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Limit Elements in the Configuration Algebra for a Cancellative Monoid†

Dedicated to Professor Heisuke Hironaka
on the occasion of his seventy-seventh birthday

by

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Abstract

Associated with the Cayley graph (Γ, G) of a cancellative monoid Γ with a finite generating system G, we introduce two compact spaces: $\Omega(\Gamma, G)$ consisting of pre-partition functions and $\Omega(P_{\Gamma,G})$ consisting of series opposite to the growth function $P_{\Gamma,G}(t) := \sum_{n=0}^{\infty} \sharp\Gamma_n \cdot t^n$ (where Γ_n is the ball of radius n centered at the unit element in the Cayley graph). Under mild assumptions on (Γ, G), we introduce a fibration $\pi : \Omega(\Gamma, G) \to \Omega(P_{\Gamma,G})$, which is equivariant with respect to a $(\tilde{\tau}, \tau)$-action. The action is transitive if it is of finite order. Then we express the finite sum of the pre-partition functions in each fiber of π as a linear combination of the ratios of the residues of the two growth functions $P_{\Gamma,G}(t)$ and $P_{\Gamma,G}\mathcal{M}(t) := \sum_{n=0}^{\infty} \mathcal{M}(\Gamma_n)t^n$ (where $\mathcal{M}(\Gamma_n)/\sharp\Gamma_n$ is the free energy of the ball Γ_n) at the poles on the circle of their convergence radius.

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Contents

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The present paper is a complete version of the announcement [Sa2] based on the preprint RIMS-726. We rewrote the introduction, left out the filtration by (p, q), divided §10 into §§10 and 11, and updated the references. §11, 12 are newly written, where, applying the results in §§2–10 to Cayley graphs (Γ, G) of cancellative monoids, we introduce a fibration $\Omega(\Gamma, G) \to \Omega(P_{\Gamma,G})$.

§1. Introduction

Replacing the square lattice $\mathbb{Z}^2$ in the classical Ising model ([Gi], [I], [O], [Ba]) by the Cayley graph (Γ, G) of a cancellative monoid Γ with a finite generating system G, we introduce the space $\Omega(\Gamma, G)$ of *pre-partition functions*. Here, the word pre-partition function is used only in the present introduction in the following sense. Namely, for any finite region T of the Cayley graph, we define the *free energy* $\mathcal{M}(T)/\sharp T$ to be the quotient of the logarithm $\mathcal{M}(T) := \log(\mathcal{A}(T))$ of the sum $\mathcal{A}(T)$ of all finite subconfigurations in T divided by the number $\sharp T$ of vertices of T ((5.1.5) and (6.1.1)). Then the space $\Omega(\Gamma, G)$ is introduced as the accumulation points set (in a suitable topological setting) of the sequence $\{\mathcal{M}(\Gamma_n)/\sharp\Gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$ of free energies of balls Γ_n of radius n centered at the unit element in (Γ, G) (11.1 Definition). In the case of $\Gamma = \mathbb{Z}^2$, $\Omega(\Gamma, G)$ consists of a single element. By inputting the data of Boltzmann weights to it, we get the most classical partition function: an elliptic function dependent on the parameters involved in the Boltzmann weights. This fact inspired the author *to use the pre-partition functions to construct functions on the moduli of* Γ ([Sa1, 3]).

In our new setting, the space $\Omega(\Gamma, G)$ is no longer a single element set in general but a compact Hausdorff space. Under mild assumptions on (Γ, G), we construct a fibration $\pi_\Omega : \Omega(\Gamma, G) \to \Omega(P_{\Gamma,G})$ (11.2.12), where $\Omega(P_{\Gamma,G})$ is another newly introduced compact space, consisting of opposite sequences of the growth function $P_{\Gamma,G}(t)$ (11.2.3). The fibration is equivariant with respect to actions $\tilde{\tau}_\Omega$ and τ_Ω (§11.2, Theorems 1–4). If the actions are of finite order, then they are transitive and the sum of the pre-partition functions in a fiber of π_Ω is given by a linear combination of the ratios of the residues of the two series $P_{\Gamma,G}\mathcal{M}(t) := \sum_{n=0}^{\infty} \mathcal{M}(\Gamma_n)\cdot t^n$ and $P_{\Gamma,G}(t) := \sum_{n=0}^{\infty} \sharp\Gamma_n \cdot t^n$ at their poles on the circle $|t| = r_{\Gamma,G}$ of convergence radius $r_{\Gamma,G}$ (§11.5, Theorem 6). We publish these results in the present paper, even though our original goal is not achieved.

The paper is divided into two parts. In the first part §§2–10, we develop a general framework on a topological Hopf algebra $\mathbb{R}[\![\mathrm{Conf}]\!]$ called the *configuration* *algebra*, where necessary concepts such as configuration sums, free energies (called equally dividing points) etc. are introduced. The algebra is equipped with two (one adic and the other classical) topologies in order to discuss carefully the limit process in it. Then, inside its subspace $\mathcal{L}_{\mathbb{R},\infty}$ of Lie-like elements at infinity, the compact set $\Omega_\infty := \overline{\log(\mathrm{EDP})}_\infty$ of all accumulation points of free energies is introduced. In the second half, §§11–12, we consider the Cayley graph (Γ, G) of a monoid. Then, the set of pre-partition functions $\Omega(\Gamma, G)$ is defined as the subset of Ω_∞ consisting of all accumulation points of the sequence of free energies of the balls Γ_n of radius $n \in \mathbb{Z}_{\geq 0}$ in (Γ, G). We also introduce another limit set $\Omega(P_{\Gamma,G}) \subset \mathbb{R}[\![s]\!]$, called *the space of opposite sequences*, depending only on the Poincaré series $P_{\Gamma,G}(t)$ of (Γ, G) (see (11.2.1–4) and (11.2.6)). The space $\Omega(P_{\Gamma,G})$ is the key to relating the space $\Omega(\Gamma, G)$ with the singularities of the Poincaré series $P_{\Gamma,G}(t)$ on the circle $|t| = r_{\Gamma,G}$ of convergence (§11.2–4, Theorems 1–5). Then, comparing the two limit spaces $\Omega(\Gamma, G)$ and $\Omega(P_{\Gamma,G})$, we arrive at the goal: a residual presentation of pre-partition functions (§11.5, Theorem 6).

Let us explain the contents of the present paper in more detail.

The isomorphism class of a colored oriented finite graph is called a *configuration.* The set of all configurations with fixed bounds of valency and colors, denoted by Conf, has an additive monoid structure (with disjoint union as product) generated by Conf_0, the isomorphism classes of connected graphs, and a partial ordering structure (§2.3). In §2.4, we introduce the basic invariant $\binom{S_1,\ldots,S_m}{S} \in \mathbb{Z}_{\geq 0}$ for $S_1, \ldots, S_m$ and $S \in \mathrm{Conf}$, called a *covering coefficient.* We denote by $\mathbb{A}[\![\mathrm{Conf}]\!]$ the completion of the semigroup ring $\mathbb{A}\cdot\mathrm{Conf}$ with respect to the grading $\deg(S) := \sharp S$, called the *configuration algebra* (§3), where $\mathbb{A}$ is the ring (commutative with unit) of coefficients. The algebra $\mathbb{A}[\![\mathrm{Conf}]\!]$ carries a topological Hopf algebra structure by taking the covering coefficients as structure constants (§4).

For a configuration $S \in \mathrm{Conf}$, let $\mathcal{A}(S)$ be the element of $\mathbb{A} \cdot \mathrm{Conf}$ given by the sum of all its subgraphs. We put $\mathcal{M}(S) := \log(\mathcal{A}(S))$. Then $\{\mathcal{M}(S)\}_{S\in\mathrm{Conf}_0}$ forms a basis of the Lie-like space of the non-complete bi-algebra $\mathbb{A} \cdot \mathrm{Conf}$ (§§5–6). However, this is not a topological basis of the Lie-like space $\mathcal{L}_{\mathbb{A}}$ of the algebra $\mathbb{A}[\![\mathrm{Conf}]\!]$. Therefore, we introduce a topological basis, denoted by $\{\varphi(S)\}_{S\in\mathrm{Conf}_0}$. The coefficients of the transformation matrix between the bases $\{\mathcal{M}(S)\}_{S\in\mathrm{Conf}_0}$ and $\{\varphi(S)\}_{S\in\mathrm{Conf}_0}$ are described by *kabi coefficients*, introduced in §7. The base change induces a linear map, called the *kabi map*, from $\mathcal{L}_{\mathbb{A}}$ to a formal module spanned by $\{\mathcal{M}(S)\}_{S\in\mathrm{Conf}_0}$. The kernel of the kabi map is denoted by $\mathcal{L}_{\mathbb{A},\infty}$ and is called the *Lie-like space at infinity* (§8).

The group-like elements $\mathfrak{G}_{\mathbb{Z},\mathrm{finite}}$ of the configuration algebra $\mathbb{Z}[\![\mathrm{Conf}]\!]$ are isomorphic to the fractional group of the monoid Conf via the correspondence $\mathcal{A}(S) \leftrightarrow S$ (§9). Thus, it contains a positive cone spanned by Conf. We are interested in *equal division points* $\mathcal{A}(S)^{1/\sharp S}$ ($S \in \mathrm{Conf}$) of the lattice points in the positive cone, and the set $\overline{\mathrm{EDP}}$ of *their accumulation points with respect to the classical topology* by specializing the coefficients $\mathbb{A}$ to $\mathbb{R}$. In §10, by taking their logarithms,[1] we define their accumulation set $\Omega := \overline{\log(\mathrm{EDP})}$ in $\mathcal{L}_{\mathbb{R}}$. The set Ω decomposes into a join of the infinite simplex spanned by the vertices $\mathcal{M}(S)/\sharp S$ for $S \in \mathrm{Conf}_0$ and $\Omega_\infty := \overline{\log(\mathrm{EDP})}_\infty$ contained in $\mathcal{L}_{\mathbb{R},\infty}$ (§10).

From §11 on, we fix a monoid Γ with a finite generating system G. The sequence of the logarithmic equal division points $\mathcal{M}(\Gamma_n)/\sharp\Gamma_n$ for the sequence of balls Γ_n of radius $n \in \mathbb{Z}_{\geq 0}$ in the Cayley graph accumulates to a compact set in $\mathcal{L}_{\mathbb{R},\infty}$, which we denote by $\Omega(\Gamma, G)$ and call the *space of limit elements* for (Γ, G). This is the main object of interest of the present article. If Γ is a group of polynomial growth, then due to results of Gromov [Gr1] and Pansu [P], for any generating system G, (Γ, G) is simple accumulating, i.e. $\sharp\Omega(\Gamma, G) = 1$.

In order to study the multi-accumulating cases, we introduce in §11 (11.2.3) another compact accumulation set $\Omega(P_{\Gamma,G})$: the *space of opposite series of* the growth series $P_{\Gamma,G}(t) := \sum_{n=0}^{\infty} \sharp\Gamma_n \cdot t^n$. Under mild Assumptions 1 in §11.1 and 2 in §11.2 on (Γ, G), we show that there exists a natural proper surjective map $\pi_\Omega : \Omega(\Gamma, G) \to \Omega(P_{\Gamma,G})$ equivariant with respect to the $(\tilde{\tau}, \tau)$ action (see 11.2 Theorems 1–4), where (i) π_Ω is a forgetful map which remembers only the portion $\lim_{n\to\infty} A(\Gamma_{n-k}, \Gamma_n)/\sharp\Gamma_n$ (here $A(\Gamma_{n-k}, \Gamma_n) := \sharp\{$subgraphs of Γ_n isomorphic to $\Gamma_{n-k}\}$) and (ii) the action $\tilde{\tau}_\Omega$ is defined by: *the limit of a subsequence* $\{n_m\}_{m\in\mathbb{Z}_{\geq 0}} \mapsto$ *the limit of the subsequence* $\{n_m - 1\}_{m\in\mathbb{Z}_{\geq 0}}$. Importantly this $\tilde{\tau}_\Omega$-action, up to an initial constant factor, has an interpretation by the fattening action on $\mathrm{Conf}(\Gamma, G)$: $S \mapsto S\Gamma_1$ (here, $S\Gamma_1$ is the equivalence class of $\mathbb{S} \cdot \Gamma_1$ for a representative $\mathbb{S}$ of S) on the level of $\Omega(\Gamma, G)$, and interpretation by the degree shift action $t^n \mapsto t^{n+1}$ on the level of $\Omega(P_{\Gamma,G})$.

Subsections 11.3 and 11.4 are devoted to the study of the space of opposite sequences $\Omega(P)$ for a power series $P(t)$ (11.2.1) in general with a loose constraint on the growth of coefficients. The main concern is to clarify a certain duality between the set $\Omega(P)$ and the set of singularities of $P(t)$ on the boundary of the convergence disc of radius r. It requires intricate analysis, and, in the present paper, we clarify only when $\Omega(P)$ is a finite set. Actually, if $\Omega(P)$ is finite, then the $\mathbb{Z}_{\geq 0}$-action τ_Ω becomes a cyclic $\mathbb{Z}/h\mathbb{Z}$ and simply-transitive action. We can determine $\Omega(P)$ explicitly as a set of rational functions in the variable s of a particular form. In particular, their common denominator $\Delta_P^{\mathrm{op}}(s)$, which is a factor of $1 - (rs)^h$, has the degree equal to the rank of the space $\mathbb{R}\Omega(P)$ spanned by $\Omega(P)$. If, further, $P(t)$ is meromorphic in a neighborhood of the convergence

[1]The logarithm $\log(\mathcal{A}(S)^{1/\sharp S}) = \mathcal{M}(S)/\sharp S$, which we call the logarithmic equal dividing point, is called the (Helmholtz) free energy in statistical mechanics ([Gi], [I], [O], [Ba]).

disc, then the top part $\Delta_P^{\mathrm{top}}(t)$ of the denominator of $P(t)$ on the convergence circle of radius r (see (11.4.1)) and the opposite denominator $\Delta_P^{\mathrm{op}}(s)$ are related by the opposite transformation $st = 1$ (11.4 Theorem 5).

If $\Omega(\Gamma, G)$ is finite, then again the $\mathbb{Z}_{\geq 0}$-action $\tilde{\tau}_\Omega$ on $\Omega(\Gamma, G)$ becomes a cyclic $\mathbb{Z}/\tilde{h}_{\Gamma,G}\mathbb{Z}$ and simply-transitive action for some multiple $\tilde{h}_{\Gamma,G} \in \mathbb{Z}_{>0}$ of the period $h_{\Gamma,G}$ of the $\mathbb{Z}_{\geq 0}$-action on $\Omega(P_{\Gamma,G})$. Therefore, the map π_Ω is equivalent to the Galois covering map $\mathbb{Z}/\tilde{h}_{\Gamma,G}\mathbb{Z} \to \mathbb{Z}/h_{\Gamma,G}\mathbb{Z}$. Let us call the kernel of the homomorphism the *inertia group* and the sum of elements in $\Omega(\Gamma, G)$ of an orbit of the inertia group a *trace element*. As the goal of the present paper, we express the trace elements as linear combinations of the ratios $\frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\big|_{t=x}$ of the residues of the meromorphic functions $P_{\Gamma,G}\mathcal{M}(t) := \sum_{n=0}^{\infty} \mathcal{M}(\Gamma_n)t^n$ and $P_{\Gamma,G}(t) := \sum_{n=0}^{\infty}(\sharp\Gamma_n)t^n$ at the roots x of $\Delta_P^{\mathrm{top}}(t) = 0$ (11.5 Theorem 6). In the proof, we essentially use the duality theory of 11.4.

Finally, in §12, we give a few concluding remarks. Since we are only at the start of the study of the limit space $\Omega(\Gamma, G)$, the questions point in various directions, of general and of specific nature.

As an immediate generalization of our final Theorem 6 to the cases when $\Omega(\Gamma, G)$ is not finite, in Problem 1.1, we ask for a *measure-theoretic approach to the duality between* $\Omega(P)$ *and* $\mathrm{Sing}(P)$, and give a conjectural formula.

Another important generalization of Theorem 6 is the *globalization* in the following sense: in many important examples, the growth function $P_{\Gamma,G}(t)$ analytically extends to a meromorphic function in covering regions of $\mathbb{C}$ (and the same for $P_{\Gamma,G}\mathcal{M}(t)$). Let x be a pole of order d of such a meromorphic function; then $\big(\frac{d^i}{dx^i}\frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\big)\big|_{t=x}$ for $0 \leq i < d$ (which we call a *higher residue* of order i at x) belongs to $\mathcal{L}_{\mathbb{C},\infty}$ (even though it is no longer a limit element). Theorem 6 treats only the extremal case $|x| = r$ and $i = 0$. Therefore, the problem is *to study all higher residues at all possible poles together with a possible action of a Galois group*; in particular, *to clarify the meaning of the (higher) residues at* $x = 1$.

We conjecture that hyperbolic groups and some groups of geometric significance (surface groups, mapping class groups and Artin groups for suitable choices of generators) are finite accumulating, i.e. $\sharp\Omega(\Gamma, G) < \infty$.

§2. Colored graphs and covering coefficients

An isomorphism class of finite graphs with a fixed color set and a bounded number of edges (valency) at each vertex is called a *configuration*. The set of all configurations carries the structure of an abelian monoid with a partial ordering. The goal of the present section is to introduce a numerical invariant, called the *covering coefficient*, and to show some of its basic properties.

§2.1. Colored graphs

We first give the definition of colored graph which is used in the present paper.

Definition. 1. A pair (Γ, B) is called a *graph* if Γ is a set and B is a subset of $\Gamma\times\Gamma\setminus\Delta$ with $\sigma(B)=B$, where σ is the involution $\sigma(\alpha,\beta):=(\beta,\alpha)$ and Δ is the diagonal. An element of Γ is called a *vertex* and a σ-orbit in B is called an *edge*. A graph is called *finite* if $\sharp\Gamma<\infty$. We sometimes denote a graph by Γ and the set of its vertices by $|\Gamma|$.

2. Two graphs are *isomorphic* if there is a bijection of vertices inducing a bijection of edges. Any subset $\mathbb{S}$ of $|\Gamma|$ carries a graph structure by taking $B\cap(\mathbb{S}\times\mathbb{S})$ as the set of edges for $\mathbb{S}$. The set $\mathbb{S}$ equipped with this graph structure is called a *subgraph* (or a *full subgraph*) of Γ and is denoted by the same $\mathbb{S}$. In the present paper, the word "subgraph" shall be used only in this sense, and the notation $\mathbb{S}\subset\Gamma$ shall mean also that $\mathbb{S}$ is a subgraph of Γ associated to the subset. Hence, we have the bijection: $\{\text{subgraphs of }\Gamma\}\ \simeq\ \{\text{subsets of }|\Gamma|\}$.

3. A pair (G,σ_G) of a set G and an involution σ_G on G (i.e. a map $\sigma_G: G\to G$ with $\sigma_G^2=\mathrm{id}_G$) is called a *color set*. For a graph (Γ,B), a map $c: B\to G$ is called a (G,σ_G)-*coloring*, or G-*coloring*, if c is equivariant with respect to the involutions: $c\circ\sigma=\sigma_G\circ c$. The pair consisting of a graph and a G-coloring is called a G-*colored graph*. Two G-colored graphs are called G-*isomorphic* if there is an isomorphism of the graphs compatible with the colorings. Subgraphs of a G-colored graph are naturally G-colored.

If all points of G are fixed by σ_G, then the graph is called *unoriented*. If G consists of one orbit of σ_G, then the graph is called *uncolored*.

The isomorphism class of a G-colored graph $\mathbb{S}$ is denoted by $[\mathbb{S}]$. Sometimes we will write $\mathbb{S}$ instead of $[\mathbb{S}]$ (for instance, we put $\sharp[\mathbb{S}]:=\sharp\mathbb{S}$, and call $[\mathbb{S}]$ *connected* if $\mathbb{S}$ is topologically connected as a simplicial complex).

Example (Colored Cayley graph of a monoid with cancellation conditions). Let Γ be a monoid satisfying the left and right cancellation conditions: if $axb=ayb$ in Γ for $a,b,x,y\in\Gamma$ then $x=y$ in Γ. In other words, for any $a,b\in\Gamma$, if there exists $g\in\Gamma$ such that $a=bg$ (resp. $a=gb$) then g is uniquely determined from a and b, and we shall denote it by $b^{-1}a$ (resp. ab^{-1}). Let G be a finite generating system of Γ with $e\notin G$. Then we equip Γ with a graph structure by taking $B:=\{(\alpha,\beta)\in\Gamma\times\Gamma\mid \alpha^{-1}\beta \text{ or } \beta^{-1}\alpha\in G\}$ as the set of edges. Due to the left cancellation condition, it becomes a colored graph by taking $G\cup G^{-1}$as the color set and by putting $c(\alpha,\beta)=\alpha^{-1}\beta$ for $(\alpha,\beta)\in B$, where G^{-1} is a formally defined set consisting of symbols α^{-1} for $\alpha\in G$ and identifying α^{-1} with $\beta\in G$ if $\alpha\beta=e$ in Γ (such a β may not always exist). Due to the right cancellation condition, for any vertex x and any $\alpha \in G$, the vertex connected with x by an edge of color α (i.e. $y \in \Gamma$ such that $y\alpha = x$) is unique. Let us call this graph, denoted by (Γ, G) or Γ for simplicity, the *colored Cayley graph* of the monoid Γ with respect to the generating system G. The left action of $g \in \Gamma$ on Γ is a color preserving graph embedding map from (Γ, G) to itself.

If $G = G^{-1}$, then Γ is a group and the above definition coincides with the usual definition of the Cayley graph of a group.

§2.2. Configuration

For the remainder of the paper, *we fix a finite color set* (G, σ_G) *(i.e.* $\sharp G < \infty$*) and a non-negative integer* $q \in \mathbb{Z}_{\geq 0}$*, and consider only the* G*-colored graphs such that the number of edges ending at any vertex (called valency) is at most* q. The isomorphism class $[\mathbb{S}]$ of such a graph $\mathbb{S}$ is called a (G, q)-*configuration* (or, a *configuration*). The set of all (connected) configurations is defined by:

(2.2.1) $$\mathrm{Conf} := \{G\text{-isomorphism classes of } G\text{-colored graphs such that the number of edges ending at any given vertex is at most } q\},$$

(2.2.2) $$\mathrm{Conf}_0 := \{S \in \mathrm{Conf} \mid S \text{ is connected}\}.$$

The isomorphism class $[\emptyset]$ of an empty graph is contained in Conf but not in Conf_0. Sometimes it is convenient to exclude $[\emptyset]$ from Conf. So put

(2.2.3) $$\mathrm{Conf}_+ := \mathrm{Conf} \setminus \{[\emptyset]\}.$$

Remark. To be exact, the set of configurations (2.2.1) should have been denoted by $\mathrm{Conf}^{G,q}$. If there is a map $G \to G'$ between two color sets compatible with their involutions and an inequality $q \leq q'$, then there is a natural map $\mathrm{Conf}^{G,q} \to \mathrm{Conf}^{G',q'}$. Thus, for any inductive system $(G_n, q_n)_{n \in \mathbb{Z}_{>0}}$ (i.e. $G_n \to G_{n+1}$ and $q_n \leq q_{n+1}$ for n), we get the inductive limit $\lim_{n\to\infty} \mathrm{Conf}^{G_n,q_n}$. In [Sa2], we used such a limit set. However, in this paper, we fix G and q, since the key limit processes (3.2.2) and (10.1.1) can be carried out for fixed G and q.

§2.3. Semigroup structure and partial ordering structure on Conf

We introduce the following two structures on Conf.

1. *The set* Conf *has a natural abelian semigroup structure* by putting

$$[\mathbb{S}] \cdot [\mathbb{T}] := [\mathbb{S} \sqcup \mathbb{T}] \quad \text{for } [\mathbb{S}], [\mathbb{T}] \in \mathrm{Conf},$$

where $\mathbb{S} \sqcup \mathbb{T}$ is the disjoint union of graphs $\mathbb{S}$ and $\mathbb{T}$ representing the isomorphism classes $[\mathbb{S}]$ and $[\mathbb{T}]$. The empty class $[\emptyset]$ plays the role of the unit and is denoted by 1. It is clear that Conf *is freely generated by* Conf_0. The power S^k or $[\mathbb{S}]^k$ $(k \geq 0)$ denotes the class of the disjoint union $\mathbb{S} \sqcup \cdots \sqcup \mathbb{S}$ of k copies of $\mathbb{S}$.

2. *The set* Conf *is partially ordered*, where we define, for $S, T \in \mathrm{Conf}$,

$$S \leq T \ \Leftrightarrow \ \text{there exist graphs } \mathbb{S} \text{ and } \mathbb{T} \text{ with } S = [\mathbb{S}],\ T = [\mathbb{T}] \text{ and } \mathbb{S} \subset \mathbb{T}.$$

The unit $1 = [\emptyset]$ *is the unique minimal element in* Conf *for this ordering.*

§2.4. Covering coefficients

For $S_1, \ldots, S_m$ and $S \in \mathrm{Conf}$, we introduce a non-negative integer

$$\binom{S_1, \ldots, S_m}{S} := \sharp \binom{S_1, \ldots, S_m}{\mathbb{S}} \in \mathbb{Z}_{\geq 0} \tag{2.4.1}$$

and call it the *covering coefficient*, where $\binom{S_1, \ldots, S_m}{\mathbb{S}}$ is defined by the following:

(i) Fix any G-graph $\mathbb{S}$ with $[\mathbb{S}] = S$.
(ii) Define

$$\binom{S_1, \ldots, S_m}{\mathbb{S}} := \{(\mathbb{S}_1, \ldots, \mathbb{S}_m) \mid \ \mathbb{S}_i \subset \mathbb{S},\ [\mathbb{S}_i] = S_i \ (i = 1, \ldots, m) \text{ and } \textstyle\bigcup_{i=1}^m |\mathbb{S}_i| = |\mathbb{S}|\}. \tag{2.4.2}$$

(iii) Show an isomorphism $\mathbb{S} \simeq \mathbb{S}'$ induces a bijection $\binom{S_1, \ldots, S_m}{\mathbb{S}} \simeq \binom{S_1, \ldots, S_m}{\mathbb{S}'}$.

Remark. In the definition (2.4.2), one should notice that:

(i) Each $\mathbb{S}_i$ in (2.4.2) should be a full subgraph of $\mathbb{S}$ (see 2.1 Def. 2).
(ii) The union of the edges of $\mathbb{S}_i$ $(i = 1, \ldots, k)$ does not have to cover all edges of $\mathbb{S}$.
(iii) The sets of vertices $|\mathbb{S}_i|$ $(i = 1, \ldots, k)$ may overlap in the set $|\mathbb{S}|$.

Example. Let X_1, X_2 be elements of Conf_0 with $\sharp X_i = i$ for $i = 1, 2$. Then $\binom{X_1 \cdot X_1}{X_2} = 0$ and $\binom{X_1, X_1}{X_2} = 2$.

The covering coefficients are the most basic tool in the present paper. We shall give their elementary properties in 2.5 and the two basic rules: *the composition rule* in 2.6 and *the decomposition rule* in 2.7.

§2.5. Elementary properties of covering coefficients

Some elementary properties of covering coefficients, as immediate consequences of the definition, are listed below. They will be used in the study of the Hopf algebra structure on the configuration algebra in §4.

(i) $\binom{S_1, \ldots, S_m}{S} = 0$ *unless* $S_i \leq S$ *for* $i = 1, \ldots, m$ *and* $\sum \sharp S_i \geq \sharp S$.
(ii) $\binom{S_1, \ldots, S_m}{S}$ *is invariant under permutations of* S_i*'s.*

(iii) *For $1 \le i \le m$, one has an elimination rule:*

$$\binom{S_1,\dots,S_{i-1},[\emptyset],S_{i+1},\dots,S_m}{S} = \binom{S_1,\dots,S_{i-1},S_{i+1},\dots,S_m}{S}. \tag{2.5.1}$$

(iv) *For the case $m = 0$, the covering coefficients are given by*

$$\binom{[\emptyset]}{S} = \begin{cases} 1 & \text{if } S = [\emptyset], \\ 0 & \text{otherwise,} \end{cases} \tag{2.5.2}$$

(v) *For the case $m = 1$, the covering coefficients are given by*

$$\binom{T}{S} = \begin{cases} 1 & \text{if } S = T, \\ 0 & \text{otherwise,} \end{cases} \tag{2.5.3}$$

(vi) *For the case $S = [\emptyset]$, the covering coefficients are given by*

$$\binom{S_1,\dots,S_m}{[\emptyset]} = \begin{cases} 1 & \text{if } \bigcup S_i = \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5.4}$$

§2.6. Composition rule

Assertion. *For $S_1,\dots,S_m,T_1,\dots,T_n,S \in \mathrm{Conf}$ $(m,n \in \mathbb{Z}_{\ge 0})$, one has*

$$\sum_{U\in\mathrm{Conf}} \binom{S_1,\dots,S_m}{U}\binom{U,T_1,\dots,T_n}{S} = \binom{S_1,\dots,S_m,T_1,\dots,T_n}{S}. \tag{2.6.1}$$

Proof. If $m = 0$, then the formula reduces to 2.5 (iii) and (iv). Assume $m \ge 1$ and consider the map

$$\binom{S_1,\dots,S_m,T_1,\dots,T_n}{\mathbb{S}} \to \bigsqcup_{U\in\mathrm{Conf}} \binom{U,T_1,\dots,T_n}{\mathbb{S}},$$

$$(\mathbb{S}_1,\dots,\mathbb{S}_m,\mathbb{T}_1,\dots,\mathbb{T}_n) \mapsto \Big(\bigcup_{i=1}^{m} \mathbb{S}_i, \mathbb{T}_1,\dots,\mathbb{T}_n\Big).$$

Here, $\bigcup_{i=1}^m \mathbb{S}_i$ means the subgraph of $\mathbb{S}$ whose vertices are the union of the vertices of the $\mathbb{S}_i$ $(i = 1,\dots,m)$ (cf. 2.1 Def. 2) and the class $[\bigcup_{i=1}^m \mathbb{S}_i]$ is denoted by U. The fiber over a point $(\mathbb{U},\mathbb{T}_1,\dots,\mathbb{T}_n)$ is bijective to the set $\binom{S_1,\dots,S_m}{\mathbb{U}}$ so that one has the bijection

$$\binom{S_1,\dots,S_m,T_1\dots,T_n}{\mathbb{S}} \simeq \bigsqcup_{U\in\mathrm{Conf}} \binom{S_1,\dots,S_m}{\mathbb{U}}\binom{U,T_1,\dots,T_n}{\mathbb{S}}. \qquad \square$$

Note. The LHS of (2.6.1) is a finite sum, since the only positive summands arise with $U \le S$ due to 2.5 (i).

§2.7. Decomposition rule

Assertion. *Let $m \in \mathbb{Z}_{\geq 0}$. For $S_1, \ldots, S_m, U$ and $V \in \mathrm{Conf}$, one has*

$$\binom{S_1, \ldots, S_m}{U \cdot V} = \sum_{\substack{R_1, T_1 \in \mathrm{Conf} \\ S_1 = R_1 \cdot T_1}} \cdots \sum_{\substack{R_m, T_m \in \mathrm{Conf} \\ S_m = R_m \cdot T_m}} \binom{R_1, \ldots, R_m}{U} \binom{T_1, \ldots, T_m}{V}. \tag{2.7.1}$$

Here $R_i, T_i \in \mathrm{Conf}$ run over all possible decompositions of S_i in Conf*.*

Proof. If $m = 0$, this is (2.5.2). Consider the map

$$\binom{S_1, \ldots, S_m}{\mathbb{U} \cdot \mathbb{V}} \to \bigcup_{S_1 = R_1 \cdot T_1} \cdots \bigcup_{S_m = R_m \cdot T_m} \binom{R_1, \ldots, R_m}{\mathbb{U}} \times \binom{T_1, \ldots, T_m}{\mathbb{V}},$$
$$(\mathbb{S}_1, \ldots, \mathbb{S}_m) \mapsto (\mathbb{S}_1 \cap \mathbb{U}, \ldots, \mathbb{S}_m \cap \mathbb{U}) \times (\mathbb{S}_1 \cap \mathbb{V}, \ldots, \mathbb{S}_m \cap \mathbb{V}).$$

One checks easily that the map is bijective. □

Note. The RHS of (2.7.1) is a finite sum, since the only positive summands arise when $R_i \leq U$ and $T_i \leq V$.

§3. Configuration algebra

We complete the semigroup ring $\mathbb{A} \cdot \mathrm{Conf}$, where $\mathbb{A}$ is a commutative associative unitary algebra, by use of the adic topology with respect to the grading $\deg(S) := \sharp S$, and call the completion the configuration algebra. It is a formal power series ring of infinitely many variables $S \in \mathrm{Conf}_0$. We discuss several basic properties of the algebra, including topological tensor products.

§3.1. The polynomial type configuration algebra $\mathbb{Z} \cdot \mathrm{Conf}$

The free abelian group generated by Conf,

$$\mathbb{Z} \cdot \mathrm{Conf}, \tag{3.1.1}$$

naturally carries the structure of an algebra by the use of the semigroup structure on Conf (recall 2.3), where $[\emptyset] = 1$ plays the role of the unit element. It is isomorphic to the free polynomial algebra generated by Conf_0, and hence is called *the polynomial type configuration algebra.* The algebra is graded by taking $\deg(S) := \sharp S$ for $S \in \mathrm{Conf}$, since one has additivity:

$$\sharp(S \cdot T) = \sharp S + \sharp T. \tag{3.1.2}$$

§3.2. The completed configuration algebra $\mathbb{Z}[\![\mathrm{Conf}]\!]$

The polynomial type algebra (3.1.1) is not sufficiently large for our purposes, since it does not contain certain limit elements which we want to investigate (cf. 4.6

Remark 3 and 6.4 Remark 2). Therefore, we localize the algebra by the completion with respect to the grading given in 3.1.

For $n \geq 0$, let us define an ideal in $\mathbb{Z} \cdot \mathrm{Conf}$ by

$$\mathcal{J}_n := \text{the ideal generated by } \{S \in \mathrm{Conf} \mid \sharp S \geq n\}. \tag{3.2.1}$$

Taking $\mathcal{J}_n$ as a fundamental system of neighborhoods of $0 \in \mathbb{Z} \cdot \mathrm{Conf}$, we define the *adic topology* on $\mathbb{Z} \cdot \mathrm{Conf}$ (see Remark below). The completion

$$\mathbb{Z}[\![\mathrm{Conf}]\!] := \varprojlim_n \mathbb{Z} \cdot \mathrm{Conf} / \mathcal{J}_n \tag{3.2.2}$$

will be called *the completed configuration algebra*, or, simply, *the configuration algebra*. More generally, for any commutative algebra $\mathbb{A}$ with unit, we put

$$\mathbb{A}[\![\mathrm{Conf}]\!] := \varprojlim_n \mathbb{A} \cdot \mathrm{Conf} / \mathbb{A}\mathcal{J}_n, \tag{3.2.3}$$

and call it the *configuration algebra* over $\mathbb{A}$, or, simply, the configuration algebra. The *augmentation ideal* of the algebra is defined as

$$\begin{aligned} \mathbb{A}[\![\mathrm{Conf}]\!]_+ &:= \text{the closed ideal generated by } \mathrm{Conf}_+ \\ &= \text{the closure of } \mathcal{J}_1 \text{ with respect to the adic topology.} \end{aligned}$$

Let us give an explicit expression of an element of the configuration algebra by an infinite series. The quotient $\mathbb{A} \cdot \mathrm{Conf} / \mathbb{A}\mathcal{J}_n$ is naturally bijective to the free module $\prod_{S \in \mathrm{Conf},\, \sharp S < n} \mathbb{A} \cdot S$ of finite rank. Taking the inverse limit of the bijection, we obtain

$$\mathbb{A}[\![\mathrm{Conf}]\!] \simeq \prod_{S \in \mathrm{Conf}} \mathbb{A} \cdot S.$$

In other words, *any element f of the configuration algebra is uniquely expressed by an infinite series*

$$f = \sum_{S \in \mathrm{Conf}} S \cdot f_S \tag{3.2.4}$$

for some constants $f_S \in \mathbb{A}$ for $S \in \mathrm{Conf}$. The coefficient $f_{[\emptyset]}$ of the unit element is called the *constant term* of f. The augmentation ideal is nothing but the collection of those f having vanishing constant term.

Remark. The topology on $\mathbb{A}[\![\mathrm{Conf}]\!]$ (except for the case $q = 0$) defined above is *not equal* to the topology defined by taking the powers of the augmentation ideal as the fundamental system of neighborhoods of 0. More precisely, for $n > 1$ and $q \neq 0$, the image of the product map

$$(\mathbb{A}[\![\mathrm{Conf}]\!]_+)^n \to \overline{\mathbb{A}\mathcal{J}}_n \tag{3.2.5}$$

(cf. (3.5.4) and (3.5.5)) does not generate (topologically) the target ideal on the RHS (= the closure in $\mathbb{A}[\![\mathrm{Conf}]\!]$ of the ideal $\mathbb{A}\mathcal{J}_n = \{f \in \mathbb{A}[\![\mathrm{Conf}]\!] \mid \deg S \geq n$ for $f_S \neq 0\}$), since there exists a connected configuration S with $\deg S = n$, but S, as an element in $\mathcal{J}_n$, cannot be expressed as a function of elements of $\mathcal{J}_m$ for $m < n$. In this sense, the name "adic topology" is *misused* here.

