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# On a Curvature Property of Effective Divisors and Its Application to Sheaf Cohomology

By

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## Abstract

Exploring a method of taming the boundary behavior of $n$-convex exhaustion functions, a curvature property of line bundles associated to effective Cartier divisors is proved. Cohomology vanishing theorems of the Serre type and the Kodaira-Nakano type are obtained as application.

## Introduction

Let $X$ be a complex analytic space of dimension $n$. It is known that $X$ is $n$-complete in the sense of Andreotti-Grauert [A-G] (see section one for the definition) if every irreducible component of $X$ is noncompact (cf. [O-3], [Dm]). This shows that the vanishing theorem for the cohomology groups of top degrees, due to Y.-T. Siu [S], is essentially contained in [A-G].

In [F], based on the 1-completeness of noncompact Riemann surfaces, an elementary proof was given to a basic fact that, for any Riemann surface $R$ and for any point $p \in R$, the line bundle $[p]$ associated to the divisor $p$, is positive.

The purpose of the present note is to extend the paper [F] to establish the following.

**Theorem 1.** *Let $X$ be a compact complex analytic space of dimension $n$ and let $D$ be an effective Cartier divisor of $X$ such that $|D|$, the support of*

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*D, intersects every n-dimensional irreducible component of X. Then the line bundle [D] is n-concave (see section one for the definition).*

Theorem 1 supplements [O-3] and [Dm]. By applying it we shall show at first the following Serre type vanishing theorem.

**Theorem 2.** *Let M be a complex manifold, let Z be a complex analytic space, let $f: M \longrightarrow Z$ be a proper holomorphic map, let D be an effective divisor of M, let $z \in f(|D|)$, and let n be any positive integer exceeding the dimension of any compact irreducible component of $(f^{-1}(z) \backslash |D|) \cup (f^{-1}(z) \cap |D|)$. Then, for any holomorphic vector bundle $E \longrightarrow M$, there exists a positive number $m_0$ such that*

$$(R^n f_* \mathcal{O}(E \otimes [D]^m)_z = 0$$

*holds if $m \geq m_0$. Here $\mathcal{O}(E \otimes [D]^m)$ denotes the sheaf of the germs of holomorphic sections of $E \otimes [D]^m$, and $R^n f_* \mathcal{O}(E \otimes [D]^m)$ the n-th direct image of $\mathcal{O}(E \otimes [D]^m)$ by f.*

For the proof of Theorem 2, we need results from [B] and [O-3].

Further, by employing a method in [O-2], we obtain a refined version of Theorem 2 when $E$ is the canonical bundle of $M$.

**Theorem 3.** *In the above situation, suppose moreover that E is the canonical bundle $K_M$ of M, and that M admits a Kähler metric. Then*

$$(R^n f_* \mathcal{O}(K_M \otimes [D]))_z = 0$$

*holds.*

Theorem 3 may well be regarded as a supplement to Theorem 3.1 and Theorem 4.5 in [O-2], which are extensions of the Kodaira-Nakano vanishing.

## §1. q-Complete Spaces and q-Concave Bundles

Let $X$ be a complex analytic space of dimension $n$. A real valued $C^2$ function $\varphi$ defined on an open set $U \subset X$ is said to be $q$-convex if, for any point $x \in U$, there exist a neighborhood $V \ni x$, a holomorphic embedding $\iota$ of $V$ into a domain $\Omega$ of $\mathbb{C}^N$ for some $N \in \mathbb{N}$, and a real valued $C^2$ function $\Phi$ on $\Omega$ such that $\iota^* \Phi = \varphi | V$ and that the complex Hessian $\partial \bar{\partial} \Phi$ has everywhere at least $N - q + 1$ positive eigenvalues on $\Omega$.

**Definition 1.** $X$ is called a *q-complete space* (in the sense of [A-G]), if there exists a $q$-convex exhaustion function on $X$.

Here, an exhaustion function on a topological space is by definition a real valued function whose sublevel sets are all relatively compact.

