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Automorphisms of a Polynomial Ring Which Admit Reductions of Type I

By

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Abstract

Recently, Shestakov-Umirbaev solved Nagata's conjecture on an automorphism of a polynomial ring. To solve the conjecture, they defined notions called reductions of types I–IV for automorphisms of a polynomial ring. An automorphism admitting a reduction of type I was first found by Shestakov-Umirbaev. Using a computer, van den Essen–Makar-Limanov–Willems gave a family of such automorphisms. In this paper, we present a new construction of such automorphisms using locally nilpotent derivations. As a consequence, we discover that there exists an automorphism admitting a reduction of type I which satisfies some degree condition for each possible value.

§1. Introduction

Let k be a field of characteristic zero, and $k[\mathbf{x}] = k[x_1, \ldots, x_n]$ the polynomial ring in n variables over k. We will identify an endomorphism $F \in \mathrm{End}_k\, k[\mathbf{x}]$ with the n-tuple $(f_1, \ldots, f_n)$ of elements of $k[\mathbf{x}]$, where $f_i = F(x_i)$ for each i. Then, F is invertible if and only if $k[f_1, \ldots, f_n] = k[\mathbf{x}]$. If this is the case, the sum $\deg F := \sum_{i=1}^{n} \deg f_i$ of the total degrees of f_i's is necessarily at least n. An automorphism $F \in \mathrm{Aut}_k\, k[\mathbf{x}]$ is said to be *affine* if $\deg F = n$, and *elementary* if there exist $i \in \{1, \ldots, n\}$ and a polynomial $\phi \in k[\mathbf{x}]$ not depending on x_i such that $f_i = x_i + \phi$ and $f_j = x_j$ for each $j \neq i$. We say that F admits an *elementary reduction* if there exists an elementary automorphism G such that $\deg(F \circ G) < \deg F$. Note that F admits an elementary

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reduction if and only if there exists $\phi \in k[f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n]$ such that $\deg(f_i + \phi) < \deg f_i$ for some i. The subgroup $\mathrm{T}_k\, k[\mathbf{x}]$ of the automorphism group $\mathrm{Aut}_k\, k[\mathbf{x}]$ generated by affine automorphisms and elementary automorphisms is called the *tame subgroup*, and each element of $\mathrm{T}_k\, k[\mathbf{x}]$ is called a *tame automorphism*.

By Jung [3], it follows that $\mathrm{T}_k\, k[\mathbf{x}] = \mathrm{Aut}_k\, k[\mathbf{x}]$ when $n = 2$. In fact, he showed that each $F \in \mathrm{Aut}_k\, k[\mathbf{x}]$ for $n = 2$ admits an elementary reduction whenever $\deg F > 2$. Thereby,

$$\deg F > \deg(F \circ G_1) > \cdots > \deg(F \circ G_1 \circ \cdots \circ G_r) = 2$$

for some elementary automorphisms $G_1, \ldots, G_r$ of $k[\mathbf{x}]$. Note that a similar result for a field k of an arbitrary characteristic was given by van der Kulk [4].

Now, assume that $n = 3$. Nagata [8] conjectured that the automorphism

$$(x_1 - 2(x_1x_3 + x_2^2)x_2 - (x_1x_3 + x_2^2)^2x_3, x_2 + (x_1x_3 + x_2^2)x_3, x_3) \tag{1.1}$$

of $k[\mathbf{x}]$ is not tame. In 2003, this well-known conjecture was finally solved in the affirmative by Shestakov-Umirbaev [9], [10]. They defined four types of reductions, said to be of types I, II, III and IV, for elements of $\mathrm{Aut}_k\, k[\mathbf{x}]$. Then, they showed that each non-affine element of $\mathrm{T}_k\, k[\mathbf{x}]$ admits an elementary reduction or one of these four types of reductions. One can easily check that Nagata's automorphism admits none of these reductions. Therefore, Nagata's conjecture holds true.

In the present paper, we study automorphisms admitting reductions of type I.

Definition 1.1 [10, Definition 1]**.** Assume that $n = 3$. We say that an automorphism $F = (f_1, f_2, f_3)$ of $k[\mathbf{x}]$ admits a *reduction of type I* if the following conditions hold:

(i) There exists an odd number $s \geq 3$ such that $\deg f_1 : \deg f_2 = 2 : s$.