The notation $\mathbb{A}[\![\mathrm{Conf}]\!]$ should not be mistaken for the algebra of formal power series in Conf. In fact, it is the set of formal series in Conf_0.

§3.3. Finite type elements in the configuration algebra

The support for the series f (3.2.4) is defined as

$$\mathrm{Supp}(f) := \{S \in \mathrm{Conf} \mid f_S \neq 0\}. \tag{3.3.1}$$

Definition. An element f of a configuration algebra is said to be of *finite type* if $\mathrm{Supp}(f)$ is contained in a finitely generated semigroup in Conf. Note that f being of finite type does not mean that f is a finite sum, but that it is expressed by a finite number of "variables". The subset of $\mathbb{A}[\![\mathrm{Conf}]\!]$ consisting of all elements of finite type is denoted by $\mathbb{A}[\![\mathrm{Conf}]\!]_{\mathrm{finite}}$. The polynomial type configuration algebra $\mathbb{A} \cdot \mathrm{Conf}$ is a subalgebra of $\mathbb{A}[\![\mathrm{Conf}]\!]_{\mathrm{finite}}$.

§3.4. Saturated subalgebras of the configuration algebra

The configuration algebra is sometimes a bit too large. For later applications, we introduce a class of its subalgebras, called saturated subalgebras.

A subset $P \subset \mathrm{Conf}$ is called *saturated* if for $S \in P$, any $T \in \mathrm{Conf}_0$ with $T \leq S$ belongs to P. For a saturated set P, let us define a subalgebra

$$\mathbb{A}[\![P]\!] := \{f \in \mathbb{A}[\![\mathrm{Conf}]\!] \mid \mathrm{Supp}(f) \subset \text{the semigroup generated by } P\}. \tag{3.4.1}$$

We shall call a subalgebra of the configuration algebra of the form (3.4.1) for some saturated P a *saturated subalgebra.* A saturated algebra R is characterized by the properties: (i) R is a closed subalgebra under the adic topology of the configuration algebra, and (ii) if $S \in \mathrm{Supp}(f)$ for $f \in R$ then any connected component of S (as a monomial) belongs to R. We call the set

$$\mathrm{Supp}(R) := \bigcup_{f \in R} \mathrm{Supp}(f) \tag{3.4.2}$$

the support of R. Obviously, $\mathrm{Supp}(R)$ is the saturated subsemigroup of Conf generated by P. The algebra R is determined $\mathrm{Supp}(R)$.

It is clear that if R is a saturated subalgebra of $\mathbb{A}[\![\mathrm{Conf}]\!]$ then $R \cap (\mathbb{A} \cdot \mathrm{Conf})$ is a dense subalgebra of R and that R is naturally isomorphic to the completion of $R \cap (\mathbb{A} \cdot \mathrm{Conf})$ with respect to the induced adic topology.

Example. We give two typical examples of saturated sets.

1. For any $S \in \mathrm{Conf}$, we define its *saturation* by

$$\langle S\rangle := \{T \in \mathrm{Conf} \mid T \le S\}. \tag{3.4.3}$$

2. Let (Γ, G) be the Cayley graph of an infinite monoid Γ with respect to a finite generating system G. Then, by choosing $G \cup G^{-1}$ as the color set and $q := \sharp(G \cup G^{-1})$ as the bound of valence, we define a saturated subset of Conf by

$$\langle \Gamma, G\rangle := \{\text{isomorphism classes of finite subgraphs of } (\Gamma, G)\}. \tag{3.4.4}$$

Obviously, the saturated subalgebra $\mathbb{A}[\![\langle S\rangle]\!]$ consists of finite type elements only, whereas the algebra $\mathbb{A}[\![\langle \Gamma, G\rangle]\!]$ contains non-finite type elements. This makes the latter algebra interesting when we study limit elements in §11.

§3.5. Completed tensor product of the configuration algebra

The tensor product over $\mathbb{A}$ of m copies of $\mathbb{A}\cdot\mathrm{Conf}$ for $m \in \mathbb{Z}_{\ge 0}$ is denoted by $\otimes^m(\mathbb{A}\cdot \mathrm{Conf})$. In this section, we describe the completed tensor product $\widehat{\otimes}^m(\mathbb{A}[\![\mathrm{Conf}]\!])$ of the completed configuration algebra,

Definition. Let $\mathbb{A}$ be a commutative algebra with unit. For $m \in \mathbb{Z}_{\ge 0}$, the *completed m-tensor product* $\widehat{\otimes}^m\mathbb{A}[\![\mathrm{Conf}]\!]$ of the configuration algebra $\mathbb{A}[\![\mathrm{Conf}]\!]$ is defined by the inverse limit

$$\widehat{\otimes}^m\mathbb{A}[\![\mathrm{Conf}]\!] := \varprojlim_n \otimes^m(\mathbb{A}\cdot \mathrm{Conf})/(\otimes^m\mathbb{A}\mathcal{J})_n, \tag{3.5.1}$$

where $(\otimes^m\mathbb{A}\mathcal{J})_n$ is the ideal in $\otimes^m(\mathbb{A}\cdot\mathrm{Conf})$ given by

$$(\otimes^m\mathbb{A}\mathcal{J})_n := \sum_{n_1+\cdots+n_m\ge n} \mathbb{A}\mathcal{J}_{n_1}\otimes\cdots\otimes\mathbb{A}\mathcal{J}_{n_m}, \tag{3.5.2}$$

where $\widehat{\otimes}^0\mathbb{A}[\![\mathrm{Conf}]\!] = \mathbb{A}$ and $\widehat{\otimes}^1\mathbb{A}[\![\mathrm{Conf}]\!] = \mathbb{A}[\![\mathrm{Conf}]\!]$.

We list some basic properties of $\widehat{\otimes}^m\mathbb{A}[\![\mathrm{Conf}]\!]$ (proofs are left to the reader).

(i) Since $\bigcap_{n=0}^{\infty}(\mathbb{A}\mathcal{J}^{\otimes m})_n = \{0\}$, we have the natural inclusion map

$$\otimes^m(\mathbb{A}\cdot\mathrm{Conf}) \subset \widehat{\otimes}^m(\mathbb{A}[\![\mathrm{Conf}]\!]) \tag{3.5.3}$$

whose image is a dense subalgebra with respect to the (3.5.2)-adic topology.

(ii) There is a natural algebra homomorphism

$$\otimes^m(\mathbb{A}[\![\mathrm{Conf}]\!]) \to \widehat{\otimes}^m(\mathbb{A}[\![\mathrm{Conf}]\!]) \tag{3.5.4}$$

with a suitable universal property. The image of an element $f_1\otimes\cdots\otimes f_m$ is denoted by $f_1\,\widehat{\otimes}\cdots\widehat{\otimes}\,f_m$. We also denote it by $f_1\otimes\cdots\otimes f_m$ if $f_i\in\mathbb{A}\cdot\mathrm{Conf}$ $(i=1,\dots,m)$ because of (i).

(iii) If $\Psi_i:\otimes^{m_i}(\mathbb{A}\cdot\mathrm{Conf})\to\otimes^{n_i}(\mathbb{A}\cdot\mathrm{Conf})$ $(i=1,\dots,l)$ are continuous homomorphisms, then one has the completed homomorphism

$$\widehat{\otimes}_{i=1}^{l}\Psi_i\ :\ \widehat{\otimes}^{\sum_{i=1}^{l}m_i}(\mathbb{A}[\![\mathrm{Conf}]\!])\to\widehat{\otimes}^{\sum_{i=1}^{l}n_i}(\mathbb{A}[\![\mathrm{Conf}]\!]) \tag{3.5.5}$$

with some natural characterizing properties.

§3.6. Exponential and logarithmic maps

Let $\varphi(t)=\sum_{n=0}^{\infty}\varphi_nt^n\in\mathbb{A}[\![t]\!]$ be a formal power series in the indeterminate t. Then the substitution for t of an element f of $\mathbb{A}[\![\mathrm{Conf}]\!]_+$ to get $\varphi(f):=\sum_{n=0}^{\infty}\varphi_nf^n\in\mathbb{A}[\![\mathrm{Conf}]\!]$ defines a map $\varphi:\mathbb{A}[\![\mathrm{Conf}]\!]_+\to\mathbb{A}[\![\mathrm{Conf}]\!]$ (cf. (3.2.5)). The map is equivariant with respect to any continuous endomorphism of the configuration algebra. The map can be restricted to any closed subalgebra of the configuration algebra to itself. If f is of finite type, then $\varphi(f)$ is also of finite type.

In particular, if $\mathbb{A}$ contains $\mathbb{Q}$, then we define the *exponential*, *logarithmic* and *power* (with an exponent $c\in\mathbb{A}$) maps as follows:

$$\exp(f):=\sum_{n=0}^{\infty}\frac{1}{n!}f^n\qquad\text{for } f\in\mathbb{A}[\![\mathrm{Conf}]\!]_+, \tag{3.6.1}$$

$$\log(1+f):=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}f^n\qquad\text{for } f\in\mathbb{A}[\![\mathrm{Conf}]\!]_+, \tag{3.6.2}$$

$$(1+f)^c:=\sum_{n=0}^{\infty}\frac{c(c-1)\cdots(c-n+1)}{n!}f^n\qquad\text{for } f\in\mathbb{A}[\![\mathrm{Conf}]\!]_+. \tag{3.6.3}$$

They satisfy the standard functional relations: $\exp(f+g)=\exp(f)\cdot\exp(g)$, $\log((1+f)(1+g))=\log(1+f)+\log(1+g)$, $(1+f)^{c_1}\cdot(1+f)^{c_2}=(1+f)_1^{c+c_2}$ and $\log((1+f)^c)=c\cdot\log(1+f)$.

Fact. *Let $\mathcal{A}=\sum_{S\in\mathrm{Conf}_+}S\cdot A_S$ and $\mathcal{M}=\sum_{S\in\mathrm{Conf}_+}S\cdot M_S\in\mathbb{A}[\![\mathrm{Conf}]\!]$ be related by $\mathcal{A}=\exp(\mathcal{M})$ ($\Leftrightarrow\mathcal{M}=\log(\mathcal{A})$). Then their coefficients are related by*

$$A_S=\sum_{m=0}^{\infty}\sum_{\substack{S_1,\dots,S_m\in\mathrm{Conf}_+\\ S=S_1^{k_1}\cdots S_m^{k_m}}}\frac{1}{k_1!\cdots k_m!}M_{S_1}^{k_1}\cdots M_{S_m}^{k_m}, \tag{3.6.4}$$

and

(3.6.5)

$$M_S = \sum_{m=0}^{\infty} \sum_{\substack{S_1,\dots,S_m \in \mathrm{Conf}_+ \\ S=S_1^{k_1}\cdots S_m^{k_m}}} \frac{(k_1+\cdots+k_m-1)!(-1)^{k_1+\cdots+k_m-1}}{k_1!\cdots k_m!} A_{S_1}^{k_1}\cdots A_{S_m}^{k_m}.$$

Here the summation is over all decompositions of S:

(3.6.6) $$S = S_1^{k_1}\cdots S_m^{k_m}$$

for pairwise distinct $S_i \in \mathrm{Conf}_+$ $(i=1,\dots,m)$ (which may not be connected) and for positive integers $k_i \in \mathbb{Z}_{>0}$. Two decompositions $S_1^{k_1}\cdots S_m^{k_m}$ and $T_1^{l_1}\cdots T_n^{l_n}$ are regarded as the same if $m=n$ and there is a permutation $\sigma \in \mathfrak{S}_n$ such that $k_i = l_{\sigma(i)}$ and $S_i = T_{\sigma(i)}$ for $i=1,\dots,m$. The RHS's of (3.6.4) *and* (3.6.5) *are finite sums, since the S_i's and k_i's are bounded by S.*

We omit the proof since it is a straightforward calculation of formal power series in the infinite generating system Conf_0.

Corollary. *Let $\mathcal{A}$ and $\mathcal{M} \in \mathbb{A}[\![\mathrm{Conf}]\!]$ be related as above. Then*

(3.6.7) $$A_S = M_S \quad \forall S \in \mathrm{Conf}_0\,.$$

§4. The Hopf algebra structure

We construct a topological commutative Hopf algebra structure on the configuration algebra $\mathbb{A}[\![\mathrm{Conf}]\!]$. More precisely, we construct in 4.1 a sequence of coproducts Φ_n $(n \in \mathbb{Z}_{\geq 0})$ by the use of the covering coefficients and, in 4.4, the antipode ι, which together satisfy the axioms of a topological Hopf algebra.

§4.1. Coproduct Φ_m for $m \in \mathbb{Z}_{\geq 0}$

For a non-negative integer $m \in \mathbb{Z}_{\geq 0}$ and $U \in \mathrm{Conf}$, define an element

(4.1.1) $$\Phi_m(U) := \sum_{S_1 \in \mathrm{Conf}} \cdots \sum_{S_m \in \mathrm{Conf}} \binom{S_1,\dots,S_m}{U} S_1 \otimes \cdots \otimes S_m$$

in the tensor product $\otimes^m(\mathbb{Z}\cdot\mathrm{Conf})$ of m copies of the polynomial type configuration algebra. Due to 2.5 (v), one has

(4.1.2) $$\Phi_m([\emptyset]) = [\emptyset] \quad (=1).$$

The map Φ_m is *multiplicative*, that is, for $U, V \in \mathrm{Conf}$,

(4.1.3) $$\Phi_m(U\cdot V) = \Phi_m(U)\cdot\Phi_m(V).$$

This follows from the decomposition rule (2.7.1).

Thus, the linear extension of Φ_m induces an algebra homomorphism from $\mathbb{Z}\cdot \mathrm{Conf}$ to its m-tensor product $\otimes^m(\mathbb{Z}\cdot \mathrm{Conf})$, which we denote by the same Φ_m and call the mth *coproduct.* The coproduct Φ_m can be further extended to a coproduct on the completed configuration algebra.

Assertion. 1. *The mth coproduct Φ_m $(m \in \mathbb{Z}_{\geq 0})$ on the polynomial type configuration algebra is continuous with respect to the adic topology. The induced homomorphism is denoted again by Φ_m and called the mth coproduct:*

$$\Phi_m : \mathbb{A}[\![\mathrm{Conf}]\!] \to \widehat{\otimes}^m \mathbb{A}[\![\mathrm{Conf}]\!] := \mathbb{A}[\![\mathrm{Conf}]\!] \,\widehat{\otimes} \cdots \widehat{\otimes}\, \mathbb{A}[\![\mathrm{Conf}]\!] \tag{4.1.4}$$

2. *The completed homomorphism Φ_m is mulitiplicative:*

$$\Phi_m(f\cdot g) = \Phi_m(f)\cdot\Phi_m(g) \tag{4.1.5}$$

for any $f, g \in \mathbb{A}[\![\mathrm{Conf}]\!]$,

3. *Any saturated subalgebra R of $\mathbb{A}[\![\mathrm{Conf}]\!]$ is preserved by Φ_m:*

$$\Phi_m(R) \subset \widehat{\otimes}^m R. \tag{4.1.6}$$

Proof. 1. Recall the fundamental system $(\otimes^m \mathbb{A}\mathcal{J})_n$ (3.5.2) of neighborhoods of the m-tensor algebra $\otimes^m(\mathbb{Z}\cdot\mathrm{Conf})$. Let us show the inclusion

$$\Phi_m(\mathbb{A}\mathcal{J}_n) \subset (\otimes^m \mathbb{A}\mathcal{J})_n \tag{4.1.7}$$

for any $m, n \in \mathbb{Z}_{\geq 0}$. The ideal $\mathcal{J}_n$ is generated by $U \in \mathrm{Conf}$ with $\deg(U) := \sharp U \geq n$, and $\Phi_m(U)$ is a sum of monomials $S_1 \otimes \cdots \otimes S_m$ for $S_i \in \mathrm{Conf}$ such that $\binom{S_1,\ldots,S_m}{U} \neq 0$. We have $\sharp S_1 + \cdots + \sharp S_m \geq \sharp U \geq n$ because of (2.5) (i), implying $\Phi_m(U) \in (\otimes^m \mathcal{J})_n$.

2. The multiplicativity of the monomials (4.1.3) implies the multiplicativity of the configuration algebra of polynomial type. This extends to multiplicativity on infinite series (3.2.4) because of the continuity of the product with respect to the adic topology.

3. Let $f \in R$ and $f = \sum_S S f_S$ be its expansion. Then $\Phi_m(f)$ is a series of the form $\sum_S S_1 \otimes \cdots \otimes S_m \binom{S_1,\ldots,S_m}{S} f_S$. Thus, $\binom{S_1,\ldots,S_m}{S} f_S \neq 0$ implies each factor S_i satisfies $S_i \leq S$ and $S \in \mathrm{Supp}(f) \subset \mathrm{Supp}(R)$. By the definition of saturation, $S_i \in \mathrm{Supp}(R)$ and $\Phi_m(f) \in \widehat{\otimes}^m R$. □

Cocommutativity of the coproduct Φ_m. The symmetric group $\mathfrak{S}_m$ acts naturally on the m-tensors (3.5.1) by permuting the tensor factors. The image of Φ_m lies in the subalgebra consisting of $\mathfrak{S}_m$-invariant elements, because of 2.5 (ii): $\Phi_m(\mathbb{A}[\![\mathrm{Conf}]\!]) \subset (\widehat{\otimes}^m \mathbb{A}[\![\mathrm{Conf}]\!])^{\mathfrak{S}_m}$. We shall call this property the *cocommutativity* of the coproduct Φ_m.

§4.2. Coassociativity

Assertion. *For $m, n \in \mathbb{Z}_{\geq 0}$, one has the formula*

$$(\underbrace{1 \widehat{\otimes} \cdots \widehat{\otimes} 1}_{n} \widehat{\otimes} \Phi_m) \circ \Phi_{n+1} = \Phi_{m+n}. \tag{4.2.1}$$

Proof. This follows immediately from the composition rule (2.6.1). □

Using the cocommutativity of Φ_2, Φ_3 can be expressed in two different ways:

$$(\Phi_2 \widehat{\otimes} 1) \circ \Phi_2 = (1 \widehat{\otimes} \Phi_2) \circ \Phi_2.$$

This equality is the *coassociativity* of the coproduct Φ_2. More generally, Φ_m is expressed by a composition of $m-1$ copies of Φ_2's in any order.

§4.3. The augmentation map Φ_0

The augmentation map for the algebra is defined by Φ_0 (recall (2.5.2)):

$$\mathrm{aug} := \Phi_0 : \mathbb{A}[\![\mathrm{Conf}]\!] \to \mathbb{A}, \quad S \in \mathrm{Conf}_+ \mapsto 0, \ [\emptyset] \mapsto 1. \tag{4.3.1}$$

Assertion. *The map* aug *is the counit with respect to the coproduct Φ_2,*

$$(\mathrm{aug} \widehat{\otimes} \mathrm{id}) \circ \Phi_2 = \mathrm{id}_{\mathbb{Z}[\![\mathrm{Conf}]\!]}. \tag{4.3.2}$$

Proof. This is the case $m = 0$ and $n = 1$ of the formula (4.2.1). Alternatively, for any $S \in \mathrm{Conf}_+$, using 2.5 (iii) and (iv), one calculates:

$$(\mathrm{aug} \widehat{\otimes} \mathrm{id}) \circ \Phi_2(S) = \sum_{T,U \in \mathrm{Conf}} \binom{T, U}{S} T \cdot \mathrm{aug}(U) = \sum_{T \in \mathrm{Conf}} \binom{T, [\emptyset]}{S} = S.$$ □

§4.4. The antipodal map ι

The coproduct and the counit exist both on the polynomial type and the completed configuration algebras. The coinverse (or antipode), which we construct in the present section, exists only on the localized configuration algebra.

Assertion. *There exists an $\mathbb{A}$-algebra endomorphism*

$$\iota : \mathbb{A}[\![\mathrm{Conf}]\!] \to \mathbb{A}[\![\mathrm{Conf}]\!] \tag{4.4.1}$$

with the following properties:

(i) *ι is an involutive automorphism. That is, $\iota^2 = \mathrm{id}_{\mathbb{A}[\![\mathrm{Conf}]\!]}$.*
(ii) *ι is the coinverse map with respect to the coproduct Φ_2, that is,*

$$M \circ (\iota \widehat{\otimes} \mathrm{id}) \circ \Phi_2 = \mathrm{aug}, \tag{4.4.2}$$

where M is the product defined on the completed tensor product (recall 3.5*).*

(iii) ι *is continuous with respect to the adic topology. More precisely,*

$$\iota(\overline{\mathcal{J}_n}) \subset \overline{\mathcal{J}_n} \tag{4.4.3}$$

for $n \in \mathbb{Z}_{\geq 0}$*, where* $\overline{\mathcal{J}_n}$ *is the closure of the ideal* $\mathcal{J}_n$ (3.2.1).

(iv) ι *leaves any saturated subalgebra of* $\mathbb{A}[\![\mathrm{Conf}]\!]$ *invariant.*

(v) *Any* $\mathbb{A}$*-endomorphism of* $\mathbb{A}[\![\mathrm{Conf}]\!]$ *satisfying* (ii) *and* (iii) *is equal to* ι.

Proof. The proof is divided into two parts. In Part 1, we construct an endomorphism φ of the algebra $\mathbb{A}[\![\mathrm{Conf}]\!]$, satisfying (i)–(iv). In Part 2, we show that any endomorphism ψ of $\mathbb{A}[\![\mathrm{Conf}]\!]$ satisfying (ii) and (iii) coincides with φ.

Part 1. Fix a bijection $i \in \mathbb{Z}_{\geq 1} \mapsto S_i \in \mathrm{Conf}_0$ such that if $S_i \leq S_j$ then $i \leq j$ (such a linearization exists since one can linearize the partial order structure on the finite set of configurations for a fixed number of vertices). Note that this condition implies that the set $\{S_1, \ldots, S_i\}$ for $i \in \mathbb{Z}_{>0}$ is saturated in the sense of 3.4. Consider the increasing sequence $R_0 := \mathbb{A}$, $R_i := \mathbb{A}[\![S_1, \ldots, S_i]\!]$ $(i \in \mathbb{Z}_{>0})$ of saturated subalgebras of $\mathbb{A}[\![\mathrm{Conf}]\!]$. Let us show:

Claim. *There exists a sequence* $\{\varphi_i\}_{i \in \mathbb{Z}_{\geq 0}}$ *of continuous endomorphisms* $\varphi_i : R_i \to R_i$ *satisfying the following relations:*

(a) $\varphi_i^2 = \mathrm{id}_{R_i}$ *for* $i \in \mathbb{Z}_{\geq 0}$.
(b) $\varphi_i|_{R_{i-1}} = \varphi_{i-1}$ *for* $i \in \mathbb{Z}_{\geq 1}$.
(c) $M \circ (\varphi_i \cdot \mathrm{id}) \circ \Phi_2|_{R_i} = \mathrm{aug}|_{R_i}$ *for* $i \in \mathbb{Z}_{\geq 0}$.
(d) $\varphi_i(\overline{\mathcal{J}_n} \cap R_i) \subset \overline{\mathcal{J}_n} \cap R_i$ *for* $i \in \mathbb{Z}_{\geq 0}$ *and* $n \in \mathbb{Z}_{\geq 0}$.
(e) $\varphi_i(S_k) \in \mathbb{Z}[\![\langle S_k \rangle]\!]$ *for* $1 \leq k \leq i$ (*see* (3.4.1) *and* (3.4.3) *for notation*).

Proof of Claim. We construct the sequence φ_i inductively. Put $\varphi_0 := \mathrm{id}_{\mathbb{A}}$. For $j \in \mathbb{Z}_{\geq 0}$, suppose that $\varphi_0, \ldots, \varphi_j$ satisfying (a)–(e) for $i \leq j$ are given.

For any given element $X \in \mathbb{Z}[\![\langle S_{j+1} \rangle]\!] \cap \overline{\mathcal{J}_{\sharp S_{j+1}}}$, define an endomorphism ψ_X of $R_j[S_{j+1}]$ by $\psi_X|R_j = \varphi_j$ and $\psi_X(S_{j+1}) = X$. Since $X \in \overline{\mathcal{J}_{\sharp S_{j+1}}}$, we have $\psi_X(\overline{\mathcal{J}_n} \cap R_j[S]) \subset \overline{\mathcal{J}_n}$ for all $n \in \mathbb{Z}$. Hence the endomorphism is continuous in the adic topology and extends to an endomorphism of $R_{j+1} = R_j[\![S]\!]$. We denote the extended homomorphism again by ψ_X. Let us show that $\varphi_{j+1} := \psi_X$ for a suitable choice of X satisfies (a)–(e) for $i = j+1$. Actually, (b), (d) and (e) are already satisfied by the construction. In order to satisfy (a) and (c), we have only to solve the following two equations on X:

$$\text{(a)}^* \ \psi_X^2(S_{j+1}) = S_{j+1} \quad \text{and} \quad \text{(c)}^* \ M \circ (\psi_X \mathbin{\widehat{\otimes}} 1) \circ \Phi_2(S_{j+1}) = 0.$$

In the following, we show the existence of a simultaneous solution of (a)* and (c)*.

(c)* Let us write down the equation (c)* explicitly by using (4.1.1):

$$M \circ (\psi_X \mathbin{\widehat{\otimes}} \mathrm{id}) \circ \Phi_2(S_{j+1}) = \sum_{U,V \in \mathrm{Conf}} \binom{U, V}{S_{j+1}} \psi_X(U) \cdot V = 0.$$

The summation index (U, V) runs over the finite set $\langle S_{j+1}\rangle \times \langle S_{j+1}\rangle$ (2.5 (i)). Decompose the set into three pieces: $\{S_{j+1}\} \times \langle S_{j+1}\rangle$, $(\langle S_{j+1}\rangle \setminus \{S_{j+1}\}) \times (\langle S_{j+1}\rangle \setminus \{S_{j+1}\})$ and $(\langle S_{j+1}\rangle \setminus \{S_{j+1}\}) \times \{S_{j+1}\}$. Since $\langle S_{j+1}\rangle \setminus \{S_{j+1}\} \subset \langle S_1, \dots, S_j\rangle$, on which ψ_X coincides with φ_j, the equation consists of three parts:

$$X \cdot \mathcal{A}(S_{j+1}) + \mathcal{B}(S_{j+1}) + \varphi_j\big(\mathcal{A}(S_{j+1}) - S_{j+1}\big) \cdot S_{j+1} = 0, \tag{4.4.4}$$

where

$$\mathcal{A}(S_{j+1}) := \sum_{V \in \langle S_{j+1}\rangle} \binom{S_{j+1}, V}{S_{j+1}} V, \tag{4.4.5}$$

$$\mathcal{B}(S_{j+1}) := \sum_{U,V \in \langle S_{j+1}\rangle \setminus \{S_{j+1}\}} \binom{U, V}{S_{j+1}} \varphi_j(U) \cdot V. \tag{4.4.5*}$$

We have the following facts concerning the equation (4.4.4).

(i) By (4.4.5), each term of $\mathcal{A}(S_{j+1}) - S_{j+1}$ is an element $\langle S_{j+1}\rangle \setminus \{S_{j+1}\}$, i.e. is a monomial in S_k's for $1 \le k \le j$. Therefore, by the induction hypothesis (e), $\varphi_j(\mathcal{A}(S_{j+1}) - S_{j+1}) \in R_j$.

(ii) By the hypothesis (d) for φ_j, $\varphi_j(U)$ belongs to $\overline{\mathcal{J}_{\sharp U}} \cap \mathbb{Z}[\![\langle U\rangle]\!]$ for all $U \le S_{j+1}$ with $U \ne S_{j+1}$. Hence, in view of 2.5 (i), $\binom{U, V}{S_{j+1}} \ne 0$ implies $\varphi_j(U) \cdot V \in \overline{\mathcal{J}_{\sharp S_{j+1}}}$, and, therefore (4.5.5)* implies $\mathcal{B}(S_{j+1}) \in \overline{\mathcal{J}_{\sharp S_{j+1}}} \cap \mathbb{Z}[\![\langle S_{j+1}\rangle]\!]$.

(iii) By (4.4.5), $\mathcal{A}(S_{j+1}) - 1$ belongs to the augmentation ideal. Hence, in view of the map (3.2.5), the sum $\sum_{m \ge 0}(1 - \mathcal{A}(S_{j+1}))^m$ converges in $\mathbb{Z}[\![\langle S_{j+1}\rangle]\!]$ to the inverse $\mathcal{A}(S_{j+1})^{-1}$.

Therefore the equation (4.4.4) for X has a unique solution in $\mathbb{Z}[\![\langle S_{j+1}\rangle]\!]$:

$$X := \frac{-1}{\mathcal{A}(S_{j+1})}\Big(\mathcal{B}(S_{j+1}) + \varphi_j\big(\mathcal{A}(S_{j+1}) - S_{j+1}\big) \cdot S_{j+1}\Big). \tag{4.4.6}$$

As a consequence of the above facts (i) and (ii), we have

($*$) *The right hand side of* (4.4.6) *belongs to* $\overline{\mathcal{J}_{\sharp S_{j+1}}} \cap \mathbb{Z}[\![\langle S_{j+1}\rangle]\!]$.

This is what we have required of X at the beginning. Thus, (c)* is solved.

(a)* We need to show $\psi_X^2(S_{j+1}) = S_{j+1}$ under the choice (4.4.6). Apply ψ_X to the equality (4.4.4). Using the induction hypothesis (a) and (b), one gets

$$\begin{aligned} \psi_X^2(S_{j+1})\big(\varphi_j(\mathcal{A}(S_{j+1}) - S_{j+1}) &\\ + \psi_X(S_{j+1})\big) + \varphi_j\mathcal{B}(S_{j+1}) &+ \big(\mathcal{A}(S_{j+1}) - S_{j+1}\big)X = 0 \end{aligned} \tag{$**$}$$

Here, we have $\varphi_j\mathcal{B}(S) = \mathcal{B}(S)$, by applying the symmetry 2.5 (ii) and the induction hypothesis (a) to the expression $(4.4.5)^*$. Subtract (4.4.4) from $(**)$:

$$(\psi_X^2(S_{j+1}) - S_{j+1})(\varphi_j(\mathcal{A}(S_{j+1}) - S_{j+1}) + \psi_X(S_{j+1})) = 0.$$

Since $\mathcal{A}(S_{j+1}) - 1 \in \mathcal{J}_1$ and $\varphi_j(\mathcal{A}(S_{j+1}) - S_{j+1}) + \psi_X(S_{j+1}) - 1 \in \mathcal{J}_1$, we see that $\varphi_j(\mathcal{A}(S_{j+1}) - S_{j+1}) + \psi_X(S_{j+1})$ is invertible in the algebra R_{j+1}. This implies $\psi_X^2(S_{j+1}) - S_{j+1} = 0$. That is, ψ_X for X as in (4.4.6) satisfies $(a)^*$.

Thus, by the choice $\varphi_{j+1} := \psi_X$, the induction step is achieved. □

We define the endomorphism φ of the subalgebra $R := \bigcup_{i=1}^{\infty} R_i$ by $\varphi|R_i := \varphi_i$ for $i \in \mathbb{Z}_{\geq 0}$. Here, we note that R consists of exactly the finite type elements, i.e. $R = \mathbb{A}[\![\mathrm{Conf}]\!]_{\mathrm{finite}}$, which is dense in the configuration algebra. Then (d) implies continuity of φ on R, and therefore φ extends to the full configuration algebra. The extended homomorphism, denoted by φ again, is involutive since it is so on the dense subalgebra R. This implies φ is invertible.

To finish Part 1, let us show that φ satisfies (iv).

We remark that, for any $S \in \mathrm{Conf}$, one has $\varphi(S) \in \mathbb{Z}[\![\langle S\rangle]\!]$ (apply (e) to each connected component of S). Let R be any saturated subalgebra of the configuration algebra. For any $f = \sum_S S f_S \in R$, by applying the above considerations, one has $\mathrm{Supp}(\varphi(f)) \subset \bigcup_{f_S \neq 0} \mathrm{Supp}(\varphi(S)) \subset \bigcup_{f_S \neq 0}$ (semigroup generated by $\langle S\rangle) \subset \mathrm{Supp}(R)$. That is, $\varphi(f) \in R$ and $\varphi(R) \subset R$.

Part 2. Uniqueness of φ. Let ψ be any endomorphism of the configuration algebra satisfying (ii) and (iii) of the Assertion. Let $S_1, S_2, \ldots$ be the linear ordering of Conf_0 used in Part 1. We show that $\psi(S_j) = \varphi(S_j)$ by induction on $j \in \mathbb{Z}_{\geq 1}$. Let $j \in \mathbb{Z}_{\geq 0}$, and assume $\varphi(S_i) = \psi(S_i)$ for $1 \leq i \leq j$ (there is no assumption if $j = 0$). By (ii), $\psi(S_{j+1})$ should satisfy the same equation (4.4.4) for X, where, by the induction hypothesis, one has the equality $\varphi(\mathcal{A}(S_{j+1}) - S_{j+1}) = \psi(\mathcal{A}(S_{j+1}) - S_{j+1})$. The uniqueness of the solution (4.4.6) implies $\psi(S_i) = \varphi(S_i)$. This implies the coincidence of φ and ψ on $\mathbb{Z} \cdot \mathrm{Conf}$. Then, by the continuity (iii), we have the coincidence of φ and ψ on the completed configuration algebra. □

Equation (4.4.4) for $n = 1$ implies that ι preserves the augmentation ideal of $\mathbb{A}[\![\mathrm{Conf}]\!]$. Hence, we have

(4.4.7) $$\mathrm{aug} \circ \iota = \mathrm{aug}.$$

Let us state an important consequence of our construction.

Assertion. *Any saturated subalgebra of the configuration algebra is a topological Hopf algebra. In particular, for any monoid* Γ *with a finite generating system* G *and commutative ring* $\mathbb{A}$ *with a unit,* $\mathbb{A}[\![\langle \Gamma, G\rangle]\!]$ *is a Hopf algebra.*

Proof. We need only remember that Φ_m $(m \geq 0)$ and ι preserve any saturated subalgebra (4.1 Assertion 3 and 4.4 Assertion (iv)). $\square$

§4.5. Some remarks on ι

Remark. 1. In §5, the functions $\mathcal{A}(S)$ $(S \in \mathrm{Conf})$ will be re-introduced and investigated. In particular, we shall show the equality

$$\iota(\mathcal{A}(S)) \cdot \mathcal{A}(S) = 1 \tag{4.5.1}$$

for $S \in \mathrm{Conf}$ (5.4.1). This can also be shown directly by use of (2.7.1) and (4.2.1). This relation gives a more natural definition of ι.

2. The polynomial ring $\mathbb{A} \cdot \mathrm{Conf}$ for any $\mathbb{A}$ is not closed under the map ι. For example, let X (resp. Y) be a graph of one (resp. two) vertices. Then

$$\iota(X) = -\frac{X}{1+X} \quad \text{and} \quad \iota(Y) = \frac{-Y + 2X^2 + XY}{(1+X)(1+2X+Y)}.$$

3. Because of the above Remark 2, the localization $(\mathbb{Z} \cdot \mathrm{Conf})_{\mathfrak{M}} = \{f/g \mid f \in \mathbb{Z} \cdot \mathrm{Conf},\ g \in \mathfrak{M}\}$ with respect to the multiplicative set $\mathfrak{M} := \{\mathcal{A}(S) \mid S \in \mathrm{Conf}\}$ is the smallest necessary extension of the algebra $\mathbb{Z} \cdot \mathrm{Conf}$ to define ι. However, the space $(\mathbb{Z} \cdot \mathrm{Conf})_{\mathfrak{M}}$ is still too small for our later applications (see 6.3 Remark).

4. There is another coalgebra structure studied in combinatorics ([JR]).

§5. Growth functions for configurations

For any $S \in \mathrm{Conf}$, the sum of the isomorphism classes of all subgraphs of a graph representing S is denoted by $\mathcal{A}(S)$. It is a group-like element in the Hopf algebra $\mathbb{A}[\![\mathrm{Conf}]\!]$ and will play a fundamental role in what follows. We shall call it a *growth function* (not to be confused with the same terminology in [M]).

§5.1. Growth functions

For $S, T \in \mathrm{Conf}$, we introduce a numerical invariant

$$A(S,T) := \sharp \mathbb{A}(S, \mathbb{T}), \tag{5.1.1}$$

by the following steps (i)–(iii).

(i) Fix a graph $\mathbb{T}$ with $[\mathbb{T}] = T$.
(ii) Put

$$\mathbb{A}(S, \mathbb{T}) := \sharp\{\mathbb{S} \mid \mathbb{S} \text{ is a subgraph of } \mathbb{T} \text{ such that } [\mathbb{S}] = S\}. \tag{5.1.2}$$

(iii) Show that $\mathbb{A}(S, \mathbb{T}) \simeq \mathbb{A}(S, \mathbb{T}')$ if $[\mathbb{T}] = [\mathbb{T}']$. (The proof is omitted.)

