad }u|^2\right\}d\mu.
\end{aligned}$$

Let $C_9=3/4$ and

$$C_{10}=\frac{(n-1)^2a^2}{4}+\frac{3}{4}+\max_{1\leq j\leq s}\left\{-\beta_j'(C_3(j))\left(\frac{1}{C_3(j)-C_2(j)}-4\beta_j'(C_3(j))\right)\right\}.$$

This proves the inequality (2.2), and Theorem 2 now follows. □

Proof of Lemma 1. For simplicity, we omit the symbol $d\mu_N$. Let v be a non-negative compactly supported smooth function on N. Since

$$\text{div }vZ=\langle\text{grad }v,Z\rangle+v\cdot\text{div }Z,$$

we have

$$0=\int_N\text{div }vZ=\int_N\langle\text{grad }v,Z\rangle+\int_N v\cdot\text{div }Z\geq\int_N\langle\text{grad }v,Z\rangle+\varepsilon\int_N v.$$

Hence

$$\varepsilon\int_N v\leq-\int_N\langle\text{grad }v,Z\rangle\leq\int_N|\text{grad }v|\cdot|Z|$$

and

$$\int_N v \le \frac{C}{\varepsilon} \int_N |\text{grad } v|.$$

For any compactly supported smooth function u on N, let $v = u^2$. Since grad $u^2 = 2u \cdot$ grad u, we have

$$\begin{aligned}\int_N u^2 &\le \frac{C}{\varepsilon} \int_N |\text{grad } u^2| \le \frac{2C}{\varepsilon} \int_N |u| \cdot |\text{grad } u| \\ &\le \frac{2C}{\varepsilon} \left(\int_N |u|^2 \right)^{1/2} \left(\int_N |\text{grad } u|^2 \right)^{1/2}\end{aligned}$$

and

$$\left(\int_N |u|^2 \right)^{1/2} \le \frac{2C}{\varepsilon} \left(\int_N |\text{grad } u|^2 \right)^{1/2}.$$

Therefore, we obtain

$$\left(\frac{\varepsilon}{2C} \right)^2 \le \frac{\int_N |\text{grad } u|^2}{\int_N |u|^2}.$$

□

§3. The Ends of Higher $\mathbb{Q}$-Rank Locally Symmetric Spaces

In this section we explain the case of higher $\mathbb{Q}$-rank locally symmetric spaces of finite volume and state a similar result to Theorem 3. The situation is slightly different from the one in Theorem 1. There are fiber bundles $\pi : Y \longrightarrow B$, cones $\mathcal{C}$, and Riemannian metrics on $Y \times \mathcal{C}$ satisfying the conditions ($\star\star$), ($*$), and ($**$). However, $Y \times \mathcal{C}$ is not entirely contained in V when B is not compact. Instead, there exist an exhaustion $\mathcal{W}_1 \subset \mathcal{W}_2 \subset \cdots \subset \mathcal{W}_i \subset \cdots$ of B by relatively compact open subsets, a corresponding nested sequence $\mathcal{C}_1 \supset \mathcal{C}_2 \supset \cdots \supset \mathcal{C}_i \supset \cdots$ of cones in $\mathcal{C}$, and an embedding $\bigcup_{i=1}^{\infty} \left(\pi^{-1}(\mathcal{W}_i) \times \mathcal{C}_i \right) \longrightarrow V$. In the sequel, for any algebraic group $\mathbf{H}$ defined over $\mathbb{Q}$, we denote by $\mathbf{H}(\mathbb{R})$, $\mathbf{H}(\mathbb{Q})$, $\mathbf{H}(\mathbb{Z})$ the group of real, rational, integral points of $\mathbf{H}$, respectively.

Let G be a connected semisimple Lie group having finite center and no compact factors. Let K be a maximal compact subgroup of G and $X = G/K$ the associated symmetric space of noncompact type with the canonical left invariant metric $\widetilde{g}$. Let Γ be an irreducible non-uniform lattice of X. Suppose that the rank of X is at least 2 and G has trivial center. Then, by the arithmeticity theorem of G.A. Margulis (see [30]), there exist a semisimple linear algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$ and an isomorphism (of Lie groups) from G to the identity component of $\mathbf{G}(\mathbb{R})$ such that the image of Γ is contained in $\mathbf{G}(\mathbb{Q})$ and is commensurable with $\mathbf{G}(\mathbb{Z})$. We suppose that Γ is torsion-free and that the $\mathbb{Q}$-rank $r_{\mathbb{Q}}(\mathbf{G})$ of $\mathbf{G}$ is at least 2. Then there is a compactification

$\overline{V}$ of the quotient manifold $V = \Gamma\backslash X$ constructed by A. Borel and J.-P. Serre such that $\overline{V}$ is a manifold with corners and its boundary $\partial\overline{V}$ is connected ([7, Corollary 8.6.2]). Hence V has only one end. Let x_0 be the coset in X of the identity element, $\Pi : X \longrightarrow V$ the natural projection, and g the metric on V such that $\Pi^* g = \widetilde{g}$.

We first recall some facts about parabolic subgroups of $\mathbf{G}$ (see [3], [29] for more details). Let $\mathbf{S}$ be a maximal $\mathbb{Q}$-split torus of $\mathbf{G}$ and $\Phi(\mathbf{G}, \mathbf{S})$ the system of rational roots of $\mathbf{G}$ with respect to $\mathbf{S}$. Let ${}_{\mathbb{R}}\mathfrak{g}$, $\mathfrak{g}$, and $\mathfrak{k}$ be the Lie algebras of G, $\mathbf{G}$, and K, respectively. We denote by exp the exponential mapping from ${}_{\mathbb{R}}\mathfrak{g}$ to G. Let ${}_{\mathbb{R}}\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition, where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in ${}_{\mathbb{R}}\mathfrak{g}$ with respect to the Killing form of ${}_{\mathbb{R}}\mathfrak{g}$. We denote by $\langle\ ,\ \rangle$ the inner product on $\mathfrak{p}$ induced from the Riemannian metric on the tangent space $T_{x_0}(X)$ of X at x_0. Then $\langle\ ,\ \rangle$ coincides with the restriction of the inner product $\langle\langle\ ,\ \rangle\rangle$ on ${}_{\mathbb{R}}\mathfrak{g}$ obtained from the Killing form of ${}_{\mathbb{R}}\mathfrak{g}$ and the Cartan involution of ${}_{\mathbb{R}}\mathfrak{g}$. Let A be the identity component of $\mathbf{S}(\mathbb{R})$ and $\mathfrak{a}$ its Lie algebra. By considering the restriction to $\mathfrak{a}$ of the differential of each rational root $\mathbf{S} \longrightarrow \mathbb{C}^*$, we can regard $\Phi(\mathbf{G}, \mathbf{S})$ as the system Σ of roots of the pair $({}_{\mathbb{R}}\mathfrak{g}, \mathfrak{a})$. For each root $\theta \in \Sigma$, let H_θ be the unique element of $\mathfrak{a}$ such that $\theta(H) = \langle H_\theta, H\rangle$ for all $H \in \mathfrak{a}$. We introduce a lexicographic order into $\Phi(\mathbf{G}, \mathbf{S})$, and denote by $\Phi^+(\mathbf{G}, \mathbf{S})$ the set of positive rational roots with respect to this order. We introduce the corresponding order into Σ, and denote by Σ^+ the set of positive roots corresponding to $\Phi^+(\mathbf{G}, \mathbf{S})$. We put

$$\mathfrak{a}^+ = \left\{ H \in \mathfrak{a} \,|\, \theta(H) \geq 0 \text{ for all } \theta \in \Sigma^+ \right\}$$

and

$$\rho = \frac{1}{2} \sum_{\theta \in \Sigma^+} H_\theta,$$

where in the sum every root occurs a number of times equal to its multiplicity. Let

$$\mathfrak{g} = \mathfrak{g}_0 + \coprod_{\chi \in \Phi(\mathbf{G}, \mathbf{S})} \mathfrak{g}_\chi$$

be the root space decomposition of $\mathfrak{g}$. Let $\mathfrak{u}$ be the subalgebra of $\mathfrak{g}$ defined by

$$\mathfrak{u} = \coprod_{\chi \in \Phi^+(\mathbf{G}, \mathbf{S})} \mathfrak{g}_\chi$$

and $\mathbf{U}$ the analytic subgroup with Lie algebra $\mathfrak{u}$. Let $\Upsilon \subset \Sigma^+$ be the set of positive simple roots of $({}_{\mathbb{R}}\mathfrak{g}, \mathfrak{a})$ and $\Upsilon(\mathbf{G}) \subset \Phi^+(\mathbf{G}, \mathbf{S})$ the corresponding set of positive simple roots. For any subset $I \subset \Upsilon$, we take the corresponding subset $I(\mathbf{G}) \subset \Upsilon(\mathbf{G})$ and define a subgroup $\mathbf{S}_I$ of $\mathbf{S}$ by

$$\mathbf{S}_I = \Big(\bigcap_{\alpha \in I(\mathbf{G})} \ker \alpha \Big)^0,$$

where $(\ )^0$ means the identity component with respect to the Zariski topology. The group $\mathbf{P}_I = Z(\mathbf{S}_I)\mathbf{U}$, where $Z(\mathbf{S}_I)$ is the centralizer of $\mathbf{S}_I$ in $\mathbf{G}$, is a rational parabolic subgroup of $\mathbf{G}$ containing $\mathbf{S}$. These are called the standard rational parabolic subgroups of $\mathbf{G}$. The unipotent radical of $\mathbf{P}_I$, that is, the greatest connected unipotent normal subgroup of $\mathbf{P}_I$, is the analytic subgroup $\mathbf{U}_I$ with Lie algebra

$$\mathfrak{u}_I = {\coprod}' \mathfrak{g}_\chi,$$

where the sum is over all the positive rational roots which are not linear combinations of elements of I. If I and J are two subsets of Υ such that $I \subset J$, then $\mathbf{P}_I \subset \mathbf{P}_J$. The rational parabolic subgroup $\mathbf{P} = \mathbf{P}_\emptyset = Z(\mathbf{S})\mathbf{U}$ is a minimal one containing $\mathbf{S}$, and $\mathbf{P}_\Upsilon = \mathbf{G}$. Each proper rational parabolic subgroup $\mathbf{Q}$ of $\mathbf{G}$ is conjugate by some element of $\mathbf{G}(\mathbb{Q})$ to one and only one of the $\mathbf{P}_I$ with $I \neq \Upsilon$; $\mathbf{Q}$ is also expressed as $k\mathbf{P}_I k^{-1}$ for some $k \in K$. For each I, the Γ-conjugacy classes of $\mathbf{P}_I$ are known to be finite ([4]).