(ii) $\deg f_1 < \deg f_3 \leq \deg f_2$.

(iii) $\bar{f}_3$ does not belong to $k[\bar{f}_1, \bar{f}_2]$, where $\bar{f}$ denotes the highest homogeneous part of f for each $f \in k[\mathbf{x}]$.

(iv) There exist $\alpha \in k \setminus \{0\}$ and $\phi \in k[f_1, f_2 - \alpha f_3]$ such that $\deg(f_3 + \phi) < \deg f_3$ and $\deg[f_1, f_3 + \phi] < \deg f_2 + \deg[f_1, f_2 - \alpha f_3]$. Here, we define

$$\deg[f, g] = \max \left\{ \deg \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i} \right) \,\Big|\, 1 \leq i < j \leq 3 \right\} + 2 \tag{1.2}$$

for each $f, g \in k[\mathbf{x}]$.

We also say that F admits a reduction of type I if $(f_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)})$ satisfies (i)–(iv) for some permutation σ of $\{1, 2, 3\}$.

We mention that the conditions listed above imply that $\deg(f_2 - \alpha f_3) = \deg f_2$ and that f_1 and $f_2 - \alpha f_3$ form a "$*$-reduced pair", which are also included in the definition of reduction of type I by Shestakov-Umirbaev.

Note that $(f_1, f_2 - \alpha f_3, f_3)$ admits an elementary reduction by (iv), while (f_1, f_2, f_3) does not (cf. [10, Proposition 1]). Shestakov-Umirbaev [10, Example 1] gave the first example of a tame automorphism which admits a reduction of type I in case of $s = 3$. Van den Essen–Makar-Limanov–Willems [1] constructed a family of such automorphisms when $s = 3, 5, 7$ using a computer. Reductions of types II, III and IV are also defined theoretically [10], but no automorphisms admitting these reductions are found. To study the structures of $\mathrm{Aut}_k\, k[\mathbf{x}]$ and $\mathrm{T}_k\, k[\mathbf{x}]$, it is of great importance to investigate automorphisms admitting reductions of these four types.

The purpose of this paper is to construct new automorphisms of $k[\mathbf{x}]$ which admit reductions of type I by employing the theory of locally nilpotent derivations. As a consequence, we discover that there exists a tame automorphism admitting a reduction of type I such that $\deg f_1 : \deg f_2 = 2 : s$ for each odd number $s \geq 3$.

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§2. Tame Automorphisms Admitting Reductions of Type I

Before stating our main result, we prove a lemma. In what follows, we assume that $n = 3$.

Lemma 2.1. *Let $s \geq 3$ be an odd number, and $H = (h_1, h_2, h_3)$ a tame automorphism of $k[\mathbf{x}]$ such that*

$$\deg h_1 : \deg h_2 : \deg h_3 = 2 : s : 1, \quad \frac{s-1}{2} \deg h_1 < \deg(ch_1^s + h_2^2) < \deg h_2$$

for some $c \in k \setminus \{0\}$. Then, $H' = (h_1, h_2', h_3')$ is a tame automorphism admitting a reduction of type I for which $\deg h_2' = \deg h_2$ and $\deg h_3' = \deg(ch_1^s + h_2^2)$. Here, $h_2' = h_2 + h_3 + ch_1^s + h_2^2$ and $h_3' = h_3 + ch_1^s + h_2^2$.

Proof. Let G_1 and G_2 be elementary automorphisms of $k[\mathbf{x}]$ defined by $G_1(x_3) = x_3 + cx_1^s + x_2^2$, $G_2(x_2) = x_2 + x_3$ and $G_i(x_j) = x_j$ for $(i, j) \neq (1, 3), (2, 2)$. Then, $H' = H \circ G_1 \circ G_2$. Hence, H' is a tame automorphism of $k[\mathbf{x}]$, since so is H. By assumption, $\deg h_2$ is greater than $\deg h_3$ and $\deg(ch_1^s + h_2^2)$, while $\deg h_3$ is less than $\deg(ch_1^s + h_2^2)$. Hence, $\deg h_2' = \deg h_2$ and $\deg h_3' = \deg(ch_1^s + h_2^2)$. It follows that

$$l \deg h_1 < \deg h_3' < \deg h_2' = \frac{s}{2} \deg h_1 < (l+1) \deg h_1, \text{ where } l = \frac{s-1}{2}.$$

