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Existence and Nonexistence of Traveling Waves for a Nonlocal Monostable Equation

By

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Abstract

We consider the nonlocal analogue of the Fisher-KPP equation

$$u_t = \mu * u - u + f(u),$$

where μ is a Borel-measure on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$ and f satisfies $f(0) = f(1) = 0$ and $f > 0$ in $(0, 1)$. We do not assume that μ is absolutely continuous with respect to the Lebesgue measure. The equation may have a standing wave solution whose profile is a monotone but discontinuous function. We show that there is a constant c_* such that it has a traveling wave solution with speed c when $c \geq c_*$ while no traveling wave solution with speed c when $c < c_*$, provided $\int_{y\in\mathbb{R}} e^{-\lambda y} d\mu(y) < +\infty$ for some positive constant λ. In order to prove it, we modify a recursive method for abstract monotone discrete dynamical systems by Weinberger. We note that the monotone semiflow generated by the equation is not compact with respect to the compact-open topology. We also show that it has no traveling wave solution, provided $f'(0) > 0$ and $\int_{y\in\mathbb{R}} e^{-\lambda y} d\mu(y) = +\infty$ for all positive constants λ.

§1. Introduction

In 1930, Fisher [8] introduced the reaction-diffusion equation $u_t = u_{xx} + u(1 - u)$ as a model for the spread of an advantageous form (allele) of a single gene in a population of diploid individuals. He [9] found that there is a constant c_* such that the equation has a traveling wave solution with speed c when $c \geq c_*$

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while it has no such solution when $c < c_*$. Kolmogorov, Petrovsky and Piskunov [16] obtained the same conclusion for a monostable equation $u_t = u_{xx} + f(u)$ with a more general nonlinearity f, and investigated long-time behavior in the model. Since the pioneering works, there have been extensive studies on traveling waves and long-time behavior for monostable evolution systems.

In this paper, we consider the following nonlocal analogue of the Fisher-KPP equation:

$$u_t = \mu * u - u + f(u). \tag{1.1}$$

Here, μ is a Borel-measure on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$ and the convolution is defined by

$$(\mu * u)(x) := \int_{y\in\mathbb{R}} u(x-y)d\mu(y)$$

for a bounded and Borel-measurable function u on $\mathbb{R}$. The nonlinearity f is a Lipschitz continuous function on $\mathbb{R}$ with $f(0) = f(1) = 0$ and $f > 0$ in $(0, 1)$. Then, $G(u) := \mu * u - u + f(u)$ is a map from the Banach space $L^\infty(\mathbb{R})$ into $L^\infty(\mathbb{R})$ and it is Lipschitz continuous. (We note that $u(x-y)$ is a Borel-measurable function on $\mathbb{R}^2$, and $\|u\|_{L^\infty(\mathbb{R})} = 0$ implies $\|\mu * u\|_{L^1(\mathbb{R})} \leq \int_{y\in\mathbb{R}}(\int_{x\in\mathbb{R}} |u(x-y)|dx)d\mu(y)=0$.) So, because the standard theory of ordinary differential equations works, we have well-posedness of the equation (1.1) and it generates a flow in $L^\infty(\mathbb{R})$.

For the nonlocal monostable equation, Atkinson and Reuter [1] first studied existence and nonexistence of traveling wave solutions. Schumacher [21, 22] showed that there is the minimal speed c_* of traveling wave solutions and it has a traveling wave solution with speed c when $c \geq c_*$, provided the extra condition $f(u) \leq f'(0)u$ and some little ones. Here, we say that the solution $u(t, x)$ is *a traveling wave solution with profile* ψ *and speed* c, if $u(t, x) \equiv \psi(x - x_0 + ct)$ holds for some constant x_0 with $0 \leq \psi \leq 1$, $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$. Further, Coville, Dávila and Martínez [6] proved the following theorem:

Theorem ([6]). *Suppose the nonlinearity* $f \in C^1(\mathbb{R})$ *satisfies* $f'(1) < 0$ *and the Borel-measure* μ *has a density function* $J \in C(\mathbb{R})$ *with*

$$\int_{y\in\mathbb{R}} (|y| + e^{-\lambda y})J(y)dy < +\infty$$

for some positive constant λ. *Then, there exists a constant* c_* *such that the equation* (1.1) *has a traveling wave solution with monotone profile and speed* c *when* $c \geq c_*$ *while it has no such solution when* $c < c_*$.

Recently, the author [29] also obtained the following:

Theorem ([29]). *Suppose there exists a positive constant λ such that*

$$\int_{y\in\mathbb{R}} e^{\lambda|y|} d\mu(y) < +\infty$$

holds. Then, there exists a constant c_* such that the equation (1.1) *has a traveling wave solution with monotone profile and speed c when $c \geq c_*$ while it has no periodic traveling wave solution with average speed c when $c < c_*$.*

Here, a solution $\{u(t,x)\}_{t\in\mathbb{R}} \subset L^\infty(\mathbb{R})$ to (1.1) *is said to be a periodic traveling wave solution with average speed c, if there exists a positive constant τ such that $u(t+\tau,x) = u(t,x+c\tau)$ holds for all t and $x \in \mathbb{R}$ with $0 \leq u(t,x) \leq 1$, $\lim_{x\to+\infty} u(t,x) = 1$ and $\|u(t,x)-1\|_{L^\infty(\mathbb{R})} \neq 0$.*

The goal of this paper is to improve this result of [29], and the following two theorems are the main results:

Theorem 1. *Suppose there exists a positive constant λ such that*

$$\int_{y\in\mathbb{R}} e^{-\lambda y} d\mu(y) < +\infty$$

holds. Then, there exists a constant c_* such that the equation (1.1) *has a traveling wave solution with monotone profile and speed c when $c \geq c_*$ while it has no periodic traveling wave solution with average speed c when $c < c_*$.*

Here, a solution $\{u(t,x)\}_{t\in\mathbb{R}} \subset L^\infty(\mathbb{R})$ to (1.1) *is said to be a traveling wave solution with monotone profile and speed c, if there exists a monotone nondecreasing function ψ on $\mathbb{R}$ with $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$ such that $u(t,x) \equiv \psi(x+ct)$ holds. Also, a solution $\{u(t,x)\}_{t\in\mathbb{R}} \subset L^\infty(\mathbb{R})$ to* (1.1) *is said to be a periodic traveling wave solution with average speed c, if there exists a positive constant τ such that $u(t+\tau,x) = u(t,x+c\tau)$ holds for all t and $x \in \mathbb{R}$ with $0 \leq u(t,x) \leq 1$, $\lim_{x\to+\infty} u(t,x) = 1$ and $\|u(t,x)-1\|_{L^\infty(\mathbb{R})} \neq 0$.*

Remark. If a solution is *a traveling wave* with speed c, then it is *a periodic traveling wave* with average speed c. The converse may not hold. In fact, if $\mu(\mathbb{Z}) = 1$ holds and there exists a positive constant λ such that $\int_{y\in\mathbb{R}} e^{-\lambda y} d\mu(y) < +\infty$ holds, then there exists a solution u to (1.1) such that u is not a traveling wave but u is a periodic traveling wave with monotone profile. Here, we represent the idea of the proof. By Theorem 1, there exist a positive constant c and a sequence $\{u_n(t)\}_{n\in\mathbb{Z}} \subset C^1(\mathbb{R})$ such that

$$\frac{du_n}{dt}(t) = \left(\sum_{m\in\mathbb{Z}} \mu(\{m\}) u_{n-m}(t)\right) - u_n(t) + f(u_n(t)),$$

$$u_n(t) \le u_{n+1}(t),\ \lim_{n\to-\infty} u_n(t) = 0,\ \lim_{n\to+\infty} u_n(t) = 1$$

and

$$u_n\left(t + \frac{1}{c}\right) = u_{n+1}(t)$$

hold. Then, the function $u(t,x) := u_{[x]}(t)$ is a periodic traveling wave with the minimal period $\frac{1}{c}$, where $[\cdot]$ is Gauss' symbol.

Theorem 2. *Suppose the measure μ satisfies*

$$\int_{y\in\mathbb{R}} e^{-\lambda y} d\mu(y) = +\infty$$

for all positive constants λ. Suppose the nonlinearity $f \in C^1(\mathbb{R})$ satisfies

$$f'(0) > 0.$$

Then, the equation (1.1) *has no periodic traveling wave solution.*

Remark. When $\int_{y\in\mathbb{R}} e^{-\lambda y} d\mu(y) = +\infty$ holds for all positive constants λ and $f'(0) = 0$ holds, we do not know whether there exists a (periodic) traveling wave. In Section 5 below, we only see Theorem 19 and Lemma 20.

In these results, we do not assume that the measure μ is absolutely continuous with respect to the Lebesgue measure. For example, not only the integro-differential equation

$$\frac{\partial u}{\partial t}(t,x) = \int_0^1 u(t, x-y)dy - u(t,x) + f(u(t,x))$$

but also the discrete equation

$$\frac{\partial u}{\partial t}(t,x) = u(t, x-1) - u(t,x) + f(u(t,x))$$

satisfies the assumption of Theorem 1 for the measure μ. In order to prove these results, we employ the recursive method for monotone dynamical systems by Weinberger [25] and Li, Weinberger and Lewis [17]. We note that the semiflow generated by the equation (1.1) is not compact with respect to the compact-open topology. See Propositions 16 and 17 of [29] for a simple condition for the profile of a standing wave solution (i.e., a traveling wave solution with speed 0) to be a discontinuous function.

Schumacher [21, 22], Carr and Chmaj [3] and Coville, Dávila and Martínez [6] also studied uniqueness of traveling wave solutions. In [6], we could see an interesting example of nonuniqueness, where the equation (1.1) admits infinitely many monotone profiles for standing wave solutions but it admits no continuous one. See, e.g., [5, 7, 10, 11, 12, 13, 14, 15, 18, 19, 23, 24, 26, 27, 28] on traveling waves and long-time behavior in various monostable evolution systems, [2, 4, 30] nonlocal bistable equations and [20] Euler equation.

In Section 2, we recall abstract results for monotone semiflows from [29]. The results give abstract conditions such that a semiflow satisfying the conditions has a traveling wave solution with speed c when $c \geq c_*$ while it has no periodic traveling wave solution with average speed c when $c < c_*$. In Section 3, we also repeat the proof given in [29] for reader's convenience. In Section 4, we give basic facts for nonlocal equations in $L^\infty(\mathbb{R})$. In Section 5, we prove Theorem 1. In Section 6, we recall a result on spreading speeds by Weinberger [25]. In Section 7, we prove Theorem 2. In a sequel [30] to this paper, the author shall refer several results from this paper.