Each group $\mathbf{P}_I$ decomposes further. We put

$$\mathbf{M}_I = \bigcap_{\chi \in \mathrm{X}(Z(\mathbf{S}_I))} \ker(\chi^2),$$

where $\mathrm{X}(Z(\mathbf{S}_I))$ is the group of rational characters of $Z(\mathbf{S}_I)$. Then $Z(\mathbf{S}_I)(\mathbb{R})$ is the direct product $\mathbf{M}_I(\mathbb{R}) \times \mathbf{S}_I(\mathbb{R})$. Let A_I be the identity component of $\mathbf{S}_I(\mathbb{R})$ and let $M_I = \mathbf{M}_I(\mathbb{R})$, $U_I = \mathbf{U}_I(\mathbb{R})$. Then we have the Langlands decomposition

$$\mathbf{P}_I(\mathbb{R}) = U_I A_I M_I. \tag{3.1}$$

The boundary $\partial\overline{V}$ of the Borel-Serre compactification $\overline{V}$ is a disjoint union of faces $e'(\mathbf{Q})$ corresponding to the Γ-conjugacy classes of proper rational parabolic subgroups of $\mathbf{G}$. Suppose that the Γ-conjugacy classes of $\mathbf{P}_I$ $(I \neq \Upsilon)$ are represented by $\mathbf{P}_{I,j} = k_{I,j}\mathbf{P}_I k_{I,j}^{-1}$, $j \in \{1, \ldots, s(I)\}$, where $k_{I,j} \in K$ for each j and $k_{I,1} = e$. In particular, $\mathbf{P}_{I,1} = \mathbf{P}_I$. We briefly describe the faces $e'(\mathbf{P}_{I,j})$ (see [7], [31] for more details).

Let

$$U_{I,j} = k_{I,j} U_I k_{I,j}^{-1}, \quad A_{I,j} = k_{I,j} A_I k_{I,j}^{-1}, \quad \text{and } M_{I,j} = k_{I,j} M_I k_{I,j}^{-1}.$$

$$\begin{array}{ccc} X \supset e(\mathbf{P}_{I,j}) & \xrightarrow{\widetilde{\pi}_{I,j}} & X_{I,j} \\ \Pi \Big\downarrow \quad \varpi_{I,j} \Big\downarrow \Gamma_{I,j}\backslash & & \Pi_{I,j} \Big\downarrow \Gamma_{M_{I,j}}\backslash \\ V \not\supset e'(\mathbf{P}_{I,j}) & \xrightarrow[\pi_{I,j}]{} & V_{I,j} \end{array}$$

Figure 4.

Let

$$e(\mathbf{P}_{I,j}) = U_{I,j}M_{I,j} \cdot x_0 = k_{I,j}U_I M_I \cdot x_0, \quad X_{I,j} = M_{I,j} \cdot x_0 = k_{I,j}M_I \cdot x_0.$$

Then $X_{I,j}$ is a product of a symmetric space of noncompact type with a possible Euclidean space, and $e(\mathbf{P}_{I,j})$ is diffeomorphic to $U_{I,j} \times X_{I,j}$. In particular, we have a fiber bundle

$$U_{I,j} \longrightarrow e(\mathbf{P}_{I,j}) \xrightarrow{\widetilde{\pi}_{I,j}} X_{I,j}, \quad \widetilde{\pi}_{I,j}(k_{I,j}um \cdot x_0) = k_{I,j}m \cdot x_0 \text{ for } u \in U_I, m \in M_I.$$

Let

$$\Gamma_{I,j} = \Gamma \cap \mathbf{P}_{I,j}, \quad \Gamma_{M_{I,j}} = (\Gamma_{I,j}U_{I,j}) \cap M_{I,j}, \quad \text{and } \Gamma_{U_{I,j}} = \Gamma \cap U_{I,j}.$$

Then $\Gamma_{I,j}$ is the semi-direct product $\Gamma_{U_{I,j}} \rtimes \Gamma_{M_{I,j}}$ and acts on $e(\mathbf{P}_{I,j})$. Let $V_{I,j} = \Gamma_{M_{I,j}}\backslash X_{I,j}$ and $\Pi_{I,j} : X_{I,j} \longrightarrow V_{I,j}$ the natural projection. The face $e'(\mathbf{P}_{I,j})$ is defined by

$$e'(\mathbf{P}_{I,j}) = \Gamma_{I,j}\backslash e(\mathbf{P}_{I,j}).$$

Let $\pi_{I,j}$ be the unique map from $e'(\mathbf{P}_{I,j})$ to $V_{I,j}$ such that the diagram in Figure 4 is commutative. In the diagram, $\varpi_{I,j} : e(\mathbf{P}_{I,j}) \longrightarrow e'(\mathbf{P}_{I,j})$ is the natural projection. Then $e'(\mathbf{P}_{I,j})$ is a fiber bundle over $V_{I,j}$:

$$F_{I,j} = \Gamma_{U_{I,j}}\backslash U_{I,j} \longrightarrow e'(\mathbf{P}_{I,j}) \xrightarrow{\pi_{I,j}} V_{I,j}. \tag{3.2}$$

The fiber $F_{I,j}$ of the fiber bundle (3.2) is a compact nilmanifold and the base space $V_{I,j}$ is a locally symmetric space of finite volume, which is compact if and only if $I = \emptyset$.

In the compactification $\overline{V}$ each face $e'(\mathbf{P}_{I,j})$ is located on the ideal boundary $\partial\overline{V}$. For $t > 0$, we put

$$A_I(t) = \{a \in A_I \,|\, \chi(a) > t \text{ for all } \chi \in \Upsilon(\mathbf{G}) - I(\mathbf{G})\}, \quad A_{I,j}(t) = k_{I,j}A_I(t)k_{I,j}^{-1}.$$

We just write $A(t)$ instead of $A_\emptyset(t)$.

Proposition 3.1 ([7, 10.3], [31, (1.5)]). *Let $\mathcal{W}$ be a relatively compact open subset of $V_{I,j}$. Then there exists $t^* = t^*(\mathcal{W}) > 0$ such that for any $t \geq t^*$ the set $E_{I,j,\mathcal{W},t} = U_{I,j}A_{I,j}(t) \cdot \Pi_{I,j}^{-1}(\mathcal{W})$ is $\Gamma_{I,j}$-invariant and*

$$\{g \in \Gamma \mid gE_{I,j,\mathcal{W},t} \cap E_{I,j,\mathcal{W},t} \neq \emptyset\} = \Gamma_{I,j}.$$

The equivalence relation defined on $E_{I,j,\mathcal{W},t}$ by Γ is the same as the one defined by $\Gamma_{I,j}$.

Let $\mathcal{W}$ be a relatively compact subset of $V_{I,j}$. We choose a positive number t^{**} such that $t^{**} \geq t^*(\mathcal{W})$ and put

$$E_{I,j,\mathcal{W}} = E_{I,j,\mathcal{W},t^{**}} = U_{I,j}A_{I,j}(t^{**}) \cdot \Pi_{I,j}^{-1}(\mathcal{W}), \quad \mathcal{E}_{I,j,\mathcal{W}} = \Pi(E_{I,j,\mathcal{W}}).$$

Let

$$Y_{I,j,\mathcal{W}} = \pi_{I,j}^{-1}(\mathcal{W}) \subset e'(\mathbf{P}_{I,j}).$$

Then the open submanifold $\mathcal{E}_{I,j,\mathcal{W}}$ of V is diffeomorphic to the product $Y_{I,j,\mathcal{W}} \times (A_{I,j}(t^{**}) \cdot x_0)$. Let

$$\mathfrak{a}_I = \{H \in \mathfrak{a} \mid \theta(H) = 0 \quad \text{for all } \theta \in I\}$$

be the Lie algebra of A_I. We denote by ρ_I the orthogonal projection of ρ on $\mathfrak{a}_I^+ := \mathfrak{a}_I \cap \mathfrak{a}^+$ and put $\rho_{I,j} = Ad(k_{I,j})\rho_I$. We define a geodesic ray $\gamma_{I,j} : [0,\infty) \longrightarrow X$ by

$$\gamma_{I,j}(t) = \exp\left(t\rho_{I,j}/|\rho_{I,j}|\right) \cdot x_0.$$

Then, for sufficiently large $t^{***} > 0$, the restriction of the geodesic $\Pi \circ \gamma_{I,j}$ to the interval $[t^{***}, \infty)$ is contained in $\mathcal{E}_{I,j,\mathcal{W}}$. If we regard $A_{I,j}(t^{**}) \cdot x_0$ as an open cone $\mathcal{C}$ in $\mathbb{R}^d$ with

$$d = \#\Upsilon - \#I = r_{\mathbb{Q}}(\mathbf{G}) - \#I,$$

the parabolic $\mathbb{Q}$-rank of $\mathbf{P}_{I,j}$, and regard $\gamma_{I,j}|_{[t^{***},\infty)}$ as a ray l in $\mathcal{C}$, we obtain an embedding $\phi_{I,j,\mathcal{W}} : Y_{I,j,\mathcal{W}} \times \mathcal{C} \longrightarrow V$ as in the introduction. In the case where $V_{I,j}$ is not compact, let $\mathcal{W}_1 \subset \mathcal{W}_2 \subset \cdots \subset \mathcal{W}_i \subset \cdots$ be an exhaustion of $V_{I,j}$ by relatively compact subsets. Then, from the above construction, we obtain an embedding $\bigcup_{i=1}^{\infty} \mathcal{E}_{I,j,\mathcal{W}_i} \longrightarrow V$.