This implies that $\bar{h}_3'$ does not belong to $k[\bar{h}_1, \bar{h}_2']$, since $\deg h_3' < \deg h_2'$. If $\alpha = 1$, then $h_2' - \alpha h_3' = h_2$, so $\phi := -ch_1^s - h_2^2$ is contained in $k[h_1, h_2' - \alpha h_3']$. The total degree of $h_3' + \phi = h_3$ is less than $\deg h_3'$. In addition,

$$\begin{aligned} \deg[h_1, h_3' + \phi] &= \deg[h_1, h_3] \leq \deg h_1 + \deg h_3 \leq \deg h_2 \\ &= \deg(h_2' - \alpha h_3') < \deg(h_2' - \alpha h_3') + \deg[h_1, h_2' - \alpha h_3']. \end{aligned}$$

Therefore, (h_1, h_2', h_3') satisfies all the conditions of Definition 1.1. □

Now, let p and q be natural numbers, and consider triangular derivations D and E of $k[\mathbf{x}]$ defined by

$$\begin{gathered} D(x_1) = x_2^{q+1}, \quad D(x_2) = 0, \quad D(x_3) = (p+1)x_1^p x_2^q, \\ E(x_1) = 2x_3, \quad E(x_2) = 2(p+1)x_1^p, \quad E(x_3) = 1. \end{gathered} \tag{2.1}$$

Here, we say that a k-derivation Δ of $k[\mathbf{x}]$ is *triangular* if $\Delta(x_{\sigma(i)})$ belongs to $k[x_{\sigma(1)}, \ldots, x_{\sigma(i-1)}]$ for each i for some permutation σ of $\{1, \ldots, n\}$. If this is the case, Δ is *locally nilpotent*, i.e., $\Delta^l(f) = 0$ for sufficiently large l for each $f \in k[\mathbf{x}]$. In particular,

$$f_i = \sum_{l=0}^{\infty} \frac{D^l(x_i)}{l!}, \qquad g_i = \sum_{l=0}^{\infty} \frac{E^l(x_i)}{l!}(-x_3)^l \tag{2.2}$$

are elements of $k[\mathbf{x}]$ for each i. We set $F = (f_1, f_2, f_3)$, $G = (g_1, g_2, x_3)$, and define $h_1 = F(g_1)$, $h_2 = F(g_2)$ and $h_3 = f_3$. Namely, $(h_1, h_2, h_3) = F \circ G$. Put

$$m = pq + p + q, \qquad c = (-2)^{p+1} \prod_{i=1}^{p} \frac{i+1}{2i+1}. \tag{2.3}$$

Here is our main result.

Theorem 2.1. *Let p and q be natural numbers. Then, (h_1, h_2, h_3) is a tame automorphism of $k[\mathbf{x}]$ for $n = 3$ such that*

$$\begin{gathered} \deg h_1 = 2m, \quad \deg h_2 = (2p+1)m, \quad \deg h_3 = m, \\ \deg(c^2 h_1^{2p+1} + h_2^2) = 2pm + p + 1. \end{gathered} \tag{2.4}$$

Note that (h_1, h_2, h_3) satisfies the assumptions of Lemma 2.1 for $s = 2p+1$. Actually, $(s-1)/2 = p$ and

$$p \deg h_1 < 2pm + p + 1 = (2p+1)m - (p+1)q + 1 < \deg h_2.$$

Therefore, we obtain the following corollary to Theorem 2.1.

Corollary 2.1. *There exists a tame automorphism (h_1', h_2', h_3') of $k[\mathbf{x}]$ admitting a reduction of type I such that*

$$\deg h_1' = 2m, \quad \deg h_2' = (2p+1)m, \quad \deg h_3' = 2pm + p + 1$$

for each $p, q \in \mathbf{N}$, where $m = pq + p + q$.