Let $L \longrightarrow X$ be a holomorphic line bundle and let $\{e_{\alpha\beta}\}_{\alpha,\beta\in\Lambda}$ be a system of transition functions of $L$ associated to an open covering $\{U_\alpha\}_{\alpha\in\Lambda}$ such that $\pi^{-1}$ $(U_\alpha)$ is equivalent to the product $U_\alpha \times \mathbb{C}$. Then a fiber metric of $L$ is naturally identified with a system of positive $C^\infty$ functions $h = \{h_\alpha\}$, $h_\alpha$ being defined on $U_\alpha$, such that $h_\alpha=|e_{\alpha\beta}|^2 h_\beta$ holds on $U_\alpha \cap U_\beta$. The pair $(L, h)$ is called a Hermitian line bundle over $X$.

**Definition 2.** A Hermitian line bundle $(L, h)$ is said to be *q-concave* if the functions $-\log\ h_\alpha$ are all $q$-convex.

## §2. Proof of Theorem 1

Let $X$ be a compact complex analytic space and let Sing$X$ be the set of singular points of $X$, with respect to the reduced structure. We put

$$X_k = \underbrace{Sing(Sing(\cdots SingX)\cdots)}_{k}$$

and $X_0 = X$. Let $X_k^j$ be the union of $j$-dimensional irreducible components of $X_k$, and let $X^j = \cup_k X_k^j$.

Let $D$ be an effective Cartier divisor of $X$, let $s = \{s_\alpha\}$ be a system of local defining functions of $D$, $s_\alpha$ being defined on an open set $U_\alpha \subset X$, and let $\{e_{\alpha\beta}\}$ be the system of transition functions of $[D]$ such that $s_\alpha = e_{\alpha\beta}s_\beta$ holds on $U_\alpha \cap U_\beta$, for every $\alpha$ and $\beta$.

A fiber metric of $[D]$ is then a system of positive $C^\infty$ functions $h = \{h_\alpha\}$, $h_\alpha$ defined on $U_\alpha$, such that $h_\alpha\ |s_\alpha|^2 = h_\beta|s_\beta|^2$ holds on $U_\alpha \cap U_\beta$. We fix a fiber metric $h$ of $[D]$ and denote by $|s|^2$ the function on $X$ defined by $h_\alpha|s_\alpha|^2$ on each $U_\alpha$.

In order to find a metric $\tilde{h} = \{\tilde{h}_\alpha\}$ of $[D]$ such that $-\log\tilde{h}_\alpha$ are $n$-convex, we shall at first find a $C^\infty$ function $\eta$ on $X$ such that $-\log|s|^2 + \eta$ is $n$-convex outside some compact subset of $X\backslash|D|$. After that, we shall modify $-\log|s|^2+\eta$ on a compact set, to obtain an $n$-convex exhaustion function $\Phi$ on $X\backslash|D|$ and put $\tilde{h}_\alpha = e^{-\Phi}|s_\alpha|^{-2}$. For this purpose, the following is crucial.

**Lemma 1.** *Let $Y$ be a complex analytic space of dimension $n$ equipped with an $n$-convex exhaustion function $\varphi$, and let $\psi$ be a $C^\infty$ exhaustion function on $Y$ such that $\psi$ is $n$-convex outside a compact subset $K$ of $Y$. Suppose that $\varphi|(Y^j\backslash K)$ and $\psi|(Y^j\backslash K)$ do not have local maximums for $j > 0$. Then there* *exist a compact set $\hat{K} \subset Y$, a positive number $\varepsilon$, a real number $C$, and a $C^\infty$ $n$-convex function $\Phi$ on $Y$ such that $\Phi|K = \varepsilon\varphi$ and $\Phi|(Y\backslash\hat{K}) = \psi - C$.*

*Proof.* Take any $c_1 \in \mathbb{R}$ such that $\varphi|K < c_1$ and that $\psi|\varphi^{-1}$ $((c_1,\infty))$ is $n$-convex.Then, take $c_2$ and $c_3$ in such a way that $\psi|\varphi^{-1}$ $((-\infty,c_1]) < c_2$ and $\varphi|\psi^{-1}$ $((-\infty,c_2]) < c_3$.