Lemma 3.1. *The metric on $Y_{I,j,\mathcal{W}} \times \mathcal{C}$ induced by $\phi_{I,j,\mathcal{W}}$ is of the form $g_{\mathbf{t}} + d\mathbf{t}^2$ and the conditions $(\star\star)$, $(*)$, $(**)$ in the introduction are satisfied. In particular,*

$$\alpha(t) = \sqrt{g_{\mathbf{t}}}/\sqrt{g_{\mathbf{0}}} = e^{-2|\rho_I|t}. \tag{3.3}$$

Proof. We only prove the case $j = 1$, since other cases are similar. In this case, $U_{I,1} = U_I$, $A_{I,1} = A_I$, and $M_{I,1} = M_I$. We write ϕ instead of $\phi_{I,1,\mathcal{W}}$. Let $(t, t_2, \ldots, t_d)$ be a Cartesian coordinate system of $\mathbb{R}^d$ such that the origin $\mathbf{0}$ is $l(0) = \gamma_{I,1}(t^{***})$ and that the positive part of the t-axis corresponds to the ray l. For each $\mathbf{t} = (t, t_2, \ldots, t_d) \in \mathbb{R}^d$, let $a(\mathbf{t}) = a(t, t_2, \ldots, t_d)$ be the unique element of A_I such that $\mathbf{t} = a(\mathbf{t}) \cdot x_0$ and let $H(\mathbf{t}) = H(t, t_2, \ldots, t_d)$ be the unique element of $\mathfrak{a}_I$ such that $\exp(H(\mathbf{t})) = a(\mathbf{t})$. We identify $e(\mathbf{P}_{I,1})$ with $U_I \times X_{I,1}$ and define a diffeomorphism Φ from $e(\mathbf{P}_{I,1}) \times \mathbb{R}^d = U_I \times X_{I,1} \times \mathbb{R}^d$ to V by

$$\Phi(um \cdot x_0, \mathbf{t}) = ua(\mathbf{t})m \cdot x_0 \quad \text{for } u \in U_I, m \in M_I. \tag{3.4}$$

Let

$$\widetilde{\mathcal{W}} = \Pi_{I,1}^{-1}(\mathcal{W}) \subset X_{I,1}, \quad \widetilde{Y}_{I,1,\mathcal{W}} = U_I \cdot \widetilde{\mathcal{W}} \subset e(\mathbf{P}_{I,1}).$$

Then $\widetilde{Y}_{I,1,\mathcal{W}}$ is diffeomorphic to $U_I \times \widetilde{\mathcal{W}}$ and the map $\widetilde{\phi} : \widetilde{Y}_{I,1,\mathcal{W}} \times \mathcal{C} \longrightarrow X$ obtained by restricting Φ on $\widetilde{Y}_{I,1,\mathcal{W}} \times \mathcal{C}$ is an embedding. If we write ϖ instead of $\varpi_{I,1}$, we have $\phi \circ (\varpi \times id_{\mathcal{C}}) = \Pi \circ \widetilde{\phi}$. Thus, in order to study the metric $\phi^* g$ it suffices to study $\widetilde{\phi}^* \widetilde{g}$.

Let $\Upsilon_I(\mathbf{G})$ be the set of maps obtained by restricting elements of $\Upsilon(\mathbf{G}) - I(\mathbf{G})$ to $\mathbf{S}_I$. For each $\alpha \in \Upsilon_I(\mathbf{G})$, let

$$\mathfrak{u}_\alpha = \{X \in {}_{\mathbb{R}}\mathfrak{g} \,|\, (\mathrm{Ad}\ a)X = \alpha(a)X \quad \text{for all } a \in A_I\}.$$

Then we also have

$$\mathfrak{u}_I = \coprod_{\alpha \in \Upsilon_I(\mathbf{G})} \mathfrak{u}_\alpha.$$

The spaces $\mathfrak{u}_\alpha$ $(\alpha \in \Upsilon_I(\mathbf{G}))$ are mutually orthogonal with respect to the inner product $\langle\langle\, ,\, \rangle\rangle$. For each $\alpha \in \Upsilon_I(\mathbf{G})$, let h_α be the left invariant tensor field of type $(0, 2)$ on U_I which is zero on $\mathfrak{u}_\beta$ for $\beta \neq \alpha$, and equal to $\langle\langle\, ,\, \rangle\rangle$ on $\mathfrak{u}_\alpha$. Then

$$du^2 := \sum_{\alpha \in \Upsilon_I(\mathbf{G})} h_\alpha \tag{3.5}$$

is a left invariant metric on U_I. For $m \in M_I$, we denote by Int m the inner automorphism of U_I given by

$$(\mathrm{Int}\ m)(u) = mum^{-1} \quad \text{for } u \in U_I.$$

Let $\widetilde{g}_{X_{I,1}}$ be the metric on $X_{I,1}$ induced from X by the natural inclusion. Since the tangent space of $\widetilde{Y}_{I,1,\mathcal{W}} \times \mathcal{C}$ at $(um \cdot x_0, \mathbf{t})$ is isomorphic to the direct sum of $T_u(U_I)$, $T_{m\cdot x_0}(X_{I,1})$, and $T_{\mathbf{t}}(\mathbb{R}^d)$, under this identification, it follows from the calculation in Proposition 4.3 of [5] that

$$(\widetilde{\phi}^*\widetilde{g})_{(um\cdot x_0,\mathbf{t})} = \frac{1}{2}\sum_{\chi\in\Upsilon_I(\mathbf{G})} \chi(a(\mathbf{t}))^{-2}\left((\mathrm{Int}\ m^{-1})^*h_\chi\right)_u + (\widetilde{g}_{X_{I,1}})_{m\cdot x_0} + (d\mathbf{t}^2)_{\mathbf{t}}. \tag{3.6}$$

From this, we can conclude that ϕ^*g is of the form $g_{\mathbf{t}} + d\mathbf{t}^2$. Let $\widetilde{g}_{\mathbf{t}} = \Pi^* g_{\mathbf{t}}$. Then, from (3.6) we have

$$(\widetilde{g}_{\mathbf{t}})_{um\cdot x_0} = \frac{1}{2}\sum_{\chi\in\Upsilon_I(\mathbf{G})} \chi(a(\mathbf{t}))^{-2}\left((\mathrm{Int}\ m^{-1})^*h_\chi\right)_u + (\widetilde{g}_{X_{I,1}})_{m\cdot x_0}. \tag{3.7}$$

This shows that the projection

$$\widetilde{\pi} : (\widetilde{Y}_{I,1,\mathcal{W}}, \widetilde{g}_{\mathbf{t}}) \longrightarrow (\widetilde{\mathcal{W}}, \widetilde{g}_{X_{I,1}}|_{\widetilde{\mathcal{W}}}) \tag{3.8}$$

is a Riemannian submersion. Let $g_{X,1}$ be the metric on $V_{I,1} = \Gamma_{M_{I,1}}\backslash X_{I,1}$ corresponding to $\widetilde{g}_{X,1}$. Since $\Gamma_I = \Gamma_{U_I} \rtimes \Gamma_{M_I}$, the bundle projection

$$\pi : (Y_{I,1,\mathcal{W}}, g_{\mathbf{t}}) \longrightarrow (\mathcal{W}, g_{X_{I,1}}|_{\mathcal{W}}) \tag{3.9}$$

is also a Riemannian submersion and the condition ($\star\star$) is satisfied.

As is shown in Corollary 4.4 of [5], two left invariant metrics $(\mathrm{Int}\ m^{-1})^*du^2$ and du^2 have the same volume element. Consequently, it follows from (3.5), (3.7) that

$$\sqrt{g_{\mathbf{t}}}/\sqrt{g_{\mathbf{0}}} = \prod_{\chi\in\Upsilon_I(\mathbf{G})} \chi(a(\mathbf{t}))^{-\dim \mathfrak{u}_\chi} \Bigg/ \prod_{\chi\in\Upsilon_I(\mathbf{G})} \chi(a(\mathbf{0}))^{-\dim \mathfrak{u}_\chi}.$$

Since

$$\prod_{\chi\in\Upsilon_I(\mathbf{G})} \chi(a(\mathbf{t}))^{-\dim \mathfrak{u}_\chi} = \exp\left(-2\langle\rho_I, H(\mathbf{t})\rangle\right) = e^{-2|\rho_I|(t+t^{***})}, \tag{3.10}$$

we obtain (3.3), which is the condition ($*$).

We show that the Riemannian submersion (3.9) is harmonic. For this, it suffices to prove that the Riemannian submersion (3.8) is harmonic. Note that the Lie group U_I acts isometrically on $\widetilde{Y}_{I,1,\mathcal{W}}$ preserving the fibers. Therefore, we can apply H.-S. Wu's theorem on *metrically homogeneously fibered submersions* to this Riemannian submersion. For precise definitions and detailed information about related concepts we refer to [19, Chapter XV, §6–§8].