§2. Abstract Results for Monotone Semiflows

In this section, we recall some abstract results for monotone semiflows from [29]. (In Section 3 below, we would prove them for reader's convenience.) Put a set of functions on $\mathbb{R}$;

$$\mathcal{M} := \{u \,|\, u \text{ is a monotone nondecreasing}$$
$$\text{and left continuous function on } \mathbb{R} \text{ with } 0 \leq u \leq 1\}.$$

The followings are basic conditions for discrete dynamical systems on $\mathcal{M}$:

Hypotheses 3. *Let Q_0 be a map from $\mathcal{M}$ into $\mathcal{M}$.*

(i) *Q_0 is continuous in the following sense: If a sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{M}$ converges to $u \in \mathcal{M}$ uniformly on every bounded interval, then the sequence $\{Q_0[u_k]\}_{k\in\mathbb{N}}$ converges to $Q_0[u]$ almost everywhere.*

(ii) *Q_0 is order preserving; i.e.,*

$$u_1 \leq u_2 \Longrightarrow Q_0[u_1] \leq Q_0[u_2]$$

for all u_1 and $u_2 \in \mathcal{M}$. Here, $u \leq v$ means that $u(x) \leq v(x)$ holds for all $x \in \mathbb{R}$.

(iii) *Q_0 is translation invariant; i.e.,*

$$T_{x_0} Q_0 = Q_0 T_{x_0}$$

for all $x_0 \in \mathbb{R}$. Here, T_{x_0} is the translation operator defined by $(T_{x_0}[u])(\cdot) := u(\cdot - x_0)$.

(iv) Q_0 *is monostable*; *i.e.*,

$$0 < \gamma < 1 \Longrightarrow \gamma < Q_0[\gamma]$$

for all constant functions γ.

Remark. 1°. If Q_0 satisfies Hypothesis 3 (iii), then Q_0 maps constant functions to constant functions. 2°. The semiflow generated by a map Q_0 satisfying Hypotheses 3 may not be compact with respect to the compact-open topology.

We add the following conditions to Hypotheses 3 for continuous dynamical systems on $\mathcal{M}$:

Hypotheses 4. *Let* $Q := \{Q^t\}_{t\in[0,+\infty)}$ *be a family of maps from* $\mathcal{M}$ *to* $\mathcal{M}$.

(i) Q *is a semigroup*; *i.e.*, $Q^t \circ Q^s = Q^{t+s}$ *for all* t *and* $s \in [0,+\infty)$.

(ii) Q *is continuous in the following sense*: *Suppose a sequence* $\{t_k\}_{k\in\mathbb{N}} \subset [0,+\infty)$ *converges to* 0, *and* $u \in \mathcal{M}$. *Then, the sequence* $\{Q^{t_k}[u]\}_{k\in\mathbb{N}}$ *converges to* u *almost everywhere.*

From [29], we recall the following two results for continuous dynamical systems on $\mathcal{M}$:

Theorem 5. *Let* Q^t *be a map from* $\mathcal{M}$ *to* $\mathcal{M}$ *for* $t \in [0,+\infty)$. *Suppose* Q^t *satisfies* Hypotheses 3 *for all* $t \in (0,+\infty)$, *and* $Q := \{Q^t\}_{t\in[0,+\infty)}$ Hypotheses 4. *Then, the following holds*:

Let $c \in \mathbb{R}$. *Suppose there exist* $\tau \in (0,+\infty)$ *and* $\phi \in \mathcal{M}$ *with* $(Q^\tau[\phi])(x - c\tau) \le \phi(x)$, $\phi \not\equiv 0$ *and* $\phi \not\equiv 1$. *Then, there exists* $\psi \in \mathcal{M}$ *with* $\psi(-\infty) = 0$ *and* $\psi(+\infty) = 1$ *such that* $(Q^t[\psi])(x - ct) \equiv \psi(x)$ *holds for all* $t \in [0,+\infty)$.

Theorem 6. *Let* Q^t *be a map from* $\mathcal{M}$ *to* $\mathcal{M}$ *for* $t \in [0,+\infty)$. *Suppose* Q^t *satisfies* Hypotheses 3 *for all* $t \in (0,+\infty)$, *and* $Q := \{Q^t\}_{t\in[0,+\infty)}$ Hypotheses 4. *Then, there exists* $c_* \in (-\infty,+\infty]$ *such that the following holds*:

Let $c \in \mathbb{R}$. *Then, there exists* $\psi \in \mathcal{M}$ *with* $\psi(-\infty) = 0$ *and* $\psi(+\infty) = 1$ *such that* $(Q^t[\psi])(x - ct) \equiv \psi(x)$ *holds for all* $t \in [0,+\infty)$ *if and only if* $c \ge c_*$.

Remark. For Theorem 6, we note that there do not exist a constant c and $\psi \in \mathcal{M}$ with $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$ such that $(Q^t[\psi])(x - ct) \equiv \psi(x)$ holds for all $t \in [0,+\infty)$ if and only if $c_* = +\infty$.

§3. Proof of the Abstract Theorems

In Section 3 of [29], the author already proved Theorems 5 and 6. He modified the recursive method introduced by Weinberger [25] and Li, Weinberger and Lewis [17]. In this section, we would repeat the proof for reader's convenience. Some results stated in this section would be also useful to a sequel [30].

Lemma 7. *Let a sequence* $\{u_k\}_{k\in\mathbb{N}}$ *of monotone nondecreasing functions on* $\mathbb{R}$ *converge to a continuous function* u *on* $\mathbb{R}$ *almost everywhere. Then,* $\{u_k\}_{k\in\mathbb{N}}$ *converges to* u *uniformly on every bounded interval.*

Proof. Let $C \in (0, +\infty)$ and $\varepsilon \in (0, +\infty)$. Then, there exists $\delta \in (0, +\infty)$ such that, for any y_1 and $y_2 \in [-C-1, +C+1]$, $|y_2 - y_1| < \delta$ implies $|u(y_2) - u(y_1)| < \varepsilon/4$. We take $N \in \mathbb{N}$ and a sequence $\{x_n\}_{n=1}^N$ such that $\lim_{k\to\infty} u_k(x_n) = u(x_n)$, $-C-1 \le x_1 \le -C$, $x_n < x_{n+1} < x_n + \delta$ and $+C \le x_N \le +C+1$ hold.

Let $k \in \mathbb{N}$ be sufficiently large. Then, $\max\{|u_k(x_n) - u(x_n)|\}_{n=1}^N < \varepsilon/4$ holds. Let $x \in [-C, +C]$. There exists n such that $x_n \le x \le x_{n+1}$ holds. So, we get $|u_k(x) - u(x)| \le |u_k(x_n) - u(x)| + |u_k(x_{n+1}) - u(x)| \le |u_k(x_n) - u(x_n)| + |u(x_n) - u(x)| + |u_k(x_{n+1}) - u(x_{n+1})| + |u(x_{n+1}) - u(x)| < \varepsilon$. □

The set of discontinuous points of a monotone function on $\mathbb{R}$ is at most countable. So, if a sequence $\{u_k\}_{k\in\mathbb{N}}$ of monotone functions on $\mathbb{R}$ converges to a monotone function u on $\mathbb{R}$ at every continuous point of u, then it converges to u almost everywhere. The converse also holds:

Lemma 8. *Let a sequence* $\{u_k\}_{k\in\mathbb{N}}$ *of monotone nondecreasing functions on* $\mathbb{R}$ *converge to a monotone nondecreasing function* u *on* $\mathbb{R}$ *almost everywhere. Then,* $\lim_{k\to\infty} u_k(x) = u(x)$ *holds for all continuous points* $x \in \mathbb{R}$ *of* u.

Proof. For $n \in \mathbb{N}$, we take $\underline{x}_n \in (x - 2^{-n}, x]$ and $\overline{x}_n \in [x, x + 2^{-n})$ satisfying $\lim_{k\to\infty} u_k(\underline{x}_n) = u(\underline{x}_n)$ and $\lim_{k\to\infty} u_k(\overline{x}_n) = u(\overline{x}_n)$. Then, $u(\underline{x}_n) \le \liminf_{k\to\infty} u_k(x) \le \limsup_{k\to\infty} u_k(x) \le u(\overline{x}_n)$ holds. Hence, we have $\lim_{k\to\infty} u_k(x) = u(x)$ as x is a continuous point of u. □

Hypotheses 3 imply more strong continuity than Hypothesis 3 (i):

Proposition 9. *Let a map* $Q_0 : \mathcal{M} \to \mathcal{M}$ *satisfy* Hypotheses 3 (i), (ii) *and* (iii). *Suppose a sequence* $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{M}$ *converges to* $u \in \mathcal{M}$ *almost everywhere. Then,* $\lim_{k\to\infty}(Q_0[u_k])(x) = (Q_0[u])(x)$ *holds for all continuous points* $x \in \mathbb{R}$ *of* $Q_0[u]$.

