

Adams Operations on Higher Arithmetic K -theory

by

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Abstract

We construct Adams operations on the rational higher arithmetic K -groups of a proper arithmetic variety. The definition applies to the higher arithmetic K -groups given by Takeda as well as to the groups suggested by Deligne and Soulé, by means of the homotopy groups of the homotopy fiber of the regulator map. They are compatible with the Adams operations on algebraic K -theory. The definition relies on the chain morphism representing Adams operations in higher algebraic K -theory given previously by the author. It is shown that this chain morphism commutes strictly with the representative of the Beilinson regulator given by Burgos and Wang.

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Introduction

This paper contributes to the development of a *higher arithmetic intersection theory* following the steps of the higher algebraic intersection theory but suitable for arithmetic varieties. In [3], the author, together with Burgos, defined the higher arithmetic Chow ring for any arithmetic variety over a field, extending the construction given by Goncharov in [14] which was valid only for proper arithmetic varieties. The question arises whether these groups are related to the higher arithmetic K -groups as given by Takeda or as suggested by Deligne and Soulé (see below). To answer this, and inspired by the algebraic analogue, in this paper we endow the higher arithmetic K -groups of an arithmetic variety (tensored by \mathbb{Q}) with a (pre)- λ -ring structure.

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Let X be an arithmetic variety over the ring of integers \mathbb{Z} . In order to define the arithmetic Chern character on hermitian vector bundles, Gillet and Soulé have introduced in [12] the arithmetic K_0 -group, denoted by $\widehat{K}_0(X)$. They endowed $\widehat{K}_0(X)$ with a pre- λ -ring structure, which was shown to be a λ -ring structure by Rössler in [19]. This group fits in an exact sequence

$$(*) \quad K_1(X) \xrightarrow{\rho} \bigoplus_{p \geq 0} \mathcal{D}^{2p-1}(X, p) / \text{im } d_{\mathcal{D}} \rightarrow \widehat{K}_0(X) \rightarrow K_0(X) \rightarrow 0,$$

with ρ the Beilinson regulator (up to a constant factor) and $\mathcal{D}^*(X, p)$ the Deligne complex of differential forms with p -twist computing Deligne–Beilinson cohomology with \mathbb{R} coefficients and twisted by p , $H_{\mathcal{D}}^*(X, \mathbb{R}(p))$.

Two different definitions for higher arithmetic K -theory have been proposed. Initially, it was suggested by Deligne and Soulé (see [20, §III.2.3.4] and [7, Remark 5.4]) that these groups should fit in a long exact sequence

$$\cdots \rightarrow K_{n+1}(X) \xrightarrow{\rho} H_{\mathcal{D}}^{2p-n-1}(X, \mathbb{R}(p)) \rightarrow \widehat{K}_n(X) \rightarrow K_n(X) \rightarrow \cdots,$$

extending the exact sequence $(*)$, with ρ the Beilinson regulator. This can be achieved by defining $\widehat{K}_n(X)$ to be the homotopy groups of the homotopy fiber of a representative of the Beilinson regulator (for instance, the representative “ch” defined by Burgos and Wang in [6]).

If X is proper, Takeda has given in [21] an alternative definition of the *higher arithmetic K -groups* of X , by means of *homotopy groups modified* by the representative of the Beilinson regulator ch. We denote these higher arithmetic K -groups by $\widehat{K}_n^T(X)$. In this case, these groups fit in exact sequences

$$K_{n+1}(X) \xrightarrow{\rho} \bigoplus_{p \geq 0} \mathcal{D}^{2p-n-1}(X, p) / \text{im } d_{\mathcal{D}} \rightarrow \widehat{K}_n^T(X) \rightarrow K_n(X) \rightarrow 0,$$

analogous to $(*)$. The two definitions do not agree, but, as proved by Takeda, they are related by a natural isomorphism:

$$\widehat{K}_n(X) \cong \ker(\text{ch} : \widehat{K}_n^T(X) \rightarrow \mathcal{D}^{2p-n}(X, p)), \quad n \geq 0.$$

In this paper we give a pre- λ -ring structure on the higher arithmetic K -groups $\widehat{K}_n(X)_{\mathbb{Q}}$ and $\widehat{K}_n^T(X)_{\mathbb{Q}}$. It is compatible with the λ -ring structure on the algebraic K -groups, $K_n(X)$, defined by Gillet and Soulé in [13]. Moreover, for $n = 0$ we recover the λ -ring structure of $\widehat{K}_0(X)$.

More concretely, we construct *Adams operations*

$$\begin{aligned} \Psi^k &: \widehat{K}_n(X) \otimes \mathbb{Q} \rightarrow \widehat{K}_n(X) \otimes \mathbb{Q}, & k \geq 0, \\ \Psi^k &: \widehat{K}_n^T(X) \otimes \mathbb{Q} \rightarrow \widehat{K}_n^T(X) \otimes \mathbb{Q}, & k \geq 0, \end{aligned}$$

which, since we have tensored by \mathbb{Q} , induce λ -operations on $\widehat{K}_n(X) \otimes \mathbb{Q}$ and $\widehat{K}_n^T(X) \otimes \mathbb{Q}$.

To this end, it is apparently necessary to have a representative of the Adams operations in algebraic K -theory, in terms of a chain morphism, which commutes, at least up to a given homotopy, with the representative of the Beilinson regulator ch . In [10], the author constructed a chain morphism representing the Adams operations in algebraic K -theory tensored by \mathbb{Q} . In this paper, we show that a slight modification of the construction of [10] commutes strictly with ch , and we deduce a pre- λ -ring structure for both $\widehat{K}_n(X) \otimes \mathbb{Q}$ and $\widehat{K}_n^T(X) \otimes \mathbb{Q}$. The modification needs to be introduced in order to deal with the fact that the Koszul complex, when endowed with its natural hermitian metrics, does not have zero Bott–Chern form. A discussion on the Bott–Chern form of the Koszul complex is found in §4.2.

In order to work with $\widehat{K}_n^T(X) \otimes \mathbb{Q}$, we introduce the *modified homology groups*, which are the homological analogue of the modified homotopy groups given by Takeda, and the dual notion of the *truncated relative cohomology groups* defined by Burgos in [2]. We show that the homology groups modified by ch give a homological description of $\widehat{K}_n^T(X) \otimes \mathbb{Q}$.

The paper is organized as follows. In the first section we review the construction of the Beilinson regulator ch given by Burgos and Wang in [6]. Next, we recall the definition of the arithmetic K -group of an arithmetic variety, $\widehat{K}_0(X)$, and proceed to the description of the higher arithmetic K -groups, in both the Deligne–Soulé version and the Takeda one. In the third section we introduce the modified homology groups and show that the Takeda higher arithmetic K -groups admit a homological description after being tensored by \mathbb{Q} . Finally, the last section is devoted to the construction of Adams operations in higher arithmetic K -theory.

Notation. If A is an abelian group, we denote $A_{\mathbb{Q}} := A \otimes \mathbb{Q}$.

We follow the conventions and definitions on (co)chain complexes and iterated (co)chain complexes as given in [5, §2]. All (co)chain complexes consist of abelian groups.

If (A_*, d_A) is a chain complex, we denote by ZA_n the group of *cycles* of degree n , that is, of $z \in A_n$ such that $d_A(z) = 0$. If $f : A_* \rightarrow B_*$ is a chain morphism, we denote by $s(f)_*$ the *simple complex* associated to f . This is the same as the cone twisted by -1 .

Given a chain complex B_* , let $\sigma_{>n}B_*$ be its *bête truncation*, that is, the complex with

$$\sigma_{>n}B_r = \begin{cases} B_r, & r > n, \\ 0, & r \leq n, \end{cases}$$

and differential induced by the differential of B_* .

§1. Higher Bott–Chern forms

The Burgos–Wang construction of the Beilinson regulator, given in [6], plays a key role in the definition of the higher arithmetic K -groups. Using the chain complex of cubes, the transgression of vector bundles, and the Chern character form of a vector bundle, they obtained a chain morphism whose induced morphism in homology is the Beilinson regulator. The construction is based on the definition of higher Bott–Chern forms. These forms are the extension to hermitian n -cubes of the Chern character form of a hermitian vector bundle.

In this section we review this construction. For further details see [6] or [4, §3.2].

We focus on the case of smooth proper complex varieties, since this will be the case in our applications. Nevertheless, most of the constructions can be adapted to the non-proper case by using hermitian metrics smooth at infinity. See [6] for details.

§1.1. Higher algebraic K -theory

Let \mathcal{P} be a small exact category and let $K_n(\mathcal{P})$ denote the n -th algebraic K -group of \mathcal{P} in the sense of Quillen [18]. Let $S(\mathcal{P})$ be the Waldhausen simplicial set, defined in [22], which computes the higher algebraic K -groups of \mathcal{P} , that is, we have

$$K_n(\mathcal{P}) \cong \pi_{i+1}(|S(\mathcal{P})|, \{0\}).$$

Denote by ∂_i, s_i the face and degeneracy maps, respectively, of $S(\mathcal{P})$ and let $\mathbb{Z}S_*(\mathcal{P})$ be the Moore complex associated to the simplicial set $S(\mathcal{P})$.

Let $\langle 0, 1, 2 \rangle$ be the category associated to the ordered set $\{0, 1, 2\}$ and let $\langle 0, 1, 2 \rangle^n$ be its n -th cartesian power. Given a functor

$$\langle 0, 1, 2 \rangle^n \xrightarrow{E} \mathcal{P},$$

the image of an n -tuple $\mathbf{j} = (j_1, \dots, j_n)$ is denoted by $E^{\mathbf{j}}$. For such a functor one defines its faces by

$$(\partial_i^k E)^{\mathbf{j}} = E^{j_1 \dots j_{i-1}, k, j_i, \dots, j_n}$$

for all $i \in \{1, \dots, n\}$, $k \in \{0, 1, 2\}$, $\mathbf{j} \in \{0, 1, 2\}^{n-1}$.

Definition 1.1. An n -cube E over \mathcal{P} is a functor $\langle 0, 1, 2 \rangle^n \xrightarrow{E} \mathcal{P}$ such that for all $\mathbf{j} \in \{0, 1, 2\}^{n-1}$ and $i = 1, \dots, n$ the sequence

$$(1.1) \quad (\partial_i^0 E)^{\mathbf{j}} \rightarrow (\partial_i^1 E)^{\mathbf{j}} \rightarrow (\partial_i^2 E)^{\mathbf{j}}$$

is a short exact sequence of \mathcal{P} .

A functor $\langle 0, 1, 2 \rangle^n \rightarrow \mathcal{P}$ is usually called a *cube*, and a functor as in Definition 1.1 an *exact cube*. Since we will only consider exact cubes, the word “exact” is dropped from the terminology.

For every $n \geq 0$, let $C_n(\mathcal{P})$ denote the set of n -cubes over \mathcal{P} . We have defined face maps

$$\partial_i^k : C_n(\mathcal{P}) \rightarrow C_{n-1}(\mathcal{P}), \quad i = 1, \dots, n, \quad k = 0, 1, 2.$$

There are as well degeneracy maps (see for instance [6, §3])

$$s_i^k : C_{n-1}(\mathcal{P}) \rightarrow C_n(\mathcal{P}), \quad i = 1, \dots, n, \quad k = 0, 1.$$

If we write $\mathbb{Z}C_n(\mathcal{P})$ for the free abelian group on the n -cubes, the alternate sum of the face maps, $d = \sum_{i=1}^n (-1)^i (\partial_i^0 - \partial_i^1 + \partial_i^2)$, endows $\mathbb{Z}C_*(\mathcal{P})$ with a chain complex structure with differential d .

Let

$$\mathbb{Z}D_n(\mathcal{P}) = \sum_{i=1}^n [s_i^0(\mathbb{Z}C_{n-1}(\mathcal{P})) + s_i^1(\mathbb{Z}C_{n-1}(\mathcal{P}))] \subset \mathbb{Z}C_n(\mathcal{P})$$

be the subgroup generated by the degenerate cubes (i.e., those that lie in the image of some degeneracy map). The differential of $\mathbb{Z}C_*(\mathcal{P})$ induces a differential on $\mathbb{Z}D_*(\mathcal{P})$ making the inclusion arrow $\mathbb{Z}D_*(\mathcal{P}) \hookrightarrow \mathbb{Z}C_*(\mathcal{P})$ a chain morphism. The quotient complex

$$\tilde{\mathbb{Z}}C_*(\mathcal{P}) = \mathbb{Z}C_*(\mathcal{P})/\mathbb{Z}D_*(\mathcal{P})$$

is called the *chain complex of cubes* in \mathcal{P} .

The Cub morphism. As shown in Wang’s thesis [23] and in [17], to every element $E \in S_n(\mathcal{P})$ one can associate an $(n - 1)$ -cube $\text{Cub}(E)$ satisfying the following property. For $i = 1, \dots, n - 1$, we have

$$\begin{aligned} \partial_i^0 \text{Cub } E &= s_{n-2}^0 \cdots s_i^0 \text{Cub } \partial_{i+1} \cdots \partial_n E, \\ \partial_i^1 \text{Cub } E &= \text{Cub } \partial_i E, \\ \partial_i^2 \text{Cub } E &= s_{i-1}^1 \cdots s_1^1 \text{Cub } \partial_0 \cdots \partial_{i-1} E. \end{aligned} \tag{1.2}$$

It follows from these equalities that Cub gives a chain morphism

$$\mathbb{Z}S_*(\mathcal{P})[-1] \xrightarrow{\text{Cub}} \tilde{\mathbb{Z}}C_*(\mathcal{P}).$$

The composition of the Hurewicz morphism with the morphism induced by Cub in homology gives a morphism

$$\text{Cub} : K_n(\mathcal{P}) = \pi_{n+1}(S(\mathcal{P})) \xrightarrow{\text{Hurewicz}} H_n(\mathbb{Z}S_*(\mathcal{P})[-1]) \xrightarrow{\text{Cub}} H_n(\tilde{\mathbb{Z}}C_*(\mathcal{P})).$$

Moreover, McCarthy showed in [17] that this morphism is an isomorphism over the field of rational numbers, that is, for all $n \geq 0$, the morphism

$$(1.3) \quad K_n(\mathcal{P})_{\mathbb{Q}} \xrightarrow{\text{Cub}} H_n(\tilde{\mathbb{Z}}C_*(\mathcal{P}), \mathbb{Q})$$

is an isomorphism.