Let $q = \dim U_I$. A q-form on $\widetilde{Y}_{I,1,\mathcal{W}}$ is called fiber null if its restriction to each fiber of (3.8) is 0. For a horizontal vector field ν on $\widetilde{Y}_{I,1,\mathcal{W}}$, we say that a q-form Ψ is ν-constant over the fibers if the Lie derivative $\mathcal{L}_\nu \Psi$ is fiber null. Consider the pull-back of the left invariant volume form on U_I determined by du^2 by the projection $U_I \times \widetilde{\mathcal{W}} \times \{\mathbf{t}\} \longrightarrow U_I$. Then, under the identification $\widetilde{Y}_{I,1,\mathcal{W}} = U_I \times \widetilde{\mathcal{W}} \times \{\mathbf{t}\}$, we obtain a left U_I-invariant q-form Ψ on $\widetilde{Y}_{I,1,\mathcal{W}}$. This form Ψ is the *Haar form* on $\widetilde{Y}_{I,1,\mathcal{W}}$ in the sense of [19, p. 142], and hence ν-constant over the fibers for any horizontal vector field ν, due to [19, Theorem 8.3]. Let Ω be the *vertical metric volume form* of (3.8) defined as follows. Let $\{\xi_1, \dots, \xi_q\}$ be an orthonormal frame of vertical vector fields on $\widetilde{Y}_{I,1,\mathcal{W}}$, suitably oriented, and let $\xi_1^*, \dots, \xi_q^*$ be the 1-forms on $\widetilde{Y}_{I,1,\mathcal{W}}$ such that $\xi_i^*(\xi) = \widetilde{g}_{\mathbf{t}}(\xi_i, \xi)$ for any vector field ξ on $\widetilde{Y}_{I,1,\mathcal{W}}$. Put $\Omega = \xi_1^* \wedge \cdots \wedge \xi_q^*$. Then Ω does not depend on the choice of $\{\xi_1, \dots, \xi_q\}$. If $\Omega_{\widetilde{Y}_{I,1,\mathcal{W}}}$, $\Omega_{\widetilde{\mathcal{W}}}$ are the volume forms on $\widetilde{Y}_{I,1,\mathcal{W}}$, $\widetilde{\mathcal{W}}$, respectively, we have $\Omega_{\widetilde{Y}_{I,1,\mathcal{W}}} = \Omega \wedge \pi^* \Omega_{\widetilde{\mathcal{W}}}$. The *Riemannian Haar density* ([19, p. 442]) $\widetilde{\omega}$ is by definition the function on $\widetilde{\mathcal{W}}$ given by

$$\Omega_{um\cdot x_0} = \widetilde{\omega}(m \cdot x_0) \Psi_{um\cdot x_0}.$$

From (3.7) and (3.10) we have

$$\widetilde{\omega}(m \cdot x_0) = e^{-2(t+t^{***})|\rho_I|}/\sqrt{2^q} \qquad \text{for any } m \in M.$$

Let $(\widetilde{\tau}_{\mathbf{t}})_{um\cdot x_0}$ be the trace of the second fundamental form of the fiber $\widetilde{\pi}^{-1}(m \cdot x_0)$ at $um \cdot x_0$. From H.-S. Wu's theorem ([19, Theorem 6.6]), $(\widetilde{\tau}_{\mathbf{t}})_{um\cdot x_0}$ is the horizontal lift of $-(\mathrm{grad}\ (\log \widetilde{\omega}))_{m\cdot x_0}$. Hence $(\widetilde{\tau}_{\mathbf{t}})_x = 0$ for all $x \in \widetilde{Y}_{I,1,\mathcal{W}}$. This shows that (3.8) is harmonic. ☐

Remark. In the $\mathbb{Q}$-rank 1, $\mathbb{R}$-rank ≥ 2 case, one can show in the same way that each boundary Y_j at infinity admits a fiber bundle structure satisfying the conditions ($\star\star$), ($**$).

From (3.3), the mean curvature $\mathcal{K}(t)$ of $Y_{I,j,\mathcal{W}} \times \mathcal{C}_t$ is

$$\mathcal{K}(t) = 2|\rho_I|/(n-1). \tag{3.11}$$

Let $C_0^\infty(V_{I,j})$ be the space of compactly supported C^∞-functions on $V_{I,j}$ and $L^2(V_{I,j})$ the space of square integrable functions on $V_{I,j}$. We denote by $\overline{\Delta}_{I,j}$ the self-adjoint extension of the Laplacian $\Delta_{I,j}$ on $V_{I,j}$ to $L^2(V_{I,j})$. We can obtain some information on the spectrum of $\overline{\Delta}$.

Theorem 4. (1) *Suppose that*

$$0 = c_{I,j,0} < c_{I,j,1} < \cdots < c_{I,j,m} < \cdots$$

are the eigenvalues of $\overline{\Delta}_{I,j}$*. Then the following holds.*

For any $r \geq 0$*, there exists a family* $\{u_{I,j,m,r,q}\}_{q \in \mathbb{N}}$ *of compactly supported smooth functions on* $\mathcal{E}_{I,j}$ *satisfying the following four conditions.*
(3.12) *For any compact subset of* V*, if we take* q *sufficiently large, then the support of* $u_{I,j,m,r,q}$ *lies outside this compact set.*
(3.13) *For some positive constant* C_{11} *independent of* q*, we have*

$$\left\| \left(\overline{\Delta} - (|\rho_I|^2 + c_{I,j,m} + r^2) \right) u_{I,j,m,r,q} \right\| \leq \frac{C_{11}}{q} \| u_{I,j,m,r,q} \|.$$

Therefore, for fixed I*,* j*,* m*, every point of* $\left[|\rho_I|^2 + c_{I,j,m},\ \infty \right)$ *belongs to the spectrum of* $\overline{\Delta}$*.*
(3.14) *If* $m \neq m'$*, then*

$$\lim_{\substack{q \longrightarrow \infty \\ q' \longrightarrow \infty}} (u_{I,j,m,r,q},\ u_{I,j,m',r',q'}) = 0.$$

(3.15) *If* $\mathbf{P}_{I,j} \cap g \mathbf{P}_{I',j'} g^{-1}$ *is not a rational parabolic subgroup of* $\mathbf{G}$ *for any* $g \in \mathbf{G}(\mathbb{Q})$*, then* $(u_{I,j,m,r,q},\ u_{I',j',m',r',q'}) = 0$*.*

(2) *The bottom of the essential spectrum of* $\overline{\Delta}$ *is*

$$\min_{I \subset \Upsilon, I \neq \Upsilon} |\rho_I|^2.$$

Remark. For any face $e'(\mathbf{P})$ we denote by $\overline{e'(\mathbf{P})}$ the closure of $e'(\mathbf{P})$ in $\overline{V}$. Then the hypothesis in (3.15) is equivalent to $\overline{e'(\mathbf{P}_{I,j})} \cap \overline{e'(\mathbf{P}_{I',j'})} = \emptyset$ (see Proposition 9.4 of [7]).

We prove (1) of Theorem 4 in the rest of this section, and (2) in the next section. We denote by $\| \ \|_{I,j}$ the L^2-norm on $L^2(V_{I,j})$.

Proof of Theorem 4 (1). Let φ_* be an eigenfunction of $\overline{\Delta}_{I,j}$ belonging to the eigenvalue $c_{I,j,m}$. For each q, we can find a function $\varphi \in C_0^\infty(V_{I,j})$ such that

$$\|(\Delta_{I,j} - c_{I,j,m})\varphi\|_{I,j} < \frac{1}{q} \|\varphi\|_{I,j}.$$

Let $\mathcal{W}$ be a relatively compact open subset containing the support of φ. For this $\mathcal{W}$, we take $t^* = t^*(\mathcal{W})$ as in Proposition 3.1, choose $t^{**} \geq t^*$, and identify $A_{I,j}(t^{**}) \cdot x_0$ with an open cone $\mathcal{C}$ in $\mathbb{R}^d$ with $d = r_{\mathbb{Q}}(\mathbf{G}) - \#I$. Apply Proposition 1.2 to the embedding $\phi_{I,j,\mathcal{W}} : Y_{I,j,\mathcal{W}} \times \mathcal{C} \longrightarrow V$. From (3.11), we have

$$\left\| \left(\Delta - (|\rho_I|^2 + c_{I,j,m} + r^2) \right) f_{\varphi,r,\delta} \right\| < \frac{C_5}{q} \| f_{\varphi,r,\delta} \|$$

for the function

$$f_{\varphi,r,\delta}(y,\mathbf{t}) = h(\delta\mathbf{t})e^{-\beta(t)+\sqrt{-1}\cdot rt}\varphi(\pi_{I,j}(y)),$$

and any positive number $\delta < \delta(1/q)$. We take one such δ and put

$$u_{I,j,m,r,q} = f_{\varphi,r,\delta}.$$

Choose the functions φ so that $\|\varphi_* - \varphi\|_{I,j}$ are sufficiently small. Then the resultant family $\{u_{I,j,m,r,q}\}_{q\in\mathbb{N}}$ of compactly supported C^∞-functions on $\mathcal{E}_{I,j}$ satisfies the conditions (3.12)–(3.14). In this construction, for fixed q, q', we can choose the functions so that the supports of $u_{I,j,m,r,q}$ and $u_{I',j',m',r',q'}$ are mutually disjoint for pairs (I,j), (I',j') such that $\mathbf{P}_{I,j} \cap g\mathbf{P}_{I',j'}g^{-1}$ is not a rational parabolic subgroup of $\mathbf{G}$ for any $g \in \mathbf{G}(\mathbb{Q})$. Hence the condition (3.15) is also satisfied. This proves (1) of Theorem 4. □

§4. Constructing Vector Fields

In this section we prove (2) of Theorem 4. We construct a vector field on the end of V and use Lemma 1. We recall Borel's construction of fundamental open sets.

Definition 4.1. An open subset $\mathcal{D}$ of X is called a fundamental open set for Γ if

$$X = \Gamma\mathcal{D} \tag{4.1}$$

and

(4.2) the set $\{g \in \Gamma \mid g\mathcal{D} \cap \mathcal{D} \neq \emptyset\}$ is finite.

Let $\mathbf{P} = \mathbf{P}_\emptyset$ be the standard minimal rational parabolic subgroup and $\mathbf{P}(\mathbb{R}) = UAM$ the Langlands decomposition of $\mathbf{P}(\mathbb{R})$ as in (3.1). Since $X = UAM \cdot x_0$, any point $x \in X$ can be represented as $x = uam \cdot x_0$ for some $u \in U$, $a \in A$, $m \in M$. In this representation the A-factor a is uniquely determined by x. We denote by $\mathcal{A}(x)$ the A-factor of x. We also recall that $am = ma$ for any $a \in A$, $m \in M$.

Definition 4.2. For any $t > 0$ and relatively compact open subset η of UM containing the identity element e, the set $\mathfrak{S}_{t\eta} := \eta A(t)K$ (resp. $\mathfrak{S}_{t\eta} \cdot x_0 = \eta A(t) \cdot x_0$) is called a (generalized) Siegel set in G (resp. X).

We remark that X is also regarded as the quotient $X = \mathbf{G}(\mathbb{R})/\widetilde{K}$, where $\widetilde{K}$ is the maximal compact subgroup of $\mathbf{G}(\mathbb{R})$ containing K. Hence we can consider the action of $\mathbf{G}(\mathbb{R})$ on X. Let $z_1, \ldots, z_\lambda$ $(z_1 = e)$ be a complete representative system of $\Gamma\backslash\mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$. The following is known.

