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Counting Lines and Conics on a Surface

Dedicated to Professor Friedrich Hirzebruch on his 80th birthday

By

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§1. Main Results

This short note is a supplementary remark to the author's article [7]. Our main results are the following two propositions concerning the number of rational curves and elliptic curves on polarized complex algebraic surfaces:

Proposition A. *Let $X \subset \mathbb{P}_{\mathbb{C}}^{\frac{h}{2}+1}$ be a $K3$ surface of degree $h \geq 4$ and let $r_d = r_d(X)$ denote the number of rational curves of degree d on X, $d \in \mathbb{Z}_{>0}$. If $h > 4N^2$ for a positive integer N, then $r_1 + 2r_2 + \cdots + Nr_N \leq \dfrac{24Nh}{h - 4N^2}$.* In particular, $r_1 \leq \dfrac{24h}{h-4}$ for $h \geq 6$ and $r_2 \leq \dfrac{24h}{h-16}$ for $h \geq 18$.

Proposition B. *Let $X \subset \mathbb{P}_{\mathbb{C}}^N$ be a canonically embedded surface of degree K^2 and put $\sigma = c_2/K^2 \geq 1/3$ (as usual, K and c_2 stand for the canonical divisor and the topological Euler number of X). Let $r_d = r_d(X)$ be the number of rational curves of degree d on X and let $s = s(X) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ denote the sum $\sum_C KC$ of the degrees of the elliptic curves $C \subset X$.*

(1) *Assume that $\sigma < 1 + \dfrac{4}{N} + \dfrac{6}{N^2}$ for some positive integer N. Then $\displaystyle\sum_{d=1}^{N} dr_d \leq \frac{(3\sigma - 1)\left(1 + \frac{2}{N}\right)}{1 - \sigma + \frac{4}{N} + \frac{6}{N^2}} K^2$. For instrance $r_1 \leq \dfrac{9\sigma - 3}{11 - \sigma} K^2$ if $\sigma < 11$ (i.e., if X is not a quintic $\subset \mathbb{P}^3$) and $r_2 \leq \dfrac{6\sigma - 2}{9 - 2\sigma} K^2$ if $\sigma < \dfrac{9}{2}$.*

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(2) *If* $\sigma < 1$, *then* $s \le \dfrac{3\sigma - 1}{1-\sigma} K^2$.

Proof of Propositions A and B. Let X be a smooth complex projective surface of non-negative Kodaira dimension. Pick a rational number $\alpha \in [0,1]$ and a finite sum $C = \sum C_i$ of distinct irreducible curves $C_i \subset X$ of geometric genera g_i. In [7], we defined an "orbibundle" $\widetilde{\mathcal{E}}_\alpha$ attached to the triplet (X, C, α) and showed the Miyaoka-Yau-Sakai inequality $3c_2(\widetilde{\mathcal{E}}_\alpha) \ge c_1^2(\widetilde{\mathcal{E}}_\alpha)$ or, more explicitly,

$$(\dagger) \qquad \frac{\alpha^2}{2}\left(C^2 + 3CK - 6\,(g-1)\right) - 2\alpha\left(CK - 3\,(g-1)\right) + 3c_2 - K^2 \ge 0.$$

Here $g-1$ is construed as the sum $\sum (g_i - 1)$; see *ibid.*, §1, Remark G.

Fix a very ample divisor H of degree $h = H^2$ on X. Assume that C_i is rational or elliptic (*i.e.*, $g_i - 1 = -1$ or 0). The formula ($\dagger$) involves four parameters α, C^2, CK and $g-1$ (= the number of rational curves with opposite sign). Denoting by δ the total degree $CH = \sum C_i H$, we bound C^2 by δ^2/h (the Hodge index theorem). This substitution simplifies ($\dagger$) in two important cases:

CASE A: (X, H) is a polarized K3 surface ($K = 0$) of degree h. Then

$$\alpha^2\left(\frac{\delta^2}{h} - 6\,(g-1)\right) + 12\alpha(g-1) + 144 \ge 0 \quad \text{for } \alpha \in [0,1].$$