Normalized complexes. Consider now the *normalized chain complex* $NC_*(\mathcal{P})$ introduced in [10, §1.5], whose n -th graded piece is given by

$$N_n C(\mathcal{P}) := \bigcap_{i=1}^n \ker \partial_i^0 \cap \bigcap_{i=1}^n \ker \partial_i^1 \subset \mathbb{Z}C_n(\mathcal{P}),$$

and its differential is the one induced by the differential of $\mathbb{Z}C_*(\mathcal{P})$. In [10] it is shown that the composition

$$N_*C(\mathcal{P}) \hookrightarrow \mathbb{Z}C_*(\mathcal{P}) \twoheadrightarrow \tilde{\mathbb{Z}}C_*(\mathcal{P})$$

is an isomorphism of chain complexes.

Let $NS_*(\mathcal{P})$ be the normalized complex associated to the simplicial abelian group $\mathbb{Z}S(\mathcal{P})$ given by

$$NS_n(\mathcal{P}) = \bigcap_{i=1}^n \ker \partial_i, \quad n \geq 0,$$

and whose differential is ∂_0 . It follows from the relations in (1.2) that the morphism Cub induces a chain morphism

$$NS_*(\mathcal{P})[-1] \xrightarrow{\text{Cub}} NC_*(\mathcal{P}).$$

§1.2. Chern character form

In this subsection, all schemes are over \mathbb{C} . As defined in [2], for every $p \geq 0$, let $\mathcal{D}^*(X, p)$ denote the Deligne complex of differential forms on X computing the Deligne–Beilinson cohomology groups with real coefficients twisted by p , $H_{\mathcal{D}}^*(X, \mathbb{R}(p))$. We will write $\mathcal{D}^{2p-*}(X, p)$ for the chain complex associated to the cochain complex $\mathcal{D}^*(X, p)[2p]$.

Let X be a smooth proper complex variety. A *hermitian vector bundle* $\overline{E} = (E, h)$ is an algebraic vector bundle E over X together with a smooth hermitian metric on E . The reader is referred to [24] for details.

For every hermitian vector bundle \overline{E} , by the Chern–Weil formulae one defines a closed differential form

$$\text{ch}(\overline{E}) \in \bigoplus_{p \geq 0} \mathcal{D}^{2p}(X, p),$$

representing the Chern character class $\text{ch}(E) = [\text{ch}(\overline{E})] \in H_{dR}^*(X)$. Although the class of $\text{ch}(\overline{E})$ is independent of the hermitian metric, the form depends on the particular choice of hermitian metric.

The following properties are satisfied:

- ▷ If $\overline{E} \cong \overline{F}$ is an isometry of hermitian vector bundles, then $\text{ch}(\overline{E}) = \text{ch}(\overline{F})$.
- ▷ Let \overline{E}_1 and \overline{E}_2 be two hermitian vector bundles. If $\overline{E}_1 \oplus \overline{E}_2$ and $\overline{E}_1 \otimes \overline{E}_2$ have the hermitian metrics induced by those on \overline{E}_1 and \overline{E}_2 , then

$$\text{ch}(\overline{E}_1 \oplus \overline{E}_2) = \text{ch}(\overline{E}_1) + \text{ch}(\overline{E}_2) \quad \text{and} \quad \text{ch}(\overline{E}_1 \otimes \overline{E}_2) = \text{ch}(\overline{E}_1) \wedge \text{ch}(\overline{E}_2).$$

§1.3. Hermitian cubes

Let X be a smooth proper complex variety. Let $\mathcal{P}(X)$ be the category of vector bundles over X . Let $\widehat{\mathcal{P}}(X)$ be the category whose objects are the hermitian vector bundles over X , and whose morphisms are given by

$$\text{Hom}_{\widehat{\mathcal{P}}(X)}((E, h), (E', h')) = \text{Hom}_{\mathcal{P}(X)}(E, E').$$

The category $\widehat{\mathcal{P}}(X)$ inherits an exact category structure from that of $\mathcal{P}(X)$.

We fix a universe \mathcal{U} so that $\widehat{\mathcal{P}}(X)$ is \mathcal{U} -small for every smooth proper complex variety X . Every vector bundle admits a smooth hermitian metric. It follows that the forgetful functor $\widehat{\mathcal{P}}(X) \rightarrow \mathcal{P}(X)$ is an equivalence of categories with its quasi-inverse constructed by choosing a hermitian metric for each vector bundle. Therefore, the algebraic K -groups of X can be computed in terms of the category $\widehat{\mathcal{P}}(X)$.

Denote by $\widehat{S}(X)$ the Waldhausen simplicial set corresponding to the exact category $\widehat{\mathcal{P}}(X)$ and let $\mathbb{Z}\widehat{C}_*(X) = \mathbb{Z}C_*(\widehat{\mathcal{P}}(X))$, $\widetilde{\mathbb{Z}}\widehat{C}_*(X) = \widetilde{\mathbb{Z}}C_*(\widehat{\mathcal{P}}(X))$ and $N\widehat{C}_*(X) = NC_*(\widehat{\mathcal{P}}(X))$. The cubes in the category $\widehat{\mathcal{P}}(X)$ are called *hermitian cubes*.

Hermitian cubes with canonical kernels. Let \overline{E} be a hermitian vector bundle and let $F \subset \overline{E}$ be an inclusion of vector bundles. Then F inherits a hermitian metric from the hermitian metric of \overline{E} . It follows that there is an induced hermitian metric on the kernel of a morphism of hermitian vector bundles. This allows one to extend the definition of cubes with canonical kernels given in [10] to hermitian cubes in the following sense.

Definition 1.2. Let \overline{E} be a hermitian n -cube and let $g_i^0 : \partial_i^0 \overline{E} \rightarrow \partial_i^1 \overline{E}$ denote the morphism in the cube. We say that \overline{E} has *canonical kernels* if for every $i = 1, \dots, n$ and $\mathbf{j} \in \{0, 1, 2\}^{n-1}$, there is an inclusion $(\partial_i^0 \overline{E})^{\mathbf{j}} \subset (\partial_i^1 \overline{E})^{\mathbf{j}}$ of sets, the morphism

$$g_i^0 : \partial_i^0 \overline{E} \rightarrow \partial_i^1 \overline{E}$$

is the canonical inclusion of cubes and the metric on $\partial_i^0 \overline{E}$ is induced by the metric of $\partial_i^1 \overline{E}$ by means of g_i^0 .

The differential of a hermitian cube with canonical kernels is again a hermitian cube with canonical kernels. Let $\mathbb{Z}K\widehat{C}_*(X)$ denote the complex of hermitian cubes with canonical kernels. As usual, the quotient of the complex of cubes with canonical kernels by the degenerate cubes with canonical kernels is denoted by $\widetilde{\mathbb{Z}}K\widehat{C}_*(X)$.

Remark 1.1. Burgos and Wang [6, Definition 3.5] introduced the notion of *emi-cubes*, in order to define the morphism ch . With the notation of the last definition, the emi-cubes are those for which the metric on $\partial_i^0 \overline{E}$ is induced by the metric of $\partial_i^1 \overline{E}$, without the need g_i^0 to be the set inclusion. In [6] the purpose was that the Chern form of the transgression bundle associated to a cube defined a chain morphism, and by the properties of ch stated in §1.2, this more relaxed condition was sufficient. Our more restrictive notion arises because we require the transgression map given in [10] to define a morphism, before being composed with the Chern form (see below).

Lemma 1.1. *There is a chain morphism $\lambda : \widetilde{\mathbb{Z}}\widehat{C}_*(X) \rightarrow \widetilde{\mathbb{Z}}K\widehat{C}_*(X)$.*

Proof. The morphism λ is defined in [10] for cubes over the category of vector bundles (not necessarily hermitian). The fact that the image by λ of a hermitian cube is a hermitian cube with canonical kernels follows from [6, Lemma 3.7]. \square

§1.4. The transgression bundle and the Chern character

Let \mathbb{P}^1 be the complex projective line. Let x and y be the global sections of the canonical bundle $\mathcal{O}(1)$ given by the projective coordinates $(x : y)$ on \mathbb{P}^1 . Let X be a complex variety and let p_0 and p_1 be the projections from $X \times \mathbb{P}^1$ to X and \mathbb{P}^1 respectively. Then, for every vector bundle E over X , we denote

$$E(k) := p_0^* E \otimes p_1^* \mathcal{O}(k), \quad \forall k.$$

The following definition is a variation of the original one from [6, §3].

Definition 1.3. Let

$$E : 0 \rightarrow E^0 \xrightarrow{f^0} E^1 \xrightarrow{f^1} E^2 \rightarrow 0$$

be a short exact sequence. The *first transgression bundle by projective lines* of E , $\text{tr}_1(E)$, is the kernel of the morphism

$$E^1(1) \oplus E^2(1) \rightarrow E^2(2), \quad (a, b) \mapsto f^1(a) \otimes x - b \otimes y.$$

Let E be an n -cube of vector bundles over X . We define the *first transgression* of E as the $(n - 1)$ -cube on $X \times (\mathbb{P}^1)^1$ given by

$$\mathrm{tr}_1(E)^j := \mathrm{tr}_1(\partial_2^{j_2} \dots \partial_n^{j_n} E) \quad \text{for all } \mathbf{j} = (j_2, \dots, j_n) \in \{0, 1, 2\}^{n-1},$$

i.e. we take the transgression of the exact sequences in the first direction. Since tr_1 is a functorial exact construction, the n -th *transgression bundle* can be defined recursively as

$$\mathrm{tr}_n(E) = \mathrm{tr}_1 \mathrm{tr}_{n-1}(E) = \mathrm{tr}_1 \dots \mathrm{tr}_1(E).$$

It is a vector bundle on $X \times (\mathbb{P}^1)^n$.

The Fubini–Study metric on \mathbb{P}^1 induces a metric on the line bundle $\mathcal{O}(1)$. We denote by $\overline{\mathcal{O}(1)}$ the corresponding hermitian line bundle. Then, given a hermitian n -cube \overline{E} , the transgression bundle $\mathrm{tr}_n(\overline{E})$ has a hermitian metric naturally induced by the metric of \overline{E} and by the metric of $\overline{\mathcal{O}(1)}$. If \overline{E} is an n -cube with canonical kernels, then

$$\mathrm{tr}_n(\overline{E})|_{x_i=0} = \mathrm{tr}_{n-1}(\partial_i^1 \overline{E}), \quad \mathrm{tr}_n(\overline{E})|_{y_i=0} \cong \mathrm{tr}_{n-1}(\partial_i^0 \overline{E}) \oplus^\perp \mathrm{tr}_{n-1}(\partial_i^2 \overline{E}),$$

where \cong is an isometry and \oplus^\perp means the orthogonal direct sum.

Consider now the differential form W_n from [6, §6]:

$$W_n = \frac{1}{2n!} \sum_{i=1}^n (-1)^i S_n^i,$$

with

$$S_n^i = \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma \log |z_{\sigma(1)}|^2 \frac{dz_{\sigma(2)}}{z_{\sigma(2)}} \wedge \dots \wedge \frac{dz_{\sigma(i)}}{z_{\sigma(i)}} \wedge \frac{d\bar{z}_{\sigma(i+1)}}{\bar{z}_{\sigma(i+1)}} \wedge \dots \wedge \frac{d\bar{z}_{\sigma(n)}}{\bar{z}_{\sigma(n)}}.$$

Theorem 1.1 (Burgos–Wang, [6]). *Let X be a smooth proper complex variety.*

(1) *The following map is a chain morphism,*

$$\begin{aligned} \widetilde{\mathbb{Z}}\widehat{C}_n(X) &\xrightarrow{\mathrm{ch}} \bigoplus_{p \geq 0} \mathcal{D}^{2p-n}(X, p) \\ \overline{E} &\mapsto \mathrm{ch}_n(\overline{E}) := \frac{(-1)^n}{(2\pi i)^n} \int_{(\mathbb{P}^1)^n} \mathrm{ch}(\mathrm{tr}_n(\lambda(\overline{E}))) \wedge W_n. \end{aligned}$$

(2) *The composition*

$$K_n(X) \xrightarrow{\mathrm{Cub}} H_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X)) \xrightarrow{\mathrm{ch}} \bigoplus_{p \geq 0} H_{\mathcal{D}}^{2p-n}(X, \mathbb{R}(p))$$

is the Beilinson regulator.

□

The form $\text{ch}_n(\overline{E})$ is called the *Bott–Chern form* of the hermitian n -cube \overline{E} .

Remark 1.2. Observe that, by means of the isomorphism $N\widehat{C}_*(X) \cong \widetilde{\mathbb{Z}}\widehat{C}_*(X)$, the Chern character is also represented by the morphism

$$N\widehat{C}_*(X) \xrightarrow{\text{ch}} \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p), \quad \overline{E} \in N\widehat{C}_n(X) \mapsto \text{ch}_n(\overline{E}).$$

Differential forms and projective lines. For some constructions in what follows, it is convenient to factor the morphism ch through a complex consisting of the Deligne complex of differential forms on $X \times (\mathbb{P}^1)^n$. This construction was introduced in [6].