Theorem 4.1 ([4]). *There exist a positive number $t_0 < 1$ and a relatively compact open subset η of UM containing the identity element e such that the set $\Omega = \bigcup_{i=1}^{\lambda} z_i \cdot \mathfrak{S}_{t_0\eta} \cdot x_0$ is a fundamental open set for Γ.*

From now on we fix such t_0 and η. The quotient space $V = X/\Gamma$ is obtained by pasting the translated Siegel sets $z_i\mathfrak{S}_{t_0\eta} \cdot x_0$, $i = 1, \ldots, \lambda$, together. In order to describe how these are pasted together, we first decompose $A(t_0)$ as in [16], [17], [18], and [22]. Let t_1 be any positive number greater than 1. We put

$$S_I = \{a \in A(t_0) \,|\, \chi(a) \le t_1 \quad \text{for all } \chi \in I(\mathbf{G})\}$$

for each nonempty subset I of Υ. Then we have

$$A(t_0) - A(t_1) = \bigcup_{\emptyset \ne I \subset \Upsilon} S_I. \tag{4.3}$$

We remark that the right-hand side is not a disjoint union. Let

$$\mathcal{S} = \eta A(t_0) \cdot x_0 = \mathfrak{S}_{t_0\eta} \cdot x_0, \quad \mathcal{S}_* = \eta A(t_1) \cdot x_0, \text{ and } \mathcal{S}_I = \eta S_I \cdot x_0.$$

The set $\mathcal{S}_\Upsilon$ is relatively compact. We also put

$$\mathcal{S}_j = z_j\mathcal{S}, \qquad \mathcal{S}_{j*} = z_j\mathcal{S}_*, \text{ and } \mathcal{S}_{j,I} = z_j\mathcal{S}_I.$$

We have a decomposition

$$\mathcal{S}_j = \mathcal{S}_{j*} \cup \left(\bigcup_{\emptyset \ne I \subset \Upsilon} \mathcal{S}_{j,I} \right). \tag{4.4}$$

Roughly speaking, if $g\mathcal{S}_i$ ($g \in \Gamma$) meets $\mathcal{S}_j$ at a point sufficiently far from x_0, then the intersection $g\mathcal{S}_i \cap \mathcal{S}_j$ is entirely contained in $\mathcal{S}_{j,I}$ for some nonempty proper subset I of Υ. More precisely,

Lemma 4.1 ([4, 12.6], [27, Lemma 2.1], see also [22, Lemmas 2.4, 2.5]). *If we take a sufficiently large t_1, then the following holds*:
(1) *Suppose that $g\mathcal{S}_i \cap \mathcal{S}_j$ is nonempty and is relatively compact for some $g \in \Gamma$. Then this intersection is contained in $g\mathcal{S}_{i,\Upsilon} \cap \mathcal{S}_{j,\Upsilon}$.*
(2) *Suppose that $g\mathcal{S}_i \cap \mathcal{S}_j$ is not empty nor relatively compact for some $g \in \Gamma$. Let I be the subset of Υ such that $I(\mathbf{G})$ consists of all $\chi \in \Upsilon(\mathbf{G})$ for which $\chi(\mathcal{A}(\mathcal{S} \cap z_j^{-1} g z_i \mathcal{S}))$ is bounded. Then $z_j^{-1} g z_i \in (U_I M_I)(\mathbb{Q}) \subset P_I$ and*

$$g\mathcal{S}_i \cap \mathcal{S}_j \subset \mathcal{S}_{j,I}, \qquad g^{-1}\mathcal{S}_j \cap \mathcal{S}_i \subset \mathcal{S}_{i,I}.$$

We also need the following.

Lemma 4.2 ([4, Lemma 12.2]). *The union* $\bigcup_{a \in A(t_0)} a^{-1}\eta a$ *is relatively compact, and hence there exists a positive number* C_{12} *such that*

$$d_X(uma \cdot x_0,\, a \cdot x_0) \leq C_{12} \qquad \text{for } \; um \in \eta,\, u \in U,\, m \in M,\, a \in A(t_0),$$

where d_X *is the distance on* X.

Next we consider Busemann functions associated with the geodesic rays corresponding to the edges of the cone $z_j A(t_0) \cdot x_0$.

Definition 4.3 (cf. [11], [2]). Let N be a complete, simply connected Riemannian manifold of nonpositive sectional curvature and let d_N be the distance on N.
(1) Two geodesic rays $\gamma_1, \gamma_2 : [0, \infty) \longrightarrow N$ are called asymptotic if the function $t \longmapsto d_N(\gamma_1(t), \gamma_2(t))$ is uniformly bounded on $[0, \infty)$. Being asymptotic is an equivalence relation on the set of all geodesic rays in N. The equivalence class represented by a geodesic ray γ is denoted by $\gamma(\infty)$.
(2) Let $\gamma : [0, \infty) \longrightarrow N$ be a geodesic ray. The Busemann function $h_\gamma : N \longrightarrow \mathbb{R}$ associated with γ is given by

$$h_\gamma(x) = \lim_{t \to \infty} \{d_N(x, \gamma(t)) - t\} \qquad \text{for } x \in N.$$

For any real number C, we call the set $h_\gamma^{-1}((-\infty, C))$ (resp. $h_\gamma^{-1}(C)$) an open horoball (resp. a horosphere) centered at $\gamma(\infty)$, or associated with γ.

Remark. If a geodesic ray γ_1 is asymptotic to γ_2, then the Busemann function h_{γ_1} differs to h_{γ_2} only by an additive constant.

Let a_0 be the unique point in A such that

$$\xi(a_0) = t_0 \quad \text{for all } \xi \in \Upsilon(\mathbf{G}).$$

In other words, $a_0 \cdot x_0$ is the apex of the cone $A(t_0) \cdot x_0$. For each $\chi \in \Upsilon(\mathbf{G})$, let H_χ be the unit vector in $\mathfrak{a}^+$ such that

$$d\xi(H_\chi) = 0 \quad \text{for all } \xi \in \Upsilon(\mathbf{G}) - \{\chi\}$$

and define a geodesic ray $c_\chi : [0, \infty) \longrightarrow X$ by

$$c_\chi(t) = a_0 \exp(tH_\chi) \cdot x_0 \quad \text{ for } t \geq 0.$$

These geodesic rays are the edges of $A(t_0)\cdot x_0$. Let $c_{j\chi}(t) = z_j c_\chi(t)$. For each j the edges of the cone $z_j A(t_0)\cdot x_0$ are $c_{j\chi}$, $\chi \in \Upsilon(\mathbf{G})$. Let $h_{j\chi}$ be the Busemann function associated with $c_{j\chi}$. We remark that if $\xi \neq \chi$, then the geodesic ray $c_{j\xi}$ is not asymptotic to $gc_{i\chi}$ for any $g \in \Gamma$, $i \in \{1,\dots,\lambda\}$. This can be seen, for example, as follows. Let I, J be the subsets of Υ such that $I(\mathbf{G}) = \Upsilon(\mathbf{G}) - \{\chi\}$, $J(\mathbf{G}) = \Upsilon(\mathbf{G}) - \{\xi\}$. If $\chi \neq \xi$, then we have $I \neq J$. From Mostow's lemma (cf. [18, Lemma 5.7]), the isotropy subgroup of $c_\chi(\infty)$ (resp. $c_\xi(\infty)$) is $\mathbf{P}_I(\mathbb{R})$ (resp. $\mathbf{P}_J(\mathbb{R})$). Suppose that $c_{j\xi}$ is asymptotic to $gc_{i\chi}$. Then, $\mathbf{P}_I$ is conjugate to $\mathbf{P}_J$ by [30, 3.1.9 Theorem]. On the other hand, $\mathbf{P}_I$ and $\mathbf{P}_J$ are standard rational parabolic subgroups of $\mathbf{G}$ with $I \neq J$, and hence they are not conjugate to each other ([3, V. 21.12 Proposition]), which is a contradiction. When $c_{j\chi}$ is asymptotic to $gc_{i\chi}$ for some $g \in \Gamma$, we have

$$h_{j\chi}(gx) = h_{i\chi}(x) + s_{ij,\chi} \qquad \text{for all } x \in X,$$

where $s_{ij,\chi}$ is a constant depending on i, j, χ but not on g (cf. [22, Proposition 3.3]).

For each $\chi \in \Upsilon(\mathbf{G})$, we renormalize the Busemann functions $h_{1\chi},\dots,h_{\lambda\chi}$ as follows. For each $i \in \{1,\dots,\lambda\}$, let $q = q(i) \in \{1,\dots,\lambda\}$ be the smallest index such that there exists an element g of Γ, for which the geodesic ray $gc_{q\chi}$ is asymptotic to $c_{i\chi}$. Let

$$\widetilde{h}_{i\chi} = h_{i\chi} - s_{q(i)i,\chi}.$$

Then we have

Lemma 4.3 ([22, Lemma 3.4])**.** *Suppose that $gc_{i\chi}$ is asymptotic to $c_{j\chi}$ for $g \in \Gamma$. Then*

$$\widetilde{h}_{j\chi}(gx) = \widetilde{h}_{i\chi}(x) \qquad \text{for all } x \in X.$$

By adding a constant to all the functions $\widetilde{h}_{i\chi}$ simultaneously if necessary, we can also assume the following for each j.

$$\left(\bigcap_{\chi\in\Upsilon(\mathbf{G})} \widetilde{h}_{j\chi}^{-1}(0)\right) \cap z_j A\cdot x_0 = \{z_j b_j \cdot x_0\}, \qquad b_j \in A(t_1). \tag{4.5}$$

By using these results, we consider another decomposition of each $\mathcal{S}_j$. In this paragraph we fix one j. Let

$$r_{j\chi} = \chi(b_j) \quad (> t_1)$$

for each $\chi \in \Upsilon(\mathbf{G})$. Let

$$F_{j*} = \{a\cdot x_0 \,|\, \xi(a) > r_{j\xi} \quad \text{for all } \xi \in \Upsilon(\mathbf{G})\}$$