For a triangular derivation Δ, it is known that the exponential map $\exp \Delta : k[\mathbf{x}] \to k[\mathbf{x}]$ is a tame automorphism of $k[\mathbf{x}]$. If furthermore $\Delta(x_n) = 1$, then $\ker \Delta = k[g_1', \ldots, g_{n-1}']$, and $(g_1', \ldots, g_{n-1}', x_n)$ is a tame automorphism of $k[\mathbf{x}]$. Here, we define

$$(\exp \Delta)(f) = \sum_{l=0}^{\infty} \frac{\Delta^l(f)}{l!}, \qquad g_i' = \sum_{l=0}^{\infty} \frac{\Delta^l(x_i)}{l!}(-x_n)^l$$

for each $f \in k[\mathbf{x}]$ and $i = 1, \ldots, n-1$ (cf. [2, Sections 1.3 and 6.1]). Hence, F and G are tame automorphisms of $k[\mathbf{x}]$, and so is $F \circ G = (h_1, h_2, h_3)$. In addition, $E(g_i) = 0$ for $i = 1, 2$. In Section 3, we will consider the polynomial

$$I = x_2 x_3 - x_1^{p+1}.$$

Since $D(I) = 0$, it follows that $F(I) = (\exp D)(I) = I$.

To conclude this section, we give explicit descriptions of f_i and g_j for $i = 1, 2, 3$ and $j = 1, 2$. By a straightforward computation, we get

$$f_1 = x_1 + x_2^{q+1}, \quad f_2 = x_2, \quad g_1 = x_1 - x_3^2. \tag{2.5}$$

We show that

$$f_3 = x_3 + \sum_{i=0}^{p} \binom{p+1}{i+1} x_1^{p-i} x_2^{(q+1)i+q}, \quad g_2 = x_2 + \sum_{i=0}^{p} c_i x_1^{p-i} x_3^{2i+1}, \tag{2.6}$$

where

$$c_i = (-2)^{i+1} \prod_{l=0}^{i} \frac{p-l+1}{2l+1}$$

for each $i \geq 0$. For the first equality of (2.6), it suffices to verify that

$$\frac{D^{i+1}(x_3)}{(p+1)!} = \frac{x_1^{p-i}x_2^{(q+1)i+q}}{(p-i)!} \tag{2.7}$$

for $i = 0, \ldots, p$. We prove (2.7) by induction on i. The case $i = 0$ follows from the definition of D. Assume that (2.7) is true if $i = l$ for some $0 \leq l < p$. Then,

$$\frac{D^{l+2}(x_3)}{(p+1)!} = D\left(\frac{D^{l+1}(x_3)}{(p+1)!}\right) = D\left(\frac{x_1^{p-l}x_2^{(q+1)l+q}}{(p-l)!}\right)$$
$$= \frac{(p-l)x_1^{p-l-1}x_2^{(q+1)l+q}D(x_1)}{(p-l)!} = \frac{x_1^{p-(l+1)}x_2^{(q+1)(l+1)+q}}{(p-(l+1))!}.$$

Hence, (2.7) holds for $i = l+1$, and thus holds for any $0 \leq i \leq p$. Therefore, we have proved the first equality of (2.6). Next, let g_2' be the right-hand side of the second equality of (2.6). Then, $g_2 - g_2' = x_3\psi$ for some $\psi \in k[\mathbf{x}]$. To conclude that $g_2 = g_2'$, it suffices to show that $E(g_2' - g_2) = 0$, since E is locally nilpotent and $E(x_3) \neq 0$ by definition. In fact, for a locally nilpotent derivation Δ of $k[\mathbf{x}]$, the condition $\Delta(\phi\psi) = 0$ implies $\psi = 0$ for $\phi, \psi \in k[\mathbf{x}]$ with $\Delta(\phi) \neq 0$ (cf. [2, Proposition 1.3.32]). It follows that $E(g_2) = 0$ as mentioned. Note that $c_0 = -2(p+1)$, and $2(p-i)c_i = -(2i+3)c_{i+1}$ for $i = 0, \ldots, p$. Hence, we have

$$E(g_2') = E(x_2) + \sum_{i=0}^{p} c_i\left((2i+1)x_1^{p-i}x_3^{2i}E(x_3) + (p-i)x_1^{p-i-1}x_3^{2i+1}E(x_1)\right)$$
$$= 2(p+1)x_1^p + \sum_{i=0}^{p}\left((2i+1)c_ix_1^{p-i}x_3^{2i} + 2(p-i)c_ix_1^{p-i-1}x_3^{2i+2}\right)$$
$$= 2(p+1)x_1^p + \sum_{i=0}^{p}\left((2i+1)c_ix_1^{p-i}x_3^{2i} - (2i+3)c_{i+1}x_1^{p-(i+1)}x_3^{2(i+1)}\right)$$
$$= 2(p+1)x_1^p + \sum_{i=0}^{p}(2i+1)c_ix_1^{p-i}x_3^{2i} - \sum_{i=1}^{p+1}(2i+1)c_ix_1^{p-i}x_3^{2i} = 0.$$

Thus, $E(g_2 - g_2') = E(g_2) - E(g_2') = 0$. Therefore, g_2 is expressed as in (2.6).