Over any base scheme, the cartesian product of projective lines $(\mathbb{P}^1)^n$ has a cubical scheme structure. Specifically, the coface and codegeneracy maps

$$\begin{aligned} \delta_j^i : (\mathbb{P}^1)^n &\rightarrow (\mathbb{P}^1)^{n+1}, & i = 1, \dots, n, \quad j = 0, 1, \\ \sigma^i : (\mathbb{P}^1)^n &\rightarrow (\mathbb{P}^1)^{n-1}, & i = 1, \dots, n, \end{aligned}$$

are defined as

$$\begin{aligned} \delta_0^i(x_1, \dots, x_n) &= (x_1, \dots, x_{i-1}, (0 : 1), x_i, \dots, x_n), \\ \delta_1^i(x_1, \dots, x_n) &= (x_1, \dots, x_{i-1}, (1 : 0), x_i, \dots, x_n), \\ \sigma^i(x_1, \dots, x_n) &= (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n). \end{aligned}$$

Let X be a smooth proper complex variety and fix $\mathbb{P}^1 = \mathbb{P}_{\mathbb{C}}^1$. The coface and codegeneracy maps induce, for every $i = 1, \dots, n$ and $l = 0, 1$, morphisms

$$\begin{aligned} \delta_i^l : \mathcal{D}^*(X \times (\mathbb{P}^1)^n, p) &\rightarrow \mathcal{D}^*(X \times (\mathbb{P}^1)^{n-1}, p), \\ \sigma_i : \mathcal{D}^*(X \times (\mathbb{P}^1)^{n-1}, p) &\rightarrow \mathcal{D}^*(X \times (\mathbb{P}^1)^n, p). \end{aligned}$$

Let $\mathcal{D}_{\mathbb{P}}^{*,*}(X, p)$ be the 2-iterated cochain complex given by

$$\mathcal{D}_{\mathbb{P}}^{r,-n}(X, p) = \mathcal{D}^r(X \times (\mathbb{P}^1)^n, p)$$

and differentials $(d_{\mathcal{D}}, \delta = \sum_{i=1}^n (-1)^i (\delta_i^0 - \delta_i^1))$, and denote by $\mathcal{D}_{\mathbb{P}}^*(X, p)$ the associated simple complex.

Let $(x : y)$ be homogeneous coordinates in \mathbb{P}^1 and consider

$$h = -\frac{1}{2} \log \frac{(x-y)\overline{(x-y)}}{x\bar{x} + y\bar{y}}.$$

It defines a function on the open set $\mathbb{P}^1 \setminus \{1\}$, with a logarithmic singularity at 1. Consider the differential $(1, 1)$ -form

$$\omega = d_{\mathcal{D}}h \in \mathcal{D}^2(\mathbb{P}^1, 1).$$

This is a smooth form all over \mathbb{P}^1 representing the class of the first Chern class of the canonical bundle of \mathbb{P}^1 , $c_1(\mathcal{O}_{\mathbb{P}^1}(1))$.

For every n , denote by $\pi : X \times (\mathbb{P}^1)^n \rightarrow X$ the projection and let $p_i : X \times (\mathbb{P}^1)^n \rightarrow \mathbb{P}^1$ be the projection onto the i -th projective line. Denote, for $i = 1, \dots, n$,

$$\omega_i = p_i^*(\omega) \in \mathcal{D}^2(X \times (\mathbb{P}^1)^n, 1).$$

Let

$$(1.4) \quad D_n^r = \sum_{i=1}^n \sigma_i(\mathcal{D}^r(X \times (\mathbb{P}^1)^{n-1}, p))$$

be the complex of degenerate elements and let \mathcal{W}_n^* be the subcomplex of $\mathcal{D}^*(X \times (\mathbb{P}^1)^n, p)$ given by

$$(1.5) \quad \mathcal{W}_n^r = \sum_{i=1}^n \omega_i \wedge \sigma_i(\mathcal{D}^{r-2}(X \times (\mathbb{P}^1)^{n-1}, p-1)).$$

This complex is meant to kill the cohomology classes coming from the projective lines. We define the 2-iterated complex

$$\tilde{\mathcal{D}}_{\mathbb{P}}^{r,-n}(X, p) := \tilde{\mathcal{D}}^r(X \times (\mathbb{P}^1)^n, p) := \frac{\mathcal{D}^r(X \times (\mathbb{P}^1)^n, p)}{D_n^r + \mathcal{W}_n^r}$$

and denote by $\tilde{\mathcal{D}}_{\mathbb{P}}^*(X, p)$ the associated simple complex.

Proposition 1.1. *The natural map*

$$\mathcal{D}^*(X, p) = \tilde{\mathcal{D}}_{\mathbb{P}}^{*,0}(X, p) \xrightarrow{i} \tilde{\mathcal{D}}_{\mathbb{P}}^*(X, p)$$

is a quasi-isomorphism.

Proof. The proof is analogous to the proof of [6, Lemma 1.3]. It follows from a spectral sequence argument together with the fact that, by the Dold–Thom isomorphism in Deligne–Beilinson cohomology,

$$H^r(\tilde{\mathcal{D}}^*(X \times (\mathbb{P}^1)^n, p)) = 0 \quad \forall n > 0. \quad \square$$

In the next proposition, \bullet denotes the product in the Deligne complex as described in [2, §3].

Proposition 1.2 (Burgos–Wang, [6]). *There is a quasi-isomorphism of complexes*

$$\tilde{\mathcal{D}}_{\mathbb{P}}^*(X, p) \xrightarrow{\varphi} \mathcal{D}^*(X, p),$$

given by

$$\alpha \in \mathcal{D}^r(X \times (\mathbb{P}^1)^n, p) \mapsto \pi_*(\alpha \bullet W_n) = \begin{cases} (2\pi i)^{-n} \int_{(\mathbb{P}^1)^n} \alpha \bullet W_n, & n > 0, \\ \alpha & n = 0. \end{cases}$$

This morphism is the quasi-inverse of the quasi-isomorphism i of Proposition 1.1. \square

Proposition 1.3 (Burgos–Wang, [6]). *The map*

$$\widetilde{\mathbb{Z}}\widehat{C}_n(X) \xrightarrow{\text{ch}} \bigoplus_{p \geq 0} \widetilde{\mathcal{D}}_{\mathbb{P}}^{2p-n}(X, p), \quad \overline{E} \mapsto \text{ch}(\text{tr}_n(\lambda(\overline{E}))),$$

is a chain morphism. Therefore, the morphism ch of (1.4) factors through the complex $\widetilde{\mathcal{D}}_{\mathbb{P}}^{2p-*}(X, p)$ in the form

$$\widetilde{\mathbb{Z}}\widehat{C}_n(X) \xrightarrow{\text{ch}} \bigoplus_{p \geq 0} \widetilde{\mathcal{D}}_{\mathbb{P}}^{2p-n}(X, p) \xrightarrow{\varphi} \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p). \quad \square$$

§2. Higher arithmetic K -theory

In this section we focus on the definition of the higher arithmetic K -groups of an arithmetic variety. We start by discussing the extension of the Chern character on complex varieties to arithmetic varieties. Then, we recall the definition of the arithmetic K -group given by Gillet and Soulé in [12]. Finally, the last two sections review the two definitions of higher arithmetic K -theory.

Following [11], by an *arithmetic variety* we mean a regular quasi-projective scheme over an arithmetic ring. In this section we restrict ourselves to proper arithmetic varieties over the arithmetic ring \mathbb{Z} . Note, however, that most of the results are valid under the less restrictive hypothesis of the variety being proper over \mathbb{C} . Moreover, one could extend the definition of higher arithmetic K -groups, $\widehat{K}_n(X)$, to quasi-projective varieties, by considering vector bundles with hermitian metrics smooth at infinity and the complex of differential forms $\widetilde{\mathcal{D}}_{\mathbb{P}}^*(X, p)$ introduced in the previous section (and by considering differential forms with log-arithmetic singularities).

§2.1. Chern character for arithmetic varieties

If X is an arithmetic variety over \mathbb{Z} , let $X(\mathbb{C})$ denote the associated complex variety, consisting of the \mathbb{C} -valued points on X . Let F_{∞} denote the complex conjugation on $X(\mathbb{C})$ and $X_{\mathbb{R}} = (X(\mathbb{C}), F_{\infty})$ the associated real variety.

The *real Deligne–Beilinson cohomology* of X is defined as the cohomology of $X_{\mathbb{R}}$, i.e.

$$H_{\mathcal{D}}^n(X, \mathbb{R}(p)) = H_{\mathcal{D}}^n(X_{\mathbb{R}}, \mathbb{R}(p)) = H_{\mathcal{D}}^n(X(\mathbb{C}), \mathbb{R}(p))^{\overline{F}_{\infty}^* = \text{id}}.$$

It is computed as the cohomology of the real Deligne complex:

$$\mathcal{D}^n(X, p) = \mathcal{D}^n(X_{\mathbb{R}}, p) = \mathcal{D}^n(X(\mathbb{C}), p)^{\overline{F}_\infty^* = \text{id}},$$

that is, we have

$$H_{\mathcal{D}}^n(X, \mathbb{R}(p)) \cong H^n(\mathcal{D}^n(X, p), d_{\mathcal{D}}).$$

Definition 2.1. Let X be a proper arithmetic variety over \mathbb{Z} . A *hermitian vector bundle* \overline{E} over X is a pair (E, h) , where E is a locally free sheaf on X and where h is an F_∞^* -invariant smooth hermitian metric on the associated vector bundle $E(\mathbb{C})$ over $X(\mathbb{C})$.

Let $\widehat{\mathcal{P}}(X)$ denote the category of hermitian vector bundles over X . The simplicial set $\widehat{S}(X)$ and the chain complexes $\mathbb{Z}\widehat{C}_*(X)$, $\widetilde{\mathbb{Z}}\widehat{C}_*(X)$ and $N\widehat{C}_*(X)$ are defined accordingly.

If \overline{E} is a hermitian vector bundle over X , the Chern character form $\text{ch}(\overline{E})$ is \overline{F}_∞^* -invariant. Therefore

$$\text{ch}(\overline{E}) \in \bigoplus_{p \geq 0} \mathcal{D}^{2p}(X, p).$$

It follows that the chain morphism of (1.4) gives a chain morphism

$$\mathbb{Z}\widehat{S}_*(X)[-1] \xrightarrow{\text{Cub}} \widetilde{\mathbb{Z}}\widehat{C}_*(X) \xrightarrow{\text{ch}} \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p).$$

§2.2. Arithmetic K_0 -group

In [12, §6] Gillet and Soulé defined the arithmetic K_0 -group of an arithmetic variety, denoted by $\widehat{K}_0(X)$. We give here a slightly different presentation using the Deligne complex of differential forms and the differential operator $-2\partial\bar{\partial}$.

Let X be an arithmetic variety and let $\widetilde{\mathcal{D}}^*(X, p) = \mathcal{D}^*(X, p)/\text{im } d_{\mathcal{D}}$. Consider pairs (\overline{F}, α) , where \overline{F} is a hermitian vector bundle over X and where $\alpha \in \bigoplus_{p \geq 0} \widetilde{\mathcal{D}}^{2p-1}(X, p)$ is a differential form. Then $\widehat{K}_0(X)$ is the quotient of the free abelian group generated by these pairs by the subgroup generated by the sums

$$(\overline{E}^0, \alpha_0) + (\overline{E}^2, \alpha_2) - (\overline{E}^1, \alpha_0 + \alpha_2 - \text{ch}(\overline{E})),$$

for every exact sequence of hermitian vector bundles over X ,

$$\overline{E} : 0 \rightarrow \overline{E}^0 \rightarrow \overline{E}^1 \rightarrow \overline{E}^2 \rightarrow 0,$$

and every $\alpha_0, \alpha_2 \in \bigoplus_{p \geq 0} \widetilde{\mathcal{D}}^{2p-1}(X, p)$.

Among other properties, this group fits in an exact sequence

$$K_1(X) \xrightarrow{\text{ch}} \bigoplus_{p \geq 0} \widetilde{\mathcal{D}}^{2p-1}(X, p) \xrightarrow{a} \widehat{K}_0(X) \xrightarrow{\zeta} K_0(X) \rightarrow 0$$

(see [12] for details).

Gillet and Soulé [12], together with Rössler [19], showed that there is a λ -ring structure on $\widehat{K}_0(X)$.

§2.3. Deligne–Soulé higher arithmetic K -theory

Although there is no reference in which the theory is developed, it has been suggested by Deligne and Soulé (see [20, §III.2.3.4] and [7, Remark 5.4]) that the higher arithmetic K -theory should be obtained as the homotopy groups of the homotopy fiber of a representative of the Beilinson regulator. We sketch here the construction, in order to show that Adams operations can be defined.

Consider the bête truncation at $n > 0$ of the complex $\mathcal{D}^{2p-*}(X, p)$, denoted by $\sigma_{>0}\mathcal{D}^{2p-*}(X, p)$. Let

$$\widehat{\text{ch}} : \widetilde{\mathbb{Z}}\widehat{C}_n(X) \rightarrow \bigoplus_{p \geq 0} \sigma_{>0}\mathcal{D}^{2p-n}(X, p),$$

be the composition of $\text{ch} : \widetilde{\mathbb{Z}}\widehat{C}_n(X) \rightarrow \bigoplus_{p \geq 0} \mathcal{D}^{2p-n}(X, p)$ with the natural map

$$\bigoplus_{p \geq 0} \mathcal{D}^{2p-n}(X, p) \rightarrow \bigoplus_{p \geq 0} \sigma_{>0}\mathcal{D}^{2p-n}(X, p).$$

Let $\mathcal{K}(\cdot)$ be the Dold–Puppe functor from the category of chain complexes of abelian groups to the category of simplicial abelian groups (see [8]). Consider the morphism

$$\mathcal{K}(\widehat{\text{ch}}) : \widehat{S}(X) \rightarrow \mathcal{K}(\mathbb{Z}\widehat{S}_*(X)) \xrightarrow{\text{Cub}} \mathcal{K}(\widetilde{\mathbb{Z}}\widehat{C}_*(X)) \xrightarrow{\widehat{\text{ch}}} \mathcal{K}\left(\bigoplus_{p \geq 0} \sigma_{>0}\mathcal{D}^{2p-*}(X, p)\right),$$

and denote by $|\mathcal{K}(\widehat{\text{ch}})|$ the morphism induced on the realization of the simplicial sets.