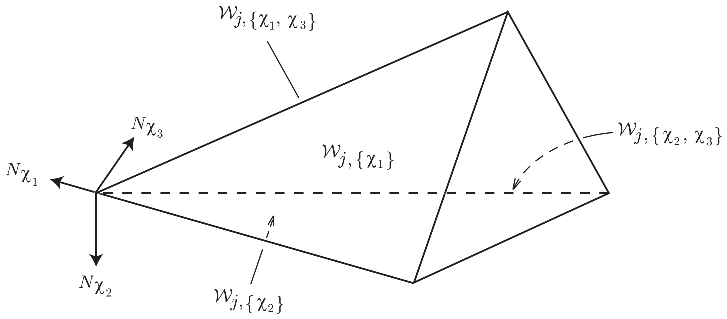


Figure 5. F_{j*} in the $\mathbb{Q}$-rank 3 case: $\Upsilon(\mathbf{G}) = \{\chi_1, \chi_2, \chi_3\}$.

be the cone in $A(t_1) \cdot x_0$ with apex $b_j \cdot x_0$. For each nonempty subset I of Υ, we define a subset $\mathcal{W}_{j,I}$ of $A \cdot x_0$ by

$$\mathcal{W}_{j,I} = \left\{ a \cdot x_0 \;\middle|\; \begin{array}{c} \chi(a) = r_{j\chi} \quad \text{for all } \chi \in I(\mathbf{G}), \\ \xi(a) > r_{j\xi} \quad \text{for all } \xi \in \Upsilon(\mathbf{G}) - I(\mathbf{G}) \end{array} \right\}.$$

These $\mathcal{W}_{j,I}$ form the boundary of the cone F_{j*} (see Figure 5). We regard $A \cdot x_0$ as the $r_{\mathbb{Q}}(\mathbf{G})$-dimensional Euclidean space. For each $\chi \in \Upsilon(\mathbf{G})$, let N_χ be the outer unit normal vector (in $\mathbb{R}^{r_{\mathbb{Q}}(\mathbf{G})}$) of the maximal face $\mathcal{W}_{j,\{\chi\}}$ of F_{j*}. We put

$$F_{j,I} = \left\{ b \cdot x_0 + \sum_{\xi \in I(\mathbf{G})} t_\xi N_\xi \in A(t_0) \cdot x_0 \;\middle|\; \begin{array}{c} b \cdot x_0 \in \mathcal{W}_{j,I}, \\ t_\xi \geq 0 \quad \text{for all } \xi \in I(\mathbf{G}) \end{array} \right\}$$

for each nonempty subset I of Υ. Then we have

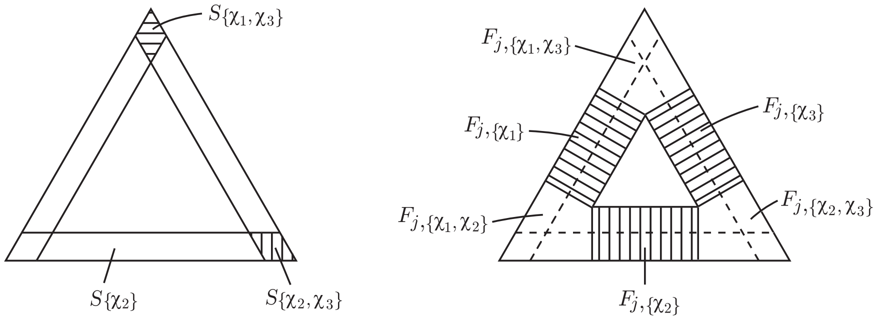
$$A(t_0) \cdot x_0 = F_{j*} \cup \left(\bigcup_{\emptyset \neq I \subset \Upsilon} F_{j,I} \right). \tag{4.6}$$

Let

$$\widetilde{\mathcal{S}}_{j*} = z_j \eta F_{j*}, \quad \widetilde{\mathcal{S}}_{j,I} = z_j \eta F_{j,I}.$$

Then we have a decomposition

$$\mathcal{S}_j = \widetilde{\mathcal{S}}_{j*} \cup \left(\bigcup_{\emptyset \neq I \subset \Upsilon} \widetilde{\mathcal{S}}_{j,I} \right). \tag{4.7}$$



The decomposition by (4.3). The decomposition by (4.6).

Figure 6. The section of $A(t_0)\cdot x_0$ by a hyperplane (in $A\cdot x_0$) transverse to the geodesic ray $\exp(t\rho/|\rho|)\cdot x_0$: $\mathbb{Q}$-rank $=3$, $\Upsilon(\mathbf{G})=\{\chi_1,\chi_2,\chi_3\}$.

We remark (see Figure 6) that $\widetilde{\mathcal{S}}_{j*}\subset\mathcal{S}_{j*}$ and

$$\widetilde{\mathcal{S}}_{j*}\cup\left(\bigcup_{I\neq\Upsilon,\emptyset}\widetilde{\mathcal{S}}_{j,I}\right)\subset\mathcal{S}_j-\mathcal{S}_{j,\Upsilon}.$$

By (4.3), (4.6) we obtained $(\lambda+1)$ different decompositions of $A\cdot x_0$. In this paragraph we fix a nonempty subset I, and compare the locations of various $\mathcal{W}_{j,I}$ when j runs through 1 to λ. We consider them in the Euclidean space $A\cdot x_0$. First of all, let $\mathcal{H}_{j,I}$ be the $(r_{\mathbb{Q}}(\mathbf{G})-\#I)$-dimensional plane containing $\mathcal{W}_{j,I}$. Then these planes $\mathcal{H}_{j,I}$ are mutually parallel. Let $v_{ij}=b_j\cdot x_0-b_i\cdot x_0$. From the definition of $\mathcal{W}_{j,I}$, the set $\{\widetilde{h}_{j\xi}\}_{\xi\in\Upsilon(\mathbf{G})-I(\mathbf{G})}$ of renormalized Busemann functions can be used as a coordinate system on $\mathcal{H}_{j,I}$. More precisely, if $a\cdot x_0$, $a'\cdot x_0\in\mathcal{H}_{j,I}$ and

$$\widetilde{h}_{j\xi}(z_ja\cdot x_0)=\widetilde{h}_{j\xi}(z_ja'\cdot x_0)\qquad\text{for all }\xi\in\Upsilon(\mathbf{G})-I(\mathbf{G}),$$

then $a=a'$. Moreover, from the choice of $b_1,\ldots,b_\lambda$ in (4.5), if $a\cdot x_0\in\mathcal{H}_{j,I}$, $a''\cdot x_0\in\mathcal{H}_{i,I}$ and

$$\widetilde{h}_{j\xi}(z_ja\cdot x_0)=\widetilde{h}_{i\xi}(z_ia''\cdot x_0)\qquad\text{for all }\xi\in\Upsilon(\mathbf{G})-I(\mathbf{G}),$$

we have $a\cdot x_0=a''\cdot x_0+v_{ij}$. Let

$$C_{13}=C_{12}+\max_{1\le j\le\lambda}d_X(x_0,\,b_j\cdot x_0).$$

Lemma 4.4. *Suppose that*

$$x = z_i uma \cdot x_0 \in \mathcal{S}_i, \quad x' = z_j u'm'a' \cdot x_0 \in \mathcal{S}_j, \quad d_X(x_0, x') > C_{13},$$

and $gx = x'$ for some $g \in \Gamma$. Then the following holds:
(1) *For some proper subset I of Υ, $x \in \mathcal{S}_{i,I}$ and $x' \in \mathcal{S}_{j,I}$.*
(2) *The line in $A \cdot x_0$ through the two points $a \cdot x_0 + v_{ij}$ and $a' \cdot x_0$ is perpendicular to $\mathcal{H}_{j,I}$.*

Proof. Since $x' \notin \mathcal{S}_{j,\Upsilon}$, (1) follows from Lemma 4.1. Moreover we have $z_j^{-1} g z_i \in (U_I M_I)(\mathbb{Q}) \subset P_I$. Then the geodesic ray $gc_{i\xi}$ is asymptotic to $c_{j\xi}$ for all $\xi \in \Upsilon(\mathbf{G}) - I(\mathbf{G})$ ([18, Proposition 5.9]). Let $\widetilde{c}_{i\xi}$ (resp. $\widetilde{c}_{j\xi}$) be the geodesic ray in $A(t_0) \cdot x_0$ which corresponds to $\mathcal{W}_{i,\{\xi\}}$ (resp. $\mathcal{W}_{j,\{\xi\}}$). Then $\widetilde{h}_{i\xi}$ (resp. $\widetilde{h}_{j,\xi}$) is the Busemann function with respect to $z_i \widetilde{c}_{i\xi}$ (resp. $z_j \widetilde{c}_{j\xi}$). Note that the values of the Busemann functions with respect to $\widetilde{c}_{i\xi}$, $\widetilde{c}_{j\xi}$ are invariant under the action of UM. Hence we have

$$\widetilde{h}_{i\xi}(z_i uma \cdot x_0) = \widetilde{h}_{i\xi}(z_i a \cdot x_0), \qquad \widetilde{h}_{j\xi}(z_j u'm'a' \cdot x_0) = \widetilde{h}_{j\xi}(z_j a' \cdot x_0). \quad (4.8)$$

On the other hand, from Lemma 4.3, we have

$$\widetilde{h}_{i\xi}(z_i uma \cdot x_0) = \widetilde{h}_{j\xi}(g z_i uma \cdot x_0) = \widetilde{h}_{j\xi}(z_j u'm'a' \cdot x_0)$$

for $\xi \in \Upsilon(\mathbf{G}) - I(\mathbf{G})$. It follows from (4.8) that

$$\widetilde{h}_{i\xi}(z_i a \cdot x_0) = \widetilde{h}_{j\xi}(z_j a' \cdot x_0) \quad (4.9)$$

for $\xi \in \Upsilon(\mathbf{G}) - I(\mathbf{G})$. We take the unique point $b \cdot x_0$ (resp. $b' \cdot x_0$) in the $(r_{\mathbb{Q}}(\mathbf{G}) - \#I)$-dimensional plane $\mathcal{H}_{i,I}$ (resp. $\mathcal{H}_{j,I}$) such that the line through $a \cdot x_0$ (resp. $a' \cdot x_0$) and $b \cdot x_0$ (resp. $b' \cdot x_0$) is perpendicular to this plane. Then we have

$$\widetilde{h}_{i\xi}(z_i b \cdot x_0) = \widetilde{h}_{i\xi}(z_i a \cdot x_0), \qquad \widetilde{h}_{j\xi}(z_j b' \cdot x_0) = \widetilde{h}_{j\xi}(z_j a' \cdot x_0) \quad (4.10)$$

for all $\xi \in \Upsilon(\mathbf{G}) - I(\mathbf{G})$. From (4.9), (4.10), we obtain

$$\widetilde{h}_{i\xi}(z_i b \cdot x_0) = \widetilde{h}_{j\xi}(z_j b' \cdot x_0) \quad \text{for all } \xi \in \Upsilon(\mathbf{G}) - I(\mathbf{G}).$$

Therefore, $b' \cdot x_0 = b \cdot x_0 + v_{ij}$ and $a' - (a + v_{ij})$ is perpendicular to $\mathcal{H}_{j,I}$. □

We can now construct a vector field on V.