Proof. We take a cutoff function $\rho \in C^\infty(\mathbb{R})$ with

$$|x| \geq 1/2 \implies \rho(x) = 0,$$

$$|x| < 1/2 \implies \rho(x) > 0$$

and

$$\int_{x\in\mathbb{R}} \rho(x)dx = 1.$$

For $n \in \mathbb{N}$, we put smooth functions

$$\rho_n(\cdot) := 2^n\rho(2^n\cdot),$$

$$\underline{u}^n(\cdot) := (\rho_n * u)(\cdot - 2^{-(n+1)})$$

and

$$\overline{u}^n(\cdot) := (\rho_n * u)(\cdot + 2^{-(n+1)})$$

Then, we obtain

$$u(\cdot - 2^{-n}) \leq \underline{u}^n(\cdot) \leq u(\cdot) \leq \overline{u}^n(\cdot) \leq u(\cdot + 2^{-n}).$$

The sequence $\{\min\{u_k, \underline{u}^n\}\}_{k\in\mathbb{N}}$ converges to $\underline{u}^n$ almost everywhere, and $\{\max\{u_k, \overline{u}^n\}\}_{k\in\mathbb{N}}$ also $\overline{u}^n$. Hence, by Lemma 7, the sequence $\{\min\{u_k, \underline{u}^n\}\}_{k\in\mathbb{N}}$ converges to $\underline{u}^n$ uniformly on every bounded interval, and $\{\max\{u_k, \overline{u}^n\}\}_{k\in\mathbb{N}}$ also $\overline{u}^n$. Then, by Hypothesis 3 (i), the sequence $\{Q_0[\min\{u_k, \underline{u}^n\}]\}_{k\in\mathbb{N}}$ converges to $Q_0[\underline{u}^n]$ almost everywhere, and $\{Q_0[\max\{u_k, \overline{u}^n\}]\}_{k\in\mathbb{N}}$ also $Q_0[\overline{u}^n]$. From Hypothesis 3 (ii), $Q_0[\min\{u_k, \underline{u}^n\}] \leq Q_0[u_k] \leq Q_0[\max\{u_k, \overline{u}^n\}]$ holds. Therefore, $Q_0[\underline{u}^n] \leq \liminf_{k\to\infty} Q_0[u_k] \leq \limsup_{k\to\infty} Q_0[u_k] \leq Q_0[\overline{u}^n]$ holds almost everywhere. So, by Hypotheses 3 (ii) and (iii), $Q_0[u](\cdot - 2^{-n}) \leq \liminf_{k\to\infty} Q_0[u_k](\cdot) \leq \limsup_{k\to\infty} Q_0[u_k](\cdot) \leq Q_0[u](\cdot + 2^{-n})$ holds almost everywhere. Hence, $\lim_{k\to\infty} Q_0[u_k](\cdot) = Q_0[u](\cdot)$ holds almost everywhere, because $\lim_{n\to\infty} Q_0[u](\cdot - 2^{-n}) = \lim_{n\to\infty} Q_0[u](\cdot + 2^{-n}) = Q_0[u](\cdot)$ holds almost everywhere. So, from Lemma 8, $\lim_{k\to\infty}(Q_0[u_k])(x) = (Q_0[u])(x)$ holds for all continuous points $x \in \mathbb{R}$ of $Q_0[u]$. ◻

Combining Proposition 9 with Helly's theorem, we can make the argument of Weinberger [25] and Li, Weinberger and Lewis [17] work to obtain the following proposition. It states that existence of suitable *super*-solutions of the form $\{v_n(x+cn)\}_{n=0}^{\infty}$ implies existence of traveling wave solutions with speed c in the discrete dynamical systems on $\mathcal{M}$:

Proposition 10. *Let a map* $Q_0 : \mathcal{M} \to \mathcal{M}$ *satisfy* Hypotheses 3, *and* $c \in \mathbb{R}$. *Suppose there exists a sequence* $\{v_n\}_{n=0}^{\infty} \subset \mathcal{M}$ *with* $(Q_0[v_n])(x-c) \leq v_{n+1}(x)$, $\inf_{n=0,1,2,\cdots} v_n(x) \not\equiv 0$ *and* $\liminf_{n\to\infty} v_n(x) \not\equiv 1$. *Then, there exists* $\psi \in \mathcal{M}$ *with* $(Q_0[\psi])(x-c) \equiv \psi(x)$, $\psi(-\infty) = 0$ *and* $\psi(+\infty) = 1$.

Proof. We put $w(\cdot) := \lim_{h\downarrow +0} \inf_{n=0,1,2,\cdots} v_n(\cdot - h)$, and $u_0^k := 2^{-k} w \in \mathcal{M}$ for $k \in \mathbb{N}$. We also take functions $u_n^k \in \mathcal{M}$ such that

$$u_n^k(\cdot) = \max\{Q_0[u_{n-1}^k](\cdot - c), 2^{-k} w(\cdot)\} \tag{3.1}$$

holds for k and $n \in \mathbb{N}$.

We show $u_n^k \leq u_{n+1}^k$. We have $u_0^k \leq u_1^k$. As $u_{n-1}^k \leq u_n^k$ holds, we get $Q_0[u_{n-1}^k] \leq Q_0[u_n^k]$ and $u_n^k \leq u_{n+1}^k$. So, we have

$$u_n^k \leq u_{n+1}^k. \tag{3.2}$$

In virtue of (3.2), we put $u^k := \lim_{n\to\infty} u_n^k \in \mathcal{M}$. Then, by (3.1) and Proposition 9,

$$u^k(\cdot) = \max\{Q_0[u^k](\cdot - c), 2^{-k} w(\cdot)\} \tag{3.3}$$

holds. Because $\lim_{m\to\infty} Q_0[u^k(\cdot + m)] = Q_0[u^k(+\infty)]$ holds from Proposition 9, we have

$$\begin{aligned} u^k(+\infty) &= \lim_{m\to\infty} \max\{Q_0[u^k](m-c), 2^{-k} w(m)\} \\ &= \lim_{m\to\infty} \max\{Q_0[u^k(\cdot + m)](-c), 2^{-k} w(m)\} \\ &= \max\{Q_0[u^k(+\infty)], 2^{-k} w(+\infty)\}. \end{aligned}$$

Hence, $u^k(+\infty) \geq Q_0[u^k(+\infty)]$ and $u^k(+\infty) \geq 2^{-k} w(+\infty) > 0$ hold. So, because $\{\gamma \in \mathbb{R} \,|\, 0 \leq \gamma \leq 1,\, \gamma \geq Q_0[\gamma]\} \subset \{0, 1\}$ holds from Hypothesis 3 (iv), we obtain

$$u^k(+\infty) = 1. \tag{3.4}$$

We show $u_n^k \leq v_n$. We get $u_0^k \leq w \leq v_0$. As $u_{n-1}^k \leq v_{n-1}$ holds, we have

$$Q_0[u_{n-1}^k](\cdot - c) \leq Q_0[v_{n-1}](\cdot - c) \leq v_n(\cdot)$$

and $u_n^k \le v_n$ because of $2^{-k} w \le w \le v_n$. So, we have

$$u_n^k \le v_n. \tag{3.5}$$

From (3.5), we see

$$u^k(-\infty) \le \lim_{m\to\infty} \liminf_{n\to\infty} v_n(-m) < 1. \tag{3.6}$$

Also, $\lim_{m\to\infty} Q_0[u^k(\cdot - m)] = Q_0[u^k(-\infty)]$ holds from Proposition 9. Hence, by (3.3), we have

$$u^k(-\infty) = \lim_{m\to\infty} \max\{Q_0[u^k](-m-c), 2^{-k}w(-m)\} \ge Q_0[u^k(-\infty)].$$

So, from Hypothesis 3 (iv) and (3.6), we obtain

$$u^k(-\infty) = 0. \tag{3.7}$$

In virtue of (3.4) and (3.7), there exists x_k such that $u^k(-x_k) \le 1/2 \le \lim_{h\downarrow+0} u^k(-x_k + h)$ for $k \in \mathbb{N}$. We put $\psi^k(\cdot) := u^k(\cdot - x_k) \in \mathcal{M}$. Then, we have

$$\psi^k(0) \le 1/2 \le \lim_{h\downarrow+0} \psi^k(h) \tag{3.8}$$

and

$$\psi^k(\cdot) = \max\{Q_0[\psi^k](\cdot - c), 2^{-k}w(\cdot - x_k)\} \tag{3.9}$$

from (3.3). By Helly's theorem, there exist a subsequence $\{k(n)\}_{n\in\mathbb{N}}$ and $\psi \in \mathcal{M}$ such that $\lim_{n\to\infty} \psi^{k(n)}(x) = \psi(x)$ holds for all continuous points $x \in \mathbb{R}$ of ψ. So, from (3.8), (3.9) and Proposition 9,

$$\psi(0) \le 1/2 \le \lim_{h\downarrow+0} \psi(h) \tag{3.10}$$

and

$$\psi(\cdot) = Q_0[\psi](\cdot - c) \tag{3.11}$$

holds. Because $\psi(-\infty) = Q_0[\psi(-\infty)]$ and $\psi(+\infty) = Q_0[\psi(+\infty)]$ also hold by (3.11) and Proposition 9, from Hypothesis 3 (iv) and (3.10), we have $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$. □

In the discrete dynamical system on $\mathcal{M}$ generated by a map Q_0 satisfying Hypotheses 3, if there is a *periodic* traveling wave *super*-solution with *average* speed c, then there is a traveling wave solution with speed c:

Theorem 11. *Let a map $Q_0 : \mathcal{M} \to \mathcal{M}$ satisfy* Hypotheses 3, *and $c \in \mathbb{R}$. Suppose there exist $\tau \in \mathbb{N}$ and $\phi \in \mathcal{M}$ with $({Q_0}^\tau[\phi])(x - c\tau) \leq \phi(x)$, $\phi \not\equiv 0$ and $\phi \not\equiv 1$. Then, there exists $\psi \in \mathcal{M}$ with $(Q_0[\psi])(x - c) \equiv \psi(x)$, $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$.*

Proof. We take functions $v_n \in \mathcal{M}$ for $n = 0, 1, 2, \cdots$ such that

$$v_{n+m\tau} = ({Q_0}^n[\phi])(\cdot - cn)$$

holds for all $n = 0, 1, 2, \cdots, \tau - 1$ and $m = 0, 1, 2, \cdots$. Then, we see

$$v_{n+1}(\cdot) \geq Q_0[v_n](\cdot - c) \tag{3.12}$$

and

$$\liminf_{n\to\infty} v_n = \inf_{n=0,1,2,\cdots} v_n = \min_{n=0,1,2,\cdots,\tau-1} v_n. \tag{3.13}$$

We show $v_n(+\infty) > 0$. We have $v_0(+\infty) > 0$. As $v_{n-1}(+\infty) > 0$ holds, we get $v_n(+\infty) \geq Q_0[v_{n-1}(+\infty)] > 0$ by (3.12), Proposition 9, Hypotheses 3 (ii) and (iv). So, we have $v_n(+\infty) > 0$. Hence, because $\lim_{m\to\infty} \min_{n=0,1,2,\cdots,\tau-1} v_n(m) > 0$ holds, from (3.13), we see $\inf_{n=0,1,2,\cdots} v_n \not\equiv 0$. Because $\min_{n=0,1,2,\cdots,\tau-1} v_n \leq \phi$ holds, by (3.13) and $\phi(-\infty) < 1$, we have $\liminf_{n\to\infty} v_n \not\equiv 1$. Therefore, by Proposition 10, there exists $\psi \in \mathcal{M}$ with $Q_0[\psi](\cdot - c) = \psi(\cdot)$, $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$. □