Then we put, for any $A > 0$,

$$\varphi_A = \begin{cases} A(\psi - c_2) & on \quad \varphi^{-1}([c_3,\infty)) \\ \max\{\varphi, A(\psi - c_2)\} & on \quad \varphi^{-1}((c_1,c_3)) \\ \varphi & on \quad \varphi^{-1}((-\infty,c_1]) \end{cases}$$

Although $\varphi_A$ is not n-convex in general, it is clearly continuous for sufficiently large $A$, and has no local maximum because so do $\varphi$ and $\psi|\varphi^{-1}$ $((c_1,\infty))$. Note that $\varphi_A|(Y^j\backslash K)$ has no local maximum if $j > 0$ by assumption.

After fixing such a number $A$, take a $C^\infty$ function $\tilde{\varphi}$ by approximating $\varphi_A$, such that $\tilde{\varphi} = \varphi_A$ on $Y\backslash\varphi^{-1}$ $((c_1,c_3))$ and $\tilde{\varphi}|(Y^j\backslash K)$ have no local maximum for $j$ >0. We may assume that the critical points of $\tilde{\varphi}|Y^j\cap\varphi^{-1}$ $([c_1,c_3])\backslash$Sing $Y^j$are isolated and non-degenerate for all $j$, and that $\tilde{\varphi}$ is $n$-convex on a neighborhood of $Y^0$.

Let $\Sigma$ be the union of the sets of critical points of $\tilde{\varphi}|(Y\cap\varphi^{-1}([c_1,c_3])\backslash$ Sing $Y^j)$ for all $j > 0$.

Since $\tilde{\varphi}|(Y^j \cap \varphi^{-1}$ $([c_1,c_3])\backslash$Sing$Y^j)$ have no local maximums, one can find an arbitrarily small neighborhood $U$ of $\Sigma$ and a $C^\infty$ diffeomorphism $F_v$ : $Y \longrightarrow Y$ fixing the points of $\Sigma$ and $Y\backslash U$, such that $\tilde{\varphi} \circ F_v$ is $n$-convex on a neighborhood of $\Sigma$. As such a diffeomorphism $F_v$, it suffices to take one with sufficiently enlarging dilation along a positive direction, compared to the complementary directions, of the Hessian of $\tilde{\varphi}|(Y^j\backslash$Sing $Y^j)$ at $\Sigma$.

Then it is obvious that one can find a $C^\infty$ convex increasing function $\lambda$ on $\mathbb{R}$ such that $\lambda \circ \tilde{\varphi} \circ F_v$ satisfies the requirements for $\Phi$ for some $\varepsilon$ and $C$. □

The following is contained in [O-3]. Although the notations are slightly different from that of [O-3], the adjustment is routine and may well be left to the reader.

**Lemma 2** (See [O-3, Proposition (in §2) and §3]). *Let $X$ be a complex analytic space whose irreducible components are noncompact, and let $\xi$: $X \longrightarrow \mathbb{R}$ be a $C^\infty$ exhaustion function such that $\xi$ is $n$-convex on a neighborhood of SingX. Then there exists a convex increasing function $\lambda$ on $\mathbb{R}$ and a $C^\infty$ $n$-convex exhaustion function $\Psi$ on $X$ such that $\Psi = \lambda\circ\xi$ holds on a neighborhood of SingX.*

Therefore, by a construction inductive on the dimension, one can find a $C^\infty$ $n$-convex exhaustion function $\varphi$ on $X\backslash|D|$ such that $\varphi|(X^j\backslash|D|)$ have no local maximums outside a compact subset, for any j>0.

*Proof of Theorem* 1. Let $\eta : X \longrightarrow \mathbb{R}$ be any $C^\infty$ function which is n-convex on a neighborhood of $|D|$. Existence of such a function $\eta$ is obvious. Replacing $\eta$ by $B\eta$ for a sufficiently large constant $B$, if necessary, we may assume that there exists a compact set $K_1 \subset X\backslash|D|$ such that $\psi := -\log|s|^2+\eta$ is $n$-convex on $X\backslash|D|\backslash K_1$.