CASE B: X is a canonical surface ($H = K$). For $\alpha \in [0,1]$, we have

$$\alpha^2\left(\frac{\delta^2}{K^2} + 3\delta - 6\,(g-1)\right) - 2\alpha\left(2\delta - 6\,(g-1)\right) + 6c_2 - 2K^2 \ge 0.$$

We view the left hand sides of these two inequalities as quadratic functions $Q_1(\alpha), Q_2(\alpha)$ of α, which attain the minima at $\alpha_1, \alpha_2 \in [0,1] \cap \mathbb{Q}$. We readily elicit Propositions A and B from the inequalities $Q_i(\alpha_i) \ge 0$, $i = 1, 2$. □

Remarks. (1) A general projective K3 surface X is known to carry countably many nodal rational curves [3] as well as a one-parameter family of nodal elliptic curves [8]; *i.e.*, $\sum_d^\infty d r_d(X) = \infty$, $s(X) = \infty$.

(2) The author has no idea how close to the best possible our estimates are. Nor does he know if there are any precedent results, apart from a handful of treatises that study either lines on surfaces in $\mathbb{P}^3_{\mathbb{C}}$ [11], [12], [2], [1] or configurations of disjoint smooth rational curves [6], [9], [10]. As shown in §§2 and 3, the inequality ($\dagger$) is optimal for countably many examples.

(3) Unfortunately, (†) does not say anything about the classical problem of counting lines on surfaces in $\mathbb{P}^3_{\mathbb{C}}$. We are luckier when dealing with lines on complete intersections of codimension two or more; elementary, but cumbersome, calculation of explicit bounds is left to the reader.

§2. Two Examples

Take four points P_1, P_2, P_3, P_4 in general position on $\mathbb{P}^2$. The line L_{ij} connecting P_i and P_j is defined by a linear form λ_{ij}, and the ratios $\lambda_{ij}/\lambda_{kl}$ are rational functions on $\mathbb{P}^2$. Fixing a positive integer $n \geq 2$, the n-th roots of these ratios define a Kummer extension K_n of $\mathbb{C}(\mathbb{P}^2)$ with Galois group $(\mathbb{Z}/n\mathbb{Z})^{\oplus 5}$ (*cf.* Hirzebruch [5]).

Let $X_1 \to \mathbb{P}^2$ be the blowing up at the four points P_i. X_1 is a del Pezzo surface of degree five (unique up to isomorphisms). Denote by $E_i \subset X_1$ the exceptional curve over P_i and let $\widetilde{L}_{ij} \subset X_1$ be the strict transform of the line $L_{ij} \subset \mathbb{P}^2$. The reduced curve $D = \bigcup_{i=1}^4 E_i \cup \bigcup_{i,j} \widetilde{L}_{ij}$ is the union of all the (-1)-curves on X_1 and linearly equivalent to $-2K_{X_1}$. D has 15 double points and its smooth part consists of 10 components, each of which isomorphic to $\mathbb{P}^1$ minus three points. The minimal model of the function field K_n is realized as a finite covering $\pi_n \colon X_n \to X_1$ with branch locus D of constant ramification index n. Standard invariants of X_n are given by:

$$K_{X_n} = \left(\frac{1}{2} - \frac{1}{n}\right)\pi_n^* D, \qquad c_2(X_n) = n^5\left(2 - \frac{10}{n} + \frac{15}{n^2}\right).$$

Specifically, X_5 is a ball quotient with $K^2 = 5^4 \times 9 = 3c_2$ (see *ibid*).

The pullback of E_i or $\widetilde{L}_{ij}$ via $\pi_n \colon X_n \to X_1$ is divisible by n and supported by n^2 disjoint curves, each of which being isomorphic to the Fermat curve $x^n + y^n + z^n = 0$ of genus $(n-1)(n-2)/2$. The half of the ramification locus $H_n = R_n/2 = \pi_n^* D/(2n)$ turns out to be an integral, very ample divisor on X_n. Thus i) (X_n, H_n) is a polarized surface of degree $5n^3$, ii) the ramification locus R_n of $\pi_n \colon X_n \to X_1$ consists of $10n^2$ irreducible components, all isomorphic to the Fermat curve of degree n, and iii) $K_{X_n} = (n-2)H_n$. By choosing 2 and 3 as values of n, we obtain two examples for which our upper bound of r_2 in Proposition A and that of s in Proposition B are respectively attained:

Example A. (X_2, H_2) is a K3 surface of degree 40, with the effective divisor $R_2 \sim 2H_2$ consisting of 40 conics. Thus $r_2(X_2) \geq 40 = \dfrac{24 \times 40}{40 - 16}$.