Lemma 12. *Let a sequence $\{u_k\}_{k\in\mathbb{N}}$ of monotone nondecreasing functions on $\mathbb{R}$ converge to a monotone nondecreasing function u on $\mathbb{R}$ almost everywhere. Then, $\lim_{k\to\infty} u_k(x_k) = u(x)$ holds for all continuous points $x \in \mathbb{R}$ of u and sequences $\{x_k\}_{k\in\mathbb{N}} \subset \mathbb{R}$ with $\lim_{k\to\infty} x_k = x$.*

Proof. We put $h_n := \sup_{k=n,n+1,n+2,\cdots} |x_k - x|$ for $n \in \mathbb{N}$. Then, $u_k(\cdot - h_n) \leq u_k(\cdot + (x_k - x)) \leq u_k(\cdot + h_n)$ holds when $k \geq n$. Hence, $u(\cdot - h_n) \leq \liminf_{k\to\infty} u_k(\cdot + (x_k - x)) \leq \limsup_{k\to\infty} u_k(\cdot + (x_k - x)) \leq u(\cdot + h_n)$ holds almost everywhere. So, $\lim_{k\to\infty} u_k(\cdot + (x_k - x)) = u(\cdot)$ holds almost everywhere, because $\lim_{n\to\infty} u(\cdot - h_n) = \lim_{n\to\infty} u(\cdot + h_n) = u(\cdot)$ holds almost everywhere. Hence, from Lemma 8, $\lim_{k\to\infty} u_k(x_k) = \lim_{k\to\infty} u_k(x + (x_k - x)) = u(x)$ holds. □

The infimum c_* of the speeds of traveling wave solutions is not $-\infty$, and there is a traveling wave solution with speed c when $c \geq c_*$:

Theorem 13. *Suppose a map* $Q_0 : \mathcal{M} \to \mathcal{M}$ *satisfies* Hypotheses 3. *Then, there exists* $c_* \in (-\infty, +\infty]$ *such that the following holds*:

Let $c \in \mathbb{R}$. *Then, there exists* $\psi \in \mathcal{M}$ *with* $(Q_0[\psi])(x - c\tau) \equiv \psi(x)$, $\psi(-\infty) = 0$ *and* $\psi(+\infty) = 1$ *if and only if* $c \geq c_*$.

Proof. [Step 1] Let $c_* \in [-\infty, +\infty]$ denote the infimum of $c \in \mathbb{R}$ such that there exists $\psi \in \mathcal{M}$ with $Q_0[\psi](\cdot - c) = \psi(\cdot)$, $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$. Then, we have the following: *Let* $c \in \mathbb{R}$. *Then, there exists* $\psi \in \mathcal{M}$ *with* $Q_0[\psi](\cdot - c) = \psi(\cdot)$, $\psi(-\infty) = 0$ *and* $\psi(+\infty) = 1$ *only if* $c \geq c_*$.

[Step 2] In this step, we show the following: *If* $c \in (c_*, +\infty)$, *then there exists* $\psi \in \mathcal{M}$ *with* $Q_0[\psi](\cdot - c) = \psi(\cdot)$, $\psi(-\infty) = 0$ *and* $\psi(+\infty) = 1$.

There exist $c_0 \in (-\infty, c)$ and $\phi \in \mathcal{M}$ with $Q_0[\phi](\cdot - c_0) = \phi(\cdot)$, $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$. Then, because we have $Q_0[\phi](\cdot - c) \leq \phi(\cdot)$, by Theorem 11, there exists $\psi \in \mathcal{M}$ with $Q_0[\psi](\cdot - c) = \psi(\cdot)$, $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$.

[Step 3] In this step, we show the following: *If* $c_* \in \mathbb{R}$, *then there exists* $\psi \in \mathcal{M}$ *with* $Q_0[\psi](\cdot - c_*) = \psi(\cdot)$, $\psi(-\infty) = 0$ *and* $\psi(+\infty) = 1$.

In virtue of Step 2, there exists $\psi_k \in \mathcal{M}$ with $Q_0[\psi_k](\cdot - (c_* + 2^{-k})) = \psi_k(\cdot)$, $\psi_k(-\infty) = 0$ and $\psi_k(+\infty) = 1$ for $k \in \mathbb{N}$. We also take x_k such that $\psi_k(-x_k) \leq 1/2 \leq \lim_{h \downarrow +0} \psi_k(-x_k + h)$, and put $\psi^k(\cdot) := \psi_k(\cdot - x_k) \in \mathcal{M}$. Then, we have

$$\psi^k(0) \leq 1/2 \leq \lim_{h \downarrow +0} \psi^k(h) \tag{3.14}$$

and

$$Q_0[\psi^k(\cdot - 2^{-k})](\cdot - c_*) = \psi^k(\cdot). \tag{3.15}$$

By Helly's theorem, there exist a subsequence $\{k(n)\}_{n \in \mathbb{N}}$ and $\psi \in \mathcal{M}$ such that $\lim_{n \to \infty} \psi^{k(n)}(x) = \psi(x)$ holds for all continuous points $x \in \mathbb{R}$ of ψ. Also, by Lemma 12, $\lim_{n \to \infty} \psi^{k(n)}(x - 2^{-k(n)}) = \psi(x)$ holds for all continuous points $x \in \mathbb{R}$ of ψ. Therefore, from (3.14), (3.15) and Proposition 9,

$$\psi(0) \leq 1/2 \leq \lim_{h \downarrow +0} \psi(h) \tag{3.16}$$

and

$$Q_0[\psi](\cdot - c_*) = \psi(\cdot) \tag{3.17}$$

holds. Because $Q_0[\psi(-\infty)] = \psi(-\infty)$ and $Q_0[\psi(+\infty)] = \psi(+\infty)$ also hold by (3.17) and Proposition 9, from Hypothesis 3 (iv) and (3.16), we have $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$.

[Step 4] Finally, we show $c_* \in (-\infty, +\infty]$.

Suppose $c_* = -\infty$. Then, in virtue of Step 2, there exists $\phi_k \in \mathcal{M}$ with $Q_0[\phi_k](\cdot + 2^k) = \phi_k(\cdot)$, $\phi_k(-\infty) = 0$ and $\phi_k(+\infty) = 1$ for $k \in \mathbb{N}$. We also take x_k such that $\phi_k(-x_k) \leq 1/2 \leq \lim_{h\downarrow+0} \phi_k(-x_k + h)$, and put $\phi^k(\cdot) := \phi_k(\cdot - x_k) \in \mathcal{M}$. Then, we have

$$\phi^k(0) \leq 1/2 \leq \lim_{h\downarrow+0} \phi^k(h) \tag{3.18}$$

and

$$Q_0[\phi^k(\cdot + 2^k)](\cdot) = \phi^k(\cdot). \tag{3.19}$$

Put $\chi \in \mathcal{M}$ such that $\chi(x) = 0\,(x \leq 0)$ and $\chi(x) = 1/2\,(0 < x)$. Then, $\chi \leq \phi^k$ holds from (3.18). Hence, by (3.18) and (3.19), we see $Q_0[\chi(\cdot + 2^k)](0) \leq 1/2$. So, from $\lim_{k\to\infty} \chi(\cdot + 2^k) = 1/2$ and Proposition 9, we obtain $Q_0[1/2] \leq 1/2$. This is a contradiction with Hypothesis 3 (iv). □

Lemma 14. *Let Q^t be a map from $\mathcal{M}$ to $\mathcal{M}$ for $t \in [0, +\infty)$. Suppose Q satisfies* Hypothesis 4 (ii). *Then, $\lim_{t\to 0}(Q^t[u])(x - ct) = u(x)$ holds for all $c \in \mathbb{R}$, $u \in \mathcal{M}$ and continuous points $x \in \mathbb{R}$ of u.*

Proof. Let a sequence $\{t_k\}_{k\in\mathbb{N}} \subset [0, +\infty)$ converge to 0. Then, by Hypothesis 4 (ii) and Lemma 12, $\lim_{k\to\infty} Q^{t_k}[u](x - ct_k) = u(x)$ holds for all continuous points $x \in \mathbb{R}$ of u. □

Proof of Theorem 5. By Theorem 11, there exists $\psi_k \in \mathcal{M}$ with $Q^{\frac{\tau}{2^k}}[\psi_k](\cdot - \frac{c\tau}{2^k}) = \psi_k(\cdot)$, $\psi_k(-\infty) = 0$ and $\psi_k(+\infty) = 1$ for $k \in \mathbb{N}$. We also take x_k such that $\psi_k(-x_k) \leq 1/2 \leq \lim_{h\downarrow+0} \psi_k(-x_k + h)$, and put $\psi^k(\cdot) := \psi_k(\cdot - x_k) \in \mathcal{M}$. Then, we have

$$\psi^k(0) \leq 1/2 \leq \lim_{h\downarrow+0} \psi^k(h) \tag{3.20}$$

and

$$Q^{\frac{\tau}{2^k}}[\psi^k]\left(\cdot - \frac{c\tau}{2^k}\right) = \psi^k(\cdot). \tag{3.21}$$

By Helly's theorem, there exist a subsequence $\{k(n)\}_{n\in\mathbb{N}}$ and $\psi \in \mathcal{M}$ such that $\lim_{n\to\infty} \psi^{k(n)}(x) = \psi(x)$ holds for all continuous points $x \in \mathbb{R}$ of ψ.

Let $k_0 \in \mathbb{N}$ and $m_0 \in \mathbb{N}$. As $n \in \mathbb{N}$ is sufficiently large,

$$Q^{\frac{m_0\tau}{2^{k_0}}}[\psi^{k(n)}]\left(\cdot - c\frac{m_0\tau}{2^{k_0}}\right)$$

$$= (Q^{\frac{\tau}{2^{k(n)}}})^{m_0 2^{k(n)-k_0}}[\psi^{k(n)}]\left(\cdot - \frac{c\tau}{2^{k(n)}} m_0 2^{k(n)-k_0}\right) = \psi^{k(n)}(\cdot)$$

holds because of $k(n) \geq k_0$ and (3.21). Therefore, by Proposition 9, we obtain

$$Q^{\frac{m_0\tau}{2^{k_0}}}[\psi]\left(\cdot - c\frac{m_0\tau}{2^{k_0}}\right) = \psi(\cdot). \tag{3.22}$$

From (3.20), we also see

$$\psi(0) \leq 1/2 \leq \lim_{h\downarrow +0} \psi(h). \tag{3.23}$$

Let $t \in [0, +\infty)$. Then, by (3.22), there exists a sequence $\{t_k\}_{k\in\mathbb{N}} \subset [0, +\infty)$ with $\lim_{k\to\infty} t_k = 0$ such that $Q^{t+t_k}[\psi](\cdot - c(t + t_k)) = \psi(\cdot)$ holds for all $k \in \mathbb{N}$. So, by $Q^{t_k}[Q^t[\psi](\cdot - ct)](\cdot - ct_k) = Q^{t+t_k}[\psi](\cdot - c(t+t_k))$ and Lemma 14, we obtain

$$Q^t[\psi](\cdot - ct) = \psi(\cdot).$$

Hence, because $Q^t[\psi(-\infty)] = \psi(-\infty)$ and $Q^t[\psi(+\infty)] = \psi(+\infty)$ hold by Proposition 9, from (3.23), we see $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$. □

Proof of Theorem 6. In virtue of Theorem 13, we take $c_* \in (-\infty, +\infty]$ such that the following holds: *Let* $c \in \mathbb{R}$. *Then, there exists* $\phi \in \mathcal{M}$ *with* $(Q^1[\phi])(\cdot - c) \equiv \phi(\cdot)$, $\phi(-\infty) = 0$ *and* $\phi(+\infty) = 1$ *if and only if* $c \geq c_*$.