§3. Proof of the Main Result

In this section, we prove the four equalities of (2.4).

Let $f = \sum_{\alpha \in \mathbf{Z}^n} c_\alpha \mathbf{x}^\alpha$ be a Laurent polynomial in $x_1, \ldots, x_n$ over k, where $c_\alpha \in k$ and $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for each $\alpha = (\alpha_1, \ldots, \alpha_n)$. Then, we set

$$|f| = \{\alpha \in \mathbf{Z}^n \mid c_\alpha \neq 0\}.$$

For $\eta \in \mathbf{R}^n$, we define $\deg_\eta f$ to be the maximum among the inner products $\alpha \cdot \eta$ for $\alpha \in |f|$, and put

$$f^\eta = \sum_{\alpha \in \mathbf{Z}^n} c'_\alpha \mathbf{x}^\alpha, \quad \text{where} \quad c'_\alpha = \begin{cases} c_\alpha & \text{if } \alpha \cdot \eta = \deg_\eta f \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\deg_\eta f = \deg_\eta f^\eta$ for each $f \in k[\mathbf{x}]$. We note that $(f+g)^\eta$ is equal to one of f^η, g^η and $f^\eta + g^\eta$ for each $f, g \in k[\mathbf{x}]$ with $|f^\eta| \cap |g^\eta| = \emptyset$. For a k-derivation Δ of $k[\mathbf{x}]$, we define a k-derivation Δ^η of $k[\mathbf{x}]$ by setting

$$\Delta^\eta(x_i) = \begin{cases} (\Delta(x_i))^\eta & \text{if } \deg_\eta(\Delta(x_i)x_i^{-1}) = \deg_\eta \Delta \\ 0 & \text{otherwise} \end{cases}$$

for each i, where $\deg_\eta \Delta$ denotes the maximum among $\deg_\eta(\Delta(x_i)x_i^{-1})$ for $i = 1, \ldots, n$. Then, we have $\Delta^\eta(f^\eta) = 0$ for each $f \in \ker \Delta$, for otherwise $0 \neq \Delta^\eta(f^\eta) = (\Delta(f))^\eta$, a contradiction.

Now, we set $\omega_i = \deg f_i$ for $i = 1, 2, 3$, and $\omega = (\omega_1, \omega_2, \omega_3)$. Then,

$$\omega_1 = q + 1, \quad \omega_2 = 1, \quad \omega_3 = pq + p + q = m.$$

For each $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we have

$$\deg F(\mathbf{x}^\alpha) = \deg f_1^{\alpha_1} f_2^{\alpha_2} f_3^{\alpha_3} = \alpha_1\omega_1 + \alpha_2\omega_2 + \alpha_3\omega_3 = \deg_\omega \mathbf{x}^\alpha.$$

Hence, $\deg F(f) \leq \deg_\omega f$ for each $f \in k[\mathbf{x}]$. The equality holds when f^ω is a term. By (2.5) and (2.6), we see that $g_1^\omega = -x_3^2$ and $g_2^\omega = c_p x_3^{2p+1}$ are terms, so $\deg h_i = \deg F(g_i) = \deg_\omega g_i$ for $i = 1, 2$. Hence, $\deg h_1 = 2m$ and $\deg h_2 = (2p+1)m$. In addition, $\deg h_3 = \deg f_3 = m$. Thus, we have proved the first three equalities of (2.4).

Next, we consider the polynomial $P := c^2 g_1^{2p+1} + g_2^2$. Our goal is to establish that $\deg F(P) = 2pm + p + 1$, which immediately implies the last equality of (2.4). Write $P = P_1 + P_2$, where $\phi = g_2 - x_2$,

$$P_1 = c^2 g_1^{2p+1} + \phi^2 \quad \text{and} \quad P_2 = x_2^2 + 2\phi x_2.$$