For each j, we regard $A \cdot x_0$ as the Euclidean space $\mathbb{R}^{r_{\mathbb{Q}}(\mathbf{G})}$ with origin $O = b_j \cdot x_0$. Let $(r, \theta_1, \ldots, \theta_{r_{\mathbb{Q}}(\mathbf{G})-1})$ be the polar coordinate of $\mathbb{R}^{r_{\mathbb{Q}}(\mathbf{G})}$. We define a vector field Z_j'' on the (open) cone F_{j*} with apex O by

$$\left(Z_j''\right)_{a \cdot x_0} = -\left(\frac{\partial}{\partial r}\right)_{a \cdot x_0} \qquad \text{for all } a \cdot x_0 \in F_{j*}.$$

Let $A_j'(t_0) \cdot x_0$ be the region obtained from $A(t_0) \cdot x_0$ by deleting a closed ball of radius R_j with center O, where R_j is an arbitrary number larger than the distance from O to the apex $a_0 \cdot x_0$ of the cone $A(t_0) \cdot x_0$. We extend the vector field Z_j'' to $A_j'(t_0) \cdot x_0 \cap F_{j,I}$ for each nonempty subset I of Υ: We define $(Z_j'')_x$ to be the vector obtained by the parallel translation in $\mathbb{R}^{r_{\mathbb{Q}}(\mathbf{G})}$ from $(Z_j'')_{b \cdot x_0}$ when

$$x = b \cdot x_0 + \sum_{\xi \in I(\mathbf{G})} t_\xi N_\xi \in A_j'(t_0) \cdot x_0 \cap F_{j,I}, \quad b \cdot x_0 \in \mathcal{W}_{j,I}.$$

Further we define $(Z_j'')_{uma \cdot x_0}$ for $uma \cdot x_0 \in \eta A_j'(t_0) \cdot x_0$ to be the horizontal vector which is mapped to the vector $(Z_j'')_{a \cdot x_0}$ by the Riemannian submersion $\eta A_j'(t_0) \cdot x_0 \longrightarrow A_j'(t_0) \cdot x_0$. We have thus obtained a vector field Z_j'' on $\eta A_j'(t_0) \cdot x_0$ which is smooth on the complement of $\bigcup_{I \neq \emptyset, \Upsilon} \eta \mathcal{W}_{j,I}$.

By using the differential of the left translation L_{z_j}, we define a vector field Z_j' on $\mathcal{S}_j' := z_j \eta A_j'(t_0) \cdot x_0$ by

$$\left(Z_j'\right)_{z_j uma \cdot x_0} = dL_{z_j}\left((Z_j'')_{uma \cdot x_0}\right) \qquad \text{for } \ uma \cdot x_0 \in \eta A_j'(t_0) \cdot x_0.$$

From Lemmas 4.1, 4.4, these vector fields $Z_1', \ldots, Z_\lambda'$ are well patched together to give a vector field Z' on the complement of some compact subset of V. By using a suitable cut-off function, we can extend this Z' to a vector field Z on V.

We recall that $\Pi : X \longrightarrow V$ is the natural projection. Let $v_0 = \Pi(x_0)$ and let $B_R(v_0)$ be the closed geodesic ball in V of radius $R > 0$ around v_0. If we take a sufficiently large R, then Z coincides with Z' on $V - B_R(v_0)$ due to Lemma 4.2. Take a submanifold with smooth boundary including $B_R(v_0)$, and let V_R be its complement. (We can find such a submanifold, for example, by using the exhaustion function constructed in [27]). We have $|Z| \equiv 1$ on the open submanifold V_R of V. Let W be the image under Π of the union of $z_j \eta \partial F_{j*}$ and $z_j \eta(\partial F_{j,I} - \partial(A(t_0) \cdot x_0))$, $1 \leq j \leq \lambda$; $I \neq \emptyset, \Upsilon$, where ∂ means the boundary. Since $\bigcup_{j=1}^{\lambda} z_j \eta A(t_0) \cdot x_0$ is a fundamental open set for $\mathit{\Gamma}$, $W \cap V_R$ is the union of a finite number of closed submanifolds of V_R of codimension 1. From the construction of Z, for any $v \in W$, $Z(v)$ is tangent to W. The complement $V_R - W$ is a disjoint union of a finite number of open submanifolds, say $V_1, \ldots, V_s$. For any given $u \in C_0^\infty(V_R)$, we have the following: If $v \in \partial V_i$, then $u(v) = 0$ or $Z(v)$ is tangent to ∂V_i. Hence, if we find a positive constant C such that div $Z \geq C$ on each V_i, we can apply the similar argument in the proof of Lemma 1 to each V_i to yield

$$\frac{C^2}{4}\left(\int_{V_i} |u|^2\right) \leq \int_{V_i} |\text{grad } u|^2.$$

By taking the sum, we obtain

$$\frac{C^2}{4}\left(\int_{V_R} |u|^2\right) \leq \int_{V_R} |\text{grad } u|^2$$

and the essential spectrum of V_R is contained in $[C^2/4, \infty)$. Therefore, it suffices to find such a constant.

Lemma 4.5. *For any $\varepsilon > 0$, there exists a number $R(\varepsilon) > 0$ such that the following holds. If $R \geq R(\varepsilon)$, then we have*

$$\text{div } Z \geq 2\left(\min_{I \neq \Upsilon} |\rho_I| - \varepsilon\right) \qquad \text{on } V - B_R(v_0). \tag{4.11}$$

Proof. It suffices to show the following for $R \geq R(\varepsilon)$:

$$\text{div } Z \geq 2\left(\min_{I \subset J \neq \Upsilon} |\rho_J| - \varepsilon\right) \qquad \text{on each } \Pi(\widetilde{\mathcal{S}}_{j,I}) - B_R(v_0). \tag{4.12}$$

For this, it suffices to estimate div Z_j'' on $\eta(F_{j,I} \cap A_j'(t_0) \cdot x_0)$.

Let $d = r_{\mathbb{Q}}(\mathbf{G}) - \#I$. We regard $A \cdot x_0$ as the Euclidean space $\mathbb{R}^{r_{\mathbb{Q}}(\mathbf{G})}$ with the origin $b_j \cdot x_0$. Let us consider the d-dimensional subspace containing $\mathcal{W}_{j,I}$. Let $(r, \theta_1, \ldots, \theta_{d-1})$ be its polar coordinate system, where

$$0 \leq r;\ 0 \leq \theta_1, \ldots, \theta_{d-2} \leq \pi/2;\ 0 \leq \theta_{d-1} \leq 2\pi$$

and the last angle θ_{d-1} is counted from the ray

$$\{\exp(t\rho_I/|\rho_I|)\, b_j \cdot x_0 \mid t \geq 0\}$$

in $\mathcal{W}_{j.I}$. (The ray is represented as $(t, \pi/2, \ldots, \pi/2, 0)$ in this coordinate system.) We take a coordinate system $(\nu_1, \ldots, \nu_{n-d})$ of the space $U_I M_I \cdot x_0$, which is diffeomorphic to the $(n-d)$-dimensional Euclidean space. Then $(r, \theta_1, \ldots, \theta_{d-1}, \nu_1, \ldots, \nu_{n-d})$ is a coordinate system of X. We can assume that this coordinate system is compatible with the orientation of X. Let

$$h(\nu_1, \ldots, \nu_{n-d}) d\nu_1 \wedge \cdots \wedge d\nu_{n-d}$$

be the volume element of $U_I M_I \cdot x_0$. Then, from [5, Corollary 4.4], the volume element of X is given by

$$\sqrt{g}\, dr \wedge d\theta_1 \wedge \cdots \wedge d\theta_{d-1} \wedge d\nu_1 \wedge \cdots \wedge d\nu_{n-d},$$

where

$$\sqrt{g} = r^{d-1} e^{-2r|\rho_I| \sin\theta_1 \cdots \sin\theta_{d-1}} f(\theta_1, \dots, \theta_{d-1}) h(\nu_1, \dots, \nu_{n-d}),$$

$$f(\theta_1, \dots, \theta_{d-1}) = e^{-2\langle \rho_I, \log b_j \rangle} \prod_{k=1}^{d-2} \sin^{d-k-1} \theta_k.$$

In this coordinate system, Z_j'' is represented as

$$Z_j'' = -1 \cdot \frac{\partial}{\partial r}.$$

Consequently, we have

$$\text{div } Z_j'' = \frac{1}{\sqrt{g}} \frac{\partial}{\partial r}(-\sqrt{g}) = -\frac{d-1}{r} + 2|\rho_I| \sin\theta_1 \cdots \sin\theta_{d-1}.$$

Thus we have, for $x = umab_j \cdot x_0 \in \eta(F_{j,I} \cap A_j'(t_0) \cdot x_0)$, $u \in U_I, m \in M_I, a \in A_I$,

$$\left(\text{div } Z_j''\right)(x) = -\frac{d-1}{|\log a|} + 2\left\langle \rho_I, \frac{\log a}{|\log a|} \right\rangle \geq -\frac{d-1}{|\log a|} + 2 \min_{I \subset J \neq \Upsilon} |\rho_J|. \quad (4.19)$$