Then, from Theorem 5, we have the conclusion of this theorem. □

§4. Basic Facts for Nonlocal Equations in $L^\infty(\mathbb{R})$

In this section, we give some basic facts for the equation

$$u_t = \hat{\mu} * u + g(u) \tag{4.1}$$

on the phase space $L^\infty(\mathbb{R})$. We do not necessarily assume $\hat{\mu}(\mathbb{R}) = 1$ or that the nonlinearity $\hat{\mu}(\mathbb{R})u + g(u)$ is monostable. So, the equation (4.1) is more general than (1.1). This slight generalization would be useful to a sequel [30].

First, we have the comparison theorem for (4.1) on $L^\infty(\mathbb{R})$:

Lemma 15. *Let* $\hat{\mu}$ *be a Borel-measure on* $\mathbb{R}$ *with* $\hat{\mu}(\mathbb{R}) < +\infty$. *Let* g *be a Lipschitz continuous function on* $\mathbb{R}$. *Let* $T \in (0, +\infty)$, *and two functions* u^1 *and* $u^2 \in C^1([0, T], L^\infty(\mathbb{R}))$. *Suppose that for any* $t \in [0, T]$, *the inequality*

$$u_t^1 - \left(\hat{\mu} * u^1 + g(u^1)\right) \leq u_t^2 - \left(\hat{\mu} * u^2 + g(u^2)\right)$$

holds almost everywhere in x. *Then, the inequality* $u^1(T, x) \leq u^2(T, x)$ *holds almost everywhere in* x *if the inequality* $u^1(0, x) \leq u^2(0, x)$ *holds almost everywhere in* x.

Proof. Put $K \in \mathbb{R}$ by

$$K := -\inf_{h>0, u\in\mathbb{R}} \frac{g(u+h)-g(u)}{h}, \tag{4.2}$$

and $v \in C^1([0,T], L^\infty(\mathbb{R}))$ by

$$v(t) := e^{Kt}(u^2 - u^1)(t). \tag{4.3}$$

Then, we have the ordinary differential equation

$$\frac{dv}{dt} = F(t,v) \tag{4.4}$$

in $L^\infty(\mathbb{R})$ with $v(0) = (u^2 - u^1)(0)$ as we define a map $F : [0,T] \times L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ by

$$F(t,w) := \hat{\mu} * w + Kw + e^{Kt}\left(g(u^1(t) + e^{-Kt}w) - g(u^1(t))\right) + e^{Kt}a(t),$$

where

$$a := \left(\frac{du^2}{dt} - \left(\hat{\mu} * u^2 + g(u^2)\right)\right) - \left(\frac{du^1}{dt} - \left(\hat{\mu} * u^1 + g(u^1)\right)\right).$$

For any $t \in [0,T]$, we see the inequality

$$a(t,x) \geq 0 \tag{4.5}$$

almost everywhere in x. Take the solution $\tilde{v} \in C^1([0,T], L^\infty(\mathbb{R}))$ to

$$\tilde{v}(t) = v(0) + \int_0^t \max\{F(s, \tilde{v}(s)), 0\} ds. \tag{4.6}$$

Then, for any $t \in [0,T]$, we have

$$\tilde{v}(t,x) \geq v(0,x) = (u^2 - u^1)(0,x) \geq 0 \tag{4.7}$$

almost everywhere in x. By using (4.2), (4.5) and (4.7), for any $t \in [0,T]$, we also have the inequality $F(t, \tilde{v}(t)) \geq 0$ almost everywhere in x. Hence, from (4.6), $\tilde{v}(t)$ is the solution to the same ordinary differential equation (4.4) in $L^\infty(\mathbb{R})$ as $v(t)$ with $\tilde{v}(0) = v(0)$. So, in virtue of (4.3) and (4.7),

$$(u^2 - u^1)(T,x) = e^{-KT}v(T,x) = e^{-KT}\tilde{v}(T,x) \geq 0$$

holds almost everywhere in x. □

The following lemma gives a invariant set and some positively invariant sets of the flow on $L^\infty(\mathbb{R})$ generated by the equation (4.1):

Lemma 16. *Let $\hat{\mu}$ be a Borel-measure on $\mathbb{R}$ with $\hat{\mu}(\mathbb{R}) < +\infty$. Let g be a Lipschitz continuous function on $\mathbb{R}$. Then, the followings hold*:

(i) *For any $u_0 \in BC(\mathbb{R})$, there exists a solution $\{u(t)\}_{t\in\mathbb{R}} \subset BC(\mathbb{R})$ to (4.1) with $u(0) = u_0$. Here, $BC(\mathbb{R})$ denote the set of bounded and continuous functions on $\mathbb{R}$.*

(ii) *Suppose a constant γ satisfies $\gamma\hat{\mu}(\mathbb{R}) + g(\gamma) = 0$. If $u_0 \in L^\infty(\mathbb{R})$ satisfies $\gamma \le u_0$, then there exists a solution $\{u(t)\}_{t\in[0,+\infty)} \subset L^\infty(\mathbb{R})$ to (4.1) with $u(0) = u_0$ and $\gamma \le u(t)$. If $u_0 \in L^\infty(\mathbb{R})$ satisfies $u_0 \le \gamma$, then there exists a solution $\{u(t)\}_{t\in[0,+\infty)} \subset L^\infty(\mathbb{R})$ to (4.1) with $u(0) = u_0$ and $u(t) \le \gamma$.*

(iii) *If u_0 is a bounded and monotone nondecreasing function on $\mathbb{R}$, then there exists a solution $\{u(t)\}_{t\in[0,+\infty)} \subset L^\infty(\mathbb{R})$ to (4.1) with $u(0) = u_0$ such that $u(t)$ is a bounded and monotone nondecreasing function on $\mathbb{R}$ for all $t \in [0, +\infty)$. If u_0 is a bounded and monotone nonincreasing function on $\mathbb{R}$, then there exists a solution $\{u(t)\}_{t\in[0,+\infty)} \subset L^\infty(\mathbb{R})$ to (4.1) with $u(0) = u_0$ such that $u(t)$ is a bounded and monotone nonincreasing function on $\mathbb{R}$ for all $t \in [0, +\infty)$.*

Proof. We could see (i), because $BC(\mathbb{R})$ is a closed sub-space of the Banach space $L^\infty(\mathbb{R})$ and $u \in BC(\mathbb{R})$ implies $\hat{\mu} * u + g(u) \in BC(\mathbb{R})$.

We could also see (ii) by using Lemma 15, because the constant γ is a solution to (4.1).

We show (iii). Suppose u_0 is a bounded and monotone nondecreasing function on $\mathbb{R}$. We take a solution $\{u(t)\}_{t\in[0,+\infty)} \subset L^\infty(\mathbb{R})$ to (4.1) with $u(0) = u_0$. Let $t \in [0, +\infty)$ and $h \in [0, +\infty)$. Then, by Lemma 15, we see $u(t, x) \le u(t, x + h)$ almost everywhere in x. We take a cutoff function $\rho \in C^\infty(\mathbb{R})$ with

$$|x| \ge 1/2 \implies \rho(x) = 0,$$

$$|x| < 1/2 \implies \rho(x) > 0$$

and

$$\int_{x\in\mathbb{R}} \rho(x)dx = 1.$$

As we put

$$v_n(x) := \int_{y\in\mathbb{R}} 2^n \rho(2^n(x - y))u(t, y)dy$$

for $n \in \mathbb{N}$, we see $v_n(x) \le v_n(x + h)$ for all $x \in \mathbb{R}$. Therefore, v_n is smooth, bounded and monotone nondecreasing. By Helly's theorem, there exist a subsequence n_k and a bounded and monotone nondecreasing function ψ on $\mathbb{R}$ such that $\lim_{k\to\infty} v_{n_k}(x) = \psi(x)$ holds for all $x \in \mathbb{R}$. Then, $\|u(t, x) -$

$\psi(x)\|_{L^1([-C,+C])} \leq \lim_{k\to\infty}(\|u(t,x) - v_{n_k}(x)\|_{L^1([-C,+C])} + \|v_{n_k}(x) - \psi(x)\|_{L^1([-C,+C])}) = 0$ holds for all $C \in (0,+\infty)$. Hence, we obtain $\|u(t,x) - \psi(x)\|_{L^\infty(\mathbb{R})} = 0$. □

Lemma 17. *Let $\hat{\mu}$ be a Borel-measure on $\mathbb{R}$ with $\hat{\mu}(\mathbb{R}) < +\infty$. Let $\{u_n\}_{n=1}^\infty$ be a sequence of bounded and continuous functions on $\mathbb{R}$ with*

$$\sup_{n\in\mathbb{N},x\in\mathbb{R}} |u_n(x)| < +\infty.$$

Suppose the sequence $\{u_n\}_{n=1}^\infty$ converges to 0 uniformly on every bounded interval. Then, the sequence $\{\hat{\mu} * u_n\}_{n=1}^\infty$ converges to 0 uniformly on every bounded interval.