Set $\epsilon = (1, 0, -2)$. Then, g_1^{2p+1} and ϕ^2 belong to $x_3^{2(2p+1)} k[\mathbf{x}^\epsilon]$, since g_1 and ϕ are in $x_3^2 k[\mathbf{x}^\epsilon]$ and $x_3^{2p+1} k[\mathbf{x}^\epsilon]$ by (2.5) and (2.6), respectively. Hence,

$$P_1^\omega = c' \mathbf{x}^{u\epsilon} x_3^{2(2p+1)} = c' x_1^u x_3^{2(2p-u+1)}, \text{ where } u \geq 0, \ c' \in k \setminus \{0\}.$$

We claim that $u \neq 0$. In fact, the monomial $x_3^{2(2p+1)}$ appears in g_1^{2p+1} and ϕ^2 with coefficients -1 and c_p^2, respectively. By definition, $c_p = c$. Hence,

$x_3^{2(2p+1)}$ does not appear in P_1, so we get $u \neq 0$. On the other hand, $P_2^\omega = x_2(x_2 + 2\phi)^\omega = 2x_2\phi^\omega = 2cx_2x_3^{2p+1}$. Clearly, $|P_1^\omega| \cap |P_2^\omega| = \emptyset$. Hence, P^ω must be equal to P_1^ω or P_2^ω or $P_1^\omega + P_2^\omega$. Recall that $E(g_i) = 0$ for $i = 1, 2$. So, $E(P) = 0$. This implies that $E^\omega(P^\omega) = 0$ as mentioned. A straightforward computation shows that

$$\begin{aligned}
&\deg_\omega(E(x_1)x_1^{-1}) = \deg_\omega x_1^{-1}x_3 = -(q+1) + m = pq + p - 1,\\
&\deg_\omega(E(x_2)x_2^{-1}) = \deg_\omega x_1^p x_2^{-1} = p(q+1) - 1 = pq + p - 1,\\
&\deg_\omega(E(x_3)x_3^{-1}) = \deg_\omega x_3^{-1} = -m = -pq - p - q < pq + p - 1.
\end{aligned}$$

Accordingly, we get $E^\omega(x_i) = (E(x_i))^\omega = E(x_i)$ for $i = 1, 2$ and $E^\omega(x_3) = 0$. Then, it follows that $E^\omega(P_i^\omega) \neq 0$ for $i = 1, 2$. Consequently,

$$\begin{aligned}
P^\omega = P_1^\omega + P_2^\omega &= c' x_1^u x_3^{2(2p-u+1)} + 2cx_2x_3^{2p+1},\\
0 = E^\omega(P^\omega) &= 2uc' x_1^{u-1} x_3^{4p-2u+3} + 4c(p+1)x_1^p x_3^{2p+1},
\end{aligned}$$

so $u = p + 1$ and $c' = -2c$. Therefore, we get

$$P^\omega = -2cx_1^{p+1}x_3^{2p} + 2cx_2x_3^{2p+1} = 2cx_3^{2p}(x_2x_3 - x_1^{p+1}) = 2cx_3^{2p} I.$$

Hence, $\deg_\omega P = 2pm + m + 1$. Since $F(I) = I$ as mentioned,

$$\deg F(P^\omega) = \deg(2cf_3^{2p} I) = 2pm + p + 1. \tag{3.1}$$

Finally, let $Q = P - P^\omega$. Since $F(P) = F(P^\omega) + F(Q)$, it remains only to verify that $\deg F(Q) < 2pm + p + 1$ by (3.1). Observe that P and Q belong to $x_2^2 k[x_2^{-1}x_3^{2p+1}, \mathbf{x}^\epsilon]$. Furthermore, $\deg_\omega \mathbf{x}^\epsilon = q + 1 - 2m < 0$, and

$$\deg_\omega x_2^{-1}x_3^{2p+1} = (2p+1)m - 1 = -(p+1)\deg_\omega \mathbf{x}^\epsilon.$$

Hence, $\deg_\omega Q \equiv \deg_\omega P \pmod{\deg_\omega \mathbf{x}^\epsilon}$. Since $\deg_\omega Q < \deg_\omega P$, we get

$$\begin{aligned}
\deg_\omega Q &\leq \deg_\omega P + \deg_\omega \mathbf{x}^\epsilon = 2pm + m + 1 + q + 1 - 2m\\
&= 2pm + p + 1 - p(q+2) + 1 < 2pm + p + 1.
\end{aligned}$$

Thus, $\deg F(Q) \leq \deg_\omega Q < 2pm + p + 1$, and thereby proving the last equality of (2.4).