We shall call $A(S,T)$ the *growth coefficient of* T *at* $S \in \mathrm{Conf}$. We have

$$A([\emptyset],T) = 1 \quad \text{for } T \in \mathrm{Conf}, \tag{5.1.3}$$

$$A(S,T) \neq 0 \quad \text{if and only if } S \in \langle T \rangle. \tag{5.1.4}$$

Let us introduce the generating polynomial of the growth coefficients:

$$\mathcal{A}(T) := \sum_{S \in \mathrm{Conf}} S \cdot A(S,T), \tag{5.1.5}$$

and call it the *growth function* of T. In fact, this is a finite sum and $\mathcal{A}(T) \in \mathbb{Z}\cdot\mathrm{Conf}$. The definition of $\mathcal{A}(T)$ can be reformulated as

$$\mathcal{A}(T) = \sum_{\mathbb{S} \in 2^{\mathbb{T}}} [\mathbb{S}], \tag{5.1.6}$$

where $2^{\mathbb{T}}$ denotes the set of all subgraphs of $\mathbb{T}$ (cf. 2.1 Definition 2).

The following multiplicativity follows immediately from the expression (5.1.6): for $T_1, T_2 \in \mathrm{Conf}$,

$$\mathcal{A}(T_1 \cdot T_2) = \mathcal{A}(T_1)\mathcal{A}(T_2). \tag{5.1.7}$$

Remark. 1. By comparing the definition (5.1.1) with (2.4.1), we see immediately that $A(S,T) = \binom{T}{T,S}$ for $S, T \in \mathrm{Conf}$. Hence the two definitions (4.4.5) and (5.1.5) for $\mathcal{A}(T)$ coincide.

2. By definition (5.1.1), we have additivity:

$$A(S, T_1 \cdot T_2) = A(S,T_1) + A(S,T_2) \tag{5.1.8}$$

for $S \in \mathrm{Conf}_0$ and $T_i \in \mathrm{Conf}$.

§5.2. A numerical bound of the growth coefficients

In our later study on the existence of limit elements in §10, the following estimates on the growth rates of growth coefficients play a crucial role.

Lemma. *For* $S, T \in \mathrm{Conf}$, *we have*

$$A(S,T) \leq \frac{1}{\sharp\,\mathrm{Aut}(S)} \cdot \sharp T^{n(S)} \cdot (q-1)^{\sharp S - n(S)}. \tag{5.2.1}$$

Here $n(S) := \sharp\{$*connected components of* $S\}$, q *is the upper bound of the number of edges at each vertex of* T (*recall* 2.2), *and* $\mathrm{Aut}(S)$ *is the isomorphism class of* $\mathrm{Aut}(\mathbb{S})$ *for a representative* $\mathbb{S}$ *of* S *and we put* $\sharp\,\mathrm{Aut}(S) := \sharp\,\mathrm{Aut}(\mathbb{S})$.

Note. In the original version [Sa2], the factor $q-1$ in (5.2.1) was q. The author is grateful to the readers for pointing out this improvement.

Proof. Let $\mathbb{S}$ and $\mathbb{T}$ be representatives of the G-colored graphs S and T respectively. We divide the proof into three steps.

(i) *Assume S is connected. Then*

$$A(S,T) \le \frac{1}{\sharp\,\mathrm{Aut}(S)} \sharp T \cdot (q-1)^{\sharp S - 1}. \tag{5.2.2}$$

Proof. Let $\mathbb{S}_1, \ldots, \mathbb{S}_a$ be an increasing sequence of connected subgraphs of $\mathbb{S}$ such that $\sharp \mathbb{S}_i = i$ ($i = 1, \ldots, a = \sharp\mathbb{S}$). Put $\mathrm{Emb}(\mathbb{S}_i, \mathbb{T}) := \{\varphi : \mathbb{S}_i \to \mathbb{T} \mid \text{embedding as a } G\text{-colored graph}\}$. Then, for $i \ge 2$, the natural restriction map $\mathrm{Emb}(\mathbb{S}_i, \mathbb{T}) \to \mathrm{Emb}(\mathbb{S}_{i-1}, \mathbb{T})$ has at most $q-1$ points in its fiber. Hence $\sharp\,\mathrm{Emb}(\mathbb{S}_i, \mathbb{T}) \le (q-1) \cdot \sharp\,\mathrm{Emb}(\mathbb{S}_{i-1}, \mathbb{T})$ ($i = 2, \ldots, a$). On the other hand, since

$$A(S,T) = \sharp\,\mathrm{Emb}(\mathbb{S}, \mathbb{T}) / \sharp\,\mathrm{Aut}(\mathbb{S}),$$

one has the inequality

$$\begin{aligned} A(S,T) &= \sharp\,\mathrm{Emb}(\mathbb{S}_a, \mathbb{T}) / \sharp\,\mathrm{Aut}(\mathbb{S}_a) \\ &\le (q-1)^{a-1} \cdot \sharp\,\mathrm{Emb}(\mathbb{S}_1, \mathbb{T}) / \sharp\,\mathrm{Aut}(\mathbb{S}_a) = (q-1)^{a-1} \cdot \sharp T / \sharp\,\mathrm{Aut}(S). \end{aligned}$$ □

(ii) *Assume that S decomposes as $S = S_1^{k_1} \sqcup \cdots \sqcup S_m^{k_m}$ for pairwise distinct $S_i \in \mathrm{Conf}_0$ ($i = 1, \ldots, m$) so that $\sum_{i=1}^m k_i = n(S)$. Then*

$$A(S,T) \le \frac{1}{k_1! \cdots k_m!} \prod_{i=1}^m A(S_i, T)^{k_i} \tag{5.2.3}$$

Proof. For $1 \le i \le m$, the subgraph of $\mathbb{S} \in \mathbb{A}(S, \mathbb{T})$ corresponding to the factor $S_i^{k_i}$, denoted by $\mathbb{S}|_{S_i^{k_i}}$, defines an off-diagonal element of $(\prod^{k_i} \mathbb{A}(S_i, \mathbb{T}))/\mathfrak{S}_{k_i}$ where $\mathfrak{S}_{k_i}$ is the symmetric group of k_i elements acting freely on the set of off-diagonal elements. Then the assignment $\mathbb{S} \mapsto (\mathbb{S}|_{S_i^{k_i}})_{i=1}^m$ defines an embedding $\mathbb{A}(S, \mathbb{T}) \to \prod_{i=1}^m ((\prod^{k_i} \mathbb{A}(S_i, \mathbb{T}))/\mathfrak{S}_{k_i})$ into the off-diagonal part. □

(iii) Let S be as in (ii). Then $\mathrm{Aut}(S) = \prod_{i=1}^m \mathrm{Aut}(S_i^{k_i})$ and each factor $\mathrm{Aut}(S_i^{k_i})$ is a wreath direct product of $\mathrm{Aut}(S_i)$ and $\mathfrak{S}_{k_i}$. Then (5.2.1) is a consequence of a combination of (5.2.2) and (5.2.3).

This completes the proof of the lemma. □

§5.3. Product-expansion formula for growth coefficients

The coefficients of a growth function of T are not algebraically independent.

Lemma. *Let $S_1, \ldots, S_m$ ($m \ge 0$) and $T \in \mathrm{Conf}$ be given. Then*

$$\prod_{i=1}^m A(S_i, T) \; = \sum_{S \in \mathrm{Conf}} \binom{S_1, \ldots, S_m}{S} A(S, T). \tag{5.3.1}$$

Proof. Let $\mathbb{T}$ be a graph representing T. For $m \in \mathbb{Z}_{\geq 0}$, consider the map

$$(\mathbb{S}_1, \dots, \mathbb{S}_m) \in \prod_{i=1}^{m} \mathbb{A}(S_i, \mathbb{T}) \mapsto \mathbb{S} := \bigcup_{i=1}^{m} \mathbb{S}_i \in 2^{\mathbb{T}},$$

whose fiber over $\mathbb{S}$ is $\binom{S_1,\dots,S_m}{\mathbb{S}}$ so that one has the decomposition

$$\prod_{i=1}^{m} \mathbb{A}(S_i, \mathbb{T}) \simeq \bigcup_{\mathbb{S} \in 2^{\mathbb{T}}} \binom{S_1, \dots, S_m}{\mathbb{S}}.$$

By counting the cardinality of both sides, one obtains the formula. □

Remark. The formula (5.3.1) is trivial for $m \in \{0, 1\}$, and can be reduced to the case $m = 2$ for $m \geq 2$ by induction on m as follows.

Multiply (5.3.1) by $A(S_{m+1}, T)$ and apply the formula for $m = 2$:

$$\begin{aligned} \prod_{i=1}^{m+1} A(S_i, T) &= \sum_{S \in \mathrm{Conf}} \binom{S_1, \dots, S_m}{S} A(S, T) A(S_{m+1}, T) \\ &= \sum_{S \in \mathrm{Conf}} \binom{S_1, \dots, S_m}{S} \sum_{U \in \mathrm{Conf}} \binom{S, S_{m+1}}{U} A(U, T) \end{aligned}$$

By the composition rule (2.6.1), this is equal to

$$\sum_{U \in \mathrm{Conf}} \binom{S_1, \dots, S_{m+1}}{U} A(U, T).$$

§5.4. Group-like property of the growth function

An element $g \in \mathbb{A}[\![\mathrm{Conf}]\!]$ is called *group-like* if it satisfies

$$\Phi_m(g) = \underbrace{g \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} g}_{m} \tag{5.4.1}$$

for all $m \in \mathbb{Z}_{\geq 0}$. This in particular implies the conditions $\Phi_0(g) = 1$ and $\iota(g) = g^{-1}$ (cf. (4.3.1) and (4.4.2)). For any group-like elements g and h, the power product $g^a h^b$ for $a, b \in \mathbb{A}$ (cf. (3.6.3)) is also group-like. We put

$$\mathfrak{G}_{\mathbb{A}} := \{\text{the group-like elements in } \mathbb{A}[\![\mathrm{Conf}]\!]\}, \tag{5.4.2}$$

$$\mathfrak{G}_{\mathbb{A},\mathrm{finite}} := \{g \in \mathfrak{G}_{\mathbb{A}} \mid g \text{ is of finite type}\}. \tag{5.4.3}$$

Lemma. *The generating polynomial $\mathcal{A}(T)$ for any $T \in \mathrm{Conf}$ is group-like. That is, for any $m \in \mathbb{Z}_{\geq 0}$ and $T \in \mathrm{Conf}$, we have*

$$\mathcal{A}(T) \otimes \cdots \otimes \mathcal{A}(T) = \Phi_m(\mathcal{A}(T)). \tag{5.4.4}$$

Proof. By the definition of $\mathcal{A}(T)$ (5.1.5), the m-fold tensor product

$$(*)\qquad \mathcal{A}(T)\otimes\cdots\otimes\mathcal{A}(T)$$

can be expanded into a sum of m variables $S_1,\dots,S_m$:

$$(**)\qquad \sum_{S_1\in\mathrm{Conf}}\cdots\sum_{S_m\in\mathrm{Conf}} S_1\otimes\cdots\otimes S_m\Big(\prod_{i=1}^{m}A(S_i,T)\Big).$$

By use of the product-expansion formula (5.3.1), this is equal to

$$\sum_{S_1\in\mathrm{Conf}}\cdots\sum_{S_m\in\mathrm{Conf}} S_1\otimes\cdots\otimes S_m\Big(\sum_{S\in\mathrm{Conf}}\binom{S_1,\dots,S_m}{S}A(S,T)\Big).$$

Recalling the definition of the map Φ_m (4.1.4), this is equal to

$$(***)\qquad \sum_{S\in\mathrm{Conf}}\Phi_m(S)\cdot A(S,T)\ =\Phi_m\Big(\sum_{S\in\mathrm{Conf}}S\cdot A(S,T)\Big)=\Phi_m\mathcal{A}(T).\qquad\square$$

§5.5. A characterization of the antipode

Equation (5.4.4) provides the formulae

$$(5.5.1)\qquad \iota(\mathcal{A}(T))\mathcal{A}(T)=1\quad\text{ for } T\in\mathrm{Conf},$$

$$(5.5.2)\qquad \Phi_m\circ\iota=(\iota\,\widehat{\otimes}\cdots\widehat{\otimes}\,\iota)\circ\Phi_m\quad\text{ for } m\in\mathbb{Z}_{\geq 0}.$$

Proof of (5.5.1). Apply (5.4.1) to $(\iota\cdot 1)\circ\Phi_2(T)=\mathrm{aug}(T)$ (4.4.2). $\square$

Proof of (5.5.2). It is enough to consider the case $m=2$ due to (4.2.1). Apply Φ_2 to (5.5.1). Recalling (5.4.1), one obtains the relation

$$\Phi_2(\iota(\mathcal{A}(T)))\cdot(\mathcal{A}(T)\otimes\mathcal{A}(T))=1.$$

Multiply by $\iota(\mathcal{A}(T))\iota(\mathcal{A}(T))$ and apply (5.5.1) again to obtain

$$\begin{aligned}\Phi_2(\iota(\mathcal{A}(T)))&=\iota(\mathcal{A}(T))\otimes\iota(\mathcal{A}(T))=(\iota\otimes\iota)(\mathcal{A}(T)\otimes\mathcal{A}(T))\\&=(\iota\otimes\iota)\Phi_2(\mathcal{A}(T)).\end{aligned}$$

Thus (5.5.2) is true for $\mathcal{A}(T)$ ($T\in\mathrm{Conf}$). Since these elements span $\mathbb{A}\cdot\mathrm{Conf}$, which is dense in the whole algebra, (5.5.2) holds on $\mathbb{A}[\![\mathrm{Conf}]\!]$. $\square$

§6. The logarithmic growth function

The growth coefficients $A(S,T)$ for $S\in\langle T\rangle$ were bounded from above in (5.2.1). However, we also need to bound its lower terms. This is achieved by introducing a logarithmic growth coefficient $M(S,T)\in\mathbb{Q}$ for $S\in\langle T\rangle$, and showing the linear relations (6.2.2) below.

§6.1. The logarithmic growth coefficient

For $T \in \mathrm{Conf}$, define the logarithm of the growth function

$$\mathcal{M}(T) := \log(\mathcal{A}(T)) \tag{6.1.1}$$

in $\mathbb{Q}[\![\langle T\rangle]\!]$ (cf. (5.1.5) and (3.6.2)). Expand $\mathcal{M}(T)$ in a series

$$\mathcal{M}(T) = \sum_{S\in \mathrm{Conf}} S \cdot M(S,T). \tag{6.1.2}$$

The coefficient $M(S,T)$ is the *logarithmic growth coefficient* for $S \in \langle T\rangle$.

By definition, $\mathcal{M}(T)$ does not have a constant term, i.e.

$$M([\emptyset],T) := 0 \quad \text{for } T \in \mathrm{Conf}\,. \tag{6.1.3}$$

For later applications, we write the explicit relations among growth functions and logarithmic growth functions (cf. (3.6.4) and (3.6.5)):

$$A(S,T) = \sum_{S=S_1^{k_1}\sqcup\cdots\sqcup S_m^{k_m}} \frac{1}{k_1!\cdots k_m!} M(S_1,T)^{k_1}\cdots M(S_m,T)^{k_m} \tag{6.1.4}$$

$$M(S,T) = \sum_{S=S_1^{k_1}\sqcup\cdots\sqcup S_m^{k_m}} \frac{(k_1+\cdots+k_m-1)!(-1)^{k_1+\cdots+k_m-1}}{k_1!\cdots k_m!} \times A(S_1,T)^{k_1}\cdots A(S_m,T)^{k_m}. \tag{6.1.5}$$

Remark. 1. From the formula, we see that for a connected $S \in \mathrm{Conf}_0$,

$$A(S,T) = M(S,T). \tag{6.1.6}$$

That is, *the logarithmic growth coefficients coincide with the growth coefficients for connected configurations.* This elementary fact will be used repeatedly.

2. The multiplicativity of $\mathcal{A}(T)$ (5.1.7) implies the additivity

$$\mathcal{M}(T_1\cdot T_2) = \mathcal{M}(T_1) + \mathcal{M}(T_2) \tag{6.1.7}$$

for $T_i \in \mathrm{Conf}$ and hence the additivity

$$M(S,T_1\cdot T_2) = M(S,T_1) + M(S,T_2) \quad \text{for } S \in \mathrm{Conf}\,. \tag{6.1.7*}$$

3. The invertibility (5.5.1) implies

$$\iota(\mathcal{M}(T)) = -\mathcal{M}(T). \tag{6.1.8}$$

§6.2. Linear dependence relations on the coefficients

Lemma. *The polynomial relation* (5.4.4) *implies the linear relation*

$$\sum_{i=1}^{m} 1 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} 1 \mathbin{\widehat{\otimes}} \overset{i\text{th}}{\mathcal{M}(T)} \mathbin{\widehat{\otimes}} 1 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} 1 = \Phi_m(\mathcal{M}(T)), \tag{6.2.1}$$

for the logarithmic growth function for $T \in \mathrm{Conf}$ and $m \in \mathbb{Z}_{\geq 0}$.

Proof. Put $\mathcal{M}_i(T) := 1 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} 1 \mathbin{\widehat{\otimes}} \overset{i\text{th}}{\mathcal{M}(T)} \mathbin{\widehat{\otimes}} 1 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} 1$ so that $\exp(\mathcal{M}_i(T)) = 1 \otimes \cdots \otimes 1 \otimes \mathcal{A}(T) \otimes 1 \otimes \cdots \otimes 1$. Then (5.4.4) can be rewritten as

$$(*) \qquad \exp(\mathcal{M}_1(T)) \cdots \exp(\mathcal{M}_m(T)) = \Phi_m(\exp(\mathcal{M}(T)))$$

where the left hand side is equal to $\exp(\mathcal{M}_1(T) + \cdots + \mathcal{M}_m(T))$ due to the commutativity of the $\mathcal{M}_i$'s and the addition rule for exp. The right hand side of $(*)$ can be rewritten as

$$\begin{aligned}\Phi_m(\exp(\mathcal{M}(T))) = \Phi_m\left(\sum_{n=0}^{\infty} \frac{1}{n!}\mathcal{M}(T)^n\right) &= \sum_{n=0}^{\infty} \frac{1}{n!}\Phi_m(\mathcal{M}(T))^n \\ &= \exp(\Phi_m(\mathcal{M}(T))).\end{aligned}$$

By taking the logarithm of both sides, we obtain (6.2.1). □

Corollary. *Let* $m \geq 2$. *For* $S_1, \ldots, S_m \in \mathrm{Conf}_+$ *and* $T \in \mathrm{Conf}$,

$$(6.2.2) \qquad \sum_{S \in \mathrm{Conf}} \binom{S_1, \ldots, S_m}{S} M(S,T) = 0.$$

Proof. Expand both sides of (6.2.1) in a series of the variables $S_i := 1 \otimes \ldots \otimes 1 \otimes S_i \otimes 1 \otimes \ldots \otimes 1$ $(i = 1, \ldots, m)$. Since the left hand side of (6.2.1) does not have a mixed term $S_1 \otimes \cdots \otimes S_m$ for $S_i \in \mathrm{Conf}_+$ and $m \geq 2$, the corresponding coefficients on the right hand side vanish. By (4.1.1) and (6.1.2), this implies the formula (6.2.2). □

Remark. 1. The formula (6.2.2) is reduced to the case $m = 2$ with $S_i \neq \emptyset$ $(i = 1, 2)$ by induction on m. Recalling the composition rule (2.6),

$$\begin{aligned}\sum_S \binom{S_1, \ldots, S_m}{S} M(S,T) &= \sum_S \Big(\sum_{U \in \mathrm{Conf}} \binom{S_1, \ldots, S_{m-1}}{U} \binom{U, S_m}{U} \Big) M(S,T) \\ &= \sum_{U \in \mathrm{Conf}_+} \binom{S_1, \ldots, S_{m-1}}{U} \Big(\sum_S \binom{U, S_m}{U} M(S,T) \Big) \\ &\quad + \binom{S_1, \ldots, S_{m-1}}{\emptyset} \Big(\sum_S \binom{\emptyset, S_m}{U} M(S,T) \Big) \\ &= 0 + 0 = 0.\end{aligned}$$

2. The linear dependence relations (6.2.2) among $M(S,T)$'s for $S \in \mathrm{Conf}$ are the key facts of the present paper. The Hopf algebra structure was introduced only to deduce this relation. We shall solve this relation in (8.3.2) by use of kabi coefficients, which we introduce in §7.

§6.3. Lie-like elements

An element $\mathcal{M}$ satisfying (6.2.1) has a name in Hopf algebra theory [9].

Definition. Let $\mathbb{A}$ be a commutative algebra with a unit. An element $\mathcal{M}$ of $\mathbb{A}[\![\mathrm{Conf}]\!]$ is called *Lie-like* if it satisfies the relation

$$\Phi_m(\mathcal{M}) = \sum_{i=1}^{m} 1 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} 1 \overset{i\text{th}}{\mathbin{\widehat{\otimes}} \mathcal{M}} \mathbin{\widehat{\otimes}} 1 \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} 1 \tag{6.3.1}$$

for all $m \in \mathbb{Z}_{\geq 0}$. This, in particular, implies the conditions $\Phi_0(\mathcal{M}) = 0$ and $\iota(\mathcal{M}) + \mathcal{M} = 0$ (cf. (4.3.1) and (4.4.2)). Linear combinations (over $\mathbb{A}$) of Lie-like elements are also Lie-like. We put

$$\mathcal{L}_{\mathbb{A}} := \{\text{all Lie-like elements in } \mathbb{A}[\![\mathrm{Conf}]\!]\}, \tag{6.3.2}$$

and

$$\mathcal{L}_{\mathbb{A},\mathrm{finite}} := \{M \in \mathcal{L}_{\mathbb{A}} \mid M \text{ is of finite type}\}. \tag{6.3.3}$$

In this terminology, 6.2 Lemma can be rewritten as: *if* $\mathbb{Q} \subset \mathbb{A}$*, then* $\mathcal{M}(T) \in \mathcal{L}_{\mathbb{A},\mathrm{finite}}$ *for* $T \in \mathrm{Conf}$*.*

Remark. We shall see in 8.4 that $\mathcal{L}_{\mathbb{R}}$ is essentially an extension of $\mathcal{L}_{\mathbb{R},\mathrm{finite}}$ by a space $\mathcal{L}_{\mathbb{R},\infty}$, which is the main objective of the present paper. On the other hand, one has $\mathcal{L}_{\mathbb{A}} \cap (\mathbb{A} \cdot \mathrm{Conf})_{\mathfrak{M}} \subset \mathcal{L}_{\mathbb{A},\ \mathrm{finite}}$ (actually equality holds, see §8), since $(\mathbb{A} \cdot \mathrm{Conf})_{\mathfrak{M}}$ consists of finite type elements only.

§7. Kabi coefficients

We describe the inverse of the infinite matrix $A := (A(S,T))_{S,T \in \mathrm{Conf}_0}$ explicitly in terms of kabi coefficients introduced in 7.2 below. The construction shows that the inverse matrix has only a bounded number of nonzero entries (7.5). This fact leads to the comparison of the two topologies on $\mathcal{L}_{\mathbb{A},\mathrm{finite}}$, which plays a key role in the construction of the infinite space $\mathcal{L}_{\mathbb{A},\infty}$.

§7.1. The unipotency of A

The matrix A is unipotent in the sense that (i) $A(S,S) = 1$ and (ii) $A(S,T) = 0$ for $S \not\leq T$ (5.1.5). Then the matrix $A^{-1} := E + A^* + A^{*2} + A^{*3} + \cdots$, where $E := (\delta(U,V))_{U,V \in \mathrm{Conf}_0}$ is the identity matrix (cf. lemma below) and $A^* := E - A$, is well defined. Precisely,

$$A^{-1}(S,T) = \begin{cases} 0 & \text{for } S \not\leq T, \\ 1 & \text{for } S = T, \\ \displaystyle\sum_{k>0} (-1)^k \sum_{S=S_0<\cdots<S_k=T} \prod_{i=1}^{k} A(S_{i-1},S_i) & \text{for } S < T. \end{cases}$$

The matrix A^{-1} is unipotent in the same sense as A, and hence the products $A^{-1} \cdot A$ and $A \cdot A^{-1}$ are well defined and are equal to E.

§7.2. Kabi coefficients

Definition. 1. A graph $\mathbb{U}$ is said to be *kabi* over its subgraph $\mathbb{S}$ if for all $x \in \mathbb{U} \setminus \mathbb{S}$, there exists $y \in \mathbb{S}$ such that (x, y) is an edge.

2. Let $U \in \mathrm{Conf}_0$ and let $\mathbb{U}$ be a graph with $[\mathbb{U}] = U$. For $S \in \mathrm{Conf}_0$, put

$$\mathbb{K}(S, \mathbb{U}) := \{\mathbb{S} \mid \mathbb{S} \subset \mathbb{U} \text{ such that } [\mathbb{S}] = S \text{ and } \mathbb{U} \text{ is kabi over } \mathbb{S}\}, \tag{7.2.1}$$

$$K(S, U) := \sharp\mathbb{K}(S, \mathbb{U}). \tag{7.2.2}$$

We call $K(S, U)$ a *kabi coefficient*. The definition of the coefficient does not depend on the choice of $\mathbb{U}$. If $K(S, U) \neq 0$, we say that U has a kabi structure over S or simply U is kabi over S.

Directly from the definition, we have

$$K(S, U) = 0 \quad \text{for } S \not\leq U, \tag{7.2.3}$$

$$K(S, S) = 1 \quad \text{for } S \in \mathrm{Conf}_0. \tag{7.2.4}$$

Note. The word "kabi" means "mold" in Japanese.

§7.3. Kabi inversion formula

Lemma. *For $S \in \mathrm{Conf}_0$ and $T \in \mathrm{Conf}$, one has the formula*

$$\sum_{U \in \mathrm{Conf}_0} (-1)^{\sharp U - \sharp S} K(S, U) \cdot A(U, T) = \delta(S, T), \tag{7.3.1}$$

where $\delta(S, T)$ means the number of connected components of T isomorphic to S.

Proof. The summation index U on the left hand side runs over the range $S \leq U \leq T$ (otherwise $K(S, U) \cdot A(U, T) = 0$). Hence if $S \not\leq T$, then the sum equals 0. If $S = T$, the only term in the sum is $K(S, S)A(S, S)$, which equals 1.

Let $S \in \mathrm{Conf}_0$ and $T \in \mathrm{Conf}$. Assume $S \leq T$ and $S \neq T$. Let $\mathbb{T}$ be a G-colored graph with $T = [\mathbb{T}]$. Applying the definition of $K(S, U)$ and $A(U, T)$ (cf. (5.1.1)), we can rewrite the left hand side of (7.3.1) as

$$\begin{aligned} &\sum_{U \in \mathrm{Conf}_0} (-1)^{\sharp U - \sharp S} K(S, U) \cdot \sharp\mathbb{A}(U, \mathbb{T}) \\ &\quad = \sum_{U \in \mathrm{Conf}_0} (-1)^{\sharp U - \sharp S} K(S, U) \cdot \sharp\{\mathbb{U} \mid \mathbb{U} \subset \mathbb{T} \text{ such that } [\mathbb{U}] = U\} \\ &\quad = \sum_{U \in \mathrm{Conf}_0} (-1)^{\sharp U - \sharp S} \sharp\left\{(\mathbb{S}, \mathbb{U}) \;\middle|\; \begin{array}{c} \mathbb{S} \subset \mathbb{U} \subset \mathbb{T} \text{ such that} \\ [\mathbb{S}] = S,\ [\mathbb{U}] = U \text{ and } \mathbb{U} \text{ is kabi over } \mathbb{S} \end{array}\right\}. \end{aligned}$$

Now resummation by fixing the subgraph $\mathbb{S}$ in $\mathbb{T}$ yields

$$= \sum_{\mathbb{S}\in A(S,\mathbb{T})} \Bigg(\sum_{U\in \mathrm{Conf}_0} (-1)^{\sharp U - \sharp S} \sharp \left\{ \mathbb{U} \;\middle|\; \begin{array}{c} \mathbb{S}\subset \mathbb{U} \subset \mathbb{T} \text{ such that} \\ [\mathbb{U}] = U \text{ and } \mathbb{U} \text{ is kabi over } \mathbb{S} \end{array} \right\} \Bigg).$$

For a fixed subgraph $\mathbb{S}$ of $\mathbb{T}$, let $\mathbb{U}_{\max}$ be the maximal subgraph of $\mathbb{T}$ such that $\mathbb{U}_{\max}$ is a kabi over $\mathbb{S}$, i.e. $\mathbb{U}_{\max}$ consists of the vertices of $\mathbb{T}$ which are either in $\mathbb{S}$ or connected to $\mathbb{S}$ by an edge. Then a subgraph $\mathbb{U}$ of $\mathbb{T}$ becomes a kabi over $\mathbb{S}$ if and only if it is a subgraph of $\mathbb{U}_{\max}$ containing $\mathbb{S}$. Hence the sum is equal to

$$\sum_{\mathbb{S}\in A(S,\mathbb{T})} \Big(\sum_{\mathbb{S}\subset\mathbb{U}\subset\mathbb{U}_{\max}} (-1)^{\sharp\mathbb{U} - \sharp\mathbb{S}} \Big) = \sum_{\mathbb{S}\in A(S,\mathbb{T})} \Big(\sum_{\mathbb{W}\subset\mathbb{U}_{\max}\setminus\mathbb{S}} (-1)^{\sharp\mathbb{W}} \Big).$$

where the last summation index $\mathbb{W}$ runs over all subsets of $\mathbb{U}_{\max}\setminus\mathbb{S}$. Hence the sum in the parenthesis becomes 1 or 0 according to whether $\mathbb{U}_{\max}\setminus\mathbb{S}$ is $\emptyset$ or not. It is clear that $\mathbb{U}_{\max}\setminus\mathbb{S}=\emptyset$ is equivalent to $\mathbb{S}$ being a connected component of $\mathbb{T}$. Hence the sum is equal to $\delta(S,T)$. □

§7.4. Corollaries to the inversion formula

The left inverse matrix of $A := (A(S,T))_{S,T\in\mathrm{Conf}_0}$ is given by

$$A^{-1} = ((-1)^{\sharp T - \sharp S} K(S,T))_{S,T\in\mathrm{Conf}_0}. \tag{7.4.1}$$

Since the left inverse matrix of A coincides with the right inverse, one has

$$\sum_{U\in\mathrm{Conf}_0} (-1)^{\sharp T - \sharp U} A(S,U)\cdot K(U,T) = \delta(S,T) \tag{7.4.2}$$

for $S\in\mathrm{Conf}_0$. Specializing S in (7.4.2) to $pt :=$ [one-point graph], one gets,

$$\sum_{U\in\mathrm{Conf}_0} (-1)^{\sharp U} \sharp U\cdot K(U,T) = (-1)^{\sharp T}\delta(pt,T). \tag{7.4.3}$$

§7.5. Boundedness of non-zero entries of K

One of the most important consequences of (7.4.1) is the boundedness of the non-zero entries of the matrix A^{-1}, as follows.

Suppose $K(S,T)\neq 0$. Then, by definition, T must have at least one structure of kabi over S. This implies that for each fixed S and $q\geq 0$, there are only a finite number of $T\in\mathrm{Conf}_0$ with $K(S,T)\neq 0$. Precisely:

Assertion. *For $S\in\mathrm{Conf}_0$, $K(S,T)=0$ unless $\sharp T\leq \sharp S\cdot(q-1)+2$.*

Proof. Let $\mathbb{T}$ be kabi over $\mathbb{S}$. Every vertex of $\mathbb{S}$ is connected to at most q points of $\mathbb{T}$. Since $\mathbb{S}$ is connected, it has at least $\sharp S-1$ edges. Hence, $\sharp T - \sharp S \leq \sharp$ {edges connecting $\mathbb{S}$ and $\mathbb{T}\setminus\mathbb{S}$} $\leq q\cdot\sharp S - 2\cdot(\sharp S-1)$. This implies the assertion. □

Remark. The above boundedness implies that K induces a continuous map between the two differently completed modules of $\mathcal{L}_{\mathbb{A},\mathrm{finite}}$ (cf. 8.4).

§8. Lie-like elements $\mathcal{L}_{\mathbb{A}}$

Under the assumption $\mathbb{Q} \subset \mathbb{A}$, we introduce two basis systems $\{\mathcal{M}(T)\}_{T\in\mathrm{Conf}_0}$ and $\{\varphi(S)\}_{S\in\mathrm{Conf}_0}$ for the module of Lie-like elements $\mathcal{L}_{\mathbb{A},\mathrm{finite}}$, where the base change between them is given by the kabi coefficients. The basis $\{\varphi(S)\}_{S\in\mathrm{Conf}_0}$ is compatible with the adic topology and gives a topological basis of $\mathcal{L}_{\mathbb{A}}$.

§8.1. The splitting map ∂

First, we introduce a useful but somewhat technical map ∂. One reason for its usefulness can be seen from the formula (9.3.6). For $S \in \mathrm{Conf}_0$, let us define an $\mathbb{A}$-linear map $\partial_S : \mathbb{A}[\![\mathrm{Conf}]\!] \to \mathbb{A}$ by associating to a series f its coefficient at S, i.e. $\partial_S f := f_S \in \mathbb{A}$ for f given by (3.2.4). Using it, we define

$$(8.1.1) \qquad \partial : \mathbb{A}[\![\mathrm{Conf}]\!] \to \prod_{S\in\mathrm{Conf}_0} \mathbb{A}\cdot e_S, \qquad f \mapsto \sum_{S\in\mathrm{Conf}_0} (\partial_S f)\cdot e_S.$$

Here, the range is an abstract direct product module of rank one modules $\mathbb{A}\cdot e_S$ with the base e_S for $S \in \mathrm{Conf}_0$. Let us verify that the map is well-defined. First, define the map ∂ from the polynomial ring $\mathbb{A}\cdot\mathrm{Conf}$ to $\oplus_{S\in\mathrm{Conf}_0}\mathbb{A}\cdot e_S$. Since $\partial(\mathcal{J}_n) \subset \oplus_{S\in\mathrm{Conf}_0,\,\sharp S\geq n}\mathbb{A}\cdot e_S$, the map is continuous with respect to the adic topology (3.2) on the LHS and the direct product topology on the RHS. Then ∂ (8.1.1) is obtained by completing this polynomial map.

We note that the restriction of ∂ induces a map

$$\partial : \mathbb{A}[\![\mathrm{Conf}]\!]_{\mathrm{finite}} \to \bigoplus_{S\in\mathrm{Conf}} \mathbb{A}\cdot e_S,$$

even though the domain of this map is not a polynomial ring but the ring of elements of finite type (recall the definition in 3.3).

§8.2. Bases $\{\varphi(S)\}_{S\in\mathrm{Conf}_0}$ of $\mathcal{L}_{\mathbb{A},\mathrm{finite}}$ and $\mathcal{L}_{\mathbb{A}}$

Lemma. *Let $\mathbb{A}$ be a commutative algebra containing $\mathbb{Q}$.*

(i) *The system $(\mathcal{M}(T))_{T\in\mathrm{Conf}_0}$ is an $\mathbb{A}$-free basis for $\mathcal{L}_{\mathbb{A},\mathrm{finite}}$:*

$$(8.2.1) \qquad \mathcal{L}_{\mathbb{A},\mathrm{finite}} \simeq \bigoplus_{S\in\mathrm{Conf}_0} \mathbb{A}\cdot\mathcal{M}(S).$$

(ii) *The map ∂ (8.1.1) induces a bijection of $\mathbb{A}$-modules:*

$$(8.2.2) \qquad \partial|_{\mathcal{L}_{\mathbb{A},\mathrm{finite}}} : \mathcal{L}_{\mathbb{A},\mathrm{finite}} \simeq \bigoplus_{S\in\mathrm{Conf}_0} \mathbb{A}\cdot e_S.$$

Put $\varphi(S) := \partial|^{-1}_{\mathcal{L}_{\mathbb{A},\mathrm{finite}}}(e_S)$ *for* $S \in \mathrm{Conf}_0$ *so that* $\{\varphi(S)\}_{S\in\mathrm{Conf}_0}$ *is another* $\mathbb{A}$*-free basis of* $\mathcal{L}_{\mathbb{A},\mathrm{finite}}$*.*

(iii) *The two basis systems* $\{\mathcal{M}(S)\}_{S\in\mathrm{Conf}_0}$ *and* $\{\varphi(S)\}_{S\in\mathrm{Conf}_0}$ *for* $\mathcal{L}_{\mathbb{A},\mathrm{finite}}$ *are related by the following formula:*

$$\mathcal{M}(T) = \sum_{S\in\mathrm{Conf}_0} \varphi(S)\cdot A(S,T), \tag{8.2.3}$$

$$\varphi(S) = \sum_{T\in\mathrm{Conf}_0} \mathcal{M}(T)\cdot(-1)^{\sharp T-\sharp S}K(T,S). \tag{8.2.4}$$

(iv) $\mathcal{L}_{\mathbb{A},\mathrm{finite}}$ *is dense in* $\mathcal{L}_{\mathbb{A}}$ *with respect to the adic topology on the configuration algebra* (*cf.* 3.2)*.*

(v) *The map* ∂ *induces an isomorphism of topological* $\mathbb{A}$*-modules:*

$$\mathcal{L}_{\mathbb{A}} \simeq \prod_{S\in\mathrm{Conf}_0} \mathbb{A}\cdot e_s. \tag{8.2.5}$$

This means that any $\mathcal{M}\in\mathcal{L}_{\mathbb{A}}$ *is expressed uniquely as an infinite sum*

$$\mathcal{M} = \sum_{S\in\mathrm{Conf}_0} \varphi(S)\cdot a_S \tag{8.2.6}$$

for $a_S\in\mathbb{A}$ $(S\in\mathrm{Conf}_0)$*. That is,* $(\varphi(S))_{S\in\mathrm{Conf}_0}$ *is a topological basis of* $\mathcal{L}_{\mathbb{A}}$*. We shall sometimes call* a_S *the coefficient of* $\mathcal{M}$ *at* $S\in\mathrm{Conf}_0$*.*

Proof. That $\mathcal{M}(T)\in\mathcal{L}_{\mathbb{A},\mathrm{finite}}$ for $T\in\mathrm{Conf}$ is shown in 6.2 Lemma.