Proof. Let $\varepsilon \in (0,+\infty)$. We take a positive constant C such that

$$\left(\sup_{n\in\mathbb{N},x\in\mathbb{R}} |u_n(x)|\right) \hat{\mu}(\mathbb{R} \setminus (-C,+C)) \leq \varepsilon$$

holds. Then, because

$$|(\hat{\mu} * u_n)(x)| \leq \int_{y\in(-C,+C)} |u_n(x-y)| d\hat{\mu}(y) + \int_{y\in\mathbb{R}\setminus(-C,+C)} |u_n(x-y)| d\hat{\mu}(y)$$

$$\leq \left(\sup_{y\in(-C,+C)} |u_n(x-y)|\right) \hat{\mu}(\mathbb{R}) + \left(\sup_{y\in\mathbb{R}} |u_n(x-y)|\right) \hat{\mu}(\mathbb{R} \setminus (-C,+C))$$

holds, we have

$$\sup_{x\in[-I,+I]} |(\hat{\mu} * u_n)(x)| \leq \left(\sup_{y\in(-(I+C),+(I+C))} |u_n(y)|\right) \hat{\mu}(\mathbb{R}) + \varepsilon$$

for all $I \in (0,+\infty)$. Hence, we obtain

$$\limsup_{n\to\infty} \sup_{x\in[-I,+I]} |(\hat{\mu} * u_n)(x)| \leq \varepsilon$$

for all $I \in (0,+\infty)$. □

Proposition 18. *Let $\hat{\mu}$ be a Borel-measure on $\mathbb{R}$ with $\hat{\mu}(\mathbb{R}) < +\infty$, g a Lipschitz continuous function on $\mathbb{R}$, and T a positive constant. Let a sequence $\{u_n\}_{n=0}^\infty \subset C^1([0,T], L^\infty(\mathbb{R}))$ of solutions to the equation (4.1) satisfy*

$$\sup_{n\in\mathbb{N},x\in\mathbb{R}} |u_n(0,x) - u_0(0,x)| < +\infty.$$

Suppose

$$\lim_{n\to\infty} \sup_{x\in[-I,+I]} |u_n(0,x) - u_0(0,x)| = 0$$

holds for all positive constants I. Then,

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|u_n(t,x) - u_0(t,x)\|_{L^\infty([-J,+J])} = 0$$

holds for all positive constants J.

Proof. First, we take a sequence $\{w_n\}_{n=1}^\infty$ of nonnegative, bounded and continuous functions on $\mathbb{R}$ with

$$\sup_{n\in\mathbb{N},x\in\mathbb{R}} |w_n(x)| < +\infty \tag{4.8}$$

such that $\{w_n\}_{n=1}^\infty$ converges to 0 uniformly on every bounded interval and

$$|u_n(0,x) - u_0(0,x)| \le w_n(x) \tag{4.9}$$

holds for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Let $\hat{A}$ denote the bounded and linear operator from the Banach space $BC(\mathbb{R})$ to $BC(\mathbb{R})$ defined by

$$\hat{A}w := \hat{\mu} * w.$$

From (4.8), we see $\sup_{n\in\mathbb{N},x\in\mathbb{R}} |(\hat{A}^k w_n)(x)| < +\infty$ for all $k = 0, 1, 2, \cdots$. Hence, because of $\lim_{n\to\infty} \sup_{x\in[-I,+I]} |w_n(x)| = 0$ for all $I \in (0, +\infty)$, by Lemma 17, we have

$$\lim_{n\to\infty} \sup_{x\in[-J,+J]} |(\hat{A}^k w_n)(x)| = 0 \tag{4.10}$$

for all $J \in (0, +\infty)$ and $k = 0, 1, 2, \cdots$.

Let γ denote the constant defined by

$$\gamma := \sup_{h>0,u\in\mathbb{R}} \frac{g(u+h) - g(u)}{h}.$$

Then, we consider the following two sequences $\{\underline{v}_n\}_{n=1}^\infty$ and $\{\overline{v}_n\}_{n=1}^\infty \subset C^1$ $([0,T], L^\infty(\mathbb{R}))$ defined by

$$\underline{v}_n(t,x) := u_0(t,x) - e^{\gamma t}(e^{\hat{A}t} w_n)(x)$$

and

$$\overline{v}_n(t,x) := u_0(t,x) + e^{\gamma t}(e^{\hat{A}t} w_n)(x).$$

Because $(e^{\hat{A}t}w_n)(x)$ is nonnegative for all $n \in \mathbb{N}$, $t \in [0, +\infty)$ and $x \in \mathbb{R}$, the function $\underline{v}_n$ is a sub-solution to (4.1) and $\overline{v}_n$ is a super-solution to (4.1) for all $n \in \mathbb{N}$. So, by Lemma 15 and (4.9), for any $n \in \mathbb{N}$ and $t \in [0, T]$,

$$|u_n(t,x) - u_0(t,x)| \leq e^{\gamma t}(e^{\hat{A}t}w_n)(x) \tag{4.11}$$

holds almost everywhere in x.

Let $\varepsilon \in (0, +\infty)$. We take $N \in \mathbb{N}$ such that

$$(1 + e^{\gamma T}) \left(\sum_{k=N}^{\infty} \frac{(T\|\hat{A}\|_{BC(\mathbb{R}) \to BC(\mathbb{R})})^k}{k!} \right) \left(\sup_{n \in \mathbb{N}, x \in \mathbb{R}} |w_n(x)| \right) \leq \varepsilon$$

holds. Then, in virtue of (4.11), we see

$$\|u_n(t,x) - u_0(t,x)\|_{L^\infty([-J,+J])} \leq \sup_{x \in [-J,+J]} |e^{\gamma t}(e^{\hat{A}t}w_n)(x)|$$

$$= e^{\gamma t} \left(\sup_{x \in [-J,+J]} \left| \left(\sum_{k=0}^{N-1} \frac{t^k}{k!} (\hat{A}^k w_n)(x) \right) + \left(\left(\sum_{k=N}^{\infty} \frac{t^k}{k!} \hat{A}^k \right) w_n \right)(x) \right| \right)$$

$$\leq (1 + e^{\gamma T}) \left(\sum_{k=0}^{N-1} \frac{T^k}{k!} \left(\sup_{x \in [-J,+J]} |(\hat{A}^k w_n)(x)| \right) \right) + \varepsilon$$

for all $J \in (0, +\infty)$, $n \in \mathbb{N}$ and $t \in [0, T]$. So, by (4.10), we obtain

$$\limsup_{n \to \infty} \sup_{t \in [0,T]} \|u_n(t,x) - u_0(t,x)\|_{L^\infty([-J,+J])} \leq \varepsilon$$

for all $J \in (0, +\infty)$. □

§5. Proof of Theorem 1

In this section, we prove Theorem 1. The argument in this section is almost similar as in [29]. First, we recall that μ is a Borel-measure on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$, f is a Lipschitz continuous function on $\mathbb{R}$ with $f(0) = f(1) = 0$ and $f > 0$ in $(0, 1)$ and the set $\mathcal{M}$ has been defined at the beginning of Section 2. Then, in virtue of Lemmas 15, 16 and Proposition 18, Q^t ($t \in (0, +\infty)$) satisfies Hypotheses 3 and Q Hypotheses 4 for the semiflow $Q = \{Q^t\}_{t \in [0,+\infty)}$ on $\mathcal{M}$ generated by (1.1). So, Theorems 5 and 6 can work for this semiflow on $\mathcal{M}$.

If the flow on $L^\infty(\mathbb{R})$ generated by (1.1) has a *periodic* traveling wave solution with *average* speed c (even if the profile is not a monotone function), then it has a traveling wave solution with *monotone* profile and speed c:

Theorem 19. *Let $c \in \mathbb{R}$. Suppose there exist a positive constant τ and a solution $\{u(t,x)\}_{t\in\mathbb{R}} \subset L^\infty(\mathbb{R})$ to* (1.1) *with $0 \le u(t,x) \le 1$, $\lim_{x\to+\infty} u(t,x) = 1$ and $\|u(t,x)-1\|_{L^\infty(\mathbb{R})} \ne 0$ such that*

$$u(t+\tau, x) = u(t, x+c\tau)$$

holds for all t and $x \in \mathbb{R}$. Then, there exists $\psi \in \mathcal{M}$ with $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$ such that $\{\psi(x+ct)\}_{t\in\mathbb{R}}$ is a solution to (1.1).

Proof. Put two monotone nondecreasing functions $\varphi(x) := \max\{\alpha \in \mathbb{R} \,|\, \alpha \le u(0,y)$ holds almost everywhere in $y \in (x,+\infty)\}$ and $\phi(x) := \lim_{h\downarrow+0} \varphi(x-h)$. Then, $\phi \in \mathcal{M}$, $\phi(-\infty) < 1$ and $\phi(+\infty) = 1$ hold. We take a cutoff function $\rho \in C^\infty(\mathbb{R})$ with

$$|x+1/2| \ge 1/2 \implies \rho(x) = 0,$$

$$|x+1/2| < 1/2 \implies \rho(x) > 0$$

and

$$\int_{x\in\mathbb{R}} \rho(x)dx = 1.$$

As we put

$$v_n(x) := \int_{y\in\mathbb{R}} 2^n \rho(2^n(x-y))u(0,y)dy$$

for $n \in \mathbb{N}$, we see $\phi \le v_n$. Let $N \in \mathbb{N}$. Because of $\lim_{n\to\infty} \|v_n(x) - u(0,x)\|_{L^1([-N,+N])} = 0$, there exists a subsequence n_k such that $\lim_{k\to\infty} v_{n_k}(x) = u(0,x)$ almost everywhere in $x \in [-N,+N]$. Therefore, we have $\phi(x) \le u(0,x)$ almost everywhere in $x \in \mathbb{R}$. So, by Lemma 15, we obtain $Q^\tau[\phi](x - c\tau) \le u(\tau, x-c\tau) = u(0,x)$ almost everywhere in x. Hence, because $Q^\tau[\phi](x-c\tau) \le \max\{\alpha \in \mathbb{R} \,|\, \alpha \le Q^\tau[\phi](y-c\tau)$ holds almost everywhere in $y \in (x,+\infty)\} \le \varphi(x)$ holds, we get $Q^\tau[\phi](x-c\tau) = \lim_{h\downarrow+0} Q^\tau[\phi]((x-h)-c\tau) \le \phi(x)$. Therefore, by Theorem 5, there exists $\psi \in \mathcal{M}$ with $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$ such that $Q^t[\psi](x-ct) \equiv \psi(x)$ holds for all $t \in [0,+\infty)$. □

The infimum c_* of the speeds of traveling wave solutions is not $-\infty$, and there is a traveling wave solution with speed c when $c \ge c_*$:

Lemma 20. *There exists $c_* \in (-\infty,+\infty]$ such that the following holds*:
Let $c \in \mathbb{R}$. Then, there exists $\psi \in \mathcal{M}$ with $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$ such that $\{\psi(x+ct)\}_{t\in\mathbb{R}}$ is a solution to (1.1) *if and only if $c \ge c_*$.*

Proof. It follows from Theorem 6. □

Proof of Theorem 1. Let c_* denote the infimum of the speeds of traveling wave solutions with monotone profile. Then, in virtue of Theorem 19 and Lemma 20, it is sufficient if we show $c_* \neq +\infty$.