Definition 2.2. For every $n \geq 0$, the (*Deligne–Soulé*) *higher arithmetic K -group* of X is defined as

$$\widehat{K}_n(X) = \pi_{n+1}(\text{homotopy fiber of } |\mathcal{K}(\widehat{\text{ch}})|).$$

Proposition 2.1. *Let X be a proper arithmetic variety.*

- (i) *The group $\widehat{K}_0(X)$ as defined in Definition 2.2 agrees with the arithmetic K -group defined by Gillet and Soulé in [12].*
- (ii) *Let $s(\widehat{\text{ch}})$ denote the simple complex associated to the chain morphism $\widehat{\text{ch}}$. If $n > 0$, there is an isomorphism $\widehat{K}_n(X)_{\mathbb{Q}} \cong H_n(s(\widehat{\text{ch}}), \mathbb{Q})$.*
- (iii) *There is a long exact sequence*

$$\cdots \rightarrow K_{n+1}(X) \xrightarrow{\text{ch}} H_{\mathcal{D}}^{2p-n-1}(X, \mathbb{R}(p)) \xrightarrow{a} \widehat{K}_n(X) \xrightarrow{\zeta} K_n(X) \rightarrow \cdots$$

with end

$$\dots \rightarrow K_1(X) \xrightarrow{\text{ch}} \widehat{\mathcal{D}}^{2p-1}(X, p) \xrightarrow{a} \widehat{K}_0(X) \xrightarrow{\zeta} K_0(X) \rightarrow 0.$$

Proof. The first and third statements follow by definition. The second statement follows from the isomorphism of (1.3) together with the following well-known fact (see [9] for a proof):

Lemma 2.1. *Let $(A_*, d_A), (B_*, d_B)$ be two chain complexes. Let $f : A_* \rightarrow B_*$ be a chain morphism and let $\mathcal{K}(f) : \mathcal{K}(A) \rightarrow \mathcal{K}(B)$ be the induced morphism. Let $\text{HoFib}(f)$ denote the homotopy fiber of the topological realization of $\mathcal{K}(f)$. Then, for every $n \geq 1$, there is an isomorphism*

$$\pi_n(\text{HoFib}(f)) \rightarrow H_n(s_*(f))$$

such that the following diagram is commutative:

$$\begin{array}{ccccc} \pi_{n+1}(\mathcal{K}(B)) & \longrightarrow & \pi_n(\text{HoFib}(f)) & \longrightarrow & \pi_n(\mathcal{K}(A)) \\ \cong \downarrow & & \downarrow & & \cong \downarrow \\ H_{n+1}(B_*) & \longrightarrow & H_n(s(f)) & \longrightarrow & H_n(A_*) \end{array} \quad \square$$

In §3.2, we will endow $\bigoplus_{n \geq 0} \widehat{K}_n(X)$ with a product structure, induced by the product structure defined by Takeda on his higher arithmetic K -groups.

§2.4. Takeda higher arithmetic K -theory

In this section we recall the definition of higher arithmetic K -groups given by Takeda in [21]. He first develops a theory of homotopy groups modified by a suitable chain morphism ρ . As a particular case, the higher arithmetic K -groups are given by the homotopy groups of $\widehat{S}(X)$ modified by the Chern character morphism ch .

Let T be a pointed CW-complex and let $C_*(T)$ be its cellular complex (see, for instance, [16]). Let (W_*, d) be a chain complex and denote $\widetilde{W}_* = W_*/\text{im } d$. Suppose that we are given a chain morphism $\rho : C_*(T) \rightarrow W_*$. Consider pairs (f, ω) where

- ▷ $f : S^n \rightarrow T$ is a pointed cellular map,
- ▷ $\omega \in \widetilde{W}_{n+1}$.

Let I be the closed unit interval $[0, 1]$ with the usual CW-complex structure. Two pairs (f, ω) and (f', ω') are *homotopy equivalent* if there exists a pointed cellular map

$$h : S^n \times I/\{*\} \times I \rightarrow T$$

such that the following conditions hold:

- (1) h is a topological homotopy between f and f' , i.e.

$$h(x, 0) = f(x) \quad \text{and} \quad h(x, 1) = f'(x).$$

- (2) Let $[S^n \times I] \in C_{n+1}(S^n \times I)$ denote the fundamental chain of $S^n \times I$. Then

$$\omega' - \omega = (-1)^{n+1} \rho(h_*([S^n \times I])).$$

Being homotopy equivalent is an equivalence relation, which we denote by \sim . Then, for every n , the *modified homotopy group* $\widehat{\pi}_n(T, \rho)$ is defined to be the set of all homotopy classes of pairs as above. Takeda proves that these are in fact abelian groups.

The higher arithmetic K -groups of a proper arithmetic variety X , as defined by Takeda, are given as the modified homotopy groups of the Waldhausen simplicial set $\widehat{\mathcal{P}}(X)$ modified by the representative of the Beilinson regulator ch given in the previous section.

Let X be a proper arithmetic variety over \mathbb{Z} . Let $|\widehat{S}\cdot(X)|$ denote the geometric realization of the simplicial set $\widehat{S}\cdot(X)$. It follows that $|\widehat{S}\cdot(X)|$ is a CW-complex.

Let $\widehat{D}_*^s(X) \subset \mathbb{Z}\widehat{S}_*(X)$ be the complex generated by the degenerate simplices of $\widehat{S}\cdot(X)$. Since the cellular complex $C_*(|\widehat{S}\cdot(X)|)$ is naturally isomorphic to the complex $\mathbb{Z}\widehat{S}_*(X)/\widehat{D}_*^s(X)$, we will identify these two complexes by this isomorphism.

As shown in [21, Theorem 4.4], the map $\text{ch} \circ \text{Cub}$ maps the degenerate simplices of $\widehat{S}\cdot(X)$ to zero. It follows that there is a well-defined chain morphism

$$\text{ch} : C_*(|\widehat{S}\cdot(X)|)[-1] \rightarrow \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p).$$

Definition 2.3 (Takeda). Let X be a proper arithmetic variety over \mathbb{Z} . For every $n \geq 0$, the *higher arithmetic K -group* of X , $\widehat{K}_n^T(X)$, is defined by

$$\begin{aligned} \widehat{K}_n^T(X) &= \widehat{\pi}_{n+1}(|\widehat{S}\cdot(X)|, \text{ch}) \\ &= \left\{ (f : S^{n+1} \rightarrow |\widehat{S}\cdot(X)|, \omega) \mid \omega \in \bigoplus_{p \geq 0} \widetilde{\mathcal{D}}^{2p-n-1}(X, p) \right\} / \sim. \end{aligned}$$

Takeda proves the following results:

- (i) For every $n \geq 0$, $\widehat{K}_n^T(X)$ is a group.
(ii) For every $n \geq 0$, there is a short exact sequence

$$K_{n+1}(X) \xrightarrow{\text{ch}} \bigoplus_{p \geq 0} \widetilde{\mathcal{D}}^{2p-n-1}(X, p) \xrightarrow{a} \widehat{K}_n^T(X) \xrightarrow{\zeta} K_n(X) \rightarrow 0.$$

The morphisms a, ζ are defined by $a(\alpha) = [(0, \alpha)]$ and $\zeta([(f, \alpha)]) = [f]$.

(iii) There is a characteristic class

$$\widehat{K}_n^T(X) \xrightarrow{\text{ch}} \bigoplus_{p \geq 0} \mathcal{D}^{2p-n}(X, p),$$

given by

$$\text{ch}([(f, \alpha)]) = \text{ch}(f_*(S^n)) + d_{\mathcal{D}}\alpha.$$

(iv) $\widehat{K}_0^T(X)$ is isomorphic to the arithmetic K -group defined by Gillet and Soulé in [12].

(v) There is a graded product on $\widehat{K}_*^T(X)$, commutative up to 2-torsion. Therefore, $\widehat{K}_*^T(X)_{\mathbb{Q}}$ is endowed with a graded commutative product.

(vi) There exist pull-back for arbitrary morphisms and push-forward for smooth and projective morphisms. A projection formula is also proved.

Lemma 2.2 ([21, Cor. 4.9]). *Let X be a proper arithmetic variety over \mathbb{Z} . Then, for every $n \geq 0$, there is a canonical isomorphism*

$$\widehat{K}_n(X) \cong \ker\left(\text{ch} : \widehat{K}_n^T(X) \rightarrow \bigoplus_{p \geq 0} \mathcal{D}^{2p-n}(X, p)\right). \quad \square$$

§3. Rational higher arithmetic K -groups

By parallelism with the algebraic situation, it is natural to expect that the higher arithmetic K -groups tensored by \mathbb{Q} can be described in homological terms. In Proposition 2.1, we saw that the Deligne–Soulé higher arithmetic K -groups are isomorphic to the homology groups of the simple complex associated to the Beilinson regulator ch , after tensoring by \mathbb{Q} . In this section we show that the higher arithmetic K -groups given by Takeda also admit, after tensoring by \mathbb{Q} , a homological description. We prove that $\widehat{K}_n^T(X)_{\mathbb{Q}}$ can be obtained by considering a variant of the complex of cubes, together with what we call modified homology groups.

§3.1. Modified homology groups

We briefly describe here the analogue, in a homological context, of the modified homotopy groups given by Takeda in [21]. The modified homology groups are the dual notion of the *truncated relative cohomology groups* defined by Burgos in [2], as one can observe by comparing both definitions and the relevant properties. These groups appear naturally in other contexts. For instance, one can express the description of hermitian-holomorphic Deligne cohomology given by Aldrovandi in [1, §2.2] in terms of modified homology groups.

Let (A_*, d_A) and (B_*, d_B) be two chain complexes and let $A_* \xrightarrow{\rho} B_*$ be a chain morphism. If $\widetilde{B}_* = B_*/\text{im } d_B$, consider pairs

$$(a, b) \in A_n \oplus \widetilde{B}_{n+1} \quad \text{with} \quad d_A a = 0.$$

We define an equivalence relation as follows. We say that $(a, b) \sim (a', b')$ if, and only if, there exists $h \in A_{n+1}$ such that

$$d_A h = a - a' \quad \text{and} \quad \rho(h) = b - b'.$$

Definition 3.1. Let $(A_*, d_A), (B_*, d_B)$ be two chain complexes and let $\rho : A_* \rightarrow B_*$ be a chain morphism. For every n , the n -th modified homology group of A_* with respect to ρ is defined as

$$\widehat{H}_n(A_*, \rho) := \{(a, b) \in ZA_n \oplus \widetilde{B}_{n+1}\} / \sim.$$

Observe that the group $\widehat{H}_n(A_*, \rho)$ can be rewritten as

$$\widehat{H}_n(A_*, \rho) = \frac{\{(a, b) \in ZA_n \oplus B_{n+1}\}}{\{(0, d_B b), (d_A a, \rho(a)) \mid a \in A_{n+1}, b \in B_{n+2}\}}.$$

The class of a pair (a, b) in $\widehat{H}_n(A_*, \rho)$ is denoted by $[(a, b)]$.

These modified homology groups can be seen as the homology groups of the simple complex of ρ truncated appropriately. Let $\rho_{>n}$ be the composition of $\rho : A_* \rightarrow B_*$ with the canonical morphism $B_* \rightarrow \sigma_{>n} B_*$. Then it follows from the definition that

$$H_r(s(\rho_{>n})) = \begin{cases} H_r(s(\rho)), & r > n, \\ \widehat{H}_n(A_*, \rho), & r = n, \\ H_r(A_*), & r < n. \end{cases}$$

Observe that, for every n , there are well-defined morphisms

$$\begin{aligned} \widetilde{B}_{n+1} &\xrightarrow{a} \widehat{H}_n(A_*, \rho), & b &\mapsto [(0, -b)], \\ \widehat{H}_n(A_*, \rho) &\xrightarrow{\zeta} H_n(A_*), & [(a, b)] &\mapsto [a], \\ \widehat{H}_n(A_*, \rho) &\xrightarrow{\rho} ZB_n, & [(a, b)] &\mapsto \rho(a) - d_B(b). \end{aligned}$$

The following proposition is the homological analogue of Theorem 3.3 together with Proposition 3.9 of [21] and the dual of Propositions 4.3 and 4.4 of [2].

Proposition 3.1. (i) Let $\rho : A_* \rightarrow B_*$ be a chain morphism. Then, for every n , there are exact sequences

$$(a) \quad 0 \rightarrow H_n(s_*(\rho)) \rightarrow \widehat{H}_n(A_*, \rho) \xrightarrow{\rho} ZB_n \rightarrow H_{n-1}(s_*(\rho)),$$

$$(b) \quad H_{n+1}(A_*) \xrightarrow{\rho} \widetilde{B}_{n+1} \xrightarrow{a} \widehat{H}_n(A_*, \rho) \xrightarrow{\zeta} H_n(A_*) \rightarrow 0.$$

(ii) Assume that there is a commutative square of chain complexes

$$\begin{array}{ccc} A_* & \xrightarrow{\rho} & B_* \\ f_1 \downarrow & & \downarrow f_2 \\ C_* & \xrightarrow{\rho'} & D_* \end{array}$$

Then, for every n , there is an induced morphism

$$\widehat{H}_n(A_*, \rho) \xrightarrow{f} \widehat{H}_n(C_*, \rho') \quad [(a, b)] \mapsto [(f_1(a), f_2(b))].$$

(iii) If f_1 is a quasi-isomorphism and f_2 is an isomorphism, then f is an isomorphism.

Proof. The exact sequences follow from the long exact sequences associated to the following short exact sequences:

$$\begin{aligned} 0 \rightarrow B_*/\sigma_{>n}B_*[-1] \rightarrow s_*(\rho) \rightarrow s_*(\rho_{>n}) \rightarrow 0, \\ 0 \rightarrow \sigma_{>n}B_*[-1] \rightarrow s_*(\rho_{>n}) \rightarrow A_* \rightarrow 0. \end{aligned}$$

The second and third statements are left to the reader. □

Corollary 3.1. For every n , there is a canonical isomorphism

$$H_n(s_*(\rho)) \cong_{\text{can}} \ker(\widehat{H}_n(A_*, \rho) \xrightarrow{\rho} B_n). \quad \square$$

§3.2. Takeda arithmetic K -theory with rational coefficients

We want to give a homological description of the rational Takeda arithmetic K -groups. Since these groups fit in the exact sequences

$$K_{n+1}(X)_{\mathbb{Q}} \xrightarrow{\text{ch}} \bigoplus_{p \geq 0} \widetilde{\mathcal{D}}^{2p-n-1}(X, p) \xrightarrow{a} \widehat{K}_n^T(X)_{\mathbb{Q}} \xrightarrow{\zeta} K_n(X)_{\mathbb{Q}} \rightarrow 0,$$

comparing them to those in Proposition 3.1(i)(b), it is natural to expect that the modified homology groups associated to the Beilinson regulator ch give the desired description.