In (a), (b) and (c) below, we prove (i)–(iii) simultaneously.

(a) *The restriction of the map* ∂ *of* (8.1.1) *to* $\mathcal{L}_{\mathbb{A}}$ *is injective.*

Proof. If $\mathcal{M} = \sum_{S\in\mathrm{Conf}} S\cdot M_S \in \mathbb{A}[\![\mathrm{Conf}]\!]_+$ is Lie-like (6.3), then one has

$$\sum_{S\in\mathrm{Conf}} \binom{S_1,\ldots,S_m}{S} M_S = 0 \tag{8.2.7}$$

for any $S_1,\ldots,S_m\neq\emptyset$ and $m\geq 2$ (the proof is the same as that for (6.2.2)). We have to prove that $\partial\mathcal{M}=0$ implies $M_S=0$ for all $S\in\mathrm{Conf}$. This will be done by induction on $n(S)=\sharp\{\text{connected components of } S\}$ as follows. The case $n(S)=1$ follows from the assumption $\partial\mathcal{M}=0$. Let $n(S)>1$ and $S=S_1^{k_1}\sqcup\cdots\sqcup S_l^{k_l}$ be an irreducible decomposition of S (so $S_i\in\mathrm{Conf}_0$ $(i=1,\ldots,l)$ are pairwise distinct). Apply (8.2.7) for $m=k_1+\cdots+k_l$ $(=n(S))$ and take $S_1,\ldots,S_1$ (k_1 times), $\ldots,S_l,\ldots,S_l$ (k_l times) for $S_1,\ldots,S_m$ to get

$$k_1!\cdots k_l!M_S + \sum_{\substack{T\in\mathrm{Conf}\\ n(T)<n(S)}} \binom{S_1,\ldots,S_m}{T} M_T = 0. \tag{$**$}$$

By the induction hypothesis, the second term in $(**)$ is 0, and hence $M_S=0$. □

(b) *For* $T \in \mathrm{Conf}$, *one has the formula*

$$\partial(\mathcal{M}(T)) \quad = \sum_{S \in \mathrm{Conf}_0} e_S \cdot A(S,T). \tag{8.2.8}$$

Proof. Recall that $\mathcal{M}(T) = \log(\mathcal{A}(T))$ and the coefficients of $\mathcal{M}(T)$ and $\mathcal{A}(T)$ at a connected $S \in \mathrm{Conf}_0$ coincide ((3.6.7) and (6.1.6)). That is, $\partial(\mathcal{M}(T)) = \partial(\mathcal{A}(T))$. By definition, $\partial(\mathcal{A}(T)) =$ RHS of (8.2.8). □

(c) *The map* (8.2.2) *is surjective, and hence bijective.*

Proof. It was shown in 7.1 that the infinite matrix $(A(S,T))_{S,T\in \mathrm{Conf}_0}$ is invertible as a unipotent matrix. Then (8.2.8) implies surjectivity.

Using again (8.2.8), we see that the system $\{\mathcal{M}(T)\}_{T\in\mathrm{Conf}_0}$ is $\mathbb{A}$-linearly independent and spans $\mathcal{L}_{\mathbb{A},\mathrm{finite}}$, i.e. (i) holds. The formula (8.2.8) can be rewritten as (8.2.3). Then (8.2.4) follows from (8.2.3) and (7.4.2). □

The proof of (iv) and (v) is done in (d), (e) and (f) below.

(d) $\mathcal{L}_{\mathbb{A}}$ *is closed in* $\mathbb{A}[\![\mathrm{Conf}]\!]$ *with respect to the adic topology*, since the coproduct Φ_m is continuous (4.1 Assertion). Thus: $(\mathcal{L}_{\mathbb{A},\mathrm{finite}})^{\mathrm{closure}} \subset \mathcal{L}_{\mathbb{A}}$.

(e) *The map* (8.2.2) *is homeomorphic with respect to the induced adic topology on the LHS and the restriction of the direct product topology on the RHS.* To show this, it is enough to show the bijection

$$\partial : (\mathcal{L}_{\mathbb{A},\mathrm{finite}}) \cap \mathcal{J}_n \simeq \bigoplus_{\substack{S\in\mathrm{Conf}_0 \\ \sharp S \geq n}} \mathbb{A}\cdot e_S, \tag{8.2.9}$$

since the sets on the RHS for $n \in \mathbb{Z}_{\geq 0}$ can be chosen as a system of fundamental neighborhoods for the direct product topology on $\oplus_{S\in\mathrm{Conf}_0} \mathbb{A}\cdot e_S$.

Proof of (8.2.9). Due to the definition (3.2.1) of the ideal $\mathcal{J}_n$, the ∂-image of the left hand side is contained in the right hand side of (8.2.9). Thus, one only has to show surjectivity. For $S \in \mathrm{Conf}_0$, let $\varphi(S)$ be the base of $\mathcal{L}_{\mathbb{A},\mathrm{finite}}$ such that $\partial(\varphi(S)) = e_S$ as introduced in (ii). It is enough to show that if $\sharp S \geq n$ and $S \in \mathrm{Conf}_0$, then $\varphi(S)$ belongs to $\mathcal{J}_n$. Expand $\varphi(S) = \sum U \cdot \varphi_U$. We show that $\varphi_U \neq 0$ implies that U is contained in the semigroup generated by $\langle S\rangle$ such that $\sharp U \geq n$. More precisely, we show $\binom{U_1,\ldots,U_m}{S} \neq 0$, where $U = U_1 \sqcup \cdots \sqcup U_m$ is an irreducible decomposition of U (cf. 2.5 (i)). The proof is by induction on $m = n(U)$. For $n(U) = 1$, $\varphi_U \neq 0$ if and only if $U = S$ by the definition of $\varphi(S)$, and hence this is trivial. If $n(U) > 1$, then apply (8.2.7) similarly to $(**)$ for the irreducible decomposition of U. We get

$$(***)\qquad k_1!\cdots k_l!\ \varphi_U + \sum_{\substack{T\in \mathrm{Conf}\\ n(T)<n(U)}} \binom{U_1,\dots,U_m}{T}\varphi_T = 0.$$

The fact that $\varphi_U \neq 0$ implies $\varphi_T \cdot \binom{U_1,\dots,U_m}{T} \neq 0$ for some T. Since $\varphi_T \neq 0$ with $n(T) < n(U)$, we apply the induction hypothesis to T, i.e. $\binom{T_1,\dots,T_p}{S} \neq 0$ for an irreducible decomposition $T = T_1 \sqcup \cdots \sqcup T_p$. Since $\binom{U_1,\dots,U_m}{T} \neq 0$, by composing the maps $U \to T \to S$, we conclude $\binom{U_1,\dots,U_m}{S} \neq 0$. In particular, $U_i \in \langle S\rangle$ and $\sharp U = \sum \sharp U_i \geq \sharp T \geq \sharp S$. This completes the proof of (e). □

(f) By completing the map (8.2.2), one sees that the composition of the two injective maps $(\mathcal{L}_{\mathbb{A},\mathrm{finite}})^{\mathrm{closure}} \subset \mathcal{L}_{\mathbb{A}} \to \varinjlim_{p,q} \prod_{S\in \mathrm{Conf}_0} \mathbb{A}\cdot e_s$ is bijective. This shows that all the maps are bijective. Hence, $\mathcal{L}_{\mathbb{A},\mathrm{finite}}$ is dense in $\mathcal{L}_{\mathbb{A}}$ and (8.2.5) holds. The formula (8.2.6) is another expression of (8.2.5).

This completes the proof of the lemma. □

Remark. 1. It was shown in the above proof that for $S \in \mathrm{Conf}_0$,

$$(8.2.10)\qquad \varphi(S) \in \mathbb{Z}[\![\langle S\rangle]\!] \cap \mathcal{J}_{\sharp S}.$$

In particular, $\varphi(U,S) = \delta(U,S)$ for $U \in \mathrm{Conf}_0$.

2. It was shown that the map $\partial|_{\mathcal{L}_{\mathbb{A},\mathrm{finite}}}$ (8.2.2) is a homeomorphism. But one should note that (8.2.1) is *not* a homeomorphism.

3. In general, an element of $\mathcal{L}_{\mathbb{A}}$ cannot be expressed by an infinite sum of $\mathcal{M}(T)$ ($T \in \mathrm{Conf}_0$) (cf. 9.4).

4. The set of Lie-like elements of the localization $\mathbb{A}[\mathrm{Conf}]_{\mathfrak{M}}$ (cf. 4.6 Remark 4) is equal to $\mathcal{L}_{\mathbb{A},\mathrm{finite}}$. This is insufficient for our later application in §10, so we employed the other localization (3.2.2).

§8.3. An explicit formula for $\varphi(S)$

Let us expand $\varphi(S)$ for $S \in \mathrm{Conf}_0$ in the series:

$$(8.3.1)\qquad \varphi(S) = \sum_{U\in \mathrm{Conf}} U\cdot \varphi(U,S)$$

for $\varphi(U,S) \in \mathbb{Q}$. The formula (8.2.3) can be rewritten as a matrix relation

$$(8.3.2)\qquad M(U,T) = \sum_{S\in \mathrm{Conf}_0} \varphi(U,S)\cdot A(S,T).$$

We remark that (8.2.3) and (8.3.2) are valid not only for $T \in \mathrm{Conf}_0$ but for all $T \in \mathrm{Conf}$, since both sides are additive with respect to T.

Formula. An explicit formula for the coefficient $\varphi(U,S)$ is

$$(8.3.3)\qquad \sum_{\substack{U=U_1^{k_1}\sqcup\cdots\sqcup U_m^{k_m}\\ V\in\mathrm{Conf},\, W\in\mathrm{Conf}_0}} \frac{(|\underline{k}|-1)!(-1)^{|\underline{k}|-1+|W|+|S|}}{k_1!\cdots k_m!}\binom{\underbrace{U_1,\dots U_1}_{k_1},\dots,\underbrace{U_m,\dots U_m}_{k_m}}{V} \times A(V,W)K(W,S).$$

Here the summation index runs over all decompositions $U=U_1^{k_1}\cdots U_m^{k_m}$ of U in the same manner explained at (3.6.6), where $|\underline{k}|=k_1+\cdots+k_m$.

Proof. By use of (6.1.5), rewrite the left hand side of (8.2.3)* into a polynomial of $A(U_i,T)$. Then apply the product expansion formula (5.3.1) to each monomial so that the left hand side is expressed linearly by $A(S,T)$'s. Using the invertibility of $\{A(S,T)\}_{S,T\in\mathrm{Conf}}$ (7.4.2), one deduces (8.3.3). □

Remark. As an application of (8.3.3), we can explicitly determine the coefficients $\{M_U\}_{U\in\mathrm{Conf}}$ of any Lie-like element $\mathcal{M}=\sum_{U\in\mathrm{Conf}}U\cdot M_U$ from the subsystem $\{M_S\}_{S\in\mathrm{Conf}_0}$ via the relation $M_U=\sum_{S\in\mathrm{Conf}_0}\varphi(U,S)\cdot M_S$. Here, the summation index S runs only over the finite set with $\sharp S\le\sharp U$.

§8.4. Lie-like elements $\mathcal{L}_{\mathbb{A},\infty}$ at infinity

We introduce the space $\mathcal{L}_{\mathbb{A},\infty}$ of Lie-like elements at infinity for use after §10.

Recall that the kabi coefficients relate the two basis systems of $\mathcal{L}_{\mathbb{A},\mathrm{finite}}$: $\{\varphi(S)\}_{S\in\mathrm{Conf}_0}$ and $\{\mathcal{M}(T)\}_{T\in\mathrm{Conf}_0}$ (cf. 8.2 Lemma). The map

$$K:\mathcal{L}_{\mathbb{A},\mathrm{finite}}\to\mathcal{L}_{\mathbb{A},\mathrm{finite}},$$
$$\sum_{S\in\mathrm{Conf}_0}\varphi(S)a_S\mapsto\sum_{T\in\mathrm{Conf}_0}\mathcal{M}(T)\sum_{S\in\mathrm{Conf}_0}(-1)^{\sharp T-\sharp S}K(T,S)a_S,$$

is the identity homomorphism. We define topologies on the modules of both sides: the fundamental system of neighborhoods of 0 consists of the linear subspaces spanned by all bases except for finite ones. Actually, the topology on the LHS coincides with the adic topology, which we have been studying (8.2 Lemma). The map K is continuous with respect to the topologies, since for any base $\mathcal{M}(T)$, there are only a finite number of bases $\varphi(S)$ whose image $K(\varphi(S))$ contains the term $\mathcal{M}(T)$, namely $K(T,S)\neq 0$ only for S satisfying $\sharp T\ge\frac{1}{q-1}(\sharp S-2)$ (7.5 Assertion). Let us denote by $\overline{K}$ the map between the completed modules and call it the *kabi map*,

$$(8.4.1)\qquad \overline{K}:\mathcal{L}_{\mathbb{A}}\ \to\prod_{T\in\mathrm{Conf}_0}\mathbb{A}\cdot\mathcal{M}(T).$$

We consider the set of Lie-like elements which are annihilated by the kabi map:

$$(8.4.2)\qquad \mathcal{L}_{\mathbb{A},\infty}:=\ker(\overline{K}),$$

and call it the space of *Lie-like elements at infinity.* In fact, $\mathcal{L}_{\mathbb{A},\infty}$ does not contain a non-trivial finite type element, i.e. $\mathcal{L}_{\mathbb{A},\mathrm{finite}} \cap \mathcal{L}_{\mathbb{A},\infty} = \{0\}$. However, the direct sum $\mathcal{L}_{\mathbb{A},\mathrm{finite}} \oplus \mathcal{L}_{\mathbb{A},\infty}$ is a small submodule of $\mathcal{L}_{\mathbb{A}}$, and *one looks for a submodule, say* $\mathcal{L}'$, *of* $\mathcal{L}_{\mathbb{A}}$ *containing* $\mathcal{L}_{\mathbb{A},\mathrm{finite}}$, *with a splitting* $\mathcal{L}_{\mathbb{A}} = \mathcal{L}' \oplus \mathcal{L}_{\mathbb{A},\infty}$. However, there is some difficulty in finding such $\mathcal{L}'$ for general $\mathbb{A}$: an *infinite sum* $\sum_{T \in \mathrm{Conf}_0} a_T \mathcal{M}(T)$ $\in \mathrm{Im}(\overline{K})$ never converges in $\mathcal{L}'$ ($\simeq \mathrm{Im}(\overline{K})$) with respect to the adic topology. We shall come back to this problem in 10.2 for the case $\mathbb{A} = \mathbb{R}$, where the classical topology plays a crucial role.

§9. Group-like elements $\mathfrak{G}_{\mathbb{A}}$

We determine the groups $\mathfrak{G}_{\mathbb{A}}$ and $\mathfrak{G}_{\mathbb{A},\mathrm{finite}}$ of group-like elements in $\mathbb{A}[\![\mathrm{Conf}]\!]$ and $\mathbb{A}[\![\mathrm{Conf}]\!]_{\mathrm{finite}}$, respectively, if $\mathbb{A}$ is $\mathbb{Z}$-torsion free. In particular, if $\mathbb{A} = \mathbb{Z}$, the group $\mathfrak{G}_{\mathbb{Z},\mathrm{finite}}$ is, via the correspondence $\mathcal{A}(S) \leftrightarrow S$, isomorphic to $\langle \mathrm{Conf} \rangle$ = the abelian group associated to the semigroup Conf, and it forms a "lattice in the continuous group" $\mathfrak{G}_{\mathbb{R}}$. Then we introduce the set EDP of equal division points inside the positive cone in $\mathfrak{G}_{\mathbb{R}}$ spanned by the basis $\{\mathcal{M}(S)\}$.

§9.1. $\mathfrak{G}_{\mathbb{A},\mathrm{finite}}$ and $\mathfrak{G}_{\mathbb{A}}$ for the case $\mathbb{Q} \subset \mathbb{A}$

We start with a general fact: *Assume* $\mathbb{Q} \subset \mathbb{A}$. *Then one has isomorphisms*

$$\exp : \mathcal{L}_{\mathbb{A}} \simeq \mathfrak{G}_{\mathbb{A}}, \quad \exp : \mathcal{L}_{\mathbb{A},\mathrm{finite}} \simeq \mathfrak{G}_{\mathbb{A},\mathrm{finite}}. \tag{9.1.1}$$

Proof. Since $\mathrm{aug}(g) = 1$, $\log(g)$ (3.6.2) is well defined for $\mathbb{Q} \subset \mathbb{A}$. That g is group-like (5.4.1) implies that $\log(g)$ is Lie-like and belongs to $\mathcal{L}_{\mathbb{A}}$ (cf. proof of 6.2 Lemma). Then g is of finite type if and only if $\log(g)$ is (cf. 3.6). Thus (9.1.1) is shown. The homeomorphism follows from that of exp (see 3.6). □

§9.2. Generators of $\mathfrak{G}_{\mathbb{A},\mathrm{finite}}$ and $\mathfrak{G}_{\mathbb{A}}$ for a $\mathbb{Z}$-torsion free $\mathbb{A}$

Lemma. *Let* $\mathbb{A}$ *be a commutative* $\mathbb{Z}$*-torsion free algebra with unit.*

(i) *Any element* g *of* $\mathfrak{G}_{\mathbb{A},\mathrm{finite}}$ *is uniquely expressed as*

$$g = \prod_{i \in I} \mathcal{A}(S_i)^{c_i} \tag{9.2.1}$$

for $S_i \in \mathrm{Conf}_0$ *and* $c_i \in \mathbb{A}$ $(i \in I)$ *with* $\sharp I < \infty$. *That is, one has an isomorphism*

$$\langle \mathrm{Conf} \rangle \otimes_{\mathbb{Z}} \mathbb{A} \simeq \mathfrak{G}_{\mathbb{A},\mathrm{finite}}, \quad S \leftrightarrow \mathcal{A}(S) \ , \tag{9.2.2}$$

where $\langle \mathrm{Conf} \rangle$ *is the group associated to the semigroup* Conf.

(ii) $\mathfrak{G}_{\mathbb{A},\mathrm{finite}}$ *is dense in* $\mathfrak{G}_{\mathbb{A}}$ *with respect to the adic topology.*

(iii) *We have the natural inclusion*

$$\{\exp(\varphi(S)) \mid S \in \mathrm{Conf}_0\} \subset \mathfrak{G}_{\mathbb{Z},\mathrm{finite}}. \tag{9.2.3}$$

The set $\{\exp(\varphi(S))\}_{S\in\mathrm{Conf}_0}$ *is a topological free generating system of* $\mathfrak{G}_{\mathbb{A}}$. *This means that any element* g *of* $\mathfrak{G}_{\mathbb{A}}$ *is uniquely expressed as an infinite product*

$$\prod_{S\in\mathrm{Conf}_0} \exp(\varphi(S)\cdot a_S) := \lim_{n\to\infty} \prod_{\substack{S\in\mathrm{Conf}_0\\ \sharp S<n}} \exp(\varphi(S)\cdot a_S) \tag{9.2.4}$$

for some $a_S \in \mathbb{A}$ $(S \in \mathrm{Conf}_0)$.

Proof. If $\mathbb{Q} \subset \mathbb{A}$, then due to the isomorphisms (9.1.1) and (6.1.1), the lemma is reduced to the corresponding statements for $\mathcal{L}_{\mathbb{A}}$ and $\mathcal{L}_{\mathbb{A},\mathrm{finite}}$ in 8.2 Lemma, where, due to (8.2.4), (8.2.10) and the integrality of kabi K, we have

$$\exp(\varphi(S)) = \prod_{T\in\mathrm{Conf}_0} \mathcal{A}(T)^{(-1)^{\sharp T-\sharp S}K(T,S)} \in \mathfrak{G}_{\mathbb{Z},\mathrm{finite}} \cap \{1+\mathcal{J}_{\sharp S}\},$$

where we note $\mathcal{A}(T) \in \mathfrak{G}_{\mathbb{Z},\mathrm{finite}}$ (cf. (5.1.6) and (5.4.4)).

Assume $\mathbb{Q} \not\subset \mathbb{A}$ and let $\tilde{\mathbb{A}}$ be the localization of $\mathbb{A}$ with respect to $\mathbb{Z}\setminus\{0\}$. Since $\mathbb{A}$ is torsion free, one has an inclusion $\mathbb{A} \subset \tilde{\mathbb{A}}$, which induces inclusions $\mathfrak{G}_{\mathbb{A}} \subset \mathfrak{G}_{\tilde{\mathbb{A}}}$ and $\mathfrak{G}_{\mathbb{A},\mathrm{finite}} \subset \mathfrak{G}_{\tilde{\mathbb{A}},\mathrm{finite}}$, and the lemma is true for $\mathfrak{G}_{\tilde{\mathbb{A}},\mathrm{finite}}$ and $\mathfrak{G}_{\tilde{\mathbb{A}}}$.

(i) Let us express an element $g \in \mathfrak{G}_{\mathbb{A},\mathrm{finite}}$ as $\prod_{i\in I} \mathcal{A}(S_i)^{c_i}$, where $c_i \in \tilde{\mathbb{A}}$ for $i \in I$ and $\sharp I < \infty$. We need to show that $c_i \in \mathbb{A}$ for $i \in I$. Suppose not. Put $I_1 := \{i \in I \mid c_i \notin \mathbb{A}\}$ and let S_1 be a maximal element of $\{S_i \mid i \in I_1\}$ with respect to the partial ordering $\leq$. Put $g_1 := \prod_{i\in I_1\setminus\{1\}} \mathcal{A}(S_i)^{c_i}$ and $g_2 := \prod_{i\in I\setminus I_1} \mathcal{A}(S_i)^{c_i}$. Then $g_1\mathcal{A}(S_1)^{c_1} = g \cdot g_2^{-1} \in \mathfrak{G}_{\mathbb{A},\mathrm{finite}}$. On the left hand side, g_1 does not contain the term S_1, whereas $\mathcal{A}(S_1)^{c_1}$ contains the term c_1S_1. Hence, the left hand side contains the term c_1S_1.

(ii) Let any $g \in \mathfrak{G}_{\mathbb{A}}$ be given. For a fixed integer $n \in \mathbb{Z}_{\geq 0}$, we calculate

$$\begin{aligned}\log(g) &= \sum_{S\in\mathrm{Conf}_0} \varphi(S)\cdot a_S \quad \text{for } a_S \in \tilde{\mathbb{A}} \quad \text{(cf. (8.2.6))}\\ &= \sum_{T\in\mathrm{Conf}_0} \mathcal{M}(T)\cdot c_{T,n} + R_n,\end{aligned}$$

where

$$c_{T,n} := \sum_{\substack{S\in\mathrm{Conf}_0\\ \sharp S<n}} (-1)^{\sharp T-\sharp S}K(T,S)\cdot a_S \in \tilde{\mathbb{A}} \quad \text{(cf. (8.2.4))}, \tag{$*$}$$

$$R_n := \sum_{\substack{S\in\mathrm{Conf}_0\\ \sharp S\geq n}} \varphi(S)\cdot a_S. \tag{$**$}$$

Here we notice that $c_{T,n} \neq 0$ implies $\sharp T < n$, since $K(T,S) \neq 0$ implies $T \leq S$ by (7.2.3), and $R_n \in \mathcal{J}_n$, since $\sharp S \geq n$ implies $\varphi(S) \in \mathcal{J}_n$ by (8.2.10). Therefore

$$g = \sum_{\substack{T \in \mathrm{Conf}_0 \\ \sharp T < n}} \mathcal{A}(T)^{c_{T,n}} \cdot \exp(R_n) = \sum_{\substack{T \in \mathrm{Conf}_0 \\ \sharp T < n}} \mathcal{A}(T)^{c_{T,n}} \bmod \mathcal{J}_n.$$

Let us show that $c_{T,n} \in \mathbb{A}$ for all $T \in \mathrm{Conf}_0$. Suppose not, and let T_1 be a maximal element of $\{T \in \mathrm{Conf}_0 \mid \sharp T < n \text{ and } c_{T,n} \notin \mathbb{A}\}$. Then similar to the proof of (i), the coefficient of g at $T_1 \equiv c_{T_1,n} \bmod \mathbb{A} \not\equiv 0 \bmod \mathbb{A}$. This is a contradiction. Therefore, $c_{T,n} \in \mathbb{A}$ for all T and hence, $g \in \mathfrak{G}_{\mathbb{A},\mathrm{finite}} \bmod \mathcal{J}_n$.

(iii) Applying (7.4.2) to the relation $(*)$ in the proof of (ii), one gets

$$a_S = \sum_{\substack{T \in \mathrm{Conf}_0 \\ \sharp T < n}} A(S,T) \cdot c_{T,n}$$

for $\sharp S < n$. Here the right hand side belongs to $\mathbb{A}$ due to the proof of (ii). On the other hand, the left hand side ($= a_S$) does not depend on n. Hence, by varying $n \in \mathbb{Z}_{\geq 0}$, one has proven that $a_S \in \mathbb{A}$ for all $S \in \mathrm{Conf}_0$. □

§9.3. Additive characters on $\mathfrak{G}_{\mathbb{A}}$

Definition. An *additive character* on $\mathfrak{G}_{\mathbb{A}}$ is an additive homomorphism

(9.3.1) $$\mathcal{X} : \mathfrak{G}_{\mathbb{A}} \to \mathbb{A},$$

which is continuous with respect to the adic topology on $\mathfrak{G}_{\mathbb{A}}$ such that

$$\mathcal{X}(g^a) = \mathcal{X}(g) \cdot a$$

for all $g \in \mathfrak{G}_{\mathbb{A}}$ and $a \in \mathbb{A}$. The continuity of $\mathcal{X}$ is equivalent to the statement that there exists $n \geq 0$ such that $\mathcal{X}(\exp(\varphi(S))) = \mathcal{X}(1) = 0$ for $S \in \mathcal{J}_n \cap \mathrm{Conf}_0$. Hence it is equivalent to $\sharp\{S \in \mathrm{Conf}_0 \mid \mathcal{X}(\exp(\varphi(S))) \neq 0\} < \infty$.

The set of all additive characters will be denoted by

(9.3.2) $$\mathrm{Hom}_{\mathbb{A}}(\mathfrak{G}_{\mathbb{A}}, \mathbb{A}).$$

Assertion. 1. *For any fixed $U \in \mathrm{Conf}_0$, the correspondence*

(9.3.3) $$\mathcal{X}_U : \mathcal{A}(S) \in \mathfrak{G}_{\mathbb{Z},\mathrm{finite}} \mapsto A(U,S) \in \mathbb{Z}$$

extends uniquely to an additive $\mathbb{A}$-character on $\mathfrak{G}_{\mathbb{A}}$, denoted by $\mathcal{X}_U$. Then

(9.3.4) $$\mathcal{X}_U(\exp(\varphi(S))) = \delta(U,S) \quad \textit{for } U, S \in \mathrm{Conf}_0.$$

2. *There is a natural isomorphism*

$$\text{(9.3.5)} \qquad \begin{aligned} \mathrm{Hom}_{\mathbb{A}}(\mathfrak{G}_{\mathbb{A}}, \mathbb{A}) \simeq \bigotimes_{U \in \mathrm{Conf}_0} \mathbb{A} \cdot \mathcal{X}_U, \\ \mathcal{X} \mapsto \textstyle\sum_{U \in \mathrm{Conf}_0} \mathcal{X}_U(\exp(\varphi(S))) \cdot \mathcal{X}_U. \end{aligned}$$

3. *If* $\mathbb{Q} \subset \mathbb{A}$, *then for any* $\mathcal{M} \in \mathcal{L}_{\mathbb{A}}$ *and* $U \in \mathrm{Conf}_0$ *one has*

$$\text{(9.3.6)} \qquad \mathcal{X}_U(\exp(\mathcal{M})) = \partial_U \mathcal{M}.$$

Proof. 1. First we note that $A(U, S)$ for fixed $U \in \mathrm{Conf}_0$ is additive in S (5.1.8), so that $\mathcal{X}_U$ naturally extends to an additive homomorphism on $\mathfrak{G}_{\mathbb{A},\text{finite}}$. For continuity (i.e. the finiteness of S with $\mathcal{X}_U(\exp(\varphi(S))) \neq 0$), it is enough to show (9.3.4). Recalling (8.2.4) and (7.4.2), this proceeds as:

$$\begin{aligned} \mathcal{X}_U(\exp(\varphi(S))) &= \mathcal{X}_U\Big(\exp\Big(\sum_{T \in \mathrm{Conf}_0} \mathcal{M}(T)(-1)^{\sharp T - \sharp S} K(T, S)\Big)\Big) \\ &= \sum_{T \in \mathrm{Conf}_0} \mathcal{X}_U(\exp(\mathcal{M}(T))) \cdot (-1)^{\sharp T - \sharp S} K(T, S) \\ &= \sum_{T \in \mathrm{Conf}_0} \mathcal{X}_U \mathcal{A}(T) \cdot (-1)^{\sharp T - \sharp S} K(T, S) \\ &= \sum_{T \in \mathrm{Conf}_0} A(U, T) \cdot (-1)^{\sharp T - \sharp S} K(T, S) = \delta(U, S). \end{aligned}$$

2. The continuity of $\mathcal{X}_U$ implies the sum in the target space is finite.
3. Both sides of (9.3.6) take the same values for the basis $(\varphi(S))_{S \in \mathrm{Conf}_0}$. □

§9.4. Equal division points of $\mathfrak{G}_{\mathbb{Z},\text{finite}}$

Recalling $\langle\mathrm{Conf}\rangle \simeq \mathfrak{G}_{\mathbb{Z},\text{finite}}$ (9.2.2), we regard $\langle\mathrm{Conf}\rangle$ as a "lattice" in $\mathfrak{G}_{\mathbb{R},\text{finite}}$. In the positive rational cone $\mathfrak{G}_{\mathbb{Q},\text{finite}} \cap \prod_{S \in \mathrm{Conf}_0} \mathcal{A}(S)^{\mathbb{R}_{\geq 0}}$, we consider a particular point, which we call the *equal division point* for $S \in \mathrm{Conf}$:

$$\text{(9.4.1)} \qquad \mathcal{A}(S)^{1/\sharp S}.$$

Here, the exponent $1/\sharp S$ is chosen so that we get the normalization

$$\text{(9.4.2)} \qquad \mathcal{X}_{pt}(\mathcal{A}(S)^{1/\sharp S}) = 1.$$

The set of all equal division points is denoted by

$$\text{(9.4.3)} \qquad \mathrm{EDP} := \{\mathcal{A}(S)^{1/\sharp S} \mid S \in \mathrm{Conf}\}.$$

The definition (9.4.1) is inspired by the free energy of Helmholtz in statistical mechanics. Instead of treating equal division points in the form (9.4.1) in $\mathfrak{G}_{\mathbb{R}}$, we shall treat their logarithms in $\mathcal{L}_{\mathbb{R}}$ in the next sections.

§9.5. A digression on $\mathcal{L}_{\mathbb{A}}$ with $\mathbb{Q} \not\subset \mathbb{A}$

We have determined the generators of $\mathfrak{G}_{\mathbb{A},\mathrm{finite}}$ and $\mathfrak{G}_{\mathbb{A}}$ without assuming $\mathbb{Q} \subset \mathbb{A}$ but assuming only $\mathbb{Z}$-torsion freeness of $\mathbb{A}$. The following assertion seems to suggest that the Lie-like elements behave differently from the group-like elements. However, we do not pursue this subject any further in the present paper.

Assertion. *Let $\mathbb{A}$ be a commutative algebra with unit. If there exists a prime number p such that $\mathbb{A}$ is p-torsion free and $1/p \notin \mathbb{A}$, then $\mathcal{L}_{\mathbb{A}}$ is divisible by p (i.e. $\mathcal{L}_{\mathbb{A}} = p\mathcal{L}_{\mathbb{A}}$). In particular, if $\mathbb{A}$ is noetherian, then $\mathcal{L}_{\mathbb{A}} = \{0\}$.*

Sketch of proof. Consider an element $\mathcal{M} = \sum_{U \in \mathrm{Conf}} U \cdot M_U \in \mathcal{L}_{\mathbb{A}}$. As an element of $\mathcal{L}_{\tilde{\mathbb{A}}}$, it can be expressed as $\sum_{S \in \mathrm{Conf}_0} \varphi(S) \cdot a_S$ where $a_S = \partial_S \mathcal{M} = M_S \in \mathbb{A}$ for $S \in \mathrm{Conf}_0$. Recall the expression (8.3.3) for $\varphi(U, S)$ ($U \in \mathrm{Conf}$) and the remark following it. We see that M_U is expressed as

$$\sum_{\substack{U=U_1^{k_1}\sqcup\cdots\sqcup U_m^{k_m}\\ V\in\mathrm{Conf},\, W\in\mathrm{Conf}_0\\ S\in\mathrm{Conf}_0}} \frac{(-1)^{|\underline{k}|-1+|W|+|S|}(|\underline{k}|-1)!}{k_1!\cdots k_m!} \binom{\overbrace{U_1,\ldots,U_1}^{k_1},\ldots,\overbrace{U_m,\ldots,U_m}^{k_m}}{V} \times A(V,W)K(W,S)a_S.$$

Apply this formula for $U = T^p$ for a fixed $T \in \mathrm{Conf}_0$. The summation index set is $\{(k_1, k_2, \ldots) \in (\mathbb{Z}_{\geq 0})^{\mathbb{Z}_{\geq 1}} \mid p = \sum_{i \geq 1} i \cdot k_i\}$, as explained in 3.6 Example. Except for the case $k_1 = p$ and $k_i = 0$ ($i > 1$), the denominator $k_1! \cdots k_m!$ is a product of prime numbers smaller than p. The coefficient $\binom{\overbrace{U_1,\ldots U_1}^{k_1},\ldots,\overbrace{U_m,\ldots U_m}^{k_m}}{V}$ for this case (i.e. $k_1 = p$, $k_i = 0$ ($i > 1$) and for $V = [\mathbb{V}]$ is equal to the cardinality of the set $\{(\mathbb{U}_1, \ldots, \mathbb{U}_p) \mid \mathbb{U}_i$ is a subgraph of $\mathbb{V}$ such that $[\mathbb{U}_i] = T$ and $\bigcup_{i=1}^p \mathbb{U}_i = \mathbb{V}\}$. Since the cyclic permutation of $\mathbb{U}_1, \ldots, \mathbb{U}_p$ acts on that set, and the action has no fixed points except for $V = T$, we see that the covering coefficient is divisible by p except for the case $V = T \in \mathrm{Conf}_0$. In that case $\sum_{W \in \mathrm{Conf}_0} (-1)^{|W|+|S|} A(T, W)K(W, S) = \delta(T, S)$. Therefore $((-1)^p/p)a_T \equiv 0 \bmod \mathbb{A}_{\mathrm{loc}}$ where $\mathbb{A}_{\mathrm{loc}}$ is the localization of the algebra $\mathbb{A}$ with respect to the prime numbers smaller than p. Hence $a_T \in p\mathbb{A}_{\mathrm{loc}} \cap \mathbb{A} = p\mathbb{A}$. □

§10. Accumulation set of logarithmic equal division points

We consider the space of Lie-like elements $\mathcal{L}_{\mathbb{R}}$ over the real number field $\mathbb{R}$ which is equipped with the classical topology. The set in $\mathcal{L}_{\mathbb{R}}$ of accumulation points of the logarithm of EDP (9.4), denoted by $\Omega := \overline{\log(\mathrm{EDP})}$, becomes a compact convex set. We decompose Ω into a join of the finite (absolutely convergent) part $\Omega_{\rm abs} := \overline{\log(\rm EDP)}_{\rm abs}$ and the infinite part $\Omega_\infty := \overline{\log(\rm EDP)}_\infty$.

§10.1. The classical topology on $\mathcal{L}_{\mathbb{R}}$

We equip the $\mathbb{R}$-vector space

$$\mathcal{L}_{\mathbb{R}} \ = \ \varprojlim_n \mathcal{L}_{\mathbb{R}}/\overline{\mathcal{J}_n} \cap \mathcal{L}_{\mathbb{R}} \tag{10.1.1}$$

with the *classical topology* defined by the projective limit of the classical topology on the finite-dimensional quotient $\mathbb{R}$-vector spaces. Since the quotient spaces are

$$\mathcal{L}_{\mathbb{R}}/\overline{\mathcal{J}_n} \cap \mathcal{L}_{\mathbb{R}} \simeq \bigoplus_{S\in \mathrm{Conf}_0,\, \sharp S<n} \mathbb{R}\cdot\varphi(S) \simeq \mathcal{L}_{\mathbb{R},\mathrm{finite}}/\mathcal{J}_n \cap \mathcal{L}_{\mathbb{R},\mathrm{finite}},$$

we see that (1) $\mathcal{L}_{\mathbb{R}}$ is homeomorphic to the direct product $\prod_{S\in\mathrm{Conf}_0}\mathbb{R}\cdot\varphi(S)$ (recall (8.2.5)), and (2) $\mathcal{L}_{\mathbb{R},\mathrm{finite}} \simeq \oplus_{S\in\mathrm{Conf}_0}\mathbb{R}\cdot\varphi(S)$ is dense in $\mathcal{L}_{\mathbb{R}}$ with respect to the classical topology. That is, *the classical topology on* $\mathcal{L}_{\mathbb{R}}$ *is the topology of coefficientwise convergence with respect to the basis* $\{\varphi(S)\}_{S\in\mathrm{Conf}_0}$. It is weaker than the adic topology.