Clearly, we may choose $K_1$ in such a way that $\psi|(X^j\backslash|D|\backslash K_1)$ have no local maximums if $j > 0$.

Hence, in view of the existence of $\varphi$ as above and Lemma 1, there exists a compact set $K_2 \subset X\backslash|D|$ such that one can extend $\psi|(X\backslash|D|\backslash K_2)$ to a $C^\infty$ $n$-convex function on $X\backslash|D|$. This was what we wanted to show, as was mentioned before stating Lemma 1.

## §3. Proof of Theorem 2

If $n >$dim $f^{-1}(z)$, then for any coherent analytic sheaf $\mathcal{F}$ over $M$ one has $(R^n f_*\mathcal{F})_z = 0$ by [A-G], since $f^{-1}(z)$ admits an $n$-complete neighborhood system (cf. [B]). If $z$ is as in the assumption and $n =$ dim $f^{-1}(z)$, then $[D]|f^{-1}(z)$ is $n$-concave by Theorem 1. Hence $[D]$ is $n$-concave on a neighborhood of $f^{-1}(z)$.

Therefore, since $f^{-1}(z)$ admits a holomorphically convex neighborhood system, the result follows from a vanishing theorem of Serre type on weakly 1-complete manifolds (cf. [O-1, Corollary 1.4]).

## §4. Proof of Theorem 3

Similarly as above, it suffices to show the assertion when $n =$ dim $f^{-1}(z)$. Let $s$ be a canonical section of $[D]$ and let $g$ be any Kähler metric on $M$. By the assumption on $z$, one can find a fiber metric h of $[D]$ and a holomorphically convex neighbourhood $W$ of $f^{-1}(z)$ in such a way that $([D], h)$ is $n$-concave on $W$. Then, since the function $\eta$ in the proof of Theorem 1 can be chosen in such a way that all the eigenvalues of $\partial\bar{\partial}\eta$ with respect to any prescribed metric are greater than $-1$ and at most $n-1$ of them are less than $n$, it is easy to see that, in the present situation, one can choose $h$ so that there exists a $C^\infty$ (weakly) convex increasing function $\lambda$ satisfying $\lambda$'(t) $= 1$ for sufficiently large $t$, such that the eigenvalues of the curvature form of

$$\hat{h} := h|s|^{-2}\exp(-\lambda(-\log|s|^2))$$

with respect to $g$, say $\gamma_1 \le \gamma_2 \le \cdots \le \gamma_N$ , satisfy $\gamma_1 + \cdots + \gamma_n \ge 0$ everywhere and $\gamma_1 + \cdots + \gamma_n > 1$ on some neighborhood of $f^{-1}(z) \cap |D|$.

Let $\varphi$ be any $C^\infty$ plurisubharmonic function on $W$ such that $g + \partial\bar{\partial}\varphi$ is a complete Kähler metric on $W$. Then there exists a positive number $L$, independent of $\varphi$, such that the sums of $n$ eigenvalues of the curvature form of $\hat{h}\exp(-L\varphi)$ with respect to $g + \partial\bar{\partial}\varphi$ are nonnegative on $W$ and greater than 1 on some neighborhood of $f^{-1}(z) \cap |D|$.

In this situation, it is clear from Nakano's inequality that there exist no nonzero square integrable $K_M \otimes [D]$-valued harmonic $(0,n)$-forms on $W$ with respect to the metrics $g + \partial\bar{\partial}\varphi$ and $\hat{h}\exp(-L\varphi)$.

Thus, the vanishing of the $L^2$ harmonic forms holds with respect to $(g + \partial\bar{\partial}\varphi^2,\ \hat{h}\exp(-L\varphi^2))$, for any $C^\infty$ nonnegative plurisubharmonic exhaustion function $\varphi$ on $W$.

Recall that the analytic sheaf cohomology groups of holomorphically convex manifolds are Hausdorff spaces by the direct image theorem of Grauert (cf. [G-2] and [F-K]).

Thus we may conclude that, from the vanishing of the $L^2$ harmonic forms, that $(R^n f_*\mathcal{O}(K_M \otimes [D]))_z = 0$.

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