There exists a positive constant $C_{14}(I)$ determined by the angle between ρ and $\mathcal{W}_{j,I}$ such that the following holds: If

$$R \geq R_j(\varepsilon) := |\log b_j| + C_{12} + C_{14}(I) \cdot \frac{d-1}{2\varepsilon},$$

then

$$|\log a| \geq (d-1)/(2\varepsilon) \quad \text{for } x = umab_j \cdot x_0 \in \eta(F_{j,I} \cap A_j'(t_0) \cdot x_0),$$
$$u \in U_I, m \in M_I, a \in A_I \ \text{ with } \Pi(x) \in V_R.$$

If we put

$$R(\varepsilon) = \max_{1 \leq j \leq \lambda,\, I \neq \Upsilon} R_j(\varepsilon),$$

then (4.12) follows from (4.19), Lemma 4.2, and the triangle inequality. □

Proof of Theorem 4 (2). For any $\varepsilon > 0$, take a submanifold V_R with $R \geq R(\varepsilon)$. Then the essential spectrum of V_R is contained in $[(\min_{I \neq \Upsilon} |\rho_I| - \varepsilon)^2, \infty)$

due to Lemma 4.5. Hence, from the decomposition principle, the essential spectrum of V is contained in

$$\bigcap_{0<\varepsilon\ll 1}\left[\left(\min_{I\neq\Upsilon}|\rho_I|-\varepsilon\right)^2,\infty\right)=\left[\min_{I\neq\Upsilon}|\rho_I|^2,\infty\right).$$

On the other hand, we have already seen (in Section 3) that $\left[\min_{I\neq\Upsilon}|\rho_I|^2,\infty\right)$ belongs to the essential spectrum of V. Therefore, $\min_{I\neq\Upsilon}|\rho_I|^2$ is the bottom of the essential spectrum of V. □

Remark. After this paper was written, the author was informed that our construction in this section might be related to the construction in [28].

§5. Manifolds with Corners at Infinity

In this section we discuss some other consequences of Sections 1 and 2.

For any given compact manifold M with boundary ∂M, we can attach $\partial M\times[0,\infty)$ to the boundary to produce a complete Riemannian manifold V and control the bottom of the essential spectrum of V. Let $Y_1,\dots,Y_s$ be the connected components of ∂M. We choose a metric on each $Y_j\times[0,\infty)$ so that the condition $(*)$ in the introduction is satisfied. For example, if

$$\alpha_j(t)=e^{-t^{1+a}},\quad a>0, \tag{5.1}$$

then, from Lemma 1 or Theorem 2, the essential spectrum of V is empty. If

$$\alpha_j(t)=e^{\pm 2\sqrt{c}t}, \tag{5.2}$$

then, from Theorem 1 (1) and Lemma 1 (or Proposition 1), the essential spectrum of V is the half-line $[c,\infty)$ $(c\geq 0)$. In particular, if $\alpha_j(t)=e^{-2\sqrt{c}t}$ for some j, V has a shrinking end. When Y_j admits a fiber bundle structure $Y_j\longrightarrow B_j$ satisfying the conditions $(\star\star)$, $(**)$ in the introduction, we can give an additional structure to the essential spectrum of V: We first deform the metric on Y_j to the metric of such a Riemannian submersion in the part $Y_j\times[0,1]$, and then apply Theorem 1 (2) to $Y_j\times(1,\infty)$. The essential spectrum of V contains a union of half-lines parametrized by the eigenvalues of B_j.

Theorem 1 can be applied to complete manifolds canonically obtained from manifolds with corners. Following [24], a manifold W with corners is a topological manifold with boundary equipped with an embedding $\iota: W\hookrightarrow\widetilde{W}$ into a closed C^∞-manifold for which there exists a finite collection of smooth functions ρ_i on $\widetilde{W}$, $i\in I$, such that

$$\iota(W)=\left\{x\in\widetilde{W}\,\middle|\,\rho_i(x)\geq 0, i\in I\right\}$$

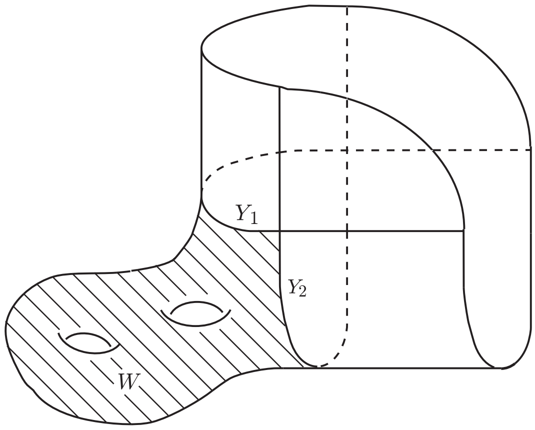


Figure 7. A manifold with corners at infinity: the case $\#I = 2$.

and for each subset $J \subset I$, the differentials $d\rho_i$, $i \in J$, are linearly independent at each point $x \in \widetilde{W}$ where all ρ_i, $i \in J$ vanish. We identify W with the image $\iota(W)$. Let

$$Y_i = W \cap \left\{ x \in \widetilde{W} \,\middle|\, \rho_i(x) = 0 \right\}$$

for each $i \in I$. Then the boundary ∂W of W is the union of the hypersurfaces Y_i, $i \in I$. For any subset $J = \{i_1, \dots, i_k\} \subset I$, we put

$$Y_J = Y_{i_1 \cdots i_k} = Y_{i_1} \cap \cdots \cap Y_{i_k}.$$

We say that Y_J is a corner of codimension k. We assume that W is endowed with a metric which is a product on a neighborhood of the form $(-\varepsilon, 0]^k \times Y_J$ for each corner Y_J of codimension k.

In this situation, we can enlarge W as follows: We first glue half-cylinders $Y_i \times [0, \infty)$ to the codimension 1 corners Y_i to obtain the space W_1. Next we glue $Y_{ij} \times [0, \infty)^2$ to each codimension 2 corner Y_{ij} to get the space W_2. After repeating this procedure, we finally fill $W_{\#I-1}$ with $Y_I \times [0, \infty)^{\#I}$ at the codimension $\#I$ corner Y_I and obtain a complete manifold V. Let us call this V the complete manifold obtained from W by gluing cylinders successively to boundary components, or briefly the complete manifold canonically obtained from W.

Proof of Corollary 1. Apply Theorem 1 (1) to the part $Y_I \times [0, \infty)^{\#I}$ under the condition that the mean curvature $\mathcal{K}(t)$ is identically zero. $\square$

We consider the case $\#I = 2$ further. Let W be an n-dimensional manifold with boundary $\partial W = Y_1 \cup Y_2$, where Y_1 and Y_2 are $(n-1)$-dimensional compact manifolds such that $Y_{12} = Y_1 \cap Y_2$ is the boundary of both Y_1 and Y_2. By deforming its metric (in a compact region) if necessary, we may assume that Y_{12} is orthogonal to both Y_1 and Y_2, and that the metric of W is a product near all Y_1, Y_2 and Y_{12}. We first glue $Y_1 \times [0, \infty)$, $Y_2 \times [0, \infty)$ to Y_1, Y_2, respectively, and then we attach $Y_{12} \times [0, \infty)^2$ to Y_{12} as in Figure 7 to get a complete manifold V. Let

$$\hat{Y}_1 = Y_1 \cup_{Y_{12}} (Y_{12} \times [0, \infty))$$

be the manifold obtained from Y_1 by attaching $Y_{12} \times [0, \infty)$ to Y_{12}, and let

$$\hat{Y}_2 = Y_2 \cup_{Y_{12}} (Y_{12} \times [0, \infty)).$$

Then $\hat{Y}_1, \hat{Y}_2$ have infinite volume and $Y_{12} = \hat{Y}_1 \cap \hat{Y}_2$,

$$V = W \cup (\hat{Y}_1 \times [0, \infty)) \cup (\hat{Y}_2 \times [0, \infty)).$$

Suppose that there exist eigenvalues of $\hat{Y}_1$ and $\hat{Y}_2$. Then we can apply Theorem 1 (2) to the three parts $\hat{Y}_1 \times [0, \infty)$, $\hat{Y}_2 \times [0, \infty)$, and $Y_{12} \times [0, \infty)^2$ under the condition that the mean curvature $\mathcal{K}(t)$ is identically zero. Let $c_{j,0} < c_{j,1} < \cdots < c_{j,m} < \cdots$ be the eigenvalues of $\hat{Y}_1$ if $j = 1$, $\hat{Y}_2$ if $j = 2$, Y_{12} if $j = 3$. We remark that $c_{3,0} = 0$. For each m and any $r \geq 0$, there exists a family $\{u_{j,m,r,q}\}_{q \in \mathbb{N}}$ of compactly supported smooth functions on V such that the following conditions are satisfied.

(5.3) For any compact subset of V, if we take q sufficiently large, then the support of $u_{j,m,r,q}$ lies outside this compact set.

(5.4) For some positive constant C_{15} independent of q, we have

$$\|(\Delta - (c_{j,m} + r^2))u_{j,m,r,q}\| \leq C_{15} \frac{1}{q} \|u_{j,m,r,q}\|.$$

Let φ be an eigenfunction belonging to the eigenvalue $c_{3,m}$. Then we can make a family of compactly supported smooth functions on $\hat{Y}_1$ (resp. $\hat{Y}_2$) satisfying the similar conditions as (5.3), (5.4) by applying Theorem 1 (2). Then, from Proposition 1.2, we can make two families of compactly supported smooth functions on V satisfying the similar conditions as (5.3), (5.4) out of these families. On the other hand, we may suppose that $\{u_{3,m,r,q}\}_{q \in \mathbb{N}}$ is made of φ. Consequently, the eigenfunction φ seems to produce three different Weyl sequences on V. However, if we use parallelograms instead of the disks $\mathcal{D}$, $\mathcal{D}'$ in $\mathbb{R}^2$ in the proof of Theorem 1, these are essentially the same. Such a phenomenon is already observed in the case of higher $\mathbb{Q}$-rank locally symmetric spaces.

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