Similarly, we equip $\mathbb{R}[\![\mathrm{Conf}]\!]$ with the classical topology defined by

$$\mathbb{R}[\![\mathrm{Conf}]\!] = \varprojlim_n \mathbb{R}\cdot\mathrm{Conf}\,/\mathcal{J}_n = \prod_{S\in\mathrm{Conf}} \mathbb{R}\cdot S. \tag{10.1.2}$$

So, the classical topology on $\mathbb{R}[\![\mathrm{Conf}]\!]$ is the same as the topology of coefficientwise convergence with respect to the basis $\{S\}_{S\in\mathrm{Conf}}$. The relation (ii) below between the two topologies (10.1.1) and (10.1.2) is a consequence of (8.3.3).

Assertion. (i) *The product and coproduct on* $\mathbb{R}[\![\mathrm{Conf}]\!]$ *are continuous with respect to the classical topology.*

(ii) *The classical topology on* $\mathcal{L}_{\mathbb{R}}$ *is homeomorphic to the topology induced from that on* $\mathbb{R}[\![\mathrm{Conf}]\!]$.

(iii) *Let us equip* $\mathfrak{G}_{\mathbb{R}}$ *with the classical topology induced from that on* $\mathbb{R}[\![\mathrm{Conf}]\!]$. *Then* $\exp: \mathcal{L}_{\mathbb{R}} \to \mathfrak{G}_{\mathbb{R}}$ *is a homeomorphism.*

Proof. (i) The product and coproduct are continuous with respect to the adic topology (cf. 3.2 and 4.2), which implies the statement.

(ii) For a sequence in $\mathcal{L}_{\mathbb{R}}$, we need to show the equivalence of convergence in $\mathcal{L}_{\mathbb{R}}$ and in $\mathbb{R}[\![\mathrm{Conf}]\!]$. This is true due to (8.3.3).

(iii) The maps exp and log are bijective (cf. 9.2 Assertion) and homeomorphic with respect to the adic topology, which implies the statement. □

§10.2. Absolutely convergent sums in $\mathcal{L}_{\mathbb{R}}$

Recall the problem posed in 8.4: find a subspace of $\mathcal{L}_{\mathbb{A}}$ containing $\mathcal{L}_{\mathbb{A},\mathrm{finite}}$ which is complementary to the subspace at infinity $\mathcal{L}_{\mathbb{A},\infty}$ (8.4.2). In the present section, we answer this problem for the case $\mathbb{A} = \mathbb{R}$ by introducing a sufficiently large submodule $\mathcal{L}_{\mathbb{R},\mathrm{abs}}$, which contains $\mathcal{L}_{\mathbb{R},\mathrm{finite}}$ but does not intersect $\mathcal{L}_{\mathbb{R},\infty}$ so that we obtain a splitting submodule $\mathcal{L}_{\mathbb{R},\mathrm{abs}} \oplus \mathcal{L}_{\mathbb{R},\infty}$ of $\mathcal{L}_{\mathbb{R}}$.

Definition. We say a formal sum $\sum_{T\in\mathrm{Conf}_0} a_T\mathcal{M}(T) \in \prod_{T\in\mathrm{Conf}_0}\mathbb{R}\cdot\mathcal{M}(T)$ is *absolutely convergent* if, for any $S \in \mathrm{Conf}$, the sum $\sum_{T\in\mathrm{Conf}_0} a_T M(S,T)$ of its coefficients at S is absolutely convergent, i.e. $\sum_{T\in\mathrm{Conf}_0} |a_T|M(S,T) < \infty$ for all $S \in \mathrm{Conf}$. Then any series $\sum_{i=1}^{\infty} a_{T_i}\mathcal{M}(T_i)$ defined by any linear ordering $T_1 < T_2 < \cdots$ of the index set Conf_0 converges in $\mathcal{L}_{\mathbb{R}}$ to the same element with respect to the classical topology. We denote the limit by $\sum_{T\in\mathrm{Conf}_0}^{\mathrm{abs}} a_T\mathcal{M}(T)$. Define the space of absolutely convergent elements:

$$\text{(10.2.1)} \qquad \mathcal{L}_{\mathbb{R},abs} := \{\text{absolutely convergent sums } \textstyle\sum_{T\in\mathrm{Conf}_0}^{\mathrm{abs}} a_T\mathcal{M}(T)\}.$$

By definition, $\mathcal{L}_{\mathbb{R},\mathrm{abs}}$ is an $\mathbb{R}$-linear subspace of $\mathcal{L}_{\mathbb{R}}$ such that $\mathcal{L}_{\mathbb{R},\mathrm{abs}} \cap \mathcal{L}_{\mathbb{R},\infty} = \{0\}$ and $\mathcal{L}_{\mathbb{R},\mathrm{abs}} \supset \mathcal{L}_{\mathbb{R},\mathrm{finite}}$. Hence, the restriction $\overline{K}|_{\mathcal{L}_{\mathbb{R},\mathrm{abs}}}$ of the kabi map (8.4.1) is injective. We give a criterion for the absolute convergence, which guarantees that $\mathcal{L}_{\mathbb{R},\mathrm{abs}}$ will be large enough for our purpose (10.4.3).

Assertion. *A formal sum $\sum_{T\in\mathrm{Conf}_0} a_T\mathcal{M}(T)$ is absolutely convergent if and only if the sum $\sum_{T\in\mathrm{Conf}_0} |a_T|\sharp T$ is convergent. $\mathcal{L}_{\mathbb{R},\mathrm{abs}}$ is a Banach space with respect to the norm $|\sum_{T\in\mathrm{Conf}_0}^{\mathrm{abs}} a_T\mathcal{M}(T)| := \sum_{T\in\mathrm{Conf}_0} |a_T|\sharp T$.*

Proof. The coefficient of $\mathcal{M}(T)$ at [one-point graph] is equal to $\sharp T$. So absolute convergence implies the convergence of $\sum_{T\in\mathrm{Conf}_0} |a_T|\sharp T$.

Conversely, under this assumption, let us show the absolute convergence of the sum $\sum_{T\in\mathrm{Conf}_0} a_T M(S,T)$ for any $S \in \mathrm{Conf}$. We prove this by induction on $n(S)$, the number of connected components of S. If S is connected (i.e. $n(S) = 1$), then $A(S,T) = M(S,T)$ and by the use of (5.2.1), we have

$$\sum_{T\in\mathrm{Conf}_0} |a_T|M(S,T) \le \Big(\sum_{T\in\mathrm{Conf}_0} |a_T|\sharp T\Big)\frac{(q-1)^{\sharp S-1}}{\sharp\,\mathrm{Aut}(S)},$$

which converges absolutely. If S is not connected, decompose it into connected components as $S = \prod_{i=1}^{m} S_i$ and apply (6.2.2). Since $\binom{S_1,\ldots,S_m}{S'} \neq 0$ implies either $n(S') < n(S)$ or $S' = S$, $M(S,T)$ is expressed as a finite linear combination of $M(S',T)$ for $n(S') < n(S)$ (independent of T). We are now done by the induction hypothesis. □

§10.3. Accumulation set $\Omega := \overline{\log(\mathrm{EDP})}$

Recall that an equal division point in $\mathfrak{G}_{\mathbb{Q}}$ (9.4.1) is, by definition, an element of the form $\mathcal{A}(S)^{1/\sharp S}$ for a $S \in \mathrm{Conf}_+$. Let us consider the set in $\mathcal{L}_{\mathbb{Q}}$ of their logarithms (by use of the homeomorphism in 10.1 Assertion (iii)):

$$\log(\mathrm{EDP}) := \{\mathcal{M}(T)/\sharp T \mid T \in \mathrm{Conf}_+\} \tag{10.3.1}$$

and its closure $\Omega := \overline{\log(\mathrm{EDP})}$ in $\mathcal{L}_{\mathbb{R}}$ with respect to the classical topology. So, any element $\omega \in \Omega$ has an expression

$$\omega := \lim_{n\to\infty}{}^{\mathrm{cl}} \frac{\mathcal{M}(T_n)}{\sharp T_n} \tag{10.3.2}$$

for a sequence $\{T_n\}_{n\in\mathbb{Z}_{>0}}$ in Conf_+, where we denote by $\lim^{\mathrm{cl}}$ the limit with respect to the classical topology. Recalling that the topology on $\mathcal{L}_{\mathbb{R}}$ is defined by coefficientwise convergence with respect to the basis $\{\varphi(S)\}_{S\in\mathrm{Conf}_0}$ and using (8.2.3), one has $\omega = \sum_{S\in\mathrm{Conf}_0} \varphi(S)\cdot a_S$ where $a_S = \lim_{n\to\infty} A(S,T_n)/\sharp T_n$.

Assertion. 1. *The set $\Omega = \overline{\log(\mathrm{EDP})}$ is compact and convex.*

2. *Expand any element $\omega \in \Omega$ as $\sum_{S\in\mathrm{Conf}_0}\varphi(S)\cdot a_S$. Then*

(i) $0 \le a_S \le (q-1)^{\sharp S-1}/\sharp\,\mathrm{Aut}(S)$ *for* $S \in \mathrm{Conf}_0$,

(ii) $(q-1)^{\sharp S-\sharp S'} a_{S'} \ge a_S$ *for* $S' \le S$. *In particular, if* $a_S \ne 0$ *then* $a_{S'} \ne 0$.

Proof. 1. Compactness: it is enough to show that the range of coefficients a_S for $\omega \in \log(\mathrm{EDP})$ is bounded for each $S \in \mathrm{Conf}_0$. Recalling the expansion formula (8.2.3), this is equivalent to the statement that $\{A(S,T)/\sharp T \mid T \in \mathrm{Conf}_0\}$ is bounded for any $S \in \mathrm{Conf}_0$. Applying the inequality (5.2.2), we have

$$0 \;\le\; A(S,T)/\sharp T \;\le\; (q-1)^{\sharp S-1}/\sharp\,\mathrm{Aut}(S),$$

which clearly gives a universal bound for $A(S,T)/\sharp T$ independent of T.

Convexity: for any T, T' ($\ne [\emptyset]$) and $r \in \mathbb{Q}$ with $0 < r < 1$, one can find positive integers p and q such that for $T'' := T^p \cdot T'^q$ one has

$$\begin{aligned}\mathcal{M}(T'')/\sharp T'' &= (p\cdot\mathcal{M}(T) + q\cdot\mathcal{M}(T'))/(p\cdot\sharp T + q\cdot\sharp T')\\ &= r\cdot\mathcal{M}(T)/\sharp T + (1-r)\cdot\mathcal{M}(T')/\sharp T'.\end{aligned}$$

2. (i) This is already shown in 1.

(ii) If $S' \le S$ and $S \in \mathrm{Conf}_0$, then for any $T \in \mathrm{Conf}$ one has the inequality $(q-1)^{\sharp S-\sharp S'} A(S',T) \ge A(S,T)$. (This can be easily seen by fixing representatives of S and S' as in the proof of (5.2.2).) Therefore $(q-1)^{\sharp S-\sharp S'} a_{S'} \ge a_S$. □

Remark. The condition (9.4.2) on EDP implies $a_{pt} = 1$ for any element $\omega \in \Omega = \overline{\log(\mathrm{EDP})}$. In particular, this implies $0 \notin \Omega$.

§10.4. Join decomposition $\Omega = \overline{\log(\mathrm{EDP})}_{\mathrm{abs}} * \overline{\log(\mathrm{EDP})}_{\infty}$

We show that $\overline{\log(\mathrm{EDP})}$ is embedded in $\mathcal{L}_{\mathbb{R},\mathrm{abs}} \oplus \mathcal{L}_{\mathbb{R},\infty}$, and, accordingly, decompose $\overline{\log(\mathrm{EDP})}$ into the join of a finite part and an infinite part, where the finite part is an infinite simplex with vertex set $\{\mathcal{M}(T)/\sharp T\}_{T\in\mathrm{Conf}_0}$.

Definition. Define the *finite part* and the *infinite part* of $\overline{\log(\mathrm{EDP})}$ by

$$\overline{\log(\mathrm{EDP})}_{\mathrm{abs}} := \overline{\log(\mathrm{EDP})} \cap \mathcal{L}_{\mathbb{R},\mathrm{abs}}, \tag{10.4.1}$$

$$\overline{\log(\mathrm{EDP})}_{\infty} := \overline{\log(\mathrm{EDP})} \cap \mathcal{L}_{\mathbb{R},\infty}. \tag{10.4.2}$$

Lemma. 1. $\overline{\log(\mathrm{EDP})}$ *is the join of the finite part and the infinite part:*

$$\overline{\log(\mathrm{EDP})} = \overline{\log(\mathrm{EDP})}_{\mathrm{abs}} * \overline{\log(\mathrm{EDP})}_{\infty}. \tag{10.4.3}$$

Here, the join of subsets A and B in real vector spaces V and W is defined by

$$A * B := \{\lambda p + (1-\lambda) q \in V \oplus W \mid p \in A,\ q \in B,\ \lambda \in [0,1]\}.$$

2. *The finite part is the infinite simplex of the vertex set $\{\mathcal{M}(S)/\sharp S\}_{S\in\mathrm{Conf}_0}$:*

$$\overline{\log(\mathrm{EDP})}_{\mathrm{abs}} = \Big\{ \sum_{S\in\mathrm{Conf}_0}^{\mathrm{abs}} \mu_S \frac{\mathcal{M}(S)}{\sharp S} \ \Big|\ \mu_S \in \mathbb{R}_{\geq 0} \text{ and } \sum_{S\in\mathrm{Conf}_0} \mu_S = 1 \Big\}.$$

Proof. We prove 1 and 2 simultaneously in two steps A and B. We show only the inclusion LHS $\subset$ RHS since the opposite inclusion is trivial due to the compactness and convexity of $\overline{\log(\mathrm{EDP})}$ (10.3 Assertion 1).

A. *Finite part.* Let $\omega \in \overline{\log(\mathrm{EDP})}$ have expression (10.3.2). For $S \in \mathrm{Conf}_0$, recall that $\delta(S, T_n)$ is the number of connected components of T_n isomorphic to S. Let us show that

$$\mu_S := \sharp S \lim_{n\to\infty} \frac{\delta(S,T_n)}{\sharp T_n} \tag{10.4.4}$$

is a finite real number such that

$$0 \leq \sum_{S\in\mathrm{Conf}_0} \mu_S \leq 1. \tag{10.4.5}$$

Note that the kabi map $\overline{K}$ (8.4.1) is also continuous with respect to the classical topology. So, it commutes with the classical limiting process $\lim^{\mathrm{cl}}_{n\to\infty} \mathcal{M}(T_n)/\sharp T_n$. Recalling the kabi inversion formula (7.3.1), we calculate

$$\overline{K}(\omega) = \overline{K}\Big(\lim^{\mathrm{cl}}_{n\to\infty} \frac{\mathcal{M}(T_n)}{\sharp T_n} \Big) = \lim^{\mathrm{cl}}_{n\to\infty} \frac{\overline{K}(\mathcal{M}(T_n))}{\sharp T_n} = \lim_{n\to\infty} \sum_{S\in\mathrm{Conf}_0} \frac{\delta(S,T_n)}{\sharp T_n} \mathcal{M}(S).$$

Here, the convergence on the RHS is the coefficientwise convergence with respect to the basis $\mathcal{M}(S)$ for $S \in \mathrm{Conf}_0$. This implies the convergence of (10.4.4).

Let C be any finite subset of Conf_0. For any $n \in \mathbb{Z}_{\geq 0}$, one has

$$\sum_{T \in C} \delta(T, T_n) \cdot \sharp T \quad \leq \quad \sharp T_n$$

since the LHS is equal to the number of vertices of the union of the connected components of T_n which are isomorphic to an element of C. Dividing both sides by $\sharp T_n$ and taking the limit $n \to \infty$, one has (10.4.5).

Define the *finite part of* ω to be the absolutely convergent sum

$$\omega_{\text{finite}} := \sum_{S \in \mathrm{Conf}_0}^{\text{abs}} \mu_S \frac{\mathcal{M}(S)}{\sharp S} \tag{10.4.6}$$

(apply 10.2 Assertion to (10.4.5)). We remark that the coefficients μ_S are uniquely determined from ω and are independent of the sequence $\{T_n\}_{n \in \mathbb{Z}_{\geq 0}}$, due to the formula

$$\overline{K}(\omega) = \sum_{S \in \mathrm{Conf}_0} \mu_S \frac{\mathcal{M}(S)}{\sharp S}. \tag{10.4.7}$$

B. *Infinite part.* Put $\mu_\infty := 1 - \sum_{S \in \mathrm{Conf}_0} \mu_S$. Let us show that

(i) *if* $\mu_\infty = 0$*, then* $\omega = \omega_{\text{finite}}$, and (ii) *if* $\mu_\infty > 0$*, then there exists a unique element* $\omega_\infty \in \mathcal{L}_{\mathbb{R},\infty}$ *such that* $\omega = \mu_\infty \omega_\infty + \omega_{\text{finite}}$.

For any $S \in \mathrm{Conf}_0$, let us denote by $T_n(S)$ the isomorphism class of the union of the connected components of T_n isomorphic to $S \in \mathrm{Conf}_0$. Thus, $\sharp T_n(S) = \delta(S, T_n)\sharp S$ and $\sharp T_n(S)/\sharp T_n \to \mu_S$ as $n \to \infty$. For any finite subset C of Conf_0, put $T_n^*(C^c) := T_n \setminus \bigcup_{S \in C} T_n(S)$ so that

$$\frac{\mathcal{M}(T_n)}{\sharp T_n} = \frac{\mathcal{M}(T_n^*(C^c))}{\sharp T_n} + \sum_{S \in C} \frac{\delta(S, T_n)\sharp S}{\sharp T_n} \cdot \frac{\mathcal{M}(S)}{\sharp S}. \tag{$*$}$$

For the given C and for $\varepsilon > 0$, there exists $n(C, \varepsilon)$ such that

$$\sum_{S \in C} |\mu_S - \sharp T_n(S)/\sharp T_n| < \varepsilon \tag{a}$$

for $n \geq n(C, \varepsilon)$. This implies

$$|\mu_\infty - \sharp T_n^*(C^c)/\sharp T_n| < \varepsilon + \sum_{S \in \mathrm{Conf}_0 \setminus C} \mu_S.$$

Let $\{\varepsilon_m\}_{m\in\mathbb{Z}_{\geq 0}}$ be any sequence of positive real numbers with $\varepsilon_m \downarrow 0$. Choose an increasing sequence $\{C_m\}_{m\in\mathbb{Z}_{\geq 0}}$ of finite subsets of Conf_0 satisfying

$$\text{(b)} \qquad \bigcup_{m\in\mathbb{Z}_{\geq 0}} C_m = \mathrm{Conf}_0 \quad \text{and} \quad \sum_{S\in \mathrm{Conf}_0 \setminus C_m} \mu_S < \varepsilon_m.$$

Put $n(m) := n(C_m, \varepsilon_m)$. Then, by definition of μ_∞ and by (a) and (b), one has

$$\text{(c)} \qquad |\mu_\infty - \sharp T^*_{n(m)}(C^c_m)/\sharp T_{n(m)}| \ < \ 2\varepsilon_m.$$

Replacing n and C in $(*)$ by $n(m)$ and C_m, respectively, we obtain a sequence of equalities indexed by $m \in \mathbb{Z}_{\geq 0}$. Let us prove:

(i) *the second term of the RHS of* $(*)$ *absolutely converges to* ω_{finite}.

(ii) *If* $\mu_\infty = 0$, *then the first term of the RHS of* $(*)$ *converges to* 0.

(iii) *If* $\mu_\infty \neq 0$, *then* $T^*_m := T^*_{n(m)}(C^c_m) \neq \emptyset$ *for large* m *and* $\mathcal{M}(T^*_m)/\sharp T^*_m$ *converges to an element* $\omega_\infty \in \overline{\log(\mathrm{EDP})} \cap \mathcal{L}_{\mathbb{R},\infty}$.

Proof of (i). For $m \in \mathbb{Z}_{\geq 0}$, the difference of ω_{finite} and the second term of the RHS of $(*)$ is $\sum_{S\in\mathrm{Conf}_0} c_S \mathcal{M}(S)/\sharp TS$ where $c_S := \mu_S - \delta(S, T_{n(m)})\sharp S/\sharp T_{n(m)}$ for $S \in C_m$ and $c_S := \mu_S$ for $S \in \mathrm{Conf}_0 \setminus C_m$. Therefore, using (a) and the inequality of (b), one sees that the sum $\sum_{S\in\mathrm{Conf}_0} |c_S|$ is bounded by $2\varepsilon_m$. Then, due to a criterion in 10.2 Assertion, the difference tends to 0 absolutely as $m \uparrow \infty$. □

Proof of (ii). Recall (c) $|\sharp T^*_{n(m)}(C^c_m)/\sharp T_{n(m)}| < 2\varepsilon_m$. The first term of the RHS $(*)$ is given by

$$\frac{\mathcal{M}(T^*_{n(m)}(C_m))}{\sharp T_{n(m)}} = \sum_{S\in\mathrm{Conf}_0} \varphi(S) \frac{A(S, T^*_{n(m)}(C_m))}{\sharp T_{n(m)}},$$

where the coefficient of $\varphi(S)$ is either 0 if $T^*_{n(m)}(C_m) = \emptyset$, or else is equal to

$$\frac{\sharp T^*_{n(m)}(C_m)}{\sharp T_{n(m)}} \frac{A(S, T^*_{n(m)}(C_m))}{\sharp T_{n(m)}(C_m)},$$

which is bounded by $2\varepsilon_m q^{\sharp S-1}/\sharp\,\mathrm{Aut}(S)$. So it converges to 0 as $m \uparrow \infty$. □

Proof of (iii). The first term of the RHS of $(*)$ converges to $\omega - \omega_{\mathrm{finite}}$, since the LHS of $(*)$ and the second term of the RHS of $(*)$ converge to ω and ω_{finite}, respectively. On the other hand, due to (c), one has $\sharp T^*_{n(m)}(C^c_m)/\sharp T_{n(m)} > \mu_\infty - 2\varepsilon_m$ for sufficiently large m, and hence $T^*_{n(m)}(C^c_m) \neq \emptyset$. The first term decomposes as

$$\frac{\mathcal{M}(T^*_{n(m)}(C^c_m))}{\sharp T_{n(m)}} \ = \ \frac{T^*_{n(m)}(C^c_m)}{\sharp T_{n(m)}} \frac{\mathcal{M}(T^*_{n(m)}(C^c_m))}{\sharp T^*_{n(m)}(C^c_m)},$$

whose first factor converges to $\mu_\infty \neq 0$ due to (c). Therefore, the second factor converges to some $\omega_\infty := (\omega - \omega_{\mathrm{finite}})/\mu_\infty$, which belongs to $\overline{\log(\mathrm{EDP})}$ by definition. Since $\overline{K}(\omega) = \overline{K}(\omega_\infty)$, ω_∞ belongs to $\ker(\overline{K})$. □

These complete the proof of the lemma. □

§10.5. Extremal points in $\Omega_\infty = \overline{\log(\mathrm{EDP})}_\infty$

A point ω in a subset A in a real vector space is called an *extremal point* of A if whenever an interval I contained in A contains ω then ω is a terminal point of I.

Assertion. *Each extremal point of* $\overline{\log(\mathrm{EDP})}$ *is one of the following:*

(i) $\mathcal{M}(S)/\sharp S$ *for an element* $S \in \mathrm{Conf}_0$,
(ii) $\lim^{\mathrm{cl}}_{n\to\infty} \mathcal{M}(T_n)/\sharp T_n$ *for a sequence* $T_n \in \mathrm{Conf}_0$ *with* $\sharp T_n \to \infty$ $(n \to \infty)$.

Proof. For $\omega \in \overline{\log(\mathrm{EDP})}$, if $\mu_\infty \neq 0, 1$, then, due to Lemma 1 of §10.4, ω cannot be extremal. If $\mu_\infty = 0$, then, due to Lemma 2, the only possibility for ω to be extremal is when it is of the form $\mathcal{M}(S)/\sharp S$ for an element $S \in \mathrm{Conf}_0$. In fact, using the uniqueness of the expression (10.4.6) (see the remark following the formula), $\mathcal{M}(S)/\sharp S$ can be shown to be extremal.

Suppose $\mu_\infty = 1$. For any fixed $S \in \mathrm{Conf}_0$ and real $\varepsilon > 0$, let $T_n^+(S,\varepsilon)$ (resp. $T_n^-(S,\varepsilon)$) be the subgraph of T_n consisting of the components T such that $A(S,T)/\sharp T \geq a_S + \varepsilon$ (resp. $\leq a_S - \varepsilon$). Let us show that $\overline{\lim}_n \sharp T_n^\pm(S,\varepsilon)/\sharp T_n = 0$. If not, then there exists a subsequence $\{\widehat{n}\}$ such that $\lim_{\widehat{n}} \sharp T_{\widehat{n}}^\pm(S,\varepsilon)/\sharp T_{\widehat{n}} = \lambda > 0$. Due to the compactness of $\overline{\log(\mathrm{EDP})}$ (10.3 Assertion 1), we can choose a subsequence such that $\mathcal{M}(T_{\widehat{n}}^\pm(S,\varepsilon))/\sharp T_{\widehat{n}}^\pm(S,\varepsilon)$ and $\mathcal{M}(T_{\widehat{n}} \setminus T_{\widehat{n}}^\pm(S,\varepsilon))/\sharp(T_{\widehat{n}} \setminus T_{\widehat{n}}^\pm(S,\varepsilon))$ converge to some $\sum_{T\in\mathrm{Conf}_0} \varphi(T)\cdot b_T$ and $\sum_{T\in\mathrm{Conf}_0} \varphi(T)\cdot c_T$, respectively, so that

$$\omega = \lambda \cdot \sum_{T\in\mathrm{Conf}_0} \varphi(T)\cdot b_T + (1-\lambda)\cdot \sum_{T\in\mathrm{Conf}_0} \varphi(T)\cdot c_T.$$

In particular, the coefficient of $\varphi(S)$ has the relation $a_S = \lambda\cdot b_S + (1-\lambda)\cdot c_S$. Since $|b_S - a_S| \geq \varepsilon$, λ cannot be 1. This contradicts the extremity of ω.

For any finite subset C of Conf_0, put $T_n^*(C,\varepsilon) := T_n \setminus \bigcup_{S\in C}\big(T_n^+(S,\varepsilon) \cup T_n^-(S,\varepsilon)\big)$. Then $T_n^*(C,\varepsilon) \neq \emptyset$ for sufficiently large n, since $\lim_{n\to\infty} \sharp T_n^*(C,\varepsilon)/\sharp T_n = 1$ due to the above fact. Let $\{C_m\}_{m\in\mathbb{Z}_{\geq 0}}$ be an increasing sequence of finite subsets of Conf_0 such that $\bigcup_{m\in\mathbb{Z}_{\geq 0}} C_m = \mathrm{Conf}_0$ and let $\{\varepsilon_m\}_{m\in\mathbb{Z}_{\geq 0}}$ be a sequence of real numbers with $\varepsilon_m \downarrow 0$. For each $m \in \mathbb{Z}_{\geq 0}$, choose any connected component of $T_n^*(C_m,\varepsilon_m)$, say T_m^*, for large n, and put $\omega_m := \mathcal{M}(T_m^*)/\sharp T_m^* = \sum_{S\in\mathrm{Conf}_0} \varphi(S)\cdot a_S^{(m)}$. By definition $|a_S - a_S^{(m)}| < \varepsilon_m$ for $S \in C_m$, which implies $\omega = \lim^{\mathrm{cl}}_{m\to\infty} \omega_m$. There are two cases to consider: (i) Suppose there is a subsequence $\{\widehat{m}\}$ such that $\sharp T_{\widehat{m}}^*$ is bounded. Since $\sharp\{T \in \mathrm{Conf}_0 \mid \sharp T \leq c\}$ for any constant c is finite, there exists $T \in \mathrm{Conf}_0$ which appears in $\{T_m^*\}_m$ infinitely often. So $\omega = \mathcal{M}(T)/\sharp T$ and $\overline{K}(\omega) = \mathcal{M}(T)/\sharp T \neq 0$. (ii) Suppose $\sharp T_m^* \to \infty$. Then the formula (10.4.4) and (10.4.7) imply $\overline{K}(\omega) = 0$. □

§10.6. Function value representation of elements of $\Omega_\infty = \overline{\log(\mathrm{EDP})}_\infty$

The coefficients a_S at $S \in \mathrm{Conf}_0$ of the sequential limit $\omega = \lim^{\mathrm{cl}}_{n\to\infty} \mathcal{M}(T_n)/\sharp T_n$ (10.3.2) are usually hard to calculate. However, in certain good cases, we can represent the coefficient as a special value of a function in one variable t.

Given an expression of the form (10.3.2) of $\omega \in \overline{\log(\mathrm{EDP})}_\infty$ and an increasing sequence of integers $\{n_m\}_{n=0}^\infty$, we consider the following two formal power series in t:

$$P(t) := \sum_{m=0}^\infty \sharp T_m \cdot t^{n_m} \in \mathbb{Z}[\![t]\!], \tag{10.6.1}$$

$$P\mathcal{M}(t) := \sum_{m=0}^\infty \mathcal{M}(T_m) \cdot t^{n_m} \in \mathcal{L}_{\mathbb{Q}}[\![t]\!] = \mathcal{L}_{\mathbb{Q}[\![t]\!]}, \tag{10.6.2}$$

where, using the basis expansion (8.2.3), the series $P\mathcal{M}(t)$ can be expanded as $P\mathcal{M}(t) = \sum_{S\in\mathrm{Conf}_0} \varphi(S) PM(S,t)$, whose coefficients at $S \in \mathrm{Conf}_0$ are given by

$$PM(S,t) := \partial_S P\mathcal{M}(t) = \sum_{m=0}^\infty A(S,T_m) \cdot t^{n_m} \in \mathbb{Q}[\![t]\!]\ . \tag{10.6.3}$$

Since $T_n \in \mathrm{Conf}_+$, one has $P(t) \neq 0$ and its radius of convergence is at most 1.

Lemma. *Suppose that the series $P(t)$ has a positive radius of convergence r. Then, for any $S \in \mathrm{Conf}_0$ (cf. Remark below), we have:*

(i) *The series $PM(S,t)$ converges at least in the radius r for $P(t)$. The radius of convergence of $PM(S,t)$ coincides with r if $a_S := \lim_{m\to\infty} M(S,T_m)/\sharp T_m \neq 0$.*

(ii) *The following two limits in the LHS and RHS give the same value:*

$$\lim_{t\uparrow r} \frac{PM(S,t)}{P(t)} = \lim_{n\to\infty} \frac{M(S,T_n)}{\sharp T_n}. \tag{10.6.4}$$

Here $t \uparrow r$ means that the real variable t tends to r from below.

(iii) *The proportion $P\mathcal{M}(t)/P(t)$ for $t \uparrow r$ converges to ω (10.3.2):*

$$\omega = \lim_{t\uparrow r}{}^{\mathrm{cl}} \frac{P\mathcal{M}(t)}{P(t)} = \sum_{S\in\mathrm{Conf}_0} \varphi(S) \lim_{t\uparrow r} \frac{PM(S,t)}{P(t)}. \tag{10.6.5}$$

Proof. Before proceeding to the proof, we recall two general properties of power series:

(A) The radius of convergence of $P(t)$ is $r := 1/\limsup_{m\to\infty} \sqrt[n_m]{\sharp T_m}$ (Hadamard).

(B) Since the coefficients $\sharp T_m$ of $P(t)$ are non-negative real numbers, $P(t)$ *is an increasing positive real function on the interval* $(0,r)$ *and* $\lim_{t\uparrow r} P(t) = +\infty$.

We now turn to the proof. Due to the linear relations among $M(S,T_m)$ for $S \in \mathrm{Conf}$ (8.3.2), it is sufficient to show the lemma only for $S \in \mathrm{Conf}_0$.

(i) Let us show that $PM(S,t)$ for $S \in \mathrm{Conf}_0$ has radius of convergence r. Since $M(S,T_m) = A(S,T_m)$ (6.1 Remark 1), using (5.2.1), we have

$$\limsup_{m\to\infty} \sqrt[n_m]{M(S,T_m)} \ \le\ \limsup_{m\to\infty} \sqrt[n_m]{\sharp T_m}\, \sqrt[n_m]{q^{\sharp S-1}/\sharp\,\mathrm{Aut}(S)} = 1/r.$$

This proves the first half of (i). The other half is shown in (ii) below.

(ii) We show that the convergence of the sequence $A(S,T_m)/\sharp T_m$ to some $a_S \in \mathbb{R}$ implies the convergence of the values of the function $PM(S,t)/P(t)$ to a_S as $t \uparrow r$. The assumption implies that for any $\varepsilon > 0$, there exists $N > 0$ such that $|A(S,T_m)/\sharp T_m - a_S| \le \varepsilon$ for all $m \ge N$. Therefore,

$$\begin{aligned}\left|\frac{PM(S,t)}{P(t)} - a_S\right| &= \frac{|Q_N(t) + \sum_{m=N}^{\infty}(A(S,T_m) - a_S \cdot \sharp T_m)t^{n_m}|}{P(t)}\\ &\le \frac{|\ Q_N(t) - \varepsilon\sum_{m=0}^{N-1}\sharp T_m t^{n_m}|}{P(t)} + \varepsilon\end{aligned}$$

where $Q_N(t) := \sum_{m<N}(A(S,T_m) - a_S \cdot \sharp T_m)t^{n_m}$ is a polynomial in t. Due to statement (B) above, the first term of the last line tends to 0 as $t \uparrow r$. Hence, $|PM(S,t)/P(t) - a_S| \le 2\varepsilon$ for t sufficiently close to r. This proves (10.6.4).

If $a_S \neq 0$, then $\lim_{t\uparrow r} PM(S,t) = \infty$ since $\lim_{t\uparrow r} P(t) = \infty$. Thus, the radius of convergence of $PM(S,t)$ is less than or equal to r. This proves the last statement of (i).

(iii) We have only to recall that the classical topology on $\mathcal{L}_{\mathbb{R}}$ is the topology of coefficientwise convergence with respect to the basis $\{\varphi(S)\}_{S\in\mathrm{Conf}_0}$. □

Corollary. *If* $P(t)$ *and* $PM(S,t)$ $(S \in \mathrm{Conf}_0)$ *extend to meromorphic functions at* $t = r$*, then* $PM(S,t)/P(t)$ *is regular at* $t = r$*, and*

$$\omega = \sum_{S\in\mathrm{Conf}_0} \varphi(S)\frac{PM(S,t)}{P(t)}\bigg|_{t=r}. \tag{10.6.6}$$

Proof. We have to show that $PM(S,t)/P(t)$ becomes holomorphic at $t = r$ under the assumption. If it were not holomorphic, it would have a pole at $t = r$ and hence $\lim_{t\uparrow r} PM(S,t)/P(t)$ diverges. On the other hand, in view of (5.2.2), one has the inequality $0 \le PM(S,t) \le P(t) \cdot q^{\sharp S-1}/\sharp\,\mathrm{Aut}(S)$ for $t \in (0,r)$. Then the positivity of $P(t)$ implies the boundedness $0 \leq PM(S,t)/P(t) \leq q^{\sharp S-1}/\sharp\,\mathrm{Aut}(S)$ for $t \in (0,r)$. This is a contradiction. □

We sometimes call (10.6.6) a *residual expression of* ω, since the coefficients are given by proportions of residues of meromorphic functions.

Remark. 1. The equality (10.6.4) gives the following important replacement. Namely, the RHS, which is a sequential limit of rational numbers and is hard to determine in general, is replaced by the LHS, which is the limit value of a function in the variable t at the special point $t = r$ where r is often a real algebraic number whose defining equation is easily calculable.

2. The convergence of the sequence $\lim^{\mathrm{cl}}_{n\to\infty} M(S,T_n)/\sharp T_n$ does not imply the convergence of the series $PM(S,t)$ and $P(t)$ in a positive radius. Conversely, the convergence of the series $PM(S,t)$ and $P(t)$ in a positive radius does not imply the convergence of the sequence $\lim^{\mathrm{cl}}_{n\to\infty} M(S,T_n)/\sharp T_n$.

§11. The limit space $\Omega(\Gamma, G)$ for a finitely generated monoid

We apply the space $\mathcal{L}_{\mathbb{R},\infty}$ to the study of finitely generated monoids.

For a pair (Γ, G) of a monoid Γ and a finite generating system G with Assumptions 1, 2, we introduce (1) the limit space $\Omega(\Gamma, G)$ as a subset of $\mathcal{L}_{\mathbb{R},\infty}$, (2) another limit space $\Omega(P_{\Gamma,G})$ associated with the Poincaré series $P_{\Gamma,G}(t)$ of (Γ, G), and (3) a proper surjective map $\pi_\Omega : \Omega(\Gamma, G) \to \Omega(P_{\Gamma,G})$ (see 11.2 Theorem).