Therefore, consider the modified homology groups $\widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X), \widehat{\text{ch}})$ associated to the chain map

$$\widetilde{\mathbb{Z}}\widehat{C}_*(X) \xrightarrow{\text{ch}} \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p)$$

given in (1.4). We want to see that there is an isomorphism

$$(3.1) \quad \widehat{K}_n^T(X)_{\mathbb{Q}} \cong \widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X), \text{ch})_{\mathbb{Q}}.$$

In order to prove this fact, considering the long exact sequences associated to $\widehat{K}_n^T(X)_{\mathbb{Q}}$ and to $\widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X), \text{ch})_{\mathbb{Q}}$ and the five lemma, it would be desirable to have a factorization of the morphism ch through Cub in the form

$$\begin{array}{ccccc} \mathbb{Z}\widehat{S}_*(X) & \xrightarrow{\text{Cub}} & \widetilde{\mathbb{Z}}\widehat{C}_*(X) & \xrightarrow{\text{ch}} & \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p) \\ & \searrow & \uparrow \text{Cub} & \nearrow \text{ch} & \\ & & C_*(|\widehat{S}_*(X)|)[-1] \cong \widehat{S}_*(X)/\widehat{D}_*^s(X)[-1] & & \end{array}$$

Let \mathcal{P} be a small exact category. If $\tau_i \in \mathfrak{S}_n$ is the permutation that interchanges i with $i + 1$, then for every $E \in S_n(\mathcal{P})$ one has

$$(3.2) \quad \begin{aligned} \text{Cub}(s_0 E) &= s_1^1 \text{Cub}(E), \\ \text{Cub}(s_n E) &= s_n^0 \text{Cub}(E), \\ \text{Cub}(s_i E) &= \tau_i \text{Cub}(s_i E), \quad i = 1, \dots, n-1. \end{aligned}$$

(See [21, Lemma 4.1].) It follows that the dotted Cub arrow of the last diagram

$$C_*(|\widehat{S}(X)|)[-1] \cong \mathbb{Z}\widehat{S}_*(X)/\widehat{D}_*^s(X)[-1] \xrightarrow{\text{Cub}} \widetilde{\mathbb{Z}}\widehat{C}_*(X)$$

does not exist, since the image under Cub of a degenerate simplex in $S_n(\mathcal{P})$ is not necessarily a degenerate cube.

Therefore, in order to prove (3.1), we should find a new complex, $\widetilde{\mathbb{Z}}\widehat{C}_*(X)$, quasi-isomorphic to the complex of hermitian cubes, admitting a factorization of ch of the form

$$\begin{array}{ccccc} \mathbb{Z}\widehat{S}_*(X) & \xrightarrow{\text{Cub}} & \widetilde{\mathbb{Z}}\widehat{C}_*(X) & \xrightarrow{\text{ch}} & \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p) \\ \downarrow & & \sim \downarrow & \nearrow \text{ch} & \\ C_*(|\widehat{S}(X)|)[-1] & \xrightarrow{\text{Cub}} & \widetilde{\mathbb{Z}}\widehat{C}_*^s(X) & & \end{array}$$

In this way, we divide the proof in two steps: we prove an isomorphism $\widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X), \text{ch}) \cong \widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}_*^s(X), \text{ch})$, and then $\widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X), \text{ch})_{\mathbb{Q}} \cong \widehat{K}_n^T(X)_{\mathbb{Q}}$. This will be shown in Theorem 3.1, once this factorization of Cub is obtained.

Factorization of Cub. Takeda factors the morphism ch through a quotient of the complex of cubes as follows. Consider the complex of cubes $\widehat{T}_n(X) \subseteq \mathbb{Z}\widehat{C}_n(X)$, generated by the n -cubes \overline{E} such that $\tau_i \overline{E} = \overline{E}$ for some index i . In the proof of Theorem 4.4 in [21], Takeda shows that if $\overline{E} \in \widehat{T}_n(X)$, then $\text{ch}(\overline{E}) = 0$. Hence ch is zero on the degenerate simplices in $\mathbb{Z}\widehat{S}_*(X)$. It follows that ch factorizes as

$$\begin{aligned} C_*(|\widehat{S}(X)|)[-1] &\xrightarrow{\cong} \mathbb{Z}\widehat{S}_*(X)/\widehat{D}_*^s(X)[-1] \xrightarrow{\text{Cub}} \widetilde{\mathbb{Z}}\widehat{C}_*(X)/\widehat{T}_*(X) \\ &\xrightarrow{\text{ch}} \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p). \end{aligned}$$

However, the complex $\widetilde{\mathbb{Z}}\widehat{C}_*(X)/\widehat{T}_*(X)$ is not quasi-isomorphic to $\widetilde{\mathbb{Z}}\widehat{C}_*(X)$. Nevertheless, since the complex $\mathbb{Z}\widehat{S}_*(X)/\widehat{D}_*^s(X)$ is quasi-isomorphic to $\mathbb{Z}\widehat{S}_*(X)$ (due to the fact that the complex of degenerate simplices of a simplicial set is acyclic), it seems reasonable to think that there exists a complex which is quasi-isomorphic to $\widetilde{\mathbb{Z}}\widehat{C}_*(X)$ and which factors the morphism ch as above. This is indeed proved below.

Let \mathcal{P} be a small exact category. The smallest complex to consider is the following. For every n , let

$$C_n^{\text{deg}}(\mathcal{P}) := \{\text{Cub}(s_i E) \mid E \in S_n(\mathcal{P}), i \in \{1, \dots, n-1\}\}.$$

Let $\mathbb{Z}C_n^{\text{deg}}(\mathcal{P})$ be the free abelian group on $C_n^{\text{deg}}(\mathcal{P})$ and let

$$\tilde{\mathbb{Z}}C_n^{\text{deg}}(\mathcal{P}) := \frac{\mathbb{Z}C_n^{\text{deg}}(\mathcal{P}) + \mathbb{Z}D_n(\mathcal{P})}{\mathbb{Z}D_n(\mathcal{P})}.$$

Lemma 3.1. *Let $E \in S_n(\mathcal{P})$.*

- (i) $d \text{Cub}(s_i E) \in \mathbb{Z}C_{n-1}^{\text{deg}}(\mathcal{P}) + \mathbb{Z}D_{n-1}(\mathcal{P})$ for all $i = 1, \dots, n-1$.
- (ii) For $i = 1, \dots, n-1$, the following equality holds in $\tilde{\mathbb{Z}}C_{n-1}^{\text{deg}}(\mathcal{P})$:

$$d \text{Cub}(s_i E) = \sum_{j=0}^{i-1} (-1)^{j+1} \text{Cub}(s_{i-1} \partial_j E) + \sum_{j=i+1}^n (-1)^j \text{Cub}(s_i \partial_j E).$$

Proof. By definition,

$$d \text{Cub}(s_i E) = \sum_{j=1}^n \sum_{l=0}^2 (-1)^{j+l} \partial_j^l \text{Cub}(s_i E).$$

Since $\partial_i^l \tau_i = \partial_{i+1}^l$ for all $l = 0, 1, 2$, by (3.2) we have

$$\partial_i^l \text{Cub}(s_i E) = \partial_{i+1}^l \text{Cub}(s_i E).$$

Hence these two terms cancel each other in the previous sum. So, assume that $j \neq i, i+1$. If $l = 1$, then, by (1.2),

$$\partial_j^1 \text{Cub}(s_i E) = \text{Cub}(\partial_j s_i E) = \begin{cases} \text{Cub}(s_{i-1} \partial_j E), & j < i, \\ \text{Cub}(s_i \partial_{j-1} E), & j > i+1. \end{cases}$$

If $l = 0$ and $j \neq n$, or $l = 2$ and $j \neq 1$, then $\partial_j^l \text{Cub}(s_i E)$ is a degenerate cube and hence it is zero in the group $\tilde{\mathbb{Z}}C_n^{\text{deg}}(\mathcal{P})$. Finally, we have

$$\partial_n^0 \text{Cub}(s_i E) = \text{Cub}(s_i \partial_n E), \quad \partial_1^2 \text{Cub}(s_i E) = \text{Cub}(s_{i-1} \partial_0 E).$$

The statements of the lemma follow from these calculations and the relations (3.2). □

It follows from the last lemma that $\tilde{\mathbb{Z}}C_*^{\text{deg}}(\mathcal{P})$ is a chain complex with the differential induced by the differential of $\tilde{\mathbb{Z}}C_*(\mathcal{P})$.

Proposition 3.2. *The complex $\tilde{\mathbb{Z}}C_*^{\text{deg}}(\mathcal{P})$ is quasi-isomorphic to zero.*

Proof. We prove this by constructing a chain of chain complexes

$$(3.3) \quad 0 = C_*^0 \subset C_*^1 \subset \cdots \subset C_*^{n-2} \subset C_*^{n-1} = \widetilde{\mathbb{Z}}C_*^{\text{deg}}(\mathcal{P})$$

such that all the quotients C_*^i/C_*^{i-1} are homotopically trivial, that is, there exists a homotopy

$$h_n : C_n^i/C_n^{i-1} \rightarrow C_{n+1}^i/C_{n+1}^{i-1}$$

such that

$$dh_n + h_{n-1}d = \text{id}.$$

This means in particular that for every i , the complex C_*^i/C_*^{i-1} is quasi-isomorphic to zero. Then, since $C_*^0 = 0$, it follows inductively that C_*^i is quasi-isomorphic to zero for all i and the proposition is proved.

For every $i = 1, \dots, n-1$, let

$$\mathbb{Z}C_n^{\text{deg},i}(\mathcal{P}) = \{\text{Cub}(s_j E) \mid E \in S_n(\mathcal{P}), j \in \{1, \dots, i\}\},$$

and let

$$C_n^i = \frac{\mathbb{Z}C_n^{\text{deg},i}(\mathcal{P}) + \mathbb{Z}D_n(\mathcal{P})}{\mathbb{Z}D_n(\mathcal{P})}.$$

By Lemma 3.1(ii), C_*^i are chain complexes with the differential induced by the differential of $\widetilde{\mathbb{Z}}C_*(\mathcal{P})$. Moreover, for every i there is an inclusion of complexes $C_*^i \subseteq C_*^{i+1}$.

Fix $E \in S_n(\mathcal{P})$ and an index i . Consider an element $\text{Cub}(s_i E) \in C_*^i/C_*^{i-1}$ and define

$$h_n(\text{Cub}(s_i E)) = (-1)^{i+1} \text{Cub}(s_i s_i E).$$

Then, by Lemma 3.1, in the complex C_*^i/C_*^{i-1} ,

$$d \text{Cub}(s_i E) = \sum_{j=i+1}^{n+1} (-1)^j \text{Cub}(s_i \partial_j E)$$

and

$$\begin{aligned} dh_n(\text{Cub}(s_i E)) &= \sum_{j=i+1}^{n+1} (-1)^{i+j+1} \text{Cub}(s_i \partial_j s_i E) \\ &= \text{Cub}(s_i E) + \sum_{j=i+2}^{n+1} (-1)^{i+j+1} \text{Cub}(s_i s_i \partial_{j-1} E) \\ &= \text{Cub}(s_i E) + h_{n-1}(d \text{Cub}(s_i E)). \end{aligned}$$

Therefore, we have proved that C_*^i/C_*^{i-1} is homotopically trivial. \square

Let

$$\tilde{\mathbb{Z}}C_*^s(\mathcal{P}) := \frac{\mathbb{Z}C_*(\mathcal{P})}{\mathbb{Z}D_*(\mathcal{P}) + \mathbb{Z}C_*^{\text{deg}}(\mathcal{P})}.$$

Corollary 3.2. *The natural chain morphism $\tilde{\mathbb{Z}}C_*(\mathcal{P}) \rightarrow \tilde{\mathbb{Z}}C_*^s(\mathcal{P})$ is a quasi-isomorphism. \square*

If $\mathcal{P} = \widehat{\mathcal{P}}(X)$, we simply write $\tilde{\mathbb{Z}}\widehat{C}_*^s(X) := \tilde{\mathbb{Z}}C_*^s(\widehat{\mathcal{P}}(X))$ and $\tilde{\mathbb{Z}}\widehat{C}_*^{\text{deg}}(X) := \tilde{\mathbb{Z}}\widehat{C}_*^{\text{deg}}(\widehat{\mathcal{P}}(X))$. Since ch is zero on $\mathbb{Z}\widehat{D}_*(X) + \mathbb{Z}\widehat{C}_*^{\text{deg}}(X)$, we have obtained the following corollary.

Corollary 3.3. *Let X be a proper arithmetic variety over \mathbb{Z} .*

(i) *The map ch admits a factorization as*

$$C_*(|\widehat{S} \cdot (X)|)[-1] \xrightarrow{\text{Cub}} \tilde{\mathbb{Z}}\widehat{C}_*^s(X) \xrightarrow{\text{ch}} \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p).$$

(ii) *The natural morphism $\tilde{\mathbb{Z}}\widehat{C}_*(X) \xrightarrow{\sim} \tilde{\mathbb{Z}}\widehat{C}_*^s(X)$ is a quasi-isomorphism. \square*

At this point, we have all the ingredients to prove that there is an isomorphism between $\widehat{K}_n^T(X)_{\mathbb{Q}}$ and $\widehat{H}_n(\tilde{\mathbb{Z}}\widehat{C}_*(X), \text{ch})_{\mathbb{Q}}$.

For the proof of the next theorem recall that the Hurewicz morphism

$$\pi_n(|\widehat{S} \cdot (X)|) \rightarrow H_n(|\widehat{S} \cdot (X)|)$$

maps the class of a pointed map $S^n \xrightarrow{f} |\widehat{S} \cdot (X)|$ to $f_*([S^n])$.