Take $K \in [0, +\infty)$ such that

$$K \geq \max\left\{\int_{y\in\mathbb{R}} e^{-\lambda y} d\mu(y), \mu(\mathbb{R})\right\} - 1 + \sup_{h>0} \frac{f(h)}{h}.$$

As we put $\phi(x) := \min\{e^{\lambda x}, 1\} \in \mathcal{M}$, we see

$$\begin{aligned}(\mu * \phi)(x) &\leq \min\left\{\left(\int_{y\in\mathbb{R}} e^{-\lambda y} d\mu(y)\right) e^{\lambda x}, \mu(\mathbb{R})\right\} \\ &\leq \max\left\{\int_{y\in\mathbb{R}} e^{-\lambda y} d\mu(y), \mu(\mathbb{R})\right\} \phi(x).\end{aligned}$$

So, $e^{Kt}\phi(x)$ is a super-solution to (1.1), because of

$$e^{Kt}(\mu * \phi) - e^{Kt}\phi + f(e^{Kt}\phi) \leq K e^{Kt}\phi.$$

Hence, by Lemma 15, we obtain $Q^1[\phi](x) \leq e^K \phi(x) \leq e^{\lambda(x+\frac{K}{\lambda})}$, and $Q^1[\phi](x - \frac{K}{\lambda}) \leq \phi(x)$. Therefore, from Theorem 5, there exists $\psi \in \mathcal{M}$ with $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$ such that $Q^t[\psi](x - \frac{K}{\lambda}t) \equiv \psi(x)$ holds for all $t \in [0, +\infty)$. So, $c_* \leq \frac{K}{\lambda}$ holds. □

§6. A Result on Spreading Speeds by Weinberger

In this section, we recall a result by Weinberger [25]. In Section 7 below, we use it to prove Theorem 2. Put a set of functions on $\mathbb{R}$;

$$\mathcal{B} := \{u \,|\, u \text{ is a continuous function on } \mathbb{R} \text{ with } 0 \leq u \leq 1\}.$$

Hypotheses 21. *Let $\tilde{Q}_0$ be a map from $\mathcal{B}$ into $\mathcal{B}$.*

(i) *$\tilde{Q}_0$ is continuous in the following sense: If a sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{B}$ converges to $u \in \mathcal{B}$ uniformly on every bounded interval, then the sequence $\{(\tilde{Q}_0[u_k])(x)\}_{k\in\mathbb{N}}$ converges to $(\tilde{Q}_0[u])(x)$ for all $x \in \mathbb{R}$.*

(ii) *$\tilde{Q}_0$ is order preserving; i.e.,*

$$u_1 \leq u_2 \Longrightarrow \tilde{Q}_0[u_1] \leq \tilde{Q}_0[u_2]$$

for all u_1 and $u_2 \in \mathcal{B}$. Here, $u \leq v$ means that $u(x) \leq v(x)$ holds for all $x \in \mathbb{R}$.

(iii) $\tilde{Q}_0$ *is translation invariant*; *i.e.*,

$$T_{x_0}\tilde{Q}_0 = \tilde{Q}_0 T_{x_0}$$

for all $x_0 \in \mathbb{R}$. *Here,* T_{x_0} *is the translation operator defined by* $(T_{x_0}[u])(\cdot) := u(\cdot - x_0)$.

(iv) $\tilde{Q}_0$ *is monostable*; *i.e.*,

$$0 < \gamma < 1 \Longrightarrow \gamma < \tilde{Q}_0[\gamma]$$

for all constant functions γ, *and* $\tilde{Q}_0[0] = 0$.

Remark. If $\tilde{Q}_0$ satisfies Hypotheses 21 (ii) and (iii), then $\tilde{Q}_0$ maps monotone functions to monotone functions.

Theorem 22. *Let a map* $\tilde{Q}_0 : \mathcal{B} \to \mathcal{B}$ *satisfy Hypotheses* 21. *Let a continuous and monotone nonincreasing function* φ *on* $\mathbb{R}$ *with* $0 < \varphi(-\infty) < 1$ *satisfy* $\varphi(x) = 0$ *for all* $x \in [0, +\infty)$. *For* $c \in \mathbb{R}$, *define the sequence* $\{a_{c,n}\}_{n=0}^{\infty}$ *of continuous and monotone nonincreasing functions on* $\mathbb{R}$ *by the recursion*

$$a_{c,n+1}(x) := \max\{(\tilde{Q}_0[a_{c,n}])(x+c), \varphi(x)\}$$

with $a_{c,0} := \varphi$. *Then, the inequality*

$$0 \le a_{c,n} \le a_{c,n+1} \le 1$$

holds for all $c \in \mathbb{R}$ *and* $n = 0, 1, 2, \cdots$. *For* $c \in \mathbb{R}$, *define the bounded and monotone nonincreasing function* a_c *on* $\mathbb{R}$ *by*

$$a_c(x) := \lim_{n\to\infty} a_{c,n}(x).$$

Let $\tilde{\nu}$ *be a Borel-measure on* $\mathbb{R}$ *with* $1 < \tilde{\nu}(\mathbb{R}) < +\infty$. *Suppose there exists a positive constant* ε *such that the inequality*

$$\tilde{\nu} * u \le \tilde{Q}_0[u]$$

holds for all $u \in \mathcal{B}$ *with* $u \le \varepsilon$. *Then, the inequality*

$$\inf_{\lambda>0} \frac{1}{\lambda} \log \int_{y\in\mathbb{R}} e^{\lambda y} d\tilde{\nu}(y) \;\le\; \sup\{c \in \mathbb{R} \,|\, a_c(+\infty) = 1\}$$

holds.

Proof. It follows from Lemma 5.4 and Theorem 6.4 in [25] with $N := 1$, $\mathcal{H} := \mathbb{R}$, $\pi_0 := 0$, $\pi_1 = \pi_+ := 1$, $S^{N-1} := \{\pm 1\}$ and $\xi := +1$. □

From Theorem 22, we have the following:

Proposition 23. *Let $\hat{\mu}$ be a Borel-measure on $\mathbb{R}$ with $\hat{\mu}(\mathbb{R}) = 1$. Let $c_0 \in \mathbb{R}$, and $\hat{\psi}$ be a monotone nonincreasing function on $\mathbb{R}$ with $\hat{\psi}(-\infty) = 1$ and $\hat{\psi}(+\infty) = 0$. Suppose $\{\hat{\psi}(x - c_0 t)\}_{t\in\mathbb{R}} \subset L^\infty(\mathbb{R})$ is a solution to*

$$u_t = \hat{\mu} * u - u + f(u). \tag{6.1}$$

Let $\tilde{Q}_0 : \mathcal{B} \to \mathcal{B}$ be the time 1 map of the semiflow on $\mathcal{B}$ generated by the equation (6.1). Let $\tilde{\nu}$ be a Borel-measure on $\mathbb{R}$ with $1 < \tilde{\nu}(\mathbb{R}) < +\infty$. Suppose there exists a positive constant ε such that the inequality

$$\tilde{\nu} * u \le \tilde{Q}_0[u]$$

holds for all $u \in \mathcal{B}$ with $u \le \varepsilon$. Then, the inequality

$$\inf_{\lambda>0} \frac{1}{\lambda} \log \int_{y\in\mathbb{R}} e^{\lambda y} d\tilde{\nu}(y) \ \le\ c_0$$

holds.

Proof. We take a continuous and monotone nonincreasing function φ on $\mathbb{R}$ with $0 < \varphi(-\infty) < 1$ and $\varphi(x) = 0$ for all $x \in [0, +\infty)$. For $c \in \mathbb{R}$, we define the sequence $\{a_{c,n}\}_{n=0}^{\infty}$ of continuous and monotone nonincreasing functions on $\mathbb{R}$ by the recursion

$$a_{c,n+1}(x) := \max\{(\tilde{Q}_0[a_{c,n}])(x + c), \varphi(x)\}$$

with $a_{c,0} := \varphi$. We also take $x_0 \in \mathbb{R}$ such that

$$\varphi(x) \le \hat{\psi}(x - x_0)$$

holds for all $x \in \mathbb{R}$.

Let $c \in [c_0, +\infty)$. Then, we show $a_{c,n}(x) \le \hat{\psi}(x - x_0)$ for all $n = 0, 1, 2, \cdots$. We have $a_{c,0}(x) = \varphi(x) \le \hat{\psi}_0(x - x_0)$. As $a_{c,n}(x) \le \hat{\psi}(x - x_0)$ holds almost everywhere in x,

$$\begin{aligned} a_{c,n+1}(x) &\le \max\{(\tilde{Q}_0[a_{c,n}])(x + c_0), \varphi(x)\} \\ &\le \max\{\hat{\psi}(x - x_0), \varphi(x)\} = \hat{\psi}(x - x_0) \end{aligned}$$

also holds almost everywhere in x, because $\hat{\psi}(x-x_0-c_0t)$ is a solution to (6.1) and so $(\tilde{Q}_0[a_{c,n}])(x) \leq \hat{\psi}(x-x_0-c_0)$ holds almost everywhere in x. So, for any $n = 0, 1, 2, \cdots$, the inequality $a_{c,n}(x) \leq \hat{\psi}(x-x_0)$ holds almost everywhere in x. Hence, because $a_{c,n}$ is continuous and $\hat{\psi}$ is monotone, we have

$$a_{c,n}(x) \leq \hat{\psi}(x-x_0) \tag{6.2}$$

for all $x \in \mathbb{R}$, $c \in [c_0, +\infty)$ and $n = 0, 1, 2, \cdots$. Therefore, by Theorem 22, (6.2) and $\hat{\psi}(+\infty) = 0$, the inequality

$$\inf_{\lambda>0} \frac{1}{\lambda} \log \int_{y\in\mathbb{R}} e^{\lambda y} d\tilde{\nu}(y) \leq \sup(\mathbb{R} \setminus [c_0, +\infty)) = c_0$$

holds. □

§7. Proof of Theorem 2

In this section, we prove Theorem 2. First, we give a basic fact for the linear equation

$$v_t = \hat{\mu} * v \tag{7.1}$$

on the phase space $BC(\mathbb{R})$:

Lemma 24. *Let $\hat{\mu}$ be a Borel-measure on $\mathbb{R}$ with $\hat{\mu}(\mathbb{R}) < +\infty$. Let $\hat{P} : BC(\mathbb{R}) \to BC(\mathbb{R})$ be the time 1 map of the flow on $BC(\mathbb{R})$ generated by the linear equation (7.1). Then, there exists a Borel-measure $\hat{\nu}$ on $\mathbb{R}$ with $\hat{\nu}(\mathbb{R}) < +\infty$ such that*

$$\hat{P}[v] = \hat{\nu} * v$$

holds for all $v \in BC(\mathbb{R})$. Further, if v is a nonnegative, bounded and continuous function on $\mathbb{R}$, then the inequality

$$v + \hat{\mu} * v \leq \hat{\nu} * v$$

holds.