The main result of the present paper is given in 11.5 Theorem, where the sum of the elements of a fiber of π_Ω is expressed by a linear combination of the proportions of residues of the Poincaré series $P_{\Gamma,G}(t)$ and $P_{\Gamma,G}\mathcal{M}(t)$ at the poles on the circle of the radius of convergence.

§11.1. The limit space $\Omega(\Gamma, G)$

Let Γ be a monoid with left and right cancellation conditions and let G be its finite generating system with $e \notin G$. We denote by (Γ, G) the associated colored oriented Cayley graph (2.1 Example). In this and the next sections, we use $G \cup G^{-1}$ as the color set and $q := \sharp(G \cup G^{-1})$ for the definition of Conf in (2.2.1). The set of all isomorphism classes of finite subgraphs of (Γ, G) is denoted by $\langle \Gamma, G\rangle$. Put $\langle \Gamma, G\rangle_0 := \langle \Gamma, G\rangle \cap \mathrm{Conf}_0$.

The *length* of $\gamma \in \Gamma$ with respect to G is defined by

$$(11.1.1) \qquad \ell_G(\gamma) := \inf\{n \in \mathbb{Z}_{\geq 0} \mid \gamma = g_1 \cdots g_n \text{ for some } g_i \in G\ (i = 1, \ldots, n)\}.$$

We remark that in (11.1.1), we admit the expressions of γ only in positive powers of elements of G (except when G itself already contains the inverse). This means that we allow only edges whose "orientation" agrees with the orientation of the path. In particular, $\ell_G(\gamma)$ may not coincide with the distance of γ from e in the Cayley graph.[2] For $n \in \mathbb{Z}_{\geq 0}$, let us consider the "balls" of radius n of (Γ, G) defined by

$$\Gamma_n := \{\gamma \in \Gamma \mid \ell_G(\gamma) \leq n\}. \tag{11.1.2}$$

We shall denote $\dot{\Gamma}_n := \Gamma_n \setminus \Gamma_{n-1}$ for $n \in \mathbb{Z}_{\geq 0}$. If there is no confusion, we shall denote by Γ_n its isomorphism class $[\Gamma_n] \in \mathrm{Conf}_0$ also.

Definition. The *set of limit elements for* (Γ, G) is defined by

$$\Omega(\Gamma, G) := \mathcal{L}_{\mathbb{R},\infty} \cap \overline{\{\mathcal{M}(\Gamma_n)/\sharp\Gamma_n \mid n \in \mathbb{Z}_{\geq 0}\}}, \tag{11.1.3}$$

where $\overline{A}$ is the closure of a subset $A \subset \mathcal{L}_{\mathbb{R}}$ with respect to the classical topology.

Fact. *The limit space* $\Omega(\Gamma, G)$ *is non-empty if and only if* Γ *is infinite.*

Proof. Since $\{\mathcal{M}(\Gamma_n)/\sharp\Gamma_n \mid n \in \mathbb{Z}_{\geq 0}\} \subset \log(\mathrm{EDP})$ and $\overline{\log(\mathrm{EDP})}$ is compact (10.3), the sequence $\{\mathcal{M}(\Gamma_n)/\sharp\Gamma_n \mid n \in \mathbb{Z}_{\geq 0}\}$ always has accumulation points. Due to (10.4.4) and (10.4.7), an accumulation point ω belongs to $\mathcal{L}_{\mathbb{R},\infty}$, i.e. it satisfies the kabi condition $\overline{K}(\omega) = 0$, if and only if $\sharp\Gamma_n \to \infty$. □

Since $\overline{\log(\mathrm{EDP})}$ is metrizable, any element ω in $\Omega(\Gamma, G)$ can be expressed as a sequential limit. That is, there exists a subsequence $n_m \uparrow \infty$ of $n \uparrow \infty$ such that

$$\omega = \lim_{n_m \to \infty}{}^{\mathrm{cl}} \frac{\mathcal{M}(\Gamma_{n_m})}{\sharp\Gamma_{n_m}} = \sum_{S \in \langle\Gamma, G\rangle_0} \varphi(S) \lim_{n_m \to \infty} \frac{A(S, \Gamma_{n_m})}{\sharp\Gamma_{n_m}} \tag{11.1.4}$$

where the coefficient of $\varphi(S)$ is convergent for all S.

Definition. We call a finitely generated monoid (Γ, G) *simple* (resp. *finite*) *accumulating* if $\Omega(\Gamma, G)$ consists of a single (resp. finite number of) element(s).

Assumption 1. From now on until the end of the present paper, we assume that the monoid Γ is embeddable into a group. That is, there exists an injective homomorphism from Γ into a group $\hat{\Gamma}$. This is obviously satisfied if Γ is a group.

In the following Examples 1 and 2, we show that any polynomial growth group and any free group are simple accumulating. We first state some general properties of the set Γ_n, which are immediate consequences of the definition.

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[2]The length ℓ_G coincides with the distance from e for the case $G = G^{-1}$ when Γ is a group. Besides this case, there is an important class of monoids where both concepts coincide, namely, when the monoid is defined by positive homogeneous relations [S-I].

Fact. 1. *For $m, n \in \mathbb{Z}_{\geq 0}$, one has a natural surjection*

$$\Gamma_m \times \Gamma_n \to \Gamma_{m+n}, \qquad \gamma \times \delta \mapsto \gamma\delta. \tag{11.1.5}$$

2. *For any $S \in \mathrm{Conf}_0$ with $S \leq \Gamma_k$ $(k \in \mathbb{Z}_{\geq 0})$ and for any $n \in \mathbb{Z}_{\geq 0}$,*

$$\sharp\Gamma_{n-k} \leq \sharp \operatorname{Aut}(S) \cdot A(S, \Gamma_n) \leq \sharp\Gamma_n. \tag{11.1.6}$$

Proof. 1. Obvious by definition.

2. By the assumption on S, there exists a subgraph $\mathbb{S} \subset \Gamma_k$ such that $S = [\mathbb{S}]$. Note that $\operatorname{Aut}(S) \simeq \operatorname{Aut}(\mathbb{S}) = \{g \in \hat{\Gamma} \mid g\mathbb{S} = \mathbb{S}\}$ *is finite and its action is fixed point free.* Consider the map p from Γ to the set of subgraphs of (Γ, G) defined by $p(g) := g\mathbb{S}$, and define an equivalence relation $\sim$ on Γ by "$g \sim h \Leftrightarrow g\mathbb{S} = h\mathbb{S} \Leftrightarrow g^{-1}h \in \operatorname{Aut}(\mathbb{S})$". Then $A(S, \Gamma_n) \geq \sharp(\mathrm{Image}(p|_{\Gamma_{n-k}})) = \sharp(\Gamma_{n-k}/\sim) \geq \sharp\Gamma_{n-k}/\sharp \operatorname{Aut}\mathbb{S}$. This implies the first inequality.

Choose a point $x \in \mathbb{S}$. Consider the set $P := \{g \in \Gamma_n \mid gx^{-1}\mathbb{S} \subset \Gamma_n\}$. Then the map $p|_P \circ x^{-1} : P \to \mathbb{A}(S, \Gamma_n)$ is surjective and P is closed under the right multiplication of $x^{-1} \operatorname{Aut}(\mathbb{S})x$. Hence $A(S, \Gamma_n) = \sharp P/\sharp \operatorname{Aut}(\mathbb{S}) \leq \sharp\Gamma_n/\sharp \operatorname{Aut}(\mathbb{S})$. This implies the second inequality. □

Let (Γ, G) be a monoid such that $\lim_{n\to\infty} \sharp\Gamma_{n-k}/\sharp\Gamma_n = 1$ for any $k \in \mathbb{Z}_{\geq 0}$. Then, as a consequence of (11.1.6),

$$\lim_{n\to\infty} \frac{A(S, \Gamma_n)}{\sharp\Gamma_n} = \frac{1}{\sharp \operatorname{Aut}(S)}. \tag{11.1.7}$$

Example. 1. *If Γ is a group of polynomial growth, then it is simple accumulating for any generating system G and the limit element is given by*

$$\omega_{\Gamma,G} := \sum_{S \in \langle \Gamma, G\rangle_0} \frac{1}{\sharp \operatorname{Aut}(S)} \varphi(S). \tag{11.1.8}$$

Proof. For a group (Γ, G) of polynomial growth (i.e. Γ contains a finitely generated nilpotent group of finite index, Wolf and Gromov [Gr1]), there exist constants $c, d \in \mathbb{Z}_{>0}$ such that $\sharp\Gamma_n = cn^d + o(n^d)$ (Pansu [P]). □

2. *Let F_f be a free group with the generating system $G = \{g_1^{\pm 1}, \dots, g_f^{\pm 1}\}$ for $f \in \mathbb{Z}_{\geq 2}$. Then (F_f, G) is simple accumulating. The limit element is given by*

$$\omega_{F_f,G} := \sum_{k=0}^{\infty} (2f-1)^{-k} \Big(\sum_{\substack{S \in \langle \Gamma, G\rangle_0 \\ d(S) = 2k}} \varphi(S) + f^{-1} \sum_{\substack{S \in \langle \Gamma, G\rangle_0 \\ d(S) = 2k+1}} \varphi(S) \Big), \tag{11.1.9}$$

where $d(S) := \max\{d(x, y) \mid x, y \in S\}$ is the diameter of $S \in \langle \Gamma, G\rangle_0$.

Proof. The induction relation $\sharp\Gamma_{n+1} - (2f-1)\sharp\Gamma_n = 2$ with the initial condition $\sharp\Gamma_0 = 1$ implies

$$\sharp\Gamma_n = \frac{f(2f-1)^n - 1}{f-1} \quad \text{for } n \in \mathbb{Z}_{\geq 0}.$$

Then, for $n \geq [d(S)/2]$,

$$A(S,\Gamma_n) = \begin{cases} \dfrac{f(2f-1)^{n-[d(S)/2]} - 1}{f-1} & \text{if } d(S) \text{ is even,} \\ \dfrac{(2f-1)^{n-[d(S)/2]} - 1}{f-1} & \text{if } d(S) \text{ is odd,} \end{cases}$$

$$\lim_{n\to\infty} \frac{A(S,\Gamma_n)}{\sharp\Gamma_n} = \begin{cases} (2f-1)^{-[d(S)/2]} & \text{if } d(S) \text{ is even,} \\ f^{-1}(2f-1)^{-[d(S)/2]} & \text{if } d(S) \text{ is odd.} \end{cases}$$

We only have to prove the first formula. Depending on whether $d(S)$ is even or odd, S has either one or two central points. Then it is easy to see the following one-to-one correspondence: *an embedding of S in Γ_n $\leftrightarrow$ an embedding of the central point(s) of S in Γ_n such that the distance from the point to the boundary of Γ_n is at least half of the diameter,* $[d(S)/2]$. Taking this into account, we can directly deduce the formula. □

§11.2. The space $\Omega(P_{\Gamma,G})$ of opposite sequences

We introduce another accumulation set $\Omega(P)$, called *the space of opposite sequences*, associated to certain real power series $P(t)$. Under a suitable assumption on (Γ, G), we have a fibration $\pi_\Omega : \Omega(\Gamma, G) \to \Omega(P_{\Gamma,G})$ for the growth function $P_{\Gamma,G}$ of (Γ, G). We construct semigroup $\mathbb{Z}_{\geq 0}$-actions on $\Omega(\Gamma, G)$ and $\Omega(P_{\Gamma,G})$ generated by $\tilde{\tau}_\Omega$ and τ_Ω, respectively, such that π_Ω is equivariant.

We start with a general definition. Consider a power series in t,

$$P(t) = \sum_{n=0}^{\infty} \gamma_n t^n, \tag{11.2.1}$$

whose coefficients are real numbers. We assume that there exist positive real numbers u, v (depending on P) such that $u \leq \gamma_{n-1}/\gamma_n \leq v$ for all $n \in \mathbb{Z}_{\geq 1}$. This, in particular, implies that P is convergent of radius r with $u \leq r \leq v$.

Example. If the sequence $\{\gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$ is increasing and semi-multiplicative, $\gamma_{m+n} \leq \gamma_m\gamma_n$, we may choose $u = 1/\gamma_1$ and $v = 1$. For example, let $\gamma_n := \sharp\Gamma_n$ ($n \in \mathbb{Z}_{\geq 0}$) in the setting of 11.1; then (11.1.5) implies semi-multiplicativity.

Associated to P, consider a sequence $\{X_n(P)\}_{n\in\mathbb{Z}_{\geq 0}}$ of polynomials

$$X_n(P) := \sum_{k=0}^{n} \frac{\gamma_{n-k}}{\gamma_n} s^k, \quad n = 0, 1, \ldots, \tag{11.2.2}$$

in the space $\mathbb{R}[\![s]\!]$ of formal power series, where $\mathbb{R}[\![s]\!]$ is equipped with the formal classical topology, i.e. the product topology of convergence of every coefficient in the classical topology. Since each coefficient of $X_n(P)$ is bounded, i.e. $u^k \le \gamma_{n-k}/\gamma_n \le v^k$, the sequence accumulates to a non-empty compact set:

$$\Omega(P) := \text{the set of accumulation points of the sequence (11.2.2).} \tag{11.2.3}$$

An element $a(s) = \sum_{k=0}^{\infty} a_k s^k$ of $\Omega(P)$ is called an *opposite series*. The sequence $\{a_k\}_{k=0}^{\infty}$ of coefficients, called an *opposite sequence*, satisfies $u^k \le a_k \le v^k$. We call a_1 the *initial* of the opposite series a, denoted by $\iota(a)$. Let us introduce the space of initials:

$$\Omega_1(P) := \text{the set of accumulation points of } \{\gamma_{n-1}/\gamma_n\}_{n\in\mathbb{Z}_{\ge 0}}, \tag{11.2.4}$$

which is a compact subset of the positive interval $[u, v]$. The projection map $a \in \Omega(P) \mapsto \iota(a) \in \Omega_1(P)$ is a continuous surjective map.

Assertion. 1. *If a sequence $\{X_{n_m}(P)\}_{m\in\mathbb{Z}_{\ge 0}}$ converges to an opposite sequence a, then $\{X_{n_m-1}(P)\}_{m\in\mathbb{Z}_{\ge 0}}$ also converges to an opposite sequence, denoted by $\tau_\Omega(a)$. We have*

$$\tau_\Omega(a) = (a-1)/(\iota(a)s). \tag{11.2.5}$$

2. *Consider the map*

$$\tau : \Omega(P) \to \overline{\mathbb{R}\Omega}(P), \qquad a \mapsto \iota(a)\tau_\Omega(a), \tag{11.2.5*}$$

where $\overline{\mathbb{R}\Omega}(P)$ is the closed $\mathbb{R}$-linear subspace of $\mathbb{R}[\![s]\!]$ generated by $\Omega(P)$. Then τ naturally extends to an endomorphism of $\overline{\mathbb{R}\Omega}(P)$.

Proof. 1. By definition, the sequence $\{\gamma_{n_m-1}/\gamma_{n_m}\}_m$ converges to the non-zero initial $\iota(a) \ne 0$. Then, for any fixed $k > 0$, the $(k-1)$th coefficient of $\tau_\Omega(a)$ is given by the limit of the sequence $\{\gamma_{n_m-k}/\gamma_{n_m-1}\}_m$ converging to a_k/a_1.

2. Let $\sum_{i\in I} c_i a_i(s) = 0$ be a linear relation among opposite sequences $a_i(s)$ $(i \in I)$ with $\sharp I < \infty$. Then we also have a linear relation $\sum_{i\in I} c_i a_{i,1}\tau_\Omega(a_i(s)) = 0$, since, using (11.2.5), this follows from the original relation $\sum_{i=1}^{\infty} c_i a_i(s) = 0$ and another one $\sum_{i=1}^{\infty} c_i = 0$, which is obtained by substituting $s = 0$ in the first relation. This implies that τ extends to a linear map $\mathbb{R}\Omega(P) \to \overline{\mathbb{R}\Omega}(P)$. On the other hand, $a(s) \in \mathbb{R}[\![s]\!] \mapsto (a(s) - a(0))/s \in \mathbb{R}[\![s]\!]$ is a well-defined continuous map, so it induces a map in $\mathrm{End}_{\mathbb{R}}(\overline{\mathbb{R}\Omega}(P))$. □

We return to the setting of 11.1 and consider a Cayley graph (Γ, G). For the sequence $\{\Gamma_n\}_{n\in\mathbb{Z}_{\ge 0}}$ (11.1.2), we consider two series (10.6.1) and (10.6.2):

$$P_{\Gamma,G}(t) := \sum_{n=0}^{\infty} \sharp\Gamma_n \cdot t^n, \tag{11.2.6}$$

$$P_{\Gamma,G}\mathcal{M}(t) := \sum_{n=0}^{\infty} \mathcal{M}(\Gamma_n) \cdot t^n. \tag{11.2.7}$$

Here (11.2.6) is well known [M] as the growth (or Poincaré) series for (Γ, G), and (11.2.7) is the series which we study in the present paper. Due to (11.1.5), it is well known that the growth series converges with positive radius:

$$r_{\Gamma,G} := 1/\lim_{n\to\infty} \sqrt[n]{\sharp\Gamma_n} \geq 1/\sharp\Gamma_1. \tag{11.2.8}$$

Due to 10.6 Lemma (i), the series $P_{\Gamma,G}\mathcal{M}(t)$ converges in the same radius as $P_{\Gamma,G}(t)$. This fact can be directly confirmed by using (11.1.6) for $S \leq [\Gamma_k]$ as

$$\lim_{n\to\infty} (\sqrt[n-k]{\sharp\Gamma_{n-k}})^{(n-k)/n} \leq \lim_{n\to\infty} \sqrt[n]{\sharp\operatorname{Aut}(S)\cdot A(S,\Gamma_n)} \leq \lim_{n\to\infty} \sqrt[n]{\sharp\Gamma_n}.$$

Let us consider the continuous linear projection map

$$\pi : \mathcal{L}_{\mathbb{R}}\langle\Gamma, G\rangle \to\ \mathbb{R}[\![s]\!], \qquad \sum_{S\in\mathrm{Conf}_0} \varphi(S)\cdot a_S \mapsto \sum_{k=0}^{\infty} a_{\Gamma_k} s^k. \tag{11.2.9}$$

In order that the map π induces the map π_Ω in (11.2.12) below, we consider the next two conditions **S** and **I** on the graph (Γ, G).

First, let us reformulate the concept of dead element (cf. [Bo], [E1]) for monoids: an element $g \in \Gamma$ is called *dead* with respect to G if $\ell_G(gx) \leq \ell_G(g)$ for all $x \in G$.[3] We denote by $D(\Gamma, G)$ the set of dead elements in Γ.

- **S:** The portion $\sharp(\Gamma_n \cap D(\Gamma, G))/\sharp\Gamma_n$ tends to 0 as $n \to \infty$.
- **I:** For any connected subgraph $\mathbb{S}$ of (Γ, G) and any element $g \in \hat{\Gamma}$, the equality $\mathbb{S}\Gamma_1 = g\mathbb{S}\Gamma_1$ implies $\mathbb{S} = g\mathbb{S}$, where $\mathbb{S}\Gamma_1 := \bigcup_{\alpha\in\mathbb{S}} \alpha\Gamma_1$.

Assumption 2. From now on until the end of the present paper, we assume the conditions **S** and **I** hold for (Γ, G).

Remark. 1. Bogopol'skiĭ ([Bo, Question (2)]) asked whether **S** holds for an arbitrary finite generating system G of a group Γ. We ask the same question for a monoid Γ satisfying Assumption 1 and any finite generating system G.

2. Since $\operatorname{Aut}(S)$ is a finite subgroup of $\hat{\Gamma}$ for $S \in \langle\Gamma, G\rangle_0$, it is trivial if $\hat{\Gamma}$ is torsion free. Then **I** holds automatically for each finite generating system.

3. If Γ has a torsion element g of order $d > 1$, define a new generating system $G' := \bigcup_{i,j=0}^{d-1}(g^i\Gamma_1 g^j) \setminus \{e\}$ for a given G. Then the new unit ball $\Gamma_1' := G' \cup \{e\}$ satisfies $\Gamma_1' = g\Gamma_1'$. That is, the condition **I** fails for $\mathbb{S} := \{e\}$. This suggests that in order to satisfy **I**, G should be small relative to torsion elements. It is an open

[3] The author is grateful to Takefumi Kondo for the information on some works on the subject.

question whether, for any finitely generated infinite group Γ, there always exists a generating system G satisfying **I**.

Notation. We define the *fattening* $S\Gamma_1$ for $S \in \langle \Gamma, G\rangle_0$ to be the isomorphism class $[\mathbb{S}\Gamma_1]$ for any representative $\mathbb{S}$ of S (the isomorphism class $[\mathbb{S}\Gamma_1]$ does not depend on the choice of $\mathbb{S}$ due to the embeddability of Γ into a group).

We regard $\mathcal{L}_{\mathbb{R}}\langle \Gamma, G\rangle$ as an $\mathbb{R}[\![s]\!]$-module by letting s act on the basis by $\varphi(S) \mapsto \varphi(S\Gamma_1)$ and extending the action formally to $\mathbb{R}[\![s]\!]$. However, *the map* π (11.2.9) *is not an* $\mathbb{R}[\![s]\!]$*-homomorphism* ($S\Gamma_1 = \Gamma_{k+1}$ does not imply $S = \Gamma_k$).

Let us state some important consequences of the assumptions **S** and **I**. Recall the notation (5.1.1) and (5.1.2).

Assertion. *For any* $S \in \langle \Gamma, G\rangle_0$*, one has the inequalities*

$$0 \le A(S\Gamma_1, \Gamma_n) - A(S, \Gamma_{n-1}) \le \sharp S \cdot \sharp(\dot{\Gamma}_n \cap D(\Gamma, G)). \tag{11.2.10}$$

Proof. Consider the map $\mathbb{S} \in \mathbb{A}(S, \Gamma_{n-1}) \mapsto \mathbb{S}\Gamma_1 \in \mathbb{A}(S\Gamma_1, \Gamma_n)$. It is injective by the condition **I**. This implies the first inequality. Any element of $\mathbb{A}(S\Gamma_1, \Gamma_n)$ is expressed as $\mathbb{S}\Gamma_1$ for a unique $\mathbb{S} \subset \Gamma_n$ with $[\mathbb{S}] = S$. If $\mathbb{S}\Gamma_1$ is not in the image of the above map (i.e. $\mathbb{S} \not\subset \Gamma_{n-1}$), then $\mathbb{S} \cap \dot{\Gamma}_n \neq \emptyset$ is a subset of $D(\Gamma, G)$. Thus, such an $\mathbb{S}$ is of the form $ds^{-1}\mathbb{S}_0$ for some $d \in \dot{\Gamma}_n \cap D(\Gamma, G)$ and some $s \in \mathbb{S}_0$ for a fixed $\mathbb{S}_0$ with $[\mathbb{S}_0] = S$. Thus the number of such $\mathbb{S}\Gamma_1$ is at most $\sharp S \cdot \sharp(\dot{\Gamma}_n \cap D(\Gamma, G))$. This implies the second inequality. $\square$

Corollary. *For* $n, k \in \mathbb{Z}_{\ge 0}$ *with* $n - k \ge 0$*, one has the inequalities*

$$0 \le A(\Gamma_k, \Gamma_n) - \sharp\Gamma_{n-k} \le \sharp\Gamma_{k-1} \cdot \sharp(\Gamma_n \cap D(\Gamma, G)). \tag{11.2.11}$$

Proof. We argue by induction on k, the case $k = 0$ being trivial (put $\sharp\Gamma_{-1} := 0$). Assume the assertion holds for $k-1$. Let $n \ge k$. Applying (11.2.10) for $S = \Gamma_{k-1}$, one has $0 \le A(\Gamma_k, \Gamma_n) - \mathbb{A}(\Gamma_{k-1}, \Gamma_{n-1}) \le \sharp\Gamma_{k-1} \cdot \sharp(\dot{\Gamma}_n \cap D(\Gamma, G))$. This together with the induction hypothesis implies (11.2.11). $\square$

Under Assumptions 1 and 2, we show the main result of the present section: the map π (11.2.9) induces a fibration map $\pi_\Omega : \Omega(\Gamma, G) \to \Omega(P_{\Gamma,G})$.

Theorem. 1. *If a sequence* $\mathcal{M}(\Gamma_{n_m})/\sharp\Gamma_{n_m}$ $(m = 1, 2, \ldots)$ *converges to an element* $\omega \in \Omega(\Gamma, G)$ *with respect to the classical topology, then the sequence* $X_{n_m}(P_{\Gamma,G})$ *converges to the element* $\pi(\omega) \in \mathbb{R}[\![s]\!]$ *with respect to the classical topology. We denote by*

$$\pi_\Omega : \Omega(\Gamma, G) \to \Omega(P_{\Gamma,G}) \tag{11.2.12}$$

the induced map. The map $\pi_\Omega := \pi|_{\Omega(\Gamma,G)}$ *is surjective and continuous.*

2. *If a sequence* $\{\mathcal{M}(\Gamma_{n_m})/\sharp\Gamma_{n_m}\}_{m\in\mathbb{Z}_{\geq 0}}$ *converges to an element* $\omega \in \Omega(\Gamma, G)$, *then* $\{\mathcal{M}(\Gamma_{n_m-1})/\sharp\Gamma_{n_m-1}\}_{m\in\mathbb{Z}_{\geq 1}}$ *also converges to an element, depending only on* ω, *denoted by* $\tilde{\tau}_\Omega(\omega)$. *For* $\omega = \sum_{S\in\langle\Gamma,G\rangle_0} a_S\varphi(S) \in \Omega(\Gamma, G)$, *one has*

$$\tilde{\tau}_\Omega\,(\omega) = \frac{1}{\iota(\pi_\Omega(\omega))} \sum_{S\in\langle\Gamma,G\rangle_0} .a_{S\Gamma_1}\varphi(S). \tag{11.2.13}$$

Using the notation ∂_S *and* $\partial_{S\Gamma_1}$ *for* $S \in \langle\Gamma, G\rangle_0$ (*see* 8.1), (11.2.13) *is equivalent to*

$$\partial_S(\tilde{\tau}_\Omega\omega) = \frac{1}{\iota(\pi_\Omega(\omega))}\partial_{S\Gamma_1}(\omega). \tag{11.2.13*}$$

Then π_Ω (11.2.12) *is equivariant with respect to the actions* $\tilde{\tau}_\Omega$ *and* τ_Ω.

3. *Denote by* $\overline{\mathbb{R}\Omega}(\Gamma, G)$ *the closed* $\mathbb{R}$*-linear subspace of* $\mathcal{L}_{\mathbb{R},\infty}$ *generated by* $\Omega(\Gamma, G)$. *Define a map* $\tilde{\tau}$ *from* $\Omega(\Gamma, G)$ *to* $\overline{\mathbb{R}\Omega}(\Gamma, G)$ *by*

$$\tilde{\tau}(\omega) := \iota(\pi_\Omega(\omega))\tilde{\tau}_\Omega(\omega). \tag{11.2.14}$$

Then $\tilde{\tau}$ *naturally extends to an* $\mathbb{R}$*-linear endomorphism of* $\overline{\mathbb{R}\Omega}(\Gamma, G)$.

4. *The restriction of* π (11.2.9) (= *the* $\mathbb{R}$*-linear extension of* π_Ω)

$$\pi : \overline{\mathbb{R}\Omega}(\Gamma, G) \to \overline{\mathbb{R}\Omega}(P_{\Gamma,G}) \tag{11.2.15}$$

is equivariant with respect to the endomorphisms $\tilde{\tau}$ *and* τ, *i.e.* $\tau\circ\pi = \pi\circ\tilde{\tau}$.

Proof. 1. Using (8.2.3), (11.2.2) and (11.2.6), we see that the difference $\pi(\mathcal{M}(\Gamma_n)/\sharp\Gamma_n) - X_n(P_{\Gamma,G})$ is a polynomial in s of degree $\leq n$ whose kth coefficient is $(A(\Gamma_k, \Gamma_n) - \sharp\Gamma_{n-k})/\sharp\Gamma_n$. Put $n = n_m$ and let $m \to \infty$. Applying (11.2.11) and the assumption **S**, we see that this tends to 0. That is, the kth coefficient of $X_{n_m}(P_{\Gamma,G})$ tends to the coefficient a_{Γ_k} at Γ_k of ω. That is, $X_{n_m}(P_{\Gamma,G})$ converge to the π-image of ω. Thus the map π_Ω (11.2.12) is defined. To show its surjectivity, for any subsequence $\{X_{n_m}(P_{\Gamma,G})\}_m$ converging to an opposite sequence, we choose a convergent sub-subsequence $\{\mathcal{M}(\Gamma_{n_{m_l}})/\sharp\Gamma_{n_{m_l}}\}_l$ due to the compactness of $\overline{\log(\mathrm{EDP})}$ (10.3).

2. For $S \in \langle\Gamma, G\rangle_0$ and $n \in \mathbb{Z}_{\geq 1}$, one has

$$\frac{A(S,\Gamma_{n-1})}{\sharp\Gamma_{n-1}} = \left(\frac{A(S\Gamma_1,\Gamma_n)}{\sharp\Gamma_n} - \frac{A(S\Gamma_1,\Gamma_n) - A(S,\Gamma_{n-1})}{\sharp\Gamma_n}\right)\Big/\frac{\sharp\Gamma_{n-1}}{\sharp\Gamma_n}. \tag{$*$}$$

Let the sequence $\mathcal{M}(\Gamma_{n_m})/\sharp\Gamma_{n_m}$ associated to a subsequence $\{n_m\}_{m\in\mathbb{Z}_{\geq 0}}$ of $\mathbb{Z}_{\geq 0}$ converge to an element $\omega = \sum_{S\in\langle\Gamma,G\rangle_0} a_S\varphi(S) \in \Omega(\Gamma, G)$. Put $n = n_m$ in $(*)$ and let $m \to \infty$. The first (resp. second) term in the brackets on the RHS of $(*)$ converges to $a_{S\Gamma_1}$ (resp. 0 due to (11.2.10) and the assumption **S**). The denominator of the RHS of $(*)$ converges to the initial $\iota(\pi(\omega))$ (11.2.4). Consequently, the RHS of $(*)$ converges to $\frac{1}{\iota(\pi(\omega))}a_{S\Gamma_1}$ for all S. This implies the convergence of $\lim^{\text{cl}}_{m\to\infty}\mathcal{M}(\Gamma_{n_m-1})/\sharp\Gamma_{n_m-1}$ and the formulae (11.2.13) and (11.2.13)*.

Let $a = \pi_\Omega(\omega)$ $(:= \sum_{k=0}^{\infty} a_{\Gamma_k}s^k)$. Comparing the formulae (11.2.5) and (11.2.13), one calculates

$$\begin{aligned}\pi_\Omega(\tilde{\tau}_\Omega(\omega)) &= \frac{1}{\iota(\pi(\omega))}\sum_{k=0}^{\infty} a_{\Gamma_k\Gamma_1}s^k = \frac{1}{\iota(\pi(\omega))}\sum_{k=0}^{\infty} a_{\Gamma_{k+1}}s^k = \frac{1}{\iota(\pi(\omega))}\sum_{l=1}^{\infty} a_{\Gamma_l}s^{l-1}\\ &= \tau_\Omega(a) = \tau_\Omega(\pi_\Omega(\omega)).\end{aligned}$$

This implies that the map π_Ω is equivariant with respect to the $(\tilde{\tau}_\Omega, \tau_\Omega)$-action.

3. Let (r): $\sum_{i\in I} c_i\omega_i = 0$ be a linear relation for $\omega_i \in \Omega(\Gamma, G)$ and $c_i \in \mathbb{R}$ $(i \in I)$ with $\sharp I < \infty$. Let us show the linear relation (s): $\sum_{i\in I} c_i\tilde{\tau}(\omega_i) = 0$. Let us expand $\omega_i = \sum_S a_{S,i}\varphi(S)$. Then the relation (r) is expressed as the relations $\sum_{i\in I} c_i a_{S,i} = 0$ among coefficients for all $S \in \langle\Gamma, G\rangle_0$. Then the relation (s) is expressed as $\sum_{i\in I} c_i a_{S\Gamma_1,i} = 0$ for all $S \in \langle\Gamma, G\rangle_0$, which is a part of the former relations among the coefficients and is automatically satisfied.

This implies that $\tilde{\tau}$ extends to a linear map $\mathbb{R}\Omega(\Gamma, G) \to \overline{\mathbb{R}\Omega}(\Gamma, G)$. On the other hand, the correspondence $\sum_{S\in\langle\Gamma,G\rangle} a_S\varphi(S) \mapsto \sum_{S\in\langle\Gamma,G\rangle} a_{S\Gamma_1}\varphi(S)$ defines a continuous linear map from a closed subspace of $\mathcal{L}_\mathbb{R}$ to itself, which induces the endomorphism $\tilde{\tau} \in \mathrm{End}_\mathbb{R}(\overline{\mathbb{R}\Omega}(\Gamma, G))$.

4. Let the notation be as in 1. Comparing (11.2.5)* and (11.2.14), one calculates: $\pi(\tilde{\tau}(\omega)) = \pi(\iota(\pi(\omega))\tilde{\tau}_\Omega(\omega)) = \iota(\pi(\omega))\pi_\Omega(\tilde{\tau}_\Omega(\omega)) = \iota(\pi(\omega))\tau_\Omega(\pi_\Omega(\omega)) = \iota(\pi(\omega))\tau_\Omega(a) = \tau(a) = \tau(\pi(\omega))$, which proves the equivariance of π. □

The map π_Ω (11.2.12) is conjecturally a finite map. In that case, the sum of the elements in a fiber is called a *trace*; in 11.5, we represent the traces by suitable "residue values" of the functions (11.2.6) and (11.2.7). The key to understanding this formula is the *"duality" between the limit space* $\Omega(P_{\Gamma,G})$ *and the space of singularities* $\mathrm{Sing}(P_{\Gamma,G})$ *of the series* $P_{\Gamma,G}(t)$ *on the circle of the convergence radius* $r_{\Gamma,G}$. In the next sections 11.3 and 11.4, we study the "duality" in case $\Omega(P_{\Gamma,G})$ is finite (see 11.4 Theorem and (11.4.3) and (11.4.4)).

In the following, we give an example of (Γ, G), where $\Omega(P_{\Gamma,G})$ consists of two elements $a^{[0]}$ and $a^{[1]}$, and τ_Ω acts on $\Omega(P_{\Gamma,G})$ as their transposition. However, we note that $\tau^2 \neq \mathrm{id}$ and $\det(t\cdot \mathrm{id} - \tau) = t^2 - 1/2$.

Example ((Machì))**.** Let $\Gamma := \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ and $G := \{a, b^{\pm 1}\}$ where a, b are the generators of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$, respectively. Machì has shown

$$P_{\Gamma,G}(t) := \sum_{k=0}^{\infty} \sharp\Gamma_k t^k = \frac{(1+t)(1+2t)}{(1-2t^2)(1-t)},$$

so that $\sharp\Gamma_{2k} = 7 \cdot 2^k - 6$ and $\sharp\Gamma_{2k+1} = 10 \cdot 2^k - 6$ for $k \in \mathbb{Z}_{\geq 0}$. Then

$$\Omega_1(P_{\Gamma,G}) = \left\{ \iota(a^{[0]}) := \lim_{n\to\infty} \frac{\sharp\Gamma_{2n-1}}{\sharp\Gamma_{2n}} = \frac{5}{7} \ \& \ \iota(a^{[1]}) := \lim_{n\to\infty} \frac{\sharp\Gamma_{2n}}{\sharp\Gamma_{2n+1}} = \frac{7}{10} \right\},$$

and hence $h_{\Gamma,G} = 2$. In fact, $\Omega(P_{\Gamma,G})$ consists of two opposite series:

$$a^{[0]}(s) := \sum_{k=0}^{\infty} 2^{-k} s^{2k} + \frac{5}{7} s \sum_{k=0}^{\infty} 2^{-k} s^{2k} = \frac{1 + \frac{5}{7}s}{1 - s^2/2} = \frac{\frac{7+5\sqrt{2}}{14}}{1 - s/\sqrt{2}} + \frac{\frac{7-5\sqrt{2}}{14}}{1 + s/\sqrt{2}}.$$

$$a^{[1]}(s) := \sum_{k=0}^{\infty} 2^{-k} s^{2k} + \frac{7}{10} s \sum_{k=0}^{\infty} 2^{-k} s^{2k} = \frac{1 + \frac{7}{10}s}{1 - s^2/2} = \frac{\frac{10+7\sqrt{2}}{20}}{1 - s/\sqrt{2}} + \frac{\frac{10-7\sqrt{2}}{20}}{1 + s/\sqrt{2}}.$$

§11.3. Finite rational accumulation

We introduce the concept of a *finite rational accumulation*, and study the series $P(t)$ (11.2.1) from that viewpoint. First, we start with preliminary definitions.

Definition. 1. A subset U of $\mathbb{Z}_{\geq 0}$ is called a *rational subset* if the sum $U(t) := \sum_{n\in U} t^n$ is the Taylor expansion at 0 of a rational function in t.