§4. Remarks

As far as we know, the answer to the following simple question is not known.

Question 4.1. Do there exist polynomials $f, g \in k[\mathbf{x}]$ for $n = 3$ as follows?

(i) $k[f, g, h] = k[\mathbf{x}]$ for some $h \in k[\mathbf{x}]$.

(ii) $\deg(f^3 + g^2) \leq \deg f$.

This question is closely related to the study of $\mathrm{Aut}_k\, k[\mathbf{x}]$ for $n = 3$. In fact, no automorphism of $k[\mathbf{x}]$ admits a reduction of type II or III or IV if the answer to Question 4.1 is negative. The reason is as follows.

Suppose that there exists an automorphism of $k[\mathbf{x}]$ admitting a reduction of type II or III or IV. Then, it follows from [10, Definitions 2, 3 and 4] that there exists an automorphism (g_1, g_2, g_3) as follows:

(1) $\deg g_1 = 2l$ and $\deg g_2 = 3l$ for some $l \in \mathbf{N}$.

(2) There exists $\phi \in k[g_1, g_2] \setminus k$ with $\deg \phi \leq 2l$ such that $\bar{\phi}$ and $\bar{g}_1$ are linearly independent over k.

Since $\deg \phi \leq \deg g_1$ and $\deg \phi < \deg g_2$, the condition (2) implies that $\bar{\phi} \notin k[\bar{g}_1, \bar{g}_2]$. Write $\phi = \sum_{i,j} c_{i,j} g_1^i g_2^j$, where $c_{i,j} \in k$ for each i and j. Let u_1 and u_2 be the maximal numbers such that $c_{u_1,j'} \neq 0$ and $c_{i',u_2} \neq 0$ for some j' and i', and let q_i and r_i respectively be the quotient and residue of u_i divided by e_i for $i = 1, 2$, where

$$e_1 = \frac{\deg g_2}{\gcd(\deg g_1, \deg g_2)} = 3, \quad e_2 = \frac{\deg g_1}{\gcd(\deg g_1, \deg g_2)} = 2.$$

Then, due to [9, Theorem 3] (see also [5], [7] and [11]), it follows that

$$\begin{aligned}\deg \phi &\geq q_i(\mathrm{lcm}(\deg g_1, \deg g_2) - \deg g_1 - \deg g_2 + \deg[g_1, g_2]) + r_i \deg g_i \\ &\geq q_i(l + 2) + r_i \deg g_i\end{aligned}$$

for $i = 1, 2$. Since $\deg g_1 = 2l$ and $\deg \phi \leq 2l$ by assumption, (q_1, r_1) must be $(0, 1)$ or $(1, 0)$. Hence, $u_1 = 3q_1 + r_1$ is equal to 1 or 3. Similarly, $(q_2, r_2) = (1, 0)$, and so $u_2 = 2$. In particular, $u_1 \leq 3$ and $u_2 = 2$. The polynomials $g_1^i g_2^j$ for $i = 0, 1, 2, 3$ and $j = 0, 1, 2$ with $(i, j) \neq (3, 0), (0, 2)$ have distinct total degrees. This implies that $c_{i,j} = 0$ for each (i, j) with $2i + 3j > 6$, while $c_{3,0} \neq 0$ and $c_{0,2} \neq 0$, for otherwise $\bar{\phi} = c_{i,j} \bar{g}_1^i \bar{g}_2^j$ for some (i, j), which contradicts that $\bar{\phi} \notin k[\bar{g}_1, \bar{g}_2]$. Hence,

$$\phi = c_{3,0} g_1^3 + c_{0,2} g_2^2 + c_{1,1} g_1 g_2 + c_{2,0} g_1^2 + c_{0,1} g_2 + c_{1,0} g_1 + c_{0,0}, \tag{4.1}$$

in which $c_{3,0} \neq 0$ and $c_{0,2} \neq 0$. Without loss of generality, we may assume that $c_{0,2} = 1$. Then, (4.1) is expressed as

$$\phi = c_{3,0} \hat{f}^3 + \hat{g}^2 + b\hat{f} + c, \text{ where } \hat{f} = g_1 + a,\ \hat{g} = g_2 + \frac{c_{1,1}}{2} g_1 + \frac{c_{0,1}}{2} \tag{4.2}$$