Theorem 3.1. *Let X be a proper arithmetic variety over \mathbb{Z} . Then, for every $n \geq 0$, there is an isomorphism*

$$\widehat{K}_n^T(X)_{\mathbb{Q}} \xrightarrow{\cong} \widehat{H}_n(\tilde{\mathbb{Z}}\widehat{C}_*(X), \text{ch})_{\mathbb{Q}}.$$

Moreover, there are commutative diagrams

$$\begin{array}{ccc} \widehat{K}_n^T(X)_{\mathbb{Q}} & \xrightarrow{\zeta} & K_n(X)_{\mathbb{Q}} & & \widehat{K}_n^T(X)_{\mathbb{Q}} & \xrightarrow{\text{ch}} & \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p) \\ \cong \downarrow & & \downarrow \cong & & \cong \downarrow & & \downarrow = \\ \widehat{H}_n(\tilde{\mathbb{Z}}\widehat{C}_*(X), \text{ch})_{\mathbb{Q}} & \xrightarrow{\zeta} & H_n(\tilde{\mathbb{Z}}\widehat{C}_*(X), \mathbb{Q}) & & \widehat{H}_n(\tilde{\mathbb{Z}}\widehat{C}_*(X), \text{ch})_{\mathbb{Q}} & \xrightarrow{\text{ch}} & \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p) \end{array}$$

Proof. Consider the chain complex $\tilde{\mathbb{Z}}\widehat{C}_*^s(X) = \frac{\mathbb{Z}\widehat{C}_*(X)}{\mathbb{Z}\widehat{D}_*(X) + \mathbb{Z}\widehat{C}_*^{\text{deg}}(X)}$, defined before Corollary 3.3. Let $\widehat{H}_*(\tilde{\mathbb{Z}}\widehat{C}_*^s(X), \text{ch})$ denote the modified homology groups with respect to the morphism

$$\tilde{\mathbb{Z}}\widehat{C}_*^s(X) \xrightarrow{\text{ch}} \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p).$$

Consider the following commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathbb{Z}}\widehat{C}_*(X) & \xrightarrow{\text{ch}} & \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p) \\ \sim \downarrow & & \downarrow = \\ \widetilde{\mathbb{Z}}\widehat{C}_*^s(X) & \xrightarrow{\text{ch}} & \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p) \end{array}$$

By Lemma 3.1, there is an induced isomorphism

$$\widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X), \text{ch}) \xrightarrow{\pi} \widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}_*^s(X), \text{ch}),$$

which commutes with ζ . It remains to prove that there is an isomorphism

$$\widehat{K}_n^T(X)_{\mathbb{Q}} \cong \widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}_*^s(X), \text{ch})_{\mathbb{Q}},$$

commuting with ζ .

Consider the chain morphism

$$C_*(|\widehat{S} \cdot (X)|)[-1] \xrightarrow{\text{Cub}} \widetilde{\mathbb{Z}}\widehat{C}_*^s(X).$$

Recall from §1.1 that the isomorphism

$$K_n(X)_{\mathbb{Q}} \xrightarrow{\text{Cub}} H_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X), \mathbb{Q})$$

is given by the composition

$$K_n(X) = \pi_{n+1}(|\widehat{S} \cdot (X)|)_{\mathbb{Q}} \xrightarrow{\text{Hurewicz}} H_n(C_*(|\widehat{S} \cdot (X)|)[-1])_{\mathbb{Q}} \xrightarrow{\text{Cub}} H_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X), \mathbb{Q})$$

which sends the class of a cellular map $[f : S^{n+1} \rightarrow |\widehat{S} \cdot (X)|]$ to $\text{Cub } f_*([S^{n+1}])$.

If $f, f' : S^{n+1} \rightarrow |\widehat{S} \cdot (X)|$ are homotopic with cellular homotopy h , then

$$dh_*[S^{n+1} \times I] = (-1)^{n+1}(f'_*[S^{n+1}] - f_*[S^{n+1}])$$

in $C_*(|\widehat{S} \cdot (X)|)[-1]$.

Let

$$\begin{aligned} \widehat{K}_n^T(X)_{\mathbb{Q}} &\xrightarrow{\text{Cub}^s} \widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}_*^s(X), \text{ch})_{\mathbb{Q}}, \\ [(f : S^{n+1} \rightarrow |\widehat{S} \cdot (X)|, \omega)] &\mapsto [(\text{Cub } f_*([S^{n+1}]), -\omega)]. \end{aligned}$$

This morphism is well defined. Indeed, let h be a cellular homotopy between (f, ω) and (f', ω') . Then, if we denote $\alpha = (-1)^{n+1} \text{Cub } h_*([S^{n+1} \times I])$, we have

$$d(\alpha) = \text{Cub } f'_*([S^{n+1}]) - \text{Cub } f_*([S^{n+1}]) \quad \text{and} \quad \text{ch}(\alpha) = \omega - \omega'.$$

Finally, consider the diagram

$$\begin{array}{ccccccccc}
 K_{n+1}(X)_{\mathbb{Q}} & \xrightarrow{\text{ch}} & \tilde{\mathcal{D}}^{2p-n-1}(X, p) & \xrightarrow{a} & \widehat{K}_n^T(X)_{\mathbb{Q}} & \xrightarrow{\zeta} & K_n(X)_{\mathbb{Q}} & \longrightarrow & 0 \\
 \cong \downarrow & & \downarrow = & & \text{Cub}^s \downarrow & & \cong \downarrow & & \\
 H_{n+1}(\tilde{\mathbb{Z}}\widehat{C}_*(X))_{\mathbb{Q}} & \xrightarrow{\text{ch}} & \tilde{\mathcal{D}}^{2p-n-1}(X, p) & \xrightarrow{a} & \widehat{H}_n(\tilde{\mathbb{Z}}\widehat{C}_*(X), \text{ch})_{\mathbb{Q}} & \xrightarrow{\zeta} & H_n(\tilde{\mathbb{Z}}\widehat{C}_*(X))_{\mathbb{Q}} & \longrightarrow & 0
 \end{array}$$

Since the rows are exact sequences, the statement of the proposition follows from the five lemma. \square

Corollary 3.4. *Let X be a proper arithmetic variety over \mathbb{Z} . Then, for every $n \geq 0$, there is an isomorphism*

$$\widehat{K}_n^T(X)_{\mathbb{Q}} \xrightarrow{\cong} \widehat{H}_n(N\widehat{C}_*(X), \text{ch})_{\mathbb{Q}}. \quad \square$$

Product structure on rational arithmetic K -theory. Takeda [21] defines a product structure for $\widehat{K}_n^T(X)$ compatible with the Loday product of algebraic K -theory, and for which the morphism

$$\text{ch} : \widehat{K}_n^T(X) \rightarrow \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p)$$

is a ring morphism (loc. cit., Proposition 6.8). Since there is a natural isomorphism

$$\widehat{K}_n(X) \cong \ker\left(\text{ch} : \widehat{K}_n^T(X) \rightarrow \bigoplus_{p \geq 0} \mathcal{D}^{2p-n}(X, p)\right),$$

there is an induced Loday product on $\widehat{K}_n(X)$.

In algebraic K -theory, the Adams operations are derived from the lambda operations by a polynomial relation. In order to do that, the product structure for $\bigoplus_{n \geq 0} K_n(X)$ is the one for which $\bigoplus_{n \geq 1} K_n(X)$ is a square zero ideal.

Therefore, we consider the product structure on $\bigoplus_{n \geq 0} \widehat{K}_n(X)$ for which $\bigoplus_{n \geq 1} \widehat{K}_n(X)$ is a square zero ideal and which agrees with the Loday product otherwise.

After tensoring with \mathbb{Q} , and using the description of $\widehat{K}_n(X)_{\mathbb{Q}}$ via the isomorphism

$$\widehat{K}_n(X)_{\mathbb{Q}} \cong H_n(s(\widehat{\text{ch}}), \mathbb{Q}),$$

the product is defined as follows.

Lemma 3.2. *Let $(\overline{E}, \alpha) \in \widehat{K}_0(X)_{\mathbb{Q}}$ and $(\overline{F}, \beta) \in \widehat{K}_n(X)_{\mathbb{Q}}$ with $n \geq 0$. Then, for the product structure on $\widehat{K}_n(X)_{\mathbb{Q}}$ induced by the product structure on $\widehat{K}_n^T(X)_{\mathbb{Q}}$, we have*

$$(\overline{E}, \alpha) \otimes (\overline{F}, \beta) = (\overline{E} \otimes \overline{F}, \alpha \bullet \text{ch}(\overline{F}) + \text{ch}(\overline{E}) \bullet \beta - \alpha \bullet d_{\mathcal{D}}(\beta)) \in \widehat{K}_n(X)_{\mathbb{Q}}. \quad \square$$

Remark 3.1. With the notation of the previous lemma, if $n > 0$ then \overline{F} is an n -cube such that $d\overline{F} = 0$ and $\beta \in \bigoplus_{p \geq 0} \mathcal{D}^{2p-n-1}(X, p)$ is a differential form such that $\text{ch}(\overline{F}) = d_{\mathcal{D}}(\beta)$. Hence,

$$\alpha \bullet \text{ch}(\overline{F}) = \alpha \bullet d_{\mathcal{D}}(\beta),$$

and therefore

$$(\overline{E}, \alpha) \otimes (\overline{F}, \beta) = (\overline{E} \otimes \overline{F}, \text{ch}(\overline{E}) \bullet \beta).$$

§4. Adams operations on higher arithmetic K -theory

In this section we construct the Adams operations on the higher arithmetic K -groups tensored by the rational numbers. The construction works for both definitions of higher arithmetic K -groups.

Let X be a proper arithmetic variety over \mathbb{Z} . In [10], we defined a chain morphism inducing Adams operations on higher algebraic K -theory (with rational coefficients), using the chain complex of cubes. The results stated in [10] for the category of locally free sheaves of finite rank over X , $\mathcal{P}(X)$, translate to the category $\widehat{\mathcal{P}}(X)$, provided we consider appropriate metrics. As will be overviewed next, the definition of the algebraic Adams operations involves the Koszul complex of a locally free sheaf and the isomorphism (4.3). Therefore, the metrics on the Koszul complex are imposed by the need for the equivalent isometry (4.5) to hold.

In order to define Adams operations on higher arithmetic K -groups tensored by \mathbb{Q} , it would be desirable to have the Chern character morphism commute with the Adams operations on the complex of cubes (diagram (4.12)). To this end, the Bott–Chern form of the Koszul complex should be zero. However, with the hermitian metric on the Koszul complex imposed by (4.5), its Bott–Chern form does not vanish. This problem can be solved by slightly modifying the definition of Adams operations of [10] (see Remark 4.1).

We start this section with an overview of the Adams operations defined in [10], together with the required modification. In the next subsection we discuss the Bott–Chern form of the Koszul complex. We finish by showing the commutativity of diagram (4.12) and by deducing the Adams operations on higher arithmetic K -theory tensored by \mathbb{Q} .

§4.1. Adams operations on higher algebraic K -theory

We recall briefly the key points of the definition of Adams operations of [10], with a slight modification. In the following definitions, \cong denotes an isometry.

The Koszul complex. For every locally free sheaf E of finite rank on any variety, the k -th Koszul complex of E is the exact sequence

$$\Psi^k(E)^* : 0 \rightarrow \Psi^k(E)^0 \xrightarrow{\varphi_0} \dots \xrightarrow{\varphi_{k-1}} \Psi^k(E)^k \rightarrow 0$$

with

$$\Psi^k(E)^p = E \cdot \dots \cdot E \otimes E \wedge^{k-p} E = S^p E \otimes \bigwedge^{k-p} E.$$

The arrows φ_p are defined as follows. Consider the inclusions

$$S^p E \xrightarrow{\iota_p} T^p E \quad \text{and} \quad \bigwedge^p E \xrightarrow{j_p} T^p E$$

defined locally by

$$(4.1) \quad \begin{aligned} \iota_p(x_{i_1} \cdot \dots \cdot x_{i_p}) &= \sum_{\sigma \in \mathfrak{S}_p} x_{\sigma(i_1)} \otimes \dots \otimes x_{\sigma(i_p)}, \\ j_p(x_{i_1} \wedge \dots \wedge x_{i_p}) &= \sum_{\tau \in \mathfrak{S}_p} (-1)^{|\tau|} x_{\tau(i_1)} \otimes \dots \otimes x_{\tau(i_p)}. \end{aligned}$$

Consider the natural projections

$$\begin{array}{ccc} T^p E \xrightarrow{\pi_p} S^p E & \text{and} & T^p E \xrightarrow{\rho_p} \bigwedge^p E \\ x_{i_1} \otimes \dots \otimes x_{i_p} \mapsto x_{i_1} \cdot \dots \cdot x_{i_p} & & x_{i_1} \otimes \dots \otimes x_{i_p} \mapsto x_{i_1} \wedge \dots \wedge x_{i_p}. \end{array}$$

For every p , the morphisms

$$\varphi_p : S^p E \otimes \bigwedge^{k-p} E \rightarrow S^{p+1} E \otimes \bigwedge^{k-p-1} E$$

in the Koszul complex are given as

$$(4.2) \quad \varphi_p = \frac{1}{p!(k-p-1)!} (\pi_{p+1} \otimes \rho_{k-p-1}) \circ (\iota_p \otimes j_{k-p})$$

(see (4.9) for the explicit computation of φ_p).

The key properties of the Koszul complex that make it suitable for the definition of Adams operations on higher algebraic K -theory are the following:

- If E and F are two locally free sheaves of finite rank, there is a canonical isomorphism of complexes

$$(4.3) \quad \Psi^k(E \oplus F)^* \cong \bigoplus_{p=0}^k \Psi^p(E)^* \otimes \Psi^{k-p}(F)^*.$$

- The secondary Euler characteristic class of the Koszul complex

$$(4.4) \quad \Psi^k(E) = \sum_{p \geq 0} (-1)^{k-p+1} (k-p) \Psi^k(E)^p$$

agrees with the k -th Adams operation of E in $K_0(X)$.

If \overline{E} is a hermitian vector bundle, there is a naturally induced metric on the tensor product $T^k \overline{E} = \overline{E} \otimes \dots \otimes \overline{E}$. Endow $\bigwedge^k \overline{E}$ with the wedge product metric, that is, the metric induced by the natural inclusion of $\bigwedge^k \overline{E}$ into $T^k \overline{E}$ given by $\frac{1}{\sqrt{k!}} j_k$. Analogously, we endow $S^k \overline{E}$ with the hermitian metric induced by the natural inclusion of $S^k \overline{E}$ into $T^k \overline{E}$ given by $\frac{1}{\sqrt{k!}} \iota_k$.