2. A *finite rational partition* of $\mathbb{Z}_{\geq 0}$ is a finite collection $\{U_a\}_{a\in\Omega}$ of rational subsets $U_a \subset \mathbb{Z}_{\geq 0}$ indexed by a finite set Ω such that there is a finite subset D of $\mathbb{Z}_{\geq 0}$ so that one has the disjoint decomposition $\mathbb{Z}_{\geq 0} \setminus D = \bigsqcup_{a\in\Omega}(U_a \setminus D)$.

Assertion. *For any rational subset U of $\mathbb{Z}_{\geq 0}$, there exist a positive integer h, a subset $u \subset \mathbb{Z}/h\mathbb{Z}$ and a finite subset $D \subset \mathbb{Z}_{\geq 0}$ such that $U \setminus D = \bigcup_{[e]\in u} U^{[e]} \setminus D$, where $[e]$ denotes the element of $\mathbb{Z}/h\mathbb{Z}$ corresponding to $e \in \mathbb{Z}$ and where $U^{[e]} := \{n \in \mathbb{Z}_{\geq 0} \mid n \equiv e \bmod h\}$. We call $\bigcup_{[e]\in u} U^{[e]}$ the* standard expression *of U.*

Proof. The fact that $U(t)$ is rational implies that the function $\chi : \mathbb{Z}_{\geq 0} \to \{0, 1\}$ ($\chi(n) = 1 \leftrightarrow n \in U$) is recursive, i.e. there exist $N \in \mathbb{Z}_{\geq 1}$ and numbers $\alpha_1, \ldots, \alpha_N$ such that one has the recursive relation $\chi(n) + \chi(n-1)\alpha_1 + \cdots + \chi(n-N)\alpha_N = 0$ for sufficiently large $n \gg 0$. Since the range of χ is finite, there exist two large numbers $n > m$ such that $\chi(n-i) = \chi(m-i)$ for $i = 0, \ldots, N$. Due to the recursive relation, this means χ is $h := n - m$-periodic after m. □

Corollary. *Any finite rational partition of $\mathbb{Z}_{\geq 0}$ has a subdivision of the form $\mathcal{U}_h := \{U^{[e]}\}_{[e]\in\mathbb{Z}/h\mathbb{Z}}$ for some $h \in \mathbb{Z}_{>0}$, called a* period *of the partition. If h is the minimal period, $\mathcal{U}_h$ is called the* standard subdivision *of the partition.*

Definition. A sequence $\{X_n\}_{n\in\mathbb{Z}_{\geq 0}}$ in a Hausdorff space is *finite rationally accumulating* if the sequence accumulates to a finite set, say Ω, such that for a system of open neighborhoods $\mathcal{V}_a$ for $a \in \Omega$ with $\mathcal{V}_a \cap \mathcal{V}_b = \emptyset$ if $a \neq b$, the system $\{U_a\}_{a\in\Omega}$

for $U_a := \{n \in \mathbb{Z}_{\geq 0} \mid X_n \in \mathcal{V}_a\}$ is a finite rational partition of $\mathbb{Z}_{\geq 0}$. We also say that Ω is a *finite rational accumulation set of period* h.

The next lemma and 11.5 Lemma are key facts, which justify the introduction of the concept of "rational accumulation". They are also the starting point of the concept of *periodicity* which is the base of the whole study to follow.

Lemma. *Let $P(t)$ be a power series in t as given in* (11.2.1). *If $\Omega(P)$ is finite, then it is a finite rational accumulation set with respect to the standard partition $\mathcal{U}_h$ of $\mathbb{Z}_{\geq 0}$ for some $h > 0$, and τ_Ω acts transitively on $\Omega(P)$ with period h.*

Proof. Recall the τ_Ω-action on the set $\Omega(P)$ in 11.2. Since $\Omega(P)$ is finite, there exists a non-empty τ_Ω-invariant subset of $\Omega(P)$. More explicitly, there exists an element $a \in \Omega(P)$ and a positive integer $h \in \mathbb{Z}_{>0}$ such that $(\tau_\Omega)^h a = a \neq (\tau_\Omega)^{h'} a$ for $0 < h' < h$. Put $U_a := \{n \in \mathbb{Z}_{\geq 0} \mid X_n(P) \in \mathcal{V}_a\}$ where $\{\mathcal{V}_a\}_{a \in \Omega(P)}$ is a system of open neighborhoods of points of $\Omega(P)$ such that $\mathcal{V}_a \cap \mathcal{V}_b = \emptyset$ for any $a \neq b \in \Omega(P)$. By the definition of τ_Ω, the relation $(\tau_\Omega)^h a = a$ implies that the sequence $\{X_{n-h}(P)\}_{n \in U_a}$ converges to a. That is, there exists a positive number N such that for any $n \in U_a$ with $n > N$, $n - h$ is contained in U_a. Consider the set $A := \{[e] \in \mathbb{Z}/h\mathbb{Z} \mid$ there are infinitely many elements of U_a which are congruent to $[e]$ modulo $h\}$. Then U_a is, up to a finite number of elements, equal to the rational set $\bigcup_{[e] \in A} U^{[e]}$. This implies $A \neq \emptyset$. Furthermore, $U_{(\tau_\Omega)^i a}$ is also, up to a finite number of elements, equal to the rational set $\bigcup_{[e] \in A} U^{[e-i]}$. Then the union $\bigcup_{i=0}^{h-1} U_{(\tau_\Omega)^i a}$ already covers $\mathbb{Z}_{\geq 0}$ up to finitely many elements. Since there should be no overlapping, $\sharp A = 1$, say $A = \{[0]\}$. If a subsequence $\{X_{n_m}(P)\}$ converges to an element in $\Omega(P)$, then there is at least one $[e] \in \mathbb{Z}/h\mathbb{Z}$ such that $\sharp(\{n_m\}_{m=0}^{\infty} \cap U^{[e]}) = \infty$, so that it converges to $(\tau_\Omega)^{h-e} a$. That is, $\Omega(P)$ is equal to the set $\{a, \tau_\Omega a, \ldots, (\tau_\Omega)^{h-1} a\}$, which is a finite rational accumulating set with the h-periodic action of τ_Ω. □

In what follows, we analyze the finite accumulation set $\Omega(P)$ in detail.

Assertion. *Let $P(t)$ be a power series in t as given in* (11.2.1).

1. *$\Omega(P)$ is a finite rational accumulation set of period $h \in \mathbb{Z}_{\geq 1}$ if and only if $\Omega_1(P)$ is. We then say P is* finite rationally accumulating of period h.

2. *Let P be finite rationally accumulating of period $h \in \mathbb{Z}_{\geq 1}$. Then the opposite series $a^{[e]} = \sum_{k=0}^{\infty} a_k^{[e]} s^k$ in $\Omega(P)$ associated to the rational subset $U^{[e]}$ for $[e] \in \mathbb{Z}/h\mathbb{Z}$ of the h-partition of $\mathbb{Z}_{\geq 0}$ converges to a rational function*

$$a^{[e]}(s) = \frac{A^{[e]}(s)}{1 - r^h s^h}, \tag{11.3.1}$$

where the numerator $A^{[e]}(s)$ is a polynomial in s of degree $h-1$ given by

$$A^{[e]}(s) := \sum_{j=0}^{h-1}\Big(\prod_{i=1}^{j} a_1^{[e-i+1]}\Big)s^j \tag{11.3.2}$$

and

$$r^h := \prod_{i=0}^{h-1} a_1^{[i]}. \tag{11.3.3}$$

The hth positive root $r>0$ of (11.3.3) is the radius of convergence of $P(t)$.

3. *If the period h is minimal, then the opposite sequences $a^{[e]}(s)$ for $[e]\in\mathbb{Z}/h\mathbb{Z}$ are mutually distinct. That is, $\Omega(P)\simeq\mathbb{Z}/h\mathbb{Z}$, $a^{[e]}(s)\leftrightarrow[e]$ and the standard partition $\mathcal{U}_h$ is the exact partition of $\mathbb{Z}_{\geq 0}$ for the opposite series $\Omega(P)$.*

Proof. 1. The necessity is obvious. To show sufficiency, assume $\{\gamma_{n-1}/\gamma_n\}_{n\in\mathbb{Z}_{\geq 0}}$ accumulates finite rationally of period h. Let the subsequence $\{\gamma_{n-1}/\gamma_n\}_{n\in U_{[e]}}$ for $[e]\in\mathbb{Z}/h\mathbb{Z}$ accumulate to a unique value $a_1^{[e]}$.

For any $k\in\mathbb{Z}_{\geq 0}$, one has the obvious relation

$$\frac{\gamma_{n-k}}{\gamma_n} = \frac{\gamma_{n-1}}{\gamma_n}\frac{\gamma_{n-2}}{\gamma_{n-1}}\cdots\frac{\gamma_{n-k}}{\gamma_{n-k+1}}.$$

For $n\in U_{[e]}=\{n\in\mathbb{Z}_{\geq 0}\mid n\equiv e \bmod h\}$ with $[e]\in\mathbb{Z}/h\mathbb{Z}$, we see that the RHS converges to $a_1^{[e]}a_1^{[e-1]}\cdots a_1^{[e-k+1]}$. Then, for $[e]\in\mathbb{Z}/h\mathbb{Z}$ and $k\in\mathbb{Z}_{\geq 0}$, by putting

$$a_k^{[e]} := a_1^{[e]}a_1^{[e-1]}\cdots a_1^{[e-k+1]}, \tag{11.3.4}$$

the sequence $\{X_n(P)\}_{n\in U_{[e]}}$ converges to $a^{[e]} := \sum_{k=0}^{\infty} a_k^{[e]}s^k$ with $a_1^{[e]}=\iota(a^{[e]})$.

2. Define r^h by (11.3.3). Then the formula (11.3.4) implies the "periodicity" $a_{mh+k}^{[e]} = r^{mh}a_k^{[e]}$ for $m\in\mathbb{Z}_{\geq 0}$. This implies (11.3.1).

To show that r is the radius of convergence of $P(t)$, it is sufficient to show:

Fact. *Let $P(t)$ be finite rationally accumulating of period h. Define $r\geq 0$ by (11.3.3). There exist positive real constants c_1 and c_2 such that for any $k\in\mathbb{Z}_{\geq 0}$ there exists $n(k)\in\mathbb{Z}_{\geq 0}$ such that for any integer $n\geq n(k)$, one has $c_1r^k\leq\gamma_{n-k}/\gamma_n\leq c_2r^k$.*

Proof. Choose $c_1,c_2\in\mathbb{R}_{>0}$ satisfying $c_1<\min\{a_i^{[e]}/r^i\mid[e]\in\mathbb{Z}/h\mathbb{Z},\ i\in\mathbb{Z}\cap[0,h-1]\}$ and $c_2>\max\{a_i^{[e]}/r^i\mid[e]\in\mathbb{Z}/h\mathbb{Z},\ i\in\mathbb{Z}\cap[0,h-1]\}$. □

3. Suppose $a^{[e]}(s)=a^{[f]}(s)$ for some $[e],[f]\in\mathbb{Z}/h\mathbb{Z}$. Then, by comparing the coefficients of $A^{[e]}(s)$ and $A^{[f]}(s)$, we get $a_1^{[e-i]}=a_1^{[f-i]}$ for $i=0,\dots,h-1$. This means $e-f$ is a period. The minimality of h implies $[e-f]=0$. □

Even if, as in the above assertion, the opposite series $a^{[e]}(s)$ for $[e] \in \mathbb{Z}/h\mathbb{Z}$ are mutually distinct for the minimal period h of $P(t)$, they may be linearly dependent. This phenomenon occurs at the zero locus of the determinant

$$D_h(a_1^{[0]}, \dots, a_1^{[h-1]}) := \det\Big(\Big(\prod_{i=1}^{f} a_1^{[e-i+1]}\Big)_{e,f\in\{0,1,\dots,h-1\}}\Big). \tag{11.3.5}$$

Regarding $a_1^{[0]}, \dots, a_1^{[h-1]}$ as indeterminates, D_h is an irreducible homogeneous polynomial of degree $h(h-1)/2$, which is neither symmetric nor anti-symmetric, but anti-invariant under a cyclic permutation (depending on the parity of h). Let us formulate more precise statements for an arbitrary field K.

Assertion. *Let $h \in \mathbb{Z}_{>0}$. For an h-tuple $\bar{a} = (a_1^{[0]}, \dots, a_1^{[h-1]}) \in (K^\times)^h$, define polynomials $A^{[e]}(s)$ ($[e] \in \mathbb{Z}/h\mathbb{Z}$) and $r^h \in K^\times$ by* (11.3.2) *and* (11.3.3)*.*

(i) *In the ring $K[s]$, the greatest common divisors $\gcd(A^{[e]}(s), 1 - r^h s^h)$ and $\gcd(A^{[e]}(s), A^{[e+1]}(s))$ for all $[e] \in \mathbb{Z}/h\mathbb{Z}$ are the same up to factors in $K^\times$. Let $\delta_{\bar{a}}(s)$ be the common divisor whose constant term is normalized to* 1*. Put*

$$\Delta_{\bar{a}}^{\mathrm{op}}(s) := (1 - r^h s^h)/\delta_{\bar{a}}(s). \tag{11.3.6}$$

(ii) *For $[e] \in \mathbb{Z}/h\mathbb{Z}$, let $a^{[e]}(s) = b^{[e]}(s)/\Delta_{\bar{a}}^{\mathrm{op}}(s)$ be the reduced expression* (*i.e. $b^{[e]}(s)$ is a polynomial of degree $< \deg(\Delta_{\bar{a}}^{\mathrm{op}})$ and $\gcd(b^{[e]}(s), \Delta_{\bar{a}}^{\mathrm{op}}(s)) = 1$*)*. Then the polynomials $b^{[e]}(s)$ for $[e] \in \mathbb{Z}/h\mathbb{Z}$ span the space $K[s]_{<\deg(\Delta_{\bar{a}}^{\mathrm{op}})}$ of polynomials of degree less than $\deg(\Delta_{\bar{a}}^{\mathrm{op}})$. One has the equality*

$$\mathrm{rank}\Big(\Big(\prod_{i=1}^{f} a_1^{[e-i+1]}\Big)_{e,f\in\{0,1,\dots,h-1\}}\Big) = \deg(\Delta_{\bar{a}}^{\mathrm{op}}). \tag{11.3.7}$$

(iii) *Let $K = \mathbb{R}$ and $\bar{a} \in (\mathbb{R}_{>0})^h$. Then $\Delta_{\bar{a}}^{\mathrm{op}}$ is divisible by $1 - rs$. Conversely, let Δ^{op} be a factor of $1 - r^h s^h$ which is divisible by $1 - rs$ for $r \in \mathbb{R}_{>0}$ with the constant term* 1*. Then there exists a smooth non-empty semialgebraic set $C_{\Delta^{\mathrm{op}}} \subset (\mathbb{R}_{>0})^h$ of dimension $\deg(\Delta^{\mathrm{op}}) - 1$ such that $\Delta^{\mathrm{op}} = \Delta_{\bar{a}}^{\mathrm{op}}$ for all $\bar{a} \in C_{\Delta^{\mathrm{op}}}$.*

Proof. (i) By the definitions (11.3.3) and (11.3.4), we have the relations

$$a_1^{[e+1]} s A^{[e]}(s) + (1 - r^h s^h) = A^{[e+1]}(s) \tag{11.3.8}$$

for $[e] \in \mathbb{Z}/h\mathbb{Z}$. This implies $\gcd(A^{[e]}(s), 1 - r^h s^h) \mid \gcd(A^{[e+1]}(s), 1 - r^h s^h)$ for $[e] \in \mathbb{Z}/h\mathbb{Z}$ so that one concludes that all the elements $\gcd(A^{[e]}(s), 1 - r^h s^h)$ $= \gcd(A^{[e]}(s), A^{[e+1]}(s))$ for $[e] \in \mathbb{Z}/h\mathbb{Z}$ are the same up to a constant factor.

(ii) Let us show that the images in $K[s]/(\Delta_{\bar{a}}^{\mathrm{op}})$ of the polynomials $A^{[e]}(s)/\delta_{\bar{a}}(s)$ for $[e] \in \mathbb{Z}/h\mathbb{Z}$ span the entire space over K. Let V be the space spanned by the images. The relation (11.3.8) implies that V is closed under multiplication by s. On the other hand, $A^{[e]}(s)/\delta_{\bar{a}}(s)$ and $\Delta_{\bar{a}}^{\mathrm{op}}$ are relatively prime so that they generate $K[s]$ as a $K[s]$-module. That is, V contains the class $[1]$ of 1, and, hence, V contains the whole $K[s]\cdot[1]$. Since $\deg(A^{[e]}(s)/\delta_{\bar{a}}(s)) < \deg(\Delta_{\bar{a}}^{\mathrm{op}})$, this means that the polynomials $A^{[e]}(s)/\delta_{\bar{a}}(s)$ for $[e]\in\mathbb{Z}/h\mathbb{Z}$ span the space of polynomials of degree less than $\deg(\Delta_{\bar{a}}^{\mathrm{op}})$. In particular, $\mathrm{rank}_K V = \deg(\Delta_{\bar{a}}^{\mathrm{op}})$.

By definition, $\mathrm{rank}(((\prod_{i=1}^{f} a_1^{[e-i+1]})_{e,f\in\{0,1,\ldots,h-1\}}))$ is equal to the rank of the space spanned by $A^{[e]}(s)$ for $[e]\in\mathbb{Z}/h\mathbb{Z}$, which is equal to the rank of the space spanned by $A^{[e]}(s)/\delta_{\bar{a}}(s)$ for $[e]\in\mathbb{Z}/h\mathbb{Z}$ and is equal to $\deg(\Delta_{\bar{a}}^{\mathrm{op}})$.

(iii) If $(1-rs)\nmid\Delta_{\bar{a}}^{\mathrm{op}}$, then $1-rs\mid\delta_{\bar{a}}\mid A^{[e]}(s)$ and $A^{[e]}(1/r)=0$. This is impossible since all coefficients of $A^{[e]}$ and $1/r$ are positive. Conversely, let Δ^{op} be a factor of $1-r^hs^h$ which is divisible by $1-rh$, whose degree is $d>0$. Put $\mathbb{R}[s]_{d-1} := \{c(s)\in\mathbb{R}[s]\mid \deg(c(s))=d-1,\ c(0)=1\}$. Consider the set

$$\overline{C}_{\Delta^{\mathrm{op}}} := \{c(s)\in\mathbb{R}[s]_{d-1}\mid \text{all coefficients of } c'(s) := c(s)(1-r^hs^h)/\Delta^{\mathrm{op}} \text{ are positive}\}.$$

Since $\overline{C}_{\Delta^{\mathrm{op}}}$ is defined by strict inequalities, it is an open subset of $\mathbb{R}[s]_{d-1}$. Further, it is non-empty since it contains $\Delta^{\mathrm{op}}/(1-rs)$. For any $c(s)\in\overline{C}_{\Delta^{\mathrm{op}}}$, we note that $\deg(c'(s))=h-1$, and hence one can find a unique $\overline{a}\in(\mathbb{R}_{>0})^h$ satisfying $c'(s)=A^{[0]}(s)$ ((11.3.2) and (11.3.3)). By this correspondence $c(s)\mapsto\overline{a}$, we embed $\overline{C}_{\Delta^{\mathrm{op}}}$ smoothly in a smooth semialgebraic subset of $(\mathbb{R}_{>0})^h$ of dimension $d-1$. If $\overline{a}$ is the image of $c(s)\in\overline{C}_{\Delta^{\mathrm{op}}}$, then $\delta_{\overline{a}} := \gcd\{c'(s),1-r^hs^h\}$ is divisible by $(1-r^hs^h)/\Delta^{\mathrm{op}}$. That is, $\Delta_{\overline{a}}^{\mathrm{op}} := (1-r^hs^h)/\delta_{\overline{a}}$ is a factor of Δ^{op}. This implies that $c(s)$ is a point of the embedded image $C_{\Delta_{\overline{a}}^{\mathrm{op}}}\to C_{\Delta^{\mathrm{op}}}$ (defined by multiplication by $\Delta^{\mathrm{op}}/\Delta_{\overline{a}}^{\mathrm{op}}$). Define the semialgebraic set $C_{\Delta^{\mathrm{op}}} := \overline{C}_{\Delta^{\mathrm{op}}}\setminus\bigcup_{\Delta'}\overline{C}_{\Delta'}$, where Δ' runs over all factors of Δ^{op} (over $\mathbb{R}$) which are not equal to Δ^{op} and are divisible by $1-rs$. Since $\dim_{\mathbb{R}}(\overline{C}_{\Delta}) = d-1 > \dim_{\mathbb{R}}(\overline{C}_{\Delta'})$, the difference C_{Δ} is non-empty. □

Suppose the characteristic of the field K is zero. Let $\tilde{K}$ be the splitting field of $\Delta_{\bar{a}}^{\mathrm{op}}$ and $\Delta_{\bar{a}}^{\mathrm{op}} = \prod_{i=1}^{d}(1-x_is)$ in $\tilde{K}$ for $d := \deg(\Delta_{\bar{a}}^{\mathrm{op}})$. Then one has the partial fraction decomposition

$$\frac{A^{[e]}(s)}{1-r^hs^h} = \sum_{i=1}^{d}\frac{\mu_{x_i}^{[e]}}{1-x_is} \tag{11.3.9}$$

for $[e]\in\mathbb{Z}/h\mathbb{Z}$, where $\mu_{x_i}^{[e]}$ is a constant in $\tilde{K}$ given by the residue

$$\mu_{x_i}^{[e]} = \left.\frac{A^{[e]}(s)(1-x_is)}{1-r^hs^h}\right|_{s=(x_i)^{-1}} = \frac{1}{h}A^{[e]}(x_i^{-1}). \tag{11.3.10}$$

Here, one has the equivariance $\sigma(\mu_{x_i}^{[e]}) = \mu_{\sigma(x_i)}^{[e]}$ with respect to the action of $\sigma \in \mathrm{Gal}(\tilde{K}, K)$. The matrix $(\mu_{x_i}^{[e]})_{[e],x_i}$ is of maximal rank $d = \deg(\Delta_{\bar{a}}^{\mathrm{op}})$.

Remark. The index x_i in (11.3.10) may run over all roots x of the equation $x^h - r^h = 0$. However, if x is not a root of $\Delta_{\bar{a}}^{\mathrm{op}}$ (i.e. $\Delta_{\bar{a}}^{\mathrm{op}}(x^{-1}) \neq 0$), then $\mu_x^{[e]} = 0$.

We return to the series $P(t)$ (11.2.1) with positive radius r of convergence. If $P(t)$ is finite rationally accumulating of period h and $a_1^{[e]} := \iota(a^{[e]})$ for $[e] \in \mathbb{Z}/h\mathbb{Z}$ (recall (11.3.1)), then $\Delta_{\bar{a}}^{\mathrm{op}}(s)$ depends only on P but not on the choice of the period h. Therefore, we shall denote it by $\Delta_P^{\mathrm{op}}(s)$. The previous Assertion (ii) says that we have the $\mathbb{R}$-isomorphism

$$\overline{\mathbb{R}\Omega}(P) \simeq \mathbb{R}[s]/\Delta_P^{\mathrm{op}}(s), \qquad a^{[e]} \;\mapsto\; \Delta_P^{\mathrm{op}} \cdot a^{[e]} \bmod \Delta_P^{\mathrm{op}}. \tag{11.3.11}$$

Since the action of τ is invertible, we define an endomorphism σ on $\overline{\mathbb{R}\Omega}(P)$ by

$$\sigma(a^{[e]}) := \tau^{-1}(a^{[e]}) = \frac{1}{a_1^{[e+1]}} a^{[e+1]}. \tag{11.3.12}$$

The isomorphism (11.3.11) *is equivariant with respect to the action of* σ *on the LHS and multiplication by* s *on the RHS.* Hence, *the linear dependence relations among the generators* $a^{[e]}$ ($[e] \in \mathbb{Z}/h\mathbb{Z}$) *are obtained from the relations* $\Delta_P^{\mathrm{op}}(\sigma)a^{[e]} = 0$ *for* $[e] \in \mathbb{Z}/h\mathbb{Z}$. However, one should note that the σ-action on $\overline{\mathbb{R}\Omega}(P)$ is not identified with multiplication by s on the subspace $\mathbb{R}[\![s]\!]$.

§11.4. Duality between $\Delta_P^{\mathrm{op}}(s)$ and $\Delta_P^{\mathrm{top}}(t)$

Assuming that $P(t)$ extends to a meromorphic function in a neighborhood of the closure of its convergence disc, we show a duality between the poles of opposite sequences of $P(t)$ and the poles of $P(t)$ on its convergence circle.

Definition. For a positive real number r, let us denote by $\mathbb{C}\{t\}_r$ the space consisting of the complex power series $P(t)$ such that (i) $P(t)$ converges (at least) on the open disc centered at 0 of radius r, and (ii) $P(t)$ analytically continues to a meromorphic function on a disc centered at 0 of radius $> r$. Let $\Delta_P(t)$ be the monic polynomial in t of minimal degree such that $\Delta_P(t)P(t)$ is holomorphic in a neighborhood of the circle $|t| = r$. Put $\Delta_P(t) = \prod_{i=1}^N (t - x_i)^{d_i}$ where x_i ($i = 1, \dots, N$, $N \in \mathbb{Z}_{\geq 0}$) are mutually distinct complex numbers with $|x_i| = r$ and $d_i \in \mathbb{Z}_{>0}$ ($i = 1, \dots, N$). Define the equation for the set of poles of highest order:

$$\Delta_P^{\mathrm{top}}(t) := \prod_{i,\, d_i = d_m} (t - x_i) \quad \text{where} \quad d_m := \max\{d_i\}_{i=1}^N. \tag{11.4.1}$$

Definition. Define an action T_U on $\mathbb{C}[\![t]\!]$ for a rational set $U \subset \mathbb{Z}_{\geq 0}$ by

$$P = \sum_{n \in \mathbb{Z}_{\geq 0}} \gamma_n t^n \mapsto T_U P := \sum_{n \in U} \gamma_n t^n. \tag{11.4.2}$$

One may regard $T_U P$ as a product of P with the function $U(t)$ in the sense of Hadamard [H]. The radius of convergence of $T_U P$ is not less than that of P.

Fact 1. *The action of T_U preserves the space $\mathbb{C}\{t\}_r$ for any $r \in \mathbb{R}_{>0}$.*

Proof. Let us expand the meromorphic function $P(t)$ into partial fractions

$$P(t) = \sum_{i=1}^{N} \sum_{j=1}^{d_i} \frac{c_{i,j}}{(t - x_i)^j} \ + \ Q(t), \tag{$*$}$$

where the coefficients $c_{i,j}$ of the principal part are constant with $c_{i,d_i} \neq 0$ for all i, and $Q(t)$ is a holomorphic function on a disc of radius $> r$. Then $T_U P = \sum_{i,j} T_U \frac{c_{i,j}}{(t-x_i)^j} + T_U Q$ where $T_U Q$ is a holomorphic function on a disc of radius $> r$. It is sufficient to show that, for any standard rational set $U^{[e]} := \{n \in \mathbb{Z}_{\geq 0} \mid n \equiv [e] \bmod h\}$ of period $h \in \mathbb{Z}_{>0}$ and $[e] \in \mathbb{Z}/h\mathbb{Z}$, one has

$$T_{U^{[e]}} \frac{1}{(t - x_i)^j} = \frac{B_{i,j}(t)}{(t^h - x_i^h)^j}$$

where $B_{i,j}(t)$ is a polynomial in t. We calculate this explicitly as follows. We use the "semi-commutativity" $T_{U^{[e]}} \cdot \frac{d}{dt} = \frac{d}{dt} \cdot T_{U^{[e+1]}}$ (the trivial proof is omitted). Then

$$\begin{aligned} T_{U^{[e]}} \frac{1}{(t - x_i)^j} &= T_{U^{[e]}} \frac{(-1)^{j-1}}{(j-1)!} \left(\frac{d}{dt}\right)^{j-1} \frac{1}{t - x_i} = \frac{(-1)^{j-1}}{(j-1)!} \left(\frac{d}{dt}\right)^{j-1} T_{U^{[e+j-1]}} \frac{1}{t - x_i} \\ &= \frac{(-1)^{j-1}}{(j-1)!} \left(\frac{d}{dt}\right)^{j-1} \frac{t^f}{t^h - x_i^h} \qquad \text{where } f := e + j - 1 - h[(e + j - 1)/h]. \end{aligned}$$

This gives the required result. □

The following is the goal of the present subsection.

Theorem. 5. (Duality) *Suppose $P(t)$ (11.2.1) belongs to $\mathbb{C}\{t\}_r$ for $r =$ the radius of convergence of P, and is finite accumulating. Then*

$$t^{\deg(\Delta_P^{\mathrm{op}})} \Delta_P^{\mathrm{op}}(t^{-1}) = \Delta_P^{\mathrm{top}}(t), \tag{11.4.3}$$

$$\mathrm{rank}(\overline{\mathbb{R}\Omega}(P)) = \deg(\Delta_P^{\mathrm{op}}) = \deg(\Delta_P^{\mathrm{top}}). \tag{11.4.4}$$

Proof. We first show some special case, and then the general case.

Fact 2. *If $P(t)$, above, is simple accumulating* (*i.e.* $\sharp\Omega(P)=1$), *then* $\Delta_P^{\mathrm{top}}=t-r$.

Proof. That $P(t)$ is simply accumulating means $\lim_{n\to\infty}\gamma_{n-1}/\gamma_n=r$ and hence, for any small $\varepsilon>0$, there exists $c>0$ such that $\gamma_n\geq c(r+\varepsilon)^{-n}$ for $n\in\mathbb{Z}_{\geq 0}$. Let δ_n be the nth Taylor coefficient of Q in $(*)$. By assumption on Q, there exist $r'>r$ and a constant $c'>0$ such that $\delta_n\leq c'r'^{-n}$ for $n\in\mathbb{Z}_{\geq 0}$. Therefore, choosing ε such that $r+\varepsilon<r'$, we have $\delta_n/\gamma_n\to 0$. Since the nth Taylor coefficient of the principal part of $(*)$ is given by $\gamma_n-\delta_n$, the principal part, say P', is also simply accumulating. That is,

$$X_n(P')=\sum_{k=0}^{n}\frac{\sum_{i=1}^{N}\sum_{j=1}^{d_i}c_{i,j}x_i^{k-n-1}(n-k;j)/(j-1)!}{\sum_{i=1}^{N}\sum_{j=1}^{d_i}c_{i,j}x_i^{-n-1}(n;j)/(j-1)!}s^k$$

converges to $\frac{1}{1-rs}=\sum_{k=0}^{\infty}r^ks^k$. Under this setting, we want to show that if $c_{i,d_m}\neq 0$ then $x_i=r$. For convenience in the proof, we may assume $r=1$ and hence $|x_i|=1$ for all i.

Consider the sequence $v_n:=\sum_{i=1}^{N}c_{i,d_m}x_i^{-n-1}$ in $n\in\mathbb{Z}_{\geq 0}$. Since $|v_n|\leq\sum_i|c_{i,d_m}|$ is bounded, the sequence accumulates to a compact set in $\mathbb{C}$. If the sequence has a unique accumulating value, say v_0, then the result is already true. (*Proof.* Consider the mean sequence $\{(\sum_{n=0}^{M-1}v_n)/M\}_{M\in\mathbb{Z}_{>0}}$. On the one hand, it converges to v_0 by assumption. On the other hand, $\sum_{i=1}^{N}c_{i,d_m}\frac{\sum_{n=0}^{M-1}x_i^{-n-1}}{M}$ converges to c_{1,d_m}, where we assume $x_1=1$. That is, the sequence $v'_n:=\sum_{i=2}^{N}c_{i,d_m}x_i^{-n-1}$ converges to 0. For a fixed $n_0\in\mathbb{Z}_{>0}$, consider the relations $v'_{n_0+k}=\sum_{i=2}^{N}(c_{i,d_m}x_i^{-n_0})x_i^{-k+1}$ for $k=1,\dots,N-1$. Regarding $c_{i,d_m}x_i^{-n_0}$ $(i=2,\dots,N)$ as the unknown, we can solve the linear equation by the use of the Vandermonde determinant for the matrix $(x_i^{-k+1})_{i=2,\dots,N,k=1,\dots,N-1}$. So, we obtain a linear approximation: $|c_{i,d_m}x_i^{-n_0}|\leq c\cdot\max\{|v'_{n_0+k}|\}_{k=1}^{N-1}$ $(i=2,\dots,N)$ for a constant $c>0$ which is independent of n_0. The RHS tends to zero as $n_0\to\infty$, whereas the LHS is unchanged. This implies $|c_{i,d_m}|=0$ $(i=2,\dots,N)$.)

Next, consider the case that the sequence v_n has more than two accumulating values. Suppose the subsequence $\{v_{n_m}\}_{m\in\mathbb{Z}_{>0}}$ converges to a non-zero value, say c. Recall the assumption that γ_{n-1}/γ_n converges to 1. So, the subsequence

$$\frac{\gamma_{n_m-1}}{\gamma_{n_m}}=\frac{v_{n_m-1}+\text{lower terms}}{v_{n_m}+\text{lower terms}}$$

should also converge to 1 as $m\to\infty$. In the denominator, the first term tends to $c\neq 0$ and the second term tends to 0. Similarly, in the numerator, the second term tends to 0. This implies that the first term in the numerator converges to c. Repeating the same argument, we see that for any $k\in\mathbb{Z}_{\geq 0}$, the subsequence $\{v_{n_m-k}\}_{m\in\mathbb{Z}_{>>0}}$ converges to the same c. Then, for each fixed $M\in\mathbb{Z}_{>0}$, the average sequence $\{(\sum_{k=0}^{M-1} v_{n_m-k})/M\}_{m\in\mathbb{Z}_{>>0}}$ converges to c, whereas, for sufficiently large M, the values are close to c_{1,d_m}. This implies $c = c_{1,d_m}$. In other words, the sequences $\{v'_{n_m-k}\}_{m\in\mathbb{Z}_{>>0}}$ for any $k \geq 0$ converge to 0. Then an argument as in the previous case implies $|c_{i,d_m}| = 0$ $(i = 2, \dots, N)$.

This ends the proof of Fact 2. □

We return to the general case, where P is finite rational accumulating of period h. For the standard partition $\{U^{[e]} \mid [e] \in \mathbb{Z}/h\mathbb{Z}\}$, put $T^{[e]} := T_{U^{[e]}}$. They decompose the unity: $\sum_{[e]\in\mathbb{Z}/h\mathbb{Z}} T^{[e]} = 1$. By the assumption, for each $0 \leq f < h$, the series $T^{[f]}P = t^f \sum_{m=0}^{\infty} \gamma_{f+mh}\tau^m$, considered as a series in $\tau = t^h$, is simple accumulating. Then Fact 2 implies that the highest order poles of $T^{[f]}P$ are only at solutions x of the equation $t^h - r^h = 0$. In view of the fact that the highest order of poles in t on the circle $|t| = r$ of $T^{[f]}P$ cannot exceed that of P (recall the explicit expression in Fact 1), and the fact $P = \sum_{[e]\in\mathbb{Z}/h\mathbb{Z}} T^{[e]}P$, the highest order poles of P are also only at solutions x of the equation $t^h - r^h = 0$. That is, $\Delta_P^{\mathrm{top}}(t)$ is a factor of $t^h - r^h$. For $0 \leq e, f < h$ and a root x of the equation $t^h - r^h$, we evaluate ((10.6.4) for $\{n_m = e + mh\}_{m=0}^{\infty}$ and $\{n_m = f + mh\}_{m=0}^{\infty}$)

$$\frac{T^{[f]}P}{T^{[e]}P}(t)\Big|_{t=x} = x^{f-e} \frac{\sum_{m=0}^{\infty} \gamma_{f+mh}\tau^m}{\sum_{m=0}^{\infty} \gamma_{e+mh}\tau^m}\Big|_{\tau=x^h=r^h} = x^{f-e} \lim_{m\to\infty} \frac{\gamma_{f+mh}}{\gamma_{e+mh}}.$$

Then a similar argument to that for (11.3.4) shows the formula

$$\text{(11.4.5)} \qquad \frac{T^{[f]}P}{T^{[e]}P}(t)\Big|_{t=x} = \begin{cases} x^{f-e}/(a_1^{[f]}a_1^{[f-1]}\cdots a_1^{[e+1]}) & \text{if } e < f, \\ 1 & \text{if } e = f, \\ x^{f-e}a_1^{[e]}a_1^{[e-1]}\cdots a_1^{[f+1]} & \text{if } e > f. \end{cases}$$

This implies that the order of poles of $T^{[e]}P(t)$ at a solution x of the equation $t^h - r^h$ is independent of $[e] \in \mathbb{Z}/h\mathbb{Z}$. On the other hand, (11.4.5) implies

$$\text{(11.4.6)} \qquad \frac{T^{[e]}P}{P}(t)\Big|_{t=x} = \frac{1}{A^{[e]}(x^{-1})}$$

(recall the $A^{[e]}(s)$ (11.3.2)). Let x be a solution of $t^h - r^h = 0$ but $\Delta_P^{\mathrm{op}}(x^{-1}) \neq 0$. Then $\delta_a(x^{-1}) = 0$ (see (11.3.6)) and $A^{[e]}(x^{-1}) = 0$ for $[e] \in \mathbb{Z}/h\mathbb{Z}$ (see 11.3 Assertion (i)). That is, $\frac{T^{[e]}P}{P}(t)$ has a pole at $t = x$. This implies that the pole of $P(t)$ at $t = x$ is of order $< d_m$ (otherwise, the pole at $t = x$ of $T^{[e]}P$ is of order d_m at most and can be canceled by dividing by P). That is, $\Delta_P^{\mathrm{top}}(t) \mid t^d\Delta_P^{\mathrm{op}}(t^{-1})$.