Proof. Put a functional $\tilde{P} : BC(\mathbb{R}) \to \mathbb{R}$ as

$$\tilde{P}[v] := (\hat{P}[v])(0).$$

Then, the functional $\tilde{P}$ is linear, bounded and positive. Hence, there exists a Borel-measure $\tilde{\nu}$ on $\mathbb{R}$ with $\tilde{\nu}(\mathbb{R}) < +\infty$ such that if a continuous function v on $\mathbb{R}$ satisfies $\lim_{|x|\to\infty} v(x) = 0$, then

$$\tilde{P}[v] = \int_{y\in\mathbb{R}} v(y) d\tilde{\nu}(y) \tag{7.2}$$

holds.

Let $v \in BC(\mathbb{R})$. Then, there exists a sequence $\{v_n\}_{n=1}^{\infty} \subset BC(\mathbb{R})$ with $\sup_{n\in\mathbb{N},x\in\mathbb{R}} |v_n(x)| < +\infty$ and $\lim_{|x|\to\infty} v_n(x) = 0$ for all $n \in \mathbb{N}$ such that $v_n \to v$ as $n \to \infty$ uniformly on every bounded interval. From Proposition 18, (7.2) and $\tilde{\nu}(\mathbb{R}) < +\infty$, we have

$$\tilde{P}[v] = \lim_{n\to\infty} \tilde{P}[v_n] = \lim_{n\to\infty} \int_{y\in\mathbb{R}} v_n(y) d\tilde{\nu}(y) = \int_{y\in\mathbb{R}} v(y) d\tilde{\nu}(y).$$

We take a Borel-measure $\hat{\nu}$ on $\mathbb{R}$ with $\hat{\nu}(\mathbb{R}) < +\infty$ such that

$$\hat{\nu}((-\infty, y)) = \tilde{\nu}((-y, +\infty))$$

holds for all $y \in \mathbb{R}$. Then, for any $v \in BC(\mathbb{R})$, we have

$$(\hat{P}[v])(x) \equiv \tilde{P}[v(\cdot + x)] \equiv \int_{y\in\mathbb{R}} v(y + x) d\tilde{\nu}(y) \equiv (\hat{\nu} * v)(x).$$

Let v be a nonnegative, bounded and continuous function on $\mathbb{R}$. Then, in $t \in [0, +\infty)$, the function

$$u(t, x) := v(x) + t(\hat{\mu} * v)(x)$$

is a sub-solution to (7.1), because of $v(x) \le u(t, x)$. Hence,

$$v + \hat{\mu} * v \le \hat{P}[v]$$

holds. □

Lemma 25. *Let $\hat{\mu}$ be a Borel-measure on $\mathbb{R}$ with $\hat{\mu}(\mathbb{R}) < +\infty$. Suppose a constant γ and a Lipschitz continuous function g on $\mathbb{R}$ with $g(0) = 0$ satisfy $\gamma < g'(0)$. Let $\tilde{P} : BC(\mathbb{R}) \to BC(\mathbb{R})$ be the time 1 map of the flow on $BC(\mathbb{R})$ generated by the linear equation*

$$v_t = \hat{\mu} * v + \gamma v. \tag{7.3}$$

Let $\tilde{P}_0 : BC(\mathbb{R}) \to BC(\mathbb{R})$ be the time 1 map of the flow on $BC(\mathbb{R})$ generated by the equation

$$v_t = \hat{\mu} * v + g(v). \tag{7.4}$$

Then, there exists a positive constant ε such that the inequality

$$\tilde{P}[v] \le \tilde{P}_0[v]$$

holds for all $v \in BC(\mathbb{R})$ with $0 \le v \le \varepsilon$.

Proof. We take a positive constant ε such that

$$h \in [0, (1 + e^{\hat{\mu}(\mathbb{R})+\gamma})\varepsilon] \implies \gamma h \leq g(h) \tag{7.5}$$

holds. Let a function $v \in BC(\mathbb{R})$ satisfy $0 \leq v \leq \varepsilon$. Then, we take the solution $\tilde{v}(t, x)$ to (7.3) with $\tilde{v}(0, x) = v(x)$. We see

$$0 \leq \tilde{v}(t, x) \leq e^{(\hat{\mu}(\mathbb{R})+\gamma)t}\varepsilon \leq (1 + e^{\hat{\mu}(\mathbb{R})+\gamma})\varepsilon$$

for all $t \in [0, 1]$. Hence, from (7.5), in $t \in [0, 1]$, the function $\tilde{v}(t, x)$ is a sub-solution to (7.4). So, the inequality

$$(\tilde{P}[v])(x) = \tilde{v}(1, x) \leq (\tilde{P}_0[v])(x)$$

holds. □

We use Proposition 23, Lemmas 24 and 25 to show the following:

Lemma 26. *Let $f'(0) > 0$. Suppose there exist $c \in \mathbb{R}$ and $\psi \in \mathcal{M}$ with $\psi(-\infty) = 0$ and $\psi(+\infty) = 1$ such that $\{\psi(x + ct)\}_{t\in\mathbb{R}}$ is a solution to (1.1). Then, there exists a positive constant λ such that*

$$\int_{y\in\mathbb{R}} e^{-\lambda y} d\mu(y) < +\infty$$

holds.

Proof. Let $\hat{\mu}$ be the Borel-measure on $\mathbb{R}$ with $\hat{\mu}(\mathbb{R}) = 1$ such that

$$\hat{\mu}((-\infty, y)) = \mu((-y, +\infty))$$

holds for all $y \in \mathbb{R}$. Let $\hat{P} : BC(\mathbb{R}) \to BC(\mathbb{R})$ be the time 1 map of the flow on $BC(\mathbb{R})$ generated by the linear equation (7.1). Then, by Lemma 24, there exists a Borel-measure $\hat{\nu}$ on $\mathbb{R}$ with $\hat{\nu}(\mathbb{R}) < +\infty$ such that for any $v \in BC(\mathbb{R})$,

$$\hat{P}[v] = \hat{\nu} * v \tag{7.6}$$

holds and for any nonnegative, bounded and continuous function v on $\mathbb{R}$,

$$\hat{\mu} * v \leq \hat{\nu} * v \tag{7.7}$$

holds. Let $\tilde{P} : BC(\mathbb{R}) \to BC(\mathbb{R})$ be the time 1 map of the flow on $BC(\mathbb{R})$ generated by the linear equation

$$v_t = \hat{\mu} * v - v + \frac{f'(0)}{2} v.$$

Then, from (7.6) and (7.7), as $\tilde{\nu}$ is the Borel-measure on $\mathbb{R}$ defined by

$$\tilde{\nu} := e^{-1+\frac{f'(0)}{2}}\hat{\nu},$$

we have

$$\tilde{P}[v] = \tilde{\nu} * v \tag{7.8}$$

for all $v \in BC(\mathbb{R})$ and

$$\hat{\mu} * v \le e^{1-\frac{f'(0)}{2}}(\tilde{\nu} * v) \tag{7.9}$$

for all nonnegative, bounded and continuous functions v on $\mathbb{R}$. Because $\tilde{\nu}(\mathbb{R}) = (\tilde{\nu} * 1)(0) = (\tilde{P}[1])(0) = e^{\frac{f'(0)}{2}}$ holds from (7.8), we also have

$$1 < \tilde{\nu}(\mathbb{R}) < +\infty. \tag{7.10}$$

Let $\tilde{Q}_0 : \mathcal{B} \to \mathcal{B}$ be the time 1 map of the semiflow on $\mathcal{B}$ generated by the equation (6.1). Then, from Lemma 25 and (7.8), there exists a positive constant ε such that the inequality

$$\tilde{\nu} * u = \tilde{P}[u] \le \tilde{Q}_0[u]$$

holds for all $u \in \mathcal{B}$ with $u \le \varepsilon$. Further, $\hat{\psi}(x - ct) := \psi(-(x - ct))$ is a solution to (6.1). Therefore, by Proposition 23 and (7.10), we obtain the inequality

$$\inf_{\lambda>0} \frac{1}{\lambda} \log \int_{y\in\mathbb{R}} e^{\lambda y} d\tilde{\nu}(y) \le c.$$

So, there exists a positive constant λ such that

$$\int_{y\in\mathbb{R}} e^{\lambda y} d\tilde{\nu}(y) \le e^{\lambda(c+1)} < +\infty$$

holds. Hence, from (7.9), the inequality

$$\begin{aligned}\int_{y\in\mathbb{R}} e^{-\lambda y} d\mu(y) &= \int_{y\in\mathbb{R}} e^{\lambda y} d\hat{\mu}(y) = \lim_{n\to\infty} \int_{y\in\mathbb{R}} \min\{e^{\lambda y}, n\} d\hat{\mu}(y) \\ &= \lim_{n\to\infty} (\hat{\mu} * \min\{e^{-\lambda x}, n\})(0) \le e^{1-\frac{f'(0)}{2}} \lim_{n\to\infty} (\tilde{\nu} * \min\{e^{-\lambda x}, n\})(0) \\ &= e^{1-\frac{f'(0)}{2}} \lim_{n\to\infty} \int_{y\in\mathbb{R}} \min\{e^{\lambda y}, n\} d\tilde{\nu}(y) = e^{1-\frac{f'(0)}{2}} \int_{y\in\mathbb{R}} e^{\lambda y} d\tilde{\nu}(y) < +\infty\end{aligned}$$

holds. □

Proof of Theorem 2. It follows from Theorem 19 and Lemma 26. □

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