In this way, if e_1, \dots, e_n is an orthonormal local frame in \overline{E} , then the set $\{e_{i_1} \otimes \dots \otimes e_{i_k}\}_{i_1, \dots, i_k \in \{1, \dots, n\}}$ forms an orthonormal basis of $T^k \overline{E}$ and the set $\{e_{i_1} \wedge \dots \wedge e_{i_k}\}_{i_1 < \dots < i_k \in \{1, \dots, n\}}$ forms an orthonormal basis of $\bigwedge^k \overline{E}$. The set $\{e_{i_1} \dots e_{i_k}\}_{i_1 \leq \dots \leq i_k \in \{1, \dots, n\}}$ forms an orthogonal basis of $S^k \overline{E}$ with the norm of each element depending on the number of repetitions among the subindices. In particular, if $i_1 < \dots < i_k$, then the norm of $e_{i_1} \dots e_{i_k}$ is one.

Denote these metrics by $h_{S^p \overline{E}}, h_{\bigwedge^p \overline{E}}, h_{T^p \overline{E}}$. The locally free sheaves $\Psi^k(\overline{E})^p$ are endowed with the tensor product metric. With these hermitian metrics, if \overline{E} and \overline{F} are hermitian vector bundles, the algebraic canonical isomorphism of complexes (4.3) is an isometry of hermitian complexes

$$(4.5) \quad \Psi^k(\overline{E} \oplus \overline{F})^* \cong \bigoplus_{p=0}^k \Psi^p(\overline{E})^* \otimes \Psi^{k-p}(\overline{F})^*.$$

Indeed, the natural inclusions

$$S^r \overline{E} \otimes \bigwedge^{p-r} \overline{E} \otimes S^l \overline{F} \otimes \bigwedge^{k-p-l} \overline{F} \hookrightarrow S^{r+l}(\overline{E} \oplus \overline{F}) \otimes \bigwedge^{k-r-l}(\overline{E} \oplus \overline{F})$$

are compatible with the hermitian metrics defined.

Hermitian split cubes

Definition 4.1 (cf. [10]). Let X be a proper arithmetic variety. Let $\{\overline{E}^j\}_{j \in \{0,2\}^n}$ be a collection of hermitian vector bundles over X , indexed by $\{0,2\}^n$. Let $[\overline{E}^j]_{j \in \{0,2\}^n}$ be the hermitian n -cube defined as follows.

▷ Let $\mathbf{j} \in \{0,1,2\}^n$ and let $u_1 < \dots < u_s$ be the indices with $j_{u_i} = 1$. We define $(v_1, \dots, v_n) = \sigma_{m_1, \dots, m_s}(\mathbf{j})$ to be the multi-index with

$$v_k = \begin{cases} j_k & \text{if } k \neq u_l \text{ for all } l, \\ m_l & \text{if } k = u_l. \end{cases}$$

Then the \mathbf{j} -component of $[\overline{E}^j]_{j \in \{0,2\}^n}$ is

$$\bigoplus_{(m_1, \dots, m_s) \in \{0,2\}^s} \overline{E}^{\sigma_{m_1, \dots, m_s}(\mathbf{j})}.$$

▷ The morphisms are compositions of the following canonical morphisms:

$$\begin{aligned} A \oplus^\perp B &\rightarrow A, & A \oplus^\perp B &\xrightarrow{\cong} B \oplus^\perp A, \\ A &\hookrightarrow A \oplus^\perp B, & A \oplus^\perp (B \oplus^\perp C) &\xrightarrow{\cong} (A \oplus^\perp B) \oplus^\perp C. \end{aligned}$$

A hermitian n -cube of this form is called a *direct sum hermitian n -cube*.

Definition 4.2 (cf. [10]). Let X be a proper arithmetic variety.

▶ Let \overline{E} be a hermitian n -cube. The *direct sum hermitian n -cube associated to \overline{E}* , $\mathrm{Sp}(\overline{E})$, is the hermitian n -cube

$$\mathrm{Sp}(\overline{E}) := [\overline{E}^j]_{j \in \{0,2\}^n}.$$

▶ A *hermitian split n -cube* is a couple (\overline{E}, f) , where \overline{E} is a hermitian n -cube and $f : \mathrm{Sp}(\overline{E}) \rightarrow \overline{E}$ is an isometry of hermitian n -cubes such that $f^j = \mathrm{id}$ if $j \in \{0,2\}^n$. The morphism f is called the *splitting* of (\overline{E}, f) .

Roughly speaking, these are the cubes which are orthogonal direct sums in all directions. Let

$$\mathbb{Z}\widehat{\mathrm{Sp}}_n(X) := \mathbb{Z}\{\text{split hermitian } n\text{-cubes in } X\}$$

and let $\mathbb{Z}\widehat{\mathrm{Sp}}_*(X) = \bigoplus_n \mathbb{Z}\widehat{\mathrm{Sp}}_n(X)$. As shown in [10], there is a differential map

$$d : \mathbb{Z}\widehat{\mathrm{Sp}}_n(X) \rightarrow \mathbb{Z}\widehat{\mathrm{Sp}}_{n-1}(X)$$

making $(\mathbb{Z}\widehat{\mathrm{Sp}}_*(X), d)$ a chain complex such that the morphism that forgets the splitting $\mathbb{Z}\widehat{\mathrm{Sp}}_*(X) \rightarrow \mathbb{Z}\widehat{C}_*(X)$ is a chain morphism.

Some intermediate chain morphisms. The construction of Adams operations factors through an intermediate complex. We recall here its construction due to the fact that a slight modification needs to be introduced.

Let $k \geq 1$. For every $n \geq 0$ and $i = 1, \dots, k-1$, we define

$$\begin{aligned} \widehat{G}_1^k(X)_n &:= \{\text{acyclic cochain complexes of length } k \text{ of hermitian } n\text{-cubes}\}, \\ \widehat{G}_2^{i,k}(X)_n &:= \{2\text{-iterated acyclic cochain complexes of lengths } (k-i, i) \\ &\quad \text{of hermitian } n\text{-cubes}\}. \end{aligned}$$

The differential of $\mathbb{Z}\widehat{C}_*(X)$ induces a differential on the graded abelian groups

$$\mathbb{Z}\widehat{G}_2^{i,k}(X)_* := \bigoplus_n \mathbb{Z}\widehat{G}_2^{i,k}(X)_n \quad \text{and} \quad \mathbb{Z}\widehat{G}_1^k(X)_* := \bigoplus_n \mathbb{Z}\widehat{G}_1^k(X)_n.$$

For every n , the simple complex associated to a 2-iterated cochain complex induces a morphism

$$\Phi^i : \mathbb{Z}\widehat{G}_2^{i,k}(X)_n \rightarrow \mathbb{Z}\widehat{G}_1^k(X)_n.$$

A new chain complex is defined by setting

$$\mathbb{Z}\widehat{G}^k(X)_n := \bigoplus_{i=1}^{k-1} \mathbb{Z}\widehat{G}_2^{i,k}(X)_{n-1} \oplus \mathbb{Z}\widehat{G}_1^k(X)_n.$$

If $\bar{B}_i \in \widehat{G}_2^{i,k}(X)_{n-1}$ for $i = 1, \dots, k-1$, and $\bar{A} \in \widehat{G}_1^k(X)_n$, the differential is given by

$$d_s(\bar{B}_1, \dots, \bar{B}_{k-1}, \bar{A}) := \left(-d\bar{B}_1, \dots, -d\bar{B}_{k-1}, \sum_{i=1}^{k-1} (-1)^i \Phi^i(\bar{B}_i) + d\bar{A} \right).$$

Definition 4.3. For any acyclic cochain complex of hermitian n -cubes

$$\bar{A} : 0 \rightarrow \bar{A}^0 \xrightarrow{f^0} \dots \xrightarrow{f^{j-1}} \bar{A}^j \xrightarrow{f^j} \dots \xrightarrow{f^{k-1}} \bar{A}^k \rightarrow 0,$$

we define:

▷ $\widehat{\varphi}_1(\bar{A})$ to be the *secondary Euler characteristic class*, i.e.

$$\widehat{\varphi}_1(\bar{A}) = \sum_{p \geq 0} (-1)^{k-p+1} (k-p) \bar{A}^p \in \mathbb{Z}\widehat{C}_n(X).$$

▷ $\mu(\bar{A}) := \sum_{j \geq 0} (-1)^{j-1} \mu^j(\bar{A})$ where $\mu^j(\bar{A})$ is the hermitian $(n+1)$ -cube defined by

$$\partial_1^0(\mu^j(\bar{A})) = \ker f^j, \quad \partial_1^1(\mu^j(\bar{A})) = \bar{A}^j, \quad \partial_1^2(\mu^j(\bar{A})) = \ker f^{j+1}.$$

▷ $\lambda_k(\bar{A})$ is the acyclic cochain complex of hermitian n -cubes

$$0 \rightarrow \bar{A}^0 \xrightarrow{\frac{1}{\sqrt{k}} f^0} \dots \xrightarrow{\frac{1}{\sqrt{k}} f^{j-1}} \bar{A}^j \xrightarrow{\frac{1}{\sqrt{k}} f^j} \dots \xrightarrow{\frac{1}{\sqrt{k}} f^{k-1}} \bar{A}^k \rightarrow 0.$$

If $\bar{B}_i \in \widehat{G}_2^{i,k}(X)_n$, then \bar{B}_i is a 2-iterated acyclic cochain complex where $\bar{B}_i^{j_1 j_2}$ is a hermitian n -cube for every j_1, j_2 . We attach to it a sum of exact sequences of hermitian n -cubes as follows:

$$\begin{aligned} \widehat{\varphi}_2(\bar{B}_i) &= \sum_{j \geq 0} (-1)^{k-j+1} ((k-i-j) \lambda_{k-i}(\bar{B}_i^{*,j}) + (i-j) \lambda_i(\bar{B}_i^{j,*})) \\ &\quad + \sum_{s \geq 1} (-1)^{k-s} (k-s) \sum_{j \geq 0} \left(\bar{B}_i^{s-j,j} \rightarrow \bigoplus_{j' \geq j} \bar{B}_i^{s-j',j'} \rightarrow \bigoplus_{j' > j} \bar{B}_i^{s-j',j'} \right). \end{aligned}$$

Roughly speaking, the first summand corresponds to the secondary Euler characteristic of the rows and the columns. The second summand is a correction factor due to the fact that direct sums are not sums in $\mathbb{Z}\widehat{C}_n(X)$.

Lemma 4.1. *The morphism given for every n by*

$$(4.6) \quad \begin{aligned} &\mathbb{Z}\widehat{G}^k(X)_n \xrightarrow{\varphi} \mathbb{Z}\widehat{C}_n(X), \\ &(\overline{B}_1, \dots, \overline{B}_{k-1}, \overline{A}) \mapsto \widehat{\varphi}_1(\overline{A}) + \sum_{i=1}^{k-1} (-1)^{i+1} \mu(\widehat{\varphi}_2(\overline{B}_i)), \end{aligned}$$

is a chain morphism.

Proof. See [10]. This follows essentially from the three equalities

$$\begin{aligned} d\widehat{\varphi}_1(\overline{A}) &= \widehat{\varphi}_1(d\overline{A}), \\ -\mu\widehat{\varphi}_2(d\overline{B}_i) &= \sum_{l=2}^n \sum_{j=0}^2 (-1)^{l+j} \partial_l^j \mu\widehat{\varphi}_2(\overline{B}_i), \quad \forall i, \\ \widehat{\varphi}_1(\Phi^i(\overline{B}_i)) &= \sum_{r \geq 0} (-1)^r \partial_1^r \mu\widehat{\varphi}_2(\overline{B}_i), \quad \forall i. \end{aligned} \quad \square$$

Remark 4.1. The above definition of $\widehat{\varphi}_2$ is a slight modification of the definition given in the original paper [10]. The change consists in the twist by $1/\sqrt{k}$ of λ_k , and has been introduced in order to have a representative of the Adams operations on higher algebraic K -theory that commutes strictly with the Chern character (the reason will become apparent in the next section; cf. Proposition 4.4). This modification does not alter the fact that the final morphism represents the Adams operations on higher rational algebraic K -theory, since the morphism at $n = 0$ remains unchanged.

Adams operations. In [10], Adams operations are constructed for every split (hermitian) n -cube, using the secondary Euler characteristic class of the Koszul complex. The following proposition follows from the construction of the Adams operations in [10].

Proposition 4.1 (cf. [10]). *Let X be a proper arithmetic variety. For every $k \geq 0$, there is a chain complex*

$$\Psi^k : \mathbb{Z}\widehat{\mathrm{Sp}}_*(X) \rightarrow \widetilde{\mathbb{Z}}\widehat{C}_*(X).$$

For every $\overline{E} \in \mathbb{Z}\widehat{\mathrm{Sp}}_n(X)$, $\Psi^k(\overline{E})$ consists of a sum of hermitian n -cubes of the following form:

- (i) *If \overline{E} is a hermitian vector bundle (i.e. $n = 0$), then $\Psi^k(\overline{E})$ is the secondary Euler characteristic class of the Koszul complex.*
- (ii) *If $n \geq 2$, the image of Ψ^k consists of hermitian n -cubes which are split in at least one direction.*

(iii) For $n = 1$, there are two types of summands. Some of the terms of $\Psi^k(\overline{E})$ are hermitian split 1-cubes. The rest are hermitian 1-cubes of the form $\mu(\overline{A})$ with $\overline{A} = \lambda_l(\Psi^l(\overline{F})^*) \otimes \overline{G}$ or $\overline{A} = \overline{F} \otimes \lambda_l(\Psi^l(\overline{G})^*)$ for some hermitian bundles $\overline{F}, \overline{G}$ in the entries of \overline{E} and $0 \leq l \leq k$. These terms arise as the image under $\mu \circ \widehat{\varphi}_2$ of 2-iterated acyclic cochain complexes of lengths $(k - i, i)$ of the form $\Psi^{k-i}(\overline{G})^* \otimes \Psi^i(\overline{F})^*$. \square

Remark 4.2. Item (ii) of the last proposition follows from the construction of the Adams operations in [10], using the isometry (4.5). These hermitian cubes appear when taking the image under φ of some elements $\widehat{A} \in \widehat{G}_1^k(X)_n, \widehat{B}_i \in \widehat{G}_2^{i,k}(X)_{n-1}$, which consist of (2-iterated) acyclic cochain complexes of hermitian n -cubes or $(n - 1)$ -cubes which are hermitian split in all directions. Note that the statement is true with the modification of λ_k introduced here since multiplication by the constants is not performed in the hermitian split directions.