Fact 3. *Let $P(t)$ (11.2.1) belong to $\mathbb{C}\{t\}_r$ and be finitely accumulating. Then*

(i) *There exists a positive constant c such that $\gamma_n \geq cr^{-n}n^{d_m-1}$ for $n \gg 0$.*
(ii) $t^d\Delta_P^{\mathrm{op}}(t^{-1}) \mid \Delta_P^{\mathrm{top}}(t)$.

Proof. (i) Consider the Taylor expansion of the function $(*)$. Using the notation v_n of the proof of Fact 2, we have $\gamma_n = -v_n \frac{r^{-n-1}(n;d_m)}{(d_m-1)!}$ + terms coming from poles of order $< d_m$ + terms coming from $Q(t)$, where $v_n = \sum_i c_{i,d_m}(x_i/r)^{-n-1}$ depends only on $n \bmod h$ since x_i is a root of the equation $t^h - r^h = 0$. Not all of these are zero (otherwise $c_{i,d_m} = 0$ for all i). Let us show that none of the v_n are zero. Suppose the contrary and $v_e = 0 \neq v_f$. Then one observes easily $\lim_{m\to\infty} \gamma_{e+mh}/\gamma_{f+mh} = 0$. This contradicts the assumption $\Omega_1(P) \subset [u,v]$ (positivity of initials).

(ii) By definition, the fractional expansion of $\Delta_P^{\mathrm{top}}(t)P(t)$ has poles of order at most $d_m - 1$. This means that its $(n-k)$th Taylor coefficient satisfies

$$(**) \qquad \gamma_{n-k}\cdot\alpha_l + \gamma_{n-k-1}\cdot\alpha_{l-1} + \cdots + \gamma_{n-k-l}\cdot 1 \sim o((n-k)^{d_m-1}r^{-(n-k)})$$

as $n-k\to\infty$ $(k,n\in\mathbb{Z}_{\geq 0})$ (here, $\Delta_P^{\mathrm{top}}(t) = t^l + \alpha_1 t^{l-1} + \cdots + \alpha_l$). Let $\sum_k a_k s^k \in \Omega(P)$ be the limit of a subsequence $\{X_{n_m}(P)\}_{m\in\mathbb{Z}_{\geq 0}}$ (11.2.2). Divide $(**)$ by γ_n. Then, using part (i), one has

$$a_k\alpha_l + a_{k+1}\alpha_{l-1} + \cdots + a_{k+l} = 0$$

for any $k \geq 0$. Thus $s^l\Delta_P^{\mathrm{top}}(1/s)a(s)$ is a polynomial in s of degree $< l$. Thus the denominator $\Delta_P^{\mathrm{op}}(s)$ of $a(s)$ divides $s^l\Delta_P^{\mathrm{top}}(s^{-1})$. So, (ii) is shown. □

We showed (11.4.3). (11.4.4) follows from (11.3.11) and (11.4.3). □

Example. Recall Machì's example 11.2 for the modular group Γ. We have

$$T_eP(t) = \sum_{k=0}^{\infty} \sharp\Gamma_{2k}t^{2k} = \frac{1+5t^2}{(1-2t^2)(1-t^2)},$$

$$T_oP(t) = \sum_{k=0}^{\infty} \sharp\Gamma_{2k+1}t^{2k+1} = \frac{2t(2+t^2)}{(1-2t^2)(1-t^2)},$$

Then the transformation matrix is given by

$$\begin{bmatrix} \frac{T_eP(t)}{P_{\Gamma,G}(t)} = \frac{1+5t^2}{(1+t)^2(1+2t)}\Big|_{t=\frac{1}{\sqrt{2}}} & \frac{T_oP(t)}{P_{\Gamma,G}(t)} = \frac{2t(2+t^2)}{(1+t)^2(1+2t)}\Big|_{t=\frac{1}{\sqrt{2}}} \\ \frac{T_eP(t)}{P_{\Gamma,G}(t)} = \frac{1+5t^2}{(1+t)^2(1+2t)}\Big|_{t=\frac{-1}{\sqrt{2}}} & \frac{T_oP(t)}{P_{\Gamma,G}(t)} = \frac{2t(2+t^2)}{(1+t)^2(1+2t)}\Big|_{t=\frac{-1}{\sqrt{2}}} \end{bmatrix}$$

$$= \begin{bmatrix} 7(5\sqrt{2}-7) & 5(10-7\sqrt{2}) \\ -7(5\sqrt{2}+7) & 5(10+7\sqrt{2}) \end{bmatrix}$$

whose determinant is equal to $5\cdot 7/\sqrt{2} \neq 0$.

§11.5. The residual representation of trace elements

As the goal of the present paper, under further asummptions (i) $\sharp\Omega(\Gamma,G)<\infty$ and (ii) $P_{\Gamma,G}\in\mathbb{C}\{t\}_{r_{\Gamma,G}}$, we show a trace formula, which states that *the sum of the limit elements in an orbit of the inertia group is expressed by a linear combination of the proportions of residues of the Poincaré series* $P_{\Gamma,G}(t)$ *and* $P_{\Gamma,G}\mathcal{M}(t)$ (11.2.6,7) *at the poles on the circle of their convergence radius, where the coefficients are given by special values of the opposite polynomials* $A^{[e]}(s)$.

We first show the following basic consequence of the finiteness $\sharp\Omega(\Gamma,G)<\infty$.

Lemma. *Let* (Γ,G) *be the pair of a monoid and its finite generating system, which satsfies Assumption* 1 *but not necessarily* 2. *If the limit space* $\Omega(\Gamma,G)$ *is finite, then it is finite rationally accumulating with respect to the standard partition* $\mathcal{U}_{\tilde h}$ *of* $\mathbb{Z}_{\geq 0}$ *for some* $\tilde h>0$, *and* $\tilde\tau_\Omega$ *acts transitively on* $\Omega(\Gamma,G)$ *of period* $\tilde h$. *In particular,* $\tilde\tau_\Omega$ *is invertible.*

Proof. Recall the action $\tilde\tau_\Omega$ on $\Omega(\Gamma,G)$ (11.2 Lemma). Then finiteness of $\Omega(\Gamma,G)$ implies that there exist $\omega\in\Omega(\Gamma,G)$ and $\tilde h\in\mathbb{Z}_{>0}$ such that $(\tilde\tau_\Omega)^{\tilde h}\omega=\omega$ and $(\tilde\tau_\Omega)^{\tilde h'}\omega\neq\omega$ for $0<\tilde h'<\tilde h$. Consider the set $U_\omega:=\{n\in\mathbb{Z}_{\geq 0}\mid \mathcal{M}(\Gamma_n)/\sharp\Gamma_n\in\mathcal{V}_\omega\}$ (here, $\mathcal{V}_\omega$ is an open neighborhood of ω in $\mathcal{L}_{\mathbb{R},\infty}$ such that $\overline{\mathcal{V}_\omega}\cap\Omega(\Gamma,G)=\{\omega\}$). Then the periodicity of the action of $\tilde\tau_\Omega$ on ω implies (using an argument similar to that found in the proof of 11.2 Lemma, replacing $a\in\Omega(P)$ by $\omega\in\Omega(\Gamma,G)$ and h by $\tilde h$) that U_ω is, up to a finite number of elements, equal to a rational set $U^{[\tilde e]}$ for some $[\tilde e]\in\mathbb{Z}/\tilde h\mathbb{Z}$, and the following equality holds:

$$\Omega(\Gamma,G)=\{\omega,\tilde\tau_\Omega\omega,\dots,(\tilde\tau_\Omega)^{\tilde h-1}\omega\}.$$

This implies the finite rationality of $\Omega(\Gamma,G)$ and the periodicity of $\tilde\tau_\Omega$. □

Let $\Omega(\Gamma,G)$ be finite rationally accumulating of period $\tilde h$, which consists of

$$\omega^{[\tilde e]}_{\Gamma,G}:=\lim_{m\to\infty}{}^{\mathrm{cl}}\,\frac{\mathcal{M}(\Gamma_{\tilde e+m\tilde h})}{\sharp\Gamma_{\tilde e+m\tilde h}} \tag{11.5.1}$$

for $[\tilde e]\in\mathbb{Z}/\tilde h\mathbb{Z}$. Then $\Omega(P_{\Gamma,G})$ is also finite rationally accumulating of period h such that $h\mid\tilde h$ (cf. 11.2 Lemma), since the sequence $\{\pi(\mathcal{M}(\Gamma_n)/\sharp\Gamma_n)=X_n(P_{\Gamma,G})\}_{n\in U^{[\tilde e]}}$ for the rational set $U^{[\tilde e]}:=\{n\in\mathbb{Z}_{\geq 0}\mid n \bmod \tilde h=[\tilde e]\}$ for any $[\tilde e]\in\mathbb{Z}/\tilde h\mathbb{Z}$ is convergent to $\pi(\omega^{[\tilde e]}_{\Gamma,G})$. Let $\tilde h_{\Gamma,G}$ and $h_{\Gamma,G}$ be the minimal periods of $\Omega(\Gamma,G)$ and $\Omega(P_{\Gamma,G})$, respectively. Then π is equivariant under the $\tilde\tau_\Omega$-action on $\Omega(\Gamma,G)$ and the τ_Ω-action on $\Omega(P_{\Gamma,G})$ so that the subgroup $h_{\Gamma,G}\mathbb{Z}/\tilde h_{\Gamma,G}\mathbb{Z}$ of $\mathbb{Z}/\tilde h_{\Gamma,G}\mathbb{Z}\simeq\langle\tilde\tau_\Omega\rangle$, called the *inertia subgroup*, acts simply transitively on the fibers of π. That is, $\Omega(\Gamma,G)/(h_{\Gamma,G}\mathbb{Z}/\tilde h_{\Gamma,G}\mathbb{Z})\simeq\Omega(P_{\Gamma,G})$. We call $m_{\Gamma,G}:=\tilde h_{\Gamma,G}/h_{\Gamma,G}$ the *inertia* of (Γ,G) so that the inertia subgroup is isomorphic to $\mathbb{Z}/m_{\Gamma,G}\mathbb{Z}$.

Definition. The *trace element* for $[e] \in \mathbb{Z}/h_{\Gamma,G}\mathbb{Z}$ is the sum of the elements in the fiber $\pi_\Omega^{-1}(a^{[e]})$ (= an orbit of the inertia group $h_{\Gamma,G}\mathbb{Z}/\tilde{h}_{\Gamma,G}\mathbb{Z}$):

$$\text{(11.5.2)} \qquad \mathrm{Trace}^{[e]}\,\Omega(\Gamma,G) := \sum_{[\tilde{e}]\in\mathbb{Z}/\tilde{h}_{\Gamma,G}\mathbb{Z},\,[\tilde{e}]\subset[e]} \omega_{\Gamma,G}^{[\tilde{e}]} = \sum_{i=1}^{m_{\Gamma,G}} \omega_{\Gamma,G}^{[\tilde{e}+ih_{\Gamma,G}]},$$

which belongs to the space $\overline{\mathbb{R}\Omega}(\Gamma,G)$.

The periodicity of $\tilde{\tau}_\Omega$ implies the invertibility of $\tilde{\tau}$ (11.2.14). As its consequence, let us introduce a $\tilde{\sigma}$-action on the module $\overline{\mathbb{R}\Omega}(\Gamma,G)$.

Definition. For any $[\tilde{e}] \in \mathbb{Z}/\tilde{h}_{\Gamma,G}\mathbb{Z}$, put $[e] \equiv [\tilde{e}] \bmod h_{\Gamma,G}$ and define

$$\text{(11.5.3)} \qquad \tilde{\sigma}(\omega_{\Gamma,G}^{[\tilde{e}]}) := \tilde{\tau}^{-1}(\omega_{\Gamma,G}^{[\tilde{e}]}) = \frac{1}{a_1^{[e+1]}}\omega_{\Gamma,G}^{[\tilde{e}+1]}.$$

The endomorphism $\tilde{\sigma}$ is semisimple since $\tilde{\sigma}^{\tilde{h}_{\Gamma,G}} = r_{\Gamma,G}^{\tilde{h}_{\Gamma,G}}\mathrm{id}_{\overline{\mathbb{R}\Omega}(\Gamma,G)}$ (cf. (11.3.3)). The $\mathbb{R}$-linear map π (11.2.15) is equivariant with respect to the endomorphisms $\tilde{\sigma}$ and σ (11.3.12). By the definition, we see that the $\tilde{\sigma}$-action takes, up to a constant factor, a trace element to the other trace element

$$\text{(11.5.4)} \qquad \tilde{\sigma}(\mathrm{Trace}^{[e]}\,\Omega(\Gamma,G)) := \frac{1}{a_1^{[e+1]}}\,\mathrm{Trace}^{[e+1]}\,\Omega(\Gamma,G)$$

for all $[e] \in \mathbb{Z}/h_{\Gamma,G}\mathbb{Z}$. In view of (11.3.3), this in particular implies

$$\text{(11.5.5)} \qquad (1-(r_{\Gamma,G}\tilde{\sigma})^h)(\mathrm{Trace}^{[e]}\,\Omega(\Gamma,G)) = 0.$$

With the results of 11.3 and 11.4, the next theorem is now straightforward.

Theorem. 6. *Let* (Γ,G) *be a pair of a monoid and its finite generating system with* $1 \notin G$, *satisfying Assumptions* 1 *and* 2. *Suppose* (i) $\Omega(\Gamma,G)$ *is finite, and* (ii) $P_{\Gamma,G} \in \mathbb{C}\{t\}_{r_{\Gamma,G}}$. *Let* $\tilde{h}_{\Gamma,G}$ *and* $h_{\Gamma,G}$ *be the minimal periods of* $\Omega(\Gamma,G)$ *and* $\Omega(P_{\Gamma,G})$, *respectively, and put* $\tilde{m}_{\Gamma,G} := \tilde{h}_{\Gamma,G}/h_{\Gamma,G}$. *Then, for any* $[e] \in \mathbb{Z}/h_{\Gamma,G}\mathbb{Z}$, *the following equality holds*

$$\text{(11.5.6)} \qquad h_{\Gamma,G}\,\mathrm{Trace}^{[e]}\,\Omega_{\Gamma,G} - \Bigg(\sum_{x^{-1}\in V(\delta_{P_{\Gamma,G}})} \frac{\delta_{P_{\Gamma,G}}(\tilde{\sigma})}{1-x\tilde{\sigma}}\Bigg)\Delta^{\mathrm{op}}_{P_{\Gamma,G}}(\tilde{\sigma})\,\mathrm{Trace}^{[e]}\,\Omega_{\Gamma,G}$$
$$= m_{\Gamma,G} \sum_{x\in V(\Delta^{\mathrm{top}}_{P_{\Gamma,G}})} A^{[e]}(x^{-1})\frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\bigg|_{t=x}.$$

where we put $\delta_{P_{\Gamma,G}}(\tilde{\sigma}) := (1-r^{h_{\Gamma,G}}\tilde{\sigma}^{h_{\Gamma,G}})/\Delta^{\mathrm{op}}_{P_{\Gamma,G}}(\tilde{\sigma})$ (*cf.* (11.3.6)) *and we denote by* $V(P)$ *the zero locus of the polynomial* P.

Proof. Let us call $\frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\big|_{t=x}$ on the RHS of (11.5.6) a *residue element*, since it is the ratio of the residues of $P_{\Gamma,G}\mathcal{M}(t)$ and $P_{\Gamma,G}(t)$ at the point $t = x$. Let us first express the residue element by a sum of trace elements. For this purpose, consider the decomposition of unity:

$$(*) \qquad \frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)} = \sum_{[\tilde f]\in\mathbb{Z}/\tilde h_{\Gamma,G}\mathbb{Z}} \frac{T^{[\tilde f]}P_{\Gamma,G}(t)}{P_{\Gamma,G}(t)} \cdot \frac{T^{[\tilde f]}P_{\Gamma,G}\mathcal{M}(t)}{T^{[\tilde f]}P_{\Gamma,G}(t)}.$$

where $T^{[\tilde f]} = T_{U^{[\tilde f]}}$ (11.4.2) is the action of the rational set $U^{[\tilde f]}$ of the standard subdivision for $\Omega(\Gamma, G)$ so that $\sum_{\tilde f\in\mathbb{Z}/\tilde h_{\Gamma,G}\mathbb{Z}} T^{[\tilde f]} = 1$. Let x be a root of $\Delta^{\mathrm{top}}_{P_{\Gamma,G}}(t) = 0$, and consider the evaluation of both sides of $(*)$ at $t = x$. The LHS gives, by definition, the residue element at x. By a slight generalization of the formula (11.3.6), the first factor on the RHS is given by $1/A^{[\tilde f]}(x^{-1}) = 1/(m_{\Gamma,G}\cdot A^{[f]}(x^{-1}))$ (note that $A^{[f]}(x^{-1}) \neq 0$ since $\delta_{P_{\Gamma,G}}(x^{-1}) \neq 0$), where $[f] := [\tilde f] \bmod h_{\Gamma,G}$. The second factor on the RHS is

$$\frac{\sum_{m=0}^{\infty}\mathcal{M}(\Gamma_{\tilde f+m\tilde h_{\Gamma,G}})t^{\tilde f+m\tilde h_{\Gamma,G}}}{\sum_{m=0}^{\infty}\sharp\Gamma_{\tilde f+m\tilde h_{\Gamma,G}}t^{\tilde f+m\tilde h_{\Gamma,G}}}\Bigg|_{t=x} = \frac{\sum_{m=0}^{\infty}\mathcal{M}(\Gamma_{\tilde f+m\tilde h_{\Gamma,G}})\tilde t^{m}}{\sum_{m=0}^{\infty}\sharp\Gamma_{\tilde f+m\tilde h_{\Gamma,G}}\tilde t^{m}}\Bigg|_{\tilde t=r^{\tilde h_{\Gamma,G}}}$$

where, on the RHS, $\tilde t := t^{\tilde h_{\Gamma,G}}$ is the new variable and $r^{\tilde h_{\Gamma,G}} = x^{\tilde h_{\Gamma,G}}$ is the common singular point of the two power series (the numerator and the denominator) in $\tilde t$ at the crossing of the positive real axis and the circle of the convergence radius (cf. 10.6 Lemma (i)). Then, since the coefficients of the series are non-negative, this ratio of the residue values is equal to the limit of the ratio of the coefficients of the series (cf. (10.6.4)) $\lim^{\mathrm{cl}}_{m\to\infty}\mathcal{M}(\Gamma_{\tilde f+m\tilde h_{\Gamma,G}})/\sharp\Gamma_{\tilde f+m\tilde h_{\Gamma,G}}$ which is nothing but the limit element $\omega^{[\tilde f]}_{\Gamma,G}$ (11.5.1). Put $\tilde f = f + ih_{\Gamma,G}$ for $0 \le f < h_{\Gamma,G}$ and $0 \le i < m_{\Gamma,G}$. Then the RHS turns into

$$\frac{1}{m_{\Gamma,G}}\sum_{[f]\in\mathbb{Z}/h_{\Gamma,G}\mathbb{Z}}\frac{1}{A^{[f]}(x^{-1})}\sum_{i=0}^{m_{\Gamma,G}-1}\omega^{[f+ih_{\Gamma,G}]}_{\Gamma,G}$$

where the second sum gives the trace $\mathrm{Trace}^{[f]}\,\Omega(\Gamma, G)$. That is,

$$(11.5.7) \qquad \frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\Bigg|_{t=x} = \frac{1}{m_{\Gamma,G}}\sum_{[f]\in\mathbb{Z}/h_{\Gamma,G}\mathbb{Z}}\frac{1}{A^{[f]}(x^{-1})}\,\mathrm{Trace}^{[f]}\,\Omega(\Gamma, G).$$

For a fixed $[e] \in \mathbb{Z}/h_{\Gamma,G}\mathbb{Z}$, we multiply both sides of (11.5.7) $A^{[e]}(x^{-1})$, and sum over the index x running over the set $V(\Delta^{\mathrm{top}}_{P_{\Gamma,G}})$ of all roots of $\Delta^{\mathrm{top}}_{P_{\Gamma,G}}(t) = 0$, whose LHS is equal to the RHS of (11.5.6). Using (11.4.6), one observes that $A^{[e]}(x^{-1})/A^{[f]}(x^{-1})$ is equal to the LHS of (11.4.5). Replace the summation index

"$[f] \in \mathbb{Z}/h_{\Gamma,G}\mathbb{Z}$" in (11.5.7) by "$[e+i]$ for $i = 0, \ldots, h_{\Gamma,G} - 1$" for fixed $[e]$. Using the first line of the RHS of (11.4.5) and repeated application of (11.5.4), the sum on the RHS of (11.5.7) turns into

$$\begin{aligned}
&\frac{1}{m_{\Gamma,G}} \sum_{x \in V(\Delta^{\mathrm{top}}_{P_{\Gamma,G}})} \sum_{i=0}^{h_{\Gamma,G}-1} \frac{A^{[e]}(x^{-1})}{A^{[e+i]}(x^{-1})} \operatorname{Trace}^{[e+i]} \Omega_{\Gamma,G} \\
&\quad = \frac{1}{m_{\Gamma,G}} \sum_{x \in V(\Delta^{\mathrm{top}}_{P_{\Gamma,G}})} \sum_{i=0}^{h_{\Gamma,G}-1} \frac{x^i}{a_1^{[e+i]} a_1^{[e+i-1]} \cdots a_1^{[e+1]}} \prod_{j=1}^{i} (a_1^{[e+j]} \tilde{\sigma}) \operatorname{Trace}^{[e]} \Omega_{\Gamma,G} \\
&\quad = \frac{1}{m_{\Gamma,G}} \sum_{x \in V(\Delta^{\mathrm{top}}_{P_{\Gamma,G}})} \sum_{i=0}^{h_{\Gamma,G}-1} x^i \tilde{\sigma}^i \operatorname{Trace}^{[e]} \Omega_{\Gamma,G}.
\end{aligned}$$

Here, we note that the sum $\sum_{i=0}^{h_{\Gamma,G}-1} x^i \tilde{\sigma}^i$ is expressed as $\frac{1-(r_{\Gamma,G}\tilde{\sigma})^{h_{\Gamma,G}}}{1-x\tilde{\sigma}}$ and that $x \in V(\Delta^{\mathrm{top}}_{P_{\Gamma,G}})$ is equivalent to $x^{-1} \in V(\Delta^{\mathrm{op}}_{P_{\Gamma,G}})$ due to the duality (11.4.3). We note further that the identity

$$\sum_{x^{-1} \in V(1-(r_{\Gamma,G}s)^{h_{\Gamma,G}})} \frac{1-(r_{\Gamma,G}s)^{h_{\Gamma,G}}}{1-xs} = h_{\Gamma,G}$$

holds (in the polynomial ring $\mathbb{R}[s]$). Therefore, recalling (11.3.6),

$$\delta_{P_{\Gamma,G}}(s) \cdot \Delta^{\mathrm{op}}_{P_{\Gamma,G}}(s) = 1 - (r_{\Gamma,G}s)^{h_{\Gamma,G}},$$

we calculate further the sum as follows:

$$\begin{aligned}
&\frac{1}{m_{\Gamma,G}} \left(\sum_{x^{-1} \in V(\Delta^{\mathrm{op}}_{P_{\Gamma,G}})} \frac{1-(r_{\Gamma,G}\tilde{\sigma})^{h_{\Gamma,G}}}{1-x\tilde{\sigma}} \right) \operatorname{Trace}^{[e]} \ \Omega_{\Gamma,G} \\
&\quad = \frac{1}{m_{\Gamma,G}} \left(h_{\Gamma,G} \cdot \mathrm{id}_{\overline{\mathbb{R}\Omega}(\Gamma,G)} - \sum_{x^{-1} \in V(\delta_{P_{\Gamma,G}})} \frac{\delta_{P_{\Gamma,G}}(\tilde{\sigma})}{1-x\tilde{\sigma}} \Delta^{\mathrm{op}}_{P_{\Gamma,G}}(\tilde{\sigma}) \right) \operatorname{Trace}^{[e]} \Omega_{\Gamma,G}.
\end{aligned}$$

This gives the LHS of (11.5.6), and hence the theorem is proven. □

Remark. 1. The second term of the LHS of (11.5.6) belongs to the kernel of π, since $\pi(\Delta^{\mathrm{op}}_{P_{\Gamma,G}}(\tilde{\sigma}) \operatorname{Trace}^{[e]} \Omega_{\Gamma,G}) = m_{\Gamma,G} \Delta^{\mathrm{op}}_{P_{\Gamma,G}}(\sigma) a^{[e]} = 0$. Therefore, we ask whether

$$\Delta^{\mathrm{op}}_{P_{\Gamma,G}}(\tilde{\sigma}) \operatorname{Trace}^{[e]} \Omega_{\Gamma,G} = 0.$$

This is equivalent to the statement that *the* $\mathbb{R}[\tilde{\sigma}]$*-module spanned by the trace elements* $\operatorname{Trace}^{[e]} \Omega_{\Gamma,G}$ *is isomorphic to the* $\mathbb{R}[\sigma]$*-module* $\overline{\mathbb{R}\Omega}(P_{\Gamma,G})$*.*

2. One can directly calculate the following formula:

$$\pi\left(\frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\right) = \frac{1}{1-st}. \tag{11.5.8}$$

Specializing t to a root x of $\Delta_P(t) = 0$ in the formula gives the Cauchy kernel $\frac{1}{1-xs}$. Therefore, the π image of (11.5.6) turns out to be the formula (11.3.9).

3. If (Γ, G) is a group of polynomial growth, then $\Delta_{P_{\Gamma,G}}(t) = (1-t)^{l+1}$ (where $l = \mathrm{rank}(\Gamma) > 0$) is never reduced. However, due to (10.6.4), one sees directly the conclusion of the theorem: $\frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\big|_{t=1} = \sum_{S\in\langle\Gamma,G\rangle_0} \frac{\varphi(S)}{\sharp\,\mathrm{Aut}(S)}$ (cf. (11.1.8)).

4. Due to D. Epstein [E2], we know that there is a wide class of groups satisfying assumption (ii) in Theorem 6. See the remarks and problems in the next section.

§12. Concluding remarks and problems

We are only at the start of the study of the space $\Omega(\Gamma, G)$ for discrete groups and monoids. Here are some problems and conjectures for further study.

1. A formula similar to (11.5.6) should be true without assuming the finiteness of $\Omega(\Gamma, G)$, where the formula should be rewritten as an integral formula.

Problem 1. Find measures ν_a on $\pi^{-1}(a)$ and μ_a on the set $\mathrm{Sing}(P_{\Gamma,G})$ of singularities of the series on the circle of radius r so that the following holds:

$$\frac{\int_{\pi^{-1}(a)} \omega_{\Gamma,G}\, d\nu_a}{\int_{\pi^{-1}(a)} d\nu_a} = \int_{\mathrm{Sing}(P_{\Gamma,G})} \frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\bigg|_{t=x} d\mu_{a,x} \tag{12.9}$$

for $a \in \Omega(P_{\Gamma,G})$, where $\omega_{\Gamma,G}$ is the tautological map from $\Omega(\Gamma, G)$ to $\mathcal{L}_{\mathbb{R},\infty}$.

2. It is known ([E2]) that, for a wide class of groups, the assumption (ii) in Theorem 6 is satisfied in a stronger (global) form: the Poincaré series $P_{\Gamma,G}(t)$ and the growth series $P_{\Gamma,G}\mathcal{M}(t)$ are rational functions, where the denominator polynomial $\Delta_{\Gamma,G}(t)$ for the rational function $P_{\Gamma,G}(t)$ is also the universal denominator for the rational functions $P_{\Gamma,G}\mathcal{M}(t)$. More generally, $P_{\Gamma,G}(t)$ analytically continues to a meromorphic function on a (branched) covering domain of $\mathbb{C}$ (in this case, $\Delta_{\Gamma,G}(t)$ is defined only up to a unit factor).

We remark that the denominator polynomial $\Delta_{P_{\Gamma,G}}(t)$ for the Poincaré series $P_{\Gamma,G}(t)$ as an element of $\mathbb{C}\{t\}_{r_{\Gamma,G}}$ (see 11.4 Definition) is the factor of $\Delta_{\Gamma,G}(t)$ involving the roots x with minimal $|x| = r_{\Gamma,G}$ (in the case of $P_{\Gamma,G}(t)$ defined on a covering of $\mathbb{C}$, whether $|x|$ of a pole x makes sense or not is unclear).

Inspired by these observations, in order to get a global understanding of the monoid (Γ, G), we propose studying the *higher residues of* $P_{\Gamma,G}\mathcal{M}(t)$ *at any root of* $\Delta_{\Gamma,G}(t)$, which are defined and shown to belong to $\mathcal{L}_{\mathbb{C},\infty}$ as follows.

Definition. Let x be a root of $\Delta_{\Gamma,G}(t) = 0$ of multiplicity $d_x > 0$. Then, for $0 \le i < d_x$, we define the *higher residue of depth i of the limit function $P_{\Gamma,G}\mathcal{M}(t)$ at x* by the formula

$$\left(\frac{d^i}{dx^i}\frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\right)\Big|_{t=x}. \tag{12.10}$$

Assertion. *The higher residues belong to the space $\mathcal{L}_{\mathbb{C},\infty}$ at infinity.*

Proof. By the definition (8.4.1),

$$\overline{K}\left(\frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\right) = \sum_{n=0}^{\infty}\mathcal{M}(\Gamma_n)\frac{t^n}{P_{\Gamma,G}(t)},$$

where the coefficients $t^n/P_{\Gamma,G}(t)$ are rational functions divisible by $\Delta_{\Gamma,G}$ and have zeros of order d_x at zeros x of $\Delta_{\Gamma,G}$. Since the kabi map $\overline{K}$ (8.4.1) is continuous with respect to the classical topology, this implies $\overline{K}\big(\big(\frac{d^i}{dt^i}\frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\big)\big|_{t=x}\big) = 0$ for $0 \le i < d_x$. □

Using the higher residues for all roots of $\Delta_{\Gamma,G}(t) = 0$, we introduce the *global module of limit elements for* (Γ, G):

$$\mathcal{L}(\Gamma,G) := \bigoplus_{0<r<\infty}\ \bigoplus_{\substack{x:\text{ a root of}\\ \Delta_{\Gamma,G}(t)=0\text{ with }|x|=r}}\ \bigoplus_{0\le i<d_x}\mathbb{C}\cdot\left(\frac{d^i}{dx^i}\frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\right)\Big|_{t=x}, \tag{12.11}$$

which is doubly filtered: one filtration is given by the absolute values $|x|$ of the roots of $\Delta_{\Gamma,G}(t) = 0$, and the other by the order i of the depth of residues at x.

Theorem 6 in §11 states relationships between the $\tilde{\tau}^{h_{\Gamma,G}}$-invariant part of the module $\overline{\mathbb{R}\Omega}(\Gamma, G)$ with the filter at $|x| = r_{\Gamma,G} := \inf\{r\}$ and the first residues part of the module $\mathcal{L}(\Gamma, G)$. We ask about its generalization.

Problem 2. What is the relationship between the modules $\overline{\mathbb{R}\Omega}(\Gamma, G)$, $\mathcal{L}(\Gamma, G)$ and $\mathcal{L}_{\mathbb{C},\infty}\langle\Gamma, G\rangle$? Find a generalization of the theorems in §11 and, in particular, of (11.4.3), (11.4.4) and (11.5.6) in this context.

3. Another important aspect of the higher residues is that the Poincaré series $P_{\Gamma,G}\mathcal{M}(t)$ and $P_{\Gamma,G}(t)$ are series, up to variables in Conf_0, defined over integers $\mathbb{Z}$. Therefore, in case $\Delta_{\Gamma,G}(t)$ is a polynomial in $\mathbb{Z}[t]$, they are rational functions defined over $\mathbb{Q}$, and hence the higher residue (12.10) is defined over the algebraic number field $\mathbb{Q}(x)$ for a root x of $\Delta_{\Gamma,G}(t) = 0$. The action of an element σ of the Galois group of the splitting field of $\Delta_{\Gamma,G}(t)$ commutes with the kabi map $\overline{K}$ (8.4.1), and seems to take the space spanned by the higher residues at x to that at $\sigma(x)$, and hence induces an action on $\mathcal{L}(\Gamma, G)_{\mathbb{Q}}$.

Actually, in several interesting examples (surface groups of Cannon [Ca], Artin monoids [Sa5,6]), we observe that the denominator polynomial $\Delta_{\Gamma,G}(t)$ is, up to the factor of a power of $1-t$, irreducible. In view of the above observation, the limit space $\mathbb{C}\cdot\Omega(\Gamma,G)$ studied in the present paper is not "isolated" but related by the action of the Galois group to the residue modules at other places x with $|x|>r_{\Gamma,G}$. However, no concrete example is known yet.

On the other hand, the higher residue module at $t=1$ is "isolated" (with respect to the Galois group action). There are a few examples of higher poles at $t=1$ (see [Sa6]), but we do not yet understand their nature and role.

Example ([Sa2]). Consider the infinite cyclic group $(\mathbb{Z},\pm 1)$. Then, the growth function is given by $P_{\mathbb{Z},\pm 1}(t)=\frac{1+t}{(1-t)^2}$ and the principal part of the singularities of $P_{\mathbb{Z},\pm 1}\mathcal{M}(t)$ is given by

$$P_{\mathbb{Z},\pm 1}\mathcal{M}(t)=\sum_{m=0}^{\infty}\varphi(I_m)\left(\frac{2}{(1-t)^2}-\frac{m}{1-t}+R_m\right)$$

where I_m is a linear graph of m vertices and R_m is a polynomial of degree $<[(m-1)/2]$ in t. Therefore, the two higher residues at $t=1$ given by

$$\left.\frac{P_{\mathbb{Z},\pm 1}\mathcal{M}}{P_{\mathbb{Z},\pm 1}}\right|_{t=1}=\sum_{m=0}^{\infty}\varphi(I_m),$$
$$\left.\left(\frac{d}{dt}\frac{P_{\mathbb{Z},\pm 1}\mathcal{M}}{P_{\mathbb{Z},\pm 1}}\right)\right|_{t=1}=\sum_{m=0}^{\infty}\frac{m-1}{2}\varphi(I_m),$$

span the space $\mathcal{L}_{\mathbb{R},\infty}\langle\mathbb{Z},\pm 1\rangle$, where the first one is the limit element in $\Omega(\mathbb{Z},\pm 1)$.

In view of these observations, we pose the following problems.

Problem 3. (i) Describe the action of the Galois group of the splitting field of $\Delta_{\Gamma,G}(t)=0$ on $\mathcal{L}(\Gamma,G)_{\mathbb{Q}}$. Clarify the role of the classical part $\mathbb{C}\cdot\Omega(\Gamma,G)$.

(ii) When is the denominator polynomial $\Delta_{\Gamma,G}(t)$, up to a factor of a power of $1-t$, irreducible over $\mathbb{Z}$?

(iii) What is the meaning of the residue module at $t=1$:

$$\mathcal{L}(\Gamma,G)_1:=\bigoplus_{0\le i<d_1}\mathbb{R}\cdot\left.\left(\frac{d^i}{dt^i}\frac{P_{\Gamma,G}\mathcal{M}(t)}{P_{\Gamma,G}(t)}\right)\right|_{t=1}?$$

4. Including Machì's example, there are a number of examples where $\Omega(P_{\Gamma,G})$ is finite. However, we do not know an example when $\Omega(\Gamma,G)$ is finite except for the simple accumulating cases (e.g. (11.1.9)). We conjecture the following.

Conjecture 4. For any hyperbolic group Γ with any finite generating system G, the limit space $\Omega(\Gamma,G)$ is finite accumulating.

Evidence is provided by Coornaert [C]: *if* Γ *is hyperbolic, then there exist positive real constants* c_1, c_2 *such that* $c_1 r_{\Gamma,G}^{-n} \leq \sharp\Gamma_n \leq c_2 r_{\Gamma,G}^{-n}$. This implies that the property in the Fact in the proof of 11.3 Assertion holds for hyperbolic groups without assuming the finite rational accumulation of $\Omega(P_{\Gamma,G})$. We further expect that Coornaert's arguments can be lifted to the level of $A(S, \Gamma_n)$.

5. The following groups are not hyperbolic. However, because of their geometric significance, it is interesting to ask the following problems.

Problem 5.1. Are the limit spaces $\Omega(\Gamma, G)$ for the following pair of a group and a system of generators simple or finite?

1. Artin groups of finite type with the generating systems given in [BS], [Sa4].
2. The fundamental groups of the complement of free divisors with respect to the generating system defining a positive homogeneous monoid structure [S-I].

In these examples, G generates a positive homogeneous monoid Γ_+ in Γ such that $\Gamma = \bigcup_{n=0}^{\infty} \Delta^{-n}\Gamma_+$, where Δ is a fundamental element.

Problem 5.2. Clarify the relationships between $\Omega(\Gamma, G)$, $\Omega(P_{\Gamma,G})$, $\Omega(\Gamma_+, G)$ and $\Omega(P_{\Gamma_+,G})$ (see [Ba, Chap. 13] for $\Gamma_+ = (\mathbb{Z}_+)^2$, and [Sa5] for Artin monoids).

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