and $a, b, c \in k$. Indeed, $\phi = c_{3,0}g_1^3 + \hat{g}^2 + c'_{2,0}g_1^2 + c'_{1,0}g_1 + c'_{0,0}$ for some $c'_{2,0}, c'_{1,0}, c'_{0,0} \in k$. Then, we have (4.2) for $a = c'_{0,2}/(3c_{3,0})$ and some $b, c \in k$. Finally, put $f = c_{3,0}\hat{f}$ and $g = c_{3,0}\hat{g}$. Clearly, $\deg f = \deg g_1 = 2l$, $\deg g = \deg g_2 = 3l$, and $k[f, g, g_3] = k[\mathbf{x}]$. Moreover,

$$\deg(f^3 + g^2) = \deg c_{3,0}^2(c_{3,0}\hat{f}^3 + \hat{g}^2) \leq 2l = \deg f$$

by (4.2), since the total degrees of ϕ and $b\hat{f} + c$ are at most $2l$. Therefore, f and g satisfy the conditions of Question 4.1.

It is worthwhile to mention that, if there exists a tame automorphism (h_1, h_2, h_3) with $\deg h_1 : \deg h_2 : \deg h_3 = 2 : 3 : 1$ and $\deg(ch_1^3 + h_2^2) \leq \deg h_1$ for some $c \in k \setminus \{0\}$, then we can construct a tame automorphism admitting a reduction of type II or III. On the other hand, if $p = 1$, then (2.4) gives that $\deg h_1 = 2m$, $\deg h_2 = 3m$, $\deg h_3 = m$ and $\deg(c^2h_1^3 + h_2^2) = 2m + 2$. In this case, we have $m = 2q + 1$ and

$$\frac{\deg h_1}{\deg(c^2h_1^3 + h_2^2)} = \frac{2m}{2m + 2} \to 1 \qquad (q \to \infty),$$

although $\deg(c^2h_1^3 + h_2^2) > \deg h_1$.

Assume that $f, g \in k[\mathbf{x}]$ are algebraically independent over k for which $\deg f : \deg g = r : s$, where $r, s \in \mathbf{N}$ with $2 \leq r < s$ and $\gcd(r, s) = 1$. Then, it easily follows from [9, Theorem 3] (see also [5, Theorem 2.1]) that

$$\deg(f^s + g^r) > \begin{cases} \deg g & \text{if } r \geq 3 \\ \deg f & \text{if } r = 2 \text{ and } s \geq 5. \end{cases}$$

Hence, $\deg(f^s + g^r) \leq \deg f$ is possible only if $(r, s) = (2, 3)$. We define $f, g \in k[x_1, x_2]$ by

$$(4.3) \qquad f = -x_1^{4l}x_2^{2(2m-1)} - 2x_1^l x_2^m, \quad g = x_1^{6l}x_2^{3(2m-1)} + 3x_1^{3l}x_2^{3m-1} + \frac{3}{2}x_2,$$

where $l, m \in \mathbf{N}$. Then, $\deg f : \deg g = 2 : 3$. Moreover, f and g are algebraically independent over k, and

$$f^3 + g^2 = x_1^{3l}x_2^{3m} + \frac{9}{4}x_2^2,$$

since $f = -x_1^l x_2^m(\mathbf{x}^\alpha + 2)$ and $g = x_2(\mathbf{x}^{2\alpha} + 3\mathbf{x}^\alpha + 3/2)$, where $\alpha = (3l, 3m - 2)$. In particular, $\deg(f^3 + g^2) = \deg f$ if $l = m = 1$, and $\deg(f^3 + g^2) < \deg f$ otherwise. If k is of characteristic $r > 0$, then $f = x_1^{rl}$ and $g = x_2 + x_1^{sl}$ satisfy $f^s - g^r = -x_2^r$ for any $l, s \in \mathbf{N}$. Hence, $\deg(f^s - g^r) \leq \deg f$ in this case.

Note: Instead of Question 4.1, the author first asked a question whether there exist $f, g \in k[\mathbf{x}]$ with $\deg(f^3 + g^2) \leq \deg f$ which are algebraically independent over k. In answer to the question, Prof. Hiraku Kawanoue informed him of an example satisfying $\deg(f^3 + g^2) = \deg f$. The example (4.3) is a modification of Kawanoue's example by the author.

Recently, the author [6] showed that no tame automorphism of $k[\mathbf{x}]$ for $n = 3$ admits a reduction of type IV.

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