Example B. (X_3, H_3) is a canonical surface with $K^2 = 3^3 \times 5$, $c_2 = 3^3 \times 3$ (*i.e.*, $\sigma = 3/5$). The divisor R_3 is a union of 90 copies of the Fermat cubic curve, so that the total degree of the elliptic curves $s(X_3)$ is at least $90 \times 3 = \dfrac{3\sigma - 1}{3(1-\sigma)} K^2$.

§3. A Concluding Remark

The *Hirzebruch proportionality principle* explains why (†) turns into an equality for the examples in the previous section.

Let $\mathbb{B}^2 \subset \mathbb{C}^2$ denote the unit ball $\mathrm{PU}(1,2)/\mathrm{P}\,(\mathrm{U}(1,1) \times \mathrm{U}(1))$ equipped with the Bergmann metric. Let $\Gamma_0 \subset \mathrm{PU}(1,2)$ be a discrete, torsion-free, cocompact subgroup of the holomorphic isometries of $\mathbb{B}^2$. Consider a Γ_0-stable curve $\Delta \subset \mathbb{B}^2$ with only normal crossing singularities (Δ may have countably many irreducible components). Let $\Gamma \subset \mathrm{PU}(1,2)$ be a subgroup which satisfies the following four conditions:

(1) Γ contains Γ_0 as a normal subgroup with $\Gamma/\Gamma_0 \simeq (\mathbb{Z}/m\mathbb{Z})^{\oplus r}$.

(2) $X_1 = \Gamma\backslash\mathbb{B}^2$ is nonsingular.

(3) The action of Γ preserves Δ.

(4) The projection $\pi\colon Y = \Gamma_0\backslash\mathbb{B}^2 \to X_1 = \Gamma\backslash\mathbb{B}^2$ is a Kummer cover branching along $D = \Gamma\backslash\Delta \subset X_1$ with constant ramification index m.

Then $D \subset X_1$ is necessarily a divisor with only normal crossings. The "orbibundle" $\widetilde{\mathcal{E}}_{1-\frac{1}{m}}$ on X constructed in [7] from the pair (X_1, D) is identified with Ω^1_Y in this case, so that $c_1^2\left(\widetilde{\mathcal{E}}_{1-\frac{1}{m}}\right) = 3c_2\left(\widetilde{\mathcal{E}}_{1-\frac{1}{m}}\right)$ by the Hirzebruch proportionality theorem [4]. If there is another Kummer cover $p_n\colon X_n \to X_1$ branching along the same divisor D, but with a smaller ramification index $n < m$, then the same orbibundle $\widetilde{\mathcal{E}}_{1-\frac{1}{m}}$ on (X_1, D) can be viewed as the orbibundle $\widetilde{\mathcal{E}}_{1-\frac{n}{m}}$ associated with (X_n, R_n), where $R_n = p_n^* D/n$ is the ramification locus of $X_n \to X_1$. This is precisely the case in Examples A and B, where $(m, n) = (5, 2)$ and $(5, 3)$. Recalling that the inequality (†) is essentially the Miyaoka-Yau-Sakai inequality $3c_2(\widetilde{\mathcal{E}}_\alpha) \geq c_1^2(\widetilde{\mathcal{E}}_\alpha)$, we see that (†) is indeed an equality when $(X, C, \alpha) = \left(X_n, R_n, \frac{m-n}{m}\right)$.

The construction above produces *countably many* examples of (X, C, α) to which the attached orbibundle $\widetilde{\mathcal{E}}_\alpha$ satisfies $3c_2(\widetilde{\mathcal{E}}_\alpha) = c_1^2(\widetilde{\mathcal{E}}_\alpha)$. It is another question, however, if we can find infinitely many such triples with C being a union of curves of small genera (a union of rational or elliptic curves, for example).

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