The transgression morphism. Once the Adams operations are defined for all split hermitian cubes, the final construction makes use of the transgression bundles introduced above. This allows us to assign to every hermitian n -cube a collection of hermitian split cubes in $X \times (\mathbb{P}^1)^*$.

Let X be a proper arithmetic variety. Let $X \times (\mathbb{P}^1)^n$ denote $X \times_{\mathbb{Z}} (\mathbb{P}^1)^n$. For $i = 1, \dots, n$ and $j = 0, 1$, consider the chain morphisms induced on the complex of hermitian cubes

$$\begin{aligned} \delta_i^j &= (\text{Id} \times \delta_j^i)^* : \mathbb{Z}\widehat{C}_*(X \times (\mathbb{P}^1)^n) \rightarrow \mathbb{Z}\widehat{C}_*(X \times (\mathbb{P}^1)^{n-1}), \\ \sigma_i &= (\text{Id} \times \sigma^i)^* : \mathbb{Z}\widehat{C}_*(X \times (\mathbb{P}^1)^{n-1}) \rightarrow \mathbb{Z}\widehat{C}_*(X \times (\mathbb{P}^1)^n). \end{aligned}$$

As before, let p_1, \dots, p_n be the projections onto the i -th coordinate of $(\mathbb{P}^1)^n$. Let $\mathbb{Z}\widehat{C}_{*,*}^{\mathbb{P}}(X)$ be the 2-iterated chain complex given by

$$\mathbb{Z}\widehat{C}_{r,n}^{\mathbb{P}}(X) := \mathbb{Z}\widehat{C}_r(X \times (\mathbb{P}^1)^n),$$

with differentials (d, δ) , with d the differential of the complex of cubes and $\delta = \sum (-1)^{i+j} \delta_i^j$. Denote by $(\mathbb{Z}\widehat{C}_*^{\mathbb{P}}(X), d_s)$ the associated simple complex.

Let

$$\mathbb{Z}\widehat{C}_{r,n}^{\mathbb{P}}(X)_{\text{deg}} = \sum_{i=1}^n [\sigma_i(\mathbb{Z}\widehat{C}_{r,n-1}^{\mathbb{P}}(X)) + p_i^* \mathcal{O}(1) \otimes \sigma_i(\mathbb{Z}\widehat{C}_{r,n-1}^{\mathbb{P}}(X))],$$

$$\widetilde{\mathbb{Z}}\widehat{C}_{r,n}^{\mathbb{P}}(X)_{\text{deg}} = \mathbb{Z}\widehat{C}_{r,n}^{\mathbb{P}}(X)_{\text{deg}} / \mathbb{Z}\widehat{D}_r(X \times (\mathbb{P}^1)^n)_{\text{deg}},$$

and let

$$\widetilde{\mathbb{Z}}\widehat{C}_{r,n}^{\mathbb{P}}(X) := \widetilde{\mathbb{Z}}\widehat{C}_{r,n}^{\mathbb{P}}(X) / \widetilde{\mathbb{Z}}\widehat{C}_{r,n}^{\mathbb{P}}(X)_{\text{deg}}.$$

Denote by $(\widetilde{\mathbb{Z}}\widehat{C}_*^{\mathbb{P}}(X), d_s)$ the simple complex associated to this 2-iterated chain complex.

Proposition 4.2 ([10, Proposition 3.2]). *If X is a regular noetherian scheme, the natural morphism of complexes*

$$\widetilde{\mathbb{Z}}\widehat{C}_*(X) = \widetilde{\mathbb{Z}}\widehat{C}_{*,0}^{\mathbb{P}}(X) \rightarrow \widetilde{\mathbb{Z}}\widehat{C}_*^{\mathbb{P}}(X)$$

induces an isomorphism on homology with coefficients in \mathbb{Q} . □

In [10, §3], a morphism

$$(4.7) \quad N\widehat{C}_*(X) \xrightarrow{T} \mathbb{Z}\widehat{C}_*^{\mathbb{P}}(X)$$

is constructed. This morphism extends the map that assigns to every hermitian n -cube \overline{E} its transgression $\text{tr}_n(\overline{E})$. Indeed, the component of $T(\overline{E})$ in $\mathbb{Z}\widehat{C}_{0,n}^{\mathbb{P}}(X)$ is $\text{tr}_n(\lambda(\overline{E}))$. Each of the components of $T(\overline{E})$ in $\mathbb{Z}\widehat{C}_{n-i,i}^{\mathbb{P}}(X)$, for $i > 0$, consists of a linear combination of split hermitian cubes, and hence the construction Ψ^k outlined above can be applied to each of these terms. The morphism T maps every hermitian cube \overline{E} to a collection of hermitian cubes with canonical kernels $\lambda(\overline{E})$ and then applies the transgression construction.

In this way, one obtains for every $k \geq 0$ a chain morphism

$$\Psi^k : N\widehat{C}_*(X) \xrightarrow{\Psi^k \circ T} \widetilde{\mathbb{Z}}\widehat{C}_*^{\mathbb{P}}(X).$$

Proposition 4.3 ([10, Theorem 4.2]). *The Adams operations on the higher algebraic K -groups of X , after tensoring by \mathbb{Q} (as given by Gillet and Soulé [13] or Grayson [15]), are represented by the chain morphism*

$$\Psi^k : N\widehat{C}_*(X) \xrightarrow{\Psi^k \circ T} \widetilde{\mathbb{Z}}\widehat{C}_*^{\mathbb{P}}(X). \quad \square$$

Remark 4.3. The fact that the image of T consists of split cubes is proved in [10, Lemma 3.15]. The fact that the image consists indeed of hermitian split cubes follows from the fact that

$$(4.8) \quad \text{tr}_n(\overline{E})|_{y_i=0} \cong \text{tr}_{n-1}(\partial_i^0 \overline{E}) \oplus^\perp \text{tr}_{n-1}(\partial_i^2 \overline{E})$$

for every hermitian n -cube \overline{E} with canonical kernels, and with \cong being an isometry.

§4.2. The Koszul complex and Bott–Chern forms

In this section we determine hermitian metrics on the Koszul complex that would make its Bott–Chern form vanish. This is the cause of the modification λ_k introduced in the definition of the algebraic Adams operations. We next perform a direct comparison of the definition of the Adams operations on locally free sheaves and the secondary Euler characteristic class of the Koszul complex.

Although it is not required for the development of this paper, we will deduce the value of the Bott–Chern form of the Koszul complex, with the hermitian metrics fixed at the beginning of this section. This will be an easy consequence of all the computations of the first part of this subsection.

The following is a known result.

Lemma 4.2. *Let E be a locally free sheaf of finite rank on any variety. For all $k \geq 1$, the k -th Koszul complex of E is split, i.e. for all $0 \leq p \leq k-1$, $\mu^p(\Psi^k(E)^*)$ is a split short exact sequence.*

Proof. Recall that the morphisms in the Koszul complex, φ_p , were defined as

$$\varphi_p = \frac{1}{p!(k-p-1)!} (\pi_{p+1} \otimes \rho_{k-p-1}) \circ (\iota_p \otimes j_{k-p}).$$

Let

$$\psi_p : S^{p+1}E \otimes \bigwedge^{k-p-1} E \rightarrow S^p E \otimes \bigwedge^{k-p} E$$

be given as

$$\psi_p = \frac{1}{kp!(k-p-1)!} (\pi_p \otimes \rho_{k-p}) \circ (\iota_{p+1} \otimes j_{k-p-1}).$$

If ψ_p is a section of φ_p over $\text{im } \varphi_p$, then the short exact sequence

$$0 \rightarrow \ker \varphi_p \rightarrow S^p E \otimes \bigwedge^{k-p} E \rightarrow \text{im } \varphi_p \rightarrow 0$$

is split for all p . That is, there is an isomorphism

$$S^p E \otimes \bigwedge^{k-p} E \cong \ker \varphi_p \oplus \text{im } \varphi_p.$$

In order to see that ψ_p is a section of φ_p over $\text{im } \varphi_p$, we have to check that for all $e \in S^p E \otimes \bigwedge^{k-p} E$, we have

$$\varphi_p \psi_p \varphi_p(e) = \varphi_p(e).$$

Assume that the rank of E is n and consider a local frame in E , $\{e_1, \dots, e_n\}$. Renaming the indices, it is sufficient to check the previous equality for an element of the form $e = e_{i_1} \cdots e_{i_p} \otimes e_1 \wedge \cdots \wedge e_{k-p}$ where $i_1, \dots, i_p \in \{1, \dots, n\}$. By definition,

$$\begin{aligned} \varphi_p(e) &= \frac{1}{p!(k-p-1)!} \sum_{\substack{\sigma \in \mathfrak{S}_p \\ \tau \in \mathfrak{S}_{k-p}}} (-1)^{|\tau|} e_{i_{\sigma(1)}} \cdots e_{i_{\sigma(p)}} \cdot e_{\tau(1)} \otimes e_{\tau(2)} \wedge \cdots \wedge e_{\tau(k-p)} \\ &= \frac{1}{p!(k-p-1)!} \sum_{\tau \in \mathfrak{S}_{k-p}} (-1)^{|\tau|} p! e_{i_1} \cdots e_{i_p} \cdot e_{\tau(1)} \otimes e_{\tau(2)} \wedge \cdots \wedge e_{\tau(k-p)}. \end{aligned}$$

Observe that if $\tau(1) = j$, then there is a decomposition $\tau = \tau' \rho$ with $\tau', \rho \in \mathfrak{S}_p$, $\rho(1, \dots, k-p) = (j, 1, \dots, \widehat{j}, \dots, k-p)$ and $\tau'(1) = 1$. The signature of ρ is

$(-1)^{j-1}$. Hence,

$$(4.9) \quad \varphi_p(e) = \sum_{j=1}^{k-p} (-1)^{j-1} e_{i_1} \cdots e_{i_p} \cdot e_j \otimes e_1 \wedge \cdots \widehat{e}_j \cdots \wedge e_{k-p}.$$

Proceeding as in the computation of $\varphi_p(e)$, we obtain

$$(4.10) \quad \psi_p \varphi_p(e) = \frac{k-p}{k} e + \sum_{j=1}^{k-p} \sum_{t=1}^p \frac{(-1)^{j-1}}{k} e_{i_1} \cdots \widehat{e}_{i_t} \cdots e_{i_p} \cdot e_j \otimes e_{i_t} \wedge e_1 \wedge \cdots \widehat{e}_j \cdots \wedge e_{k-p}.$$

Therefore,

$$\varphi_p \psi_p \varphi_p(e) = \frac{k-p}{k} \varphi_p(e) + \frac{1}{k} \varphi_p(y)$$

where

$$y = \sum_{j=1}^{k-p} \sum_{t=1}^p (-1)^{j-1} e_{i_1} \cdots \widehat{e}_{i_t} \cdots e_{i_p} \cdot e_j \otimes e_{i_t} \wedge e_1 \wedge \cdots \widehat{e}_j \cdots \wedge e_{k-p}.$$

Using (4.9), we have

$$\begin{aligned} \varphi_p(y) &= \sum_{j=1}^{k-p} \sum_{t=1}^p (-1)^{j-1} e_{i_1} \cdots e_{i_p} \cdot e_j \otimes e_1 \wedge \cdots \widehat{e}_j \cdots \wedge e_{k-p} \\ &\quad + \sum_{j=1}^{k-p} \sum_{t=1}^p \sum_{l=1}^{j-1} (-1)^{j-1+l} e_{i_1} \cdots \widehat{e}_{i_t} \cdots e_{i_p} \cdot e_j \cdot e_l \\ &\quad \quad \quad \otimes e_{i_t} \wedge e_1 \wedge \cdots \widehat{e}_l \cdots \widehat{e}_j \cdots \wedge e_{k-p} \\ &\quad + \sum_{j=1}^{k-p} \sum_{t=1}^p \sum_{l=j+1}^{k-p} (-1)^{j+l} e_{i_1} \cdots \widehat{e}_{i_t} \cdots e_{i_p} \cdot e_j \cdot e_l \\ &\quad \quad \quad \otimes e_{i_t} \wedge e_1 \wedge \cdots \widehat{e}_j \cdots \widehat{e}_l \cdots \wedge e_{k-p} \\ &= p \varphi_p(e) \end{aligned}$$

since the last two summands cancel each other. Hence, we see that

$$\varphi_p \psi_p \varphi_p(e) = \frac{k-p}{k} \varphi_p(e) + \frac{p}{k} \varphi_p(e) = \varphi_p(e). \quad \square$$

Although the Koszul complex is algebraic split, the short exact sequences $\mu^j(\Psi^k(\overline{E})^*)$ are not orthogonal split, and hence its Bott–Chern form is not zero, as would be desirable. In order to achieve this, we need to multiply each morphism φ_p of the Koszul complex by $1/\sqrt{k}$.

Proposition 4.4. *Let \overline{E} be a hermitian vector bundle over a smooth proper complex variety and consider the Koszul complex $\Psi^k(\overline{E})^*$, $k \geq 1$. Then the acyclic cochain complex $\lambda_k(\Psi^k(\overline{E})^*)$ has the property that $\mu^p(\lambda_k(\Psi^k(\overline{E})^*))$ is hermitian split for all p .*

Proof. Let ι_p and j_p be the inclusions defined in (4.1), φ_p be as defined in (4.2) and ψ_p be defined as in the proof of the previous lemma.

Let us compute explicitly the squared norm of an element of $\text{im } \varphi_p$ under the two hermitian metrics: the one induced by the inclusion into $S^{p+1}\overline{E} \otimes \bigwedge^{k-p-1}\overline{E}$ and the quotient metric induced by that of $S^p\overline{E} \otimes \bigwedge^{k-p}\overline{E}$. To prove the statement, we need to see that the two norms are related by the factor $1/k$.

Denote by $\|\varphi_p(e)\|_i$ the norm of $\varphi_p(e)$ in $S^{p+1}\overline{E} \otimes \bigwedge^{k-p-1}\overline{E}$ and by $\|\varphi_p(e)\|_q$ the norm of $\varphi_p(e)$ given by considering it as a quotient of $S^p\overline{E} \otimes \bigwedge^{k-p}\overline{E}$.

Assume that the rank of \overline{E} is n and consider an orthonormal local frame e_1, \dots, e_n in \overline{E} . Then $\{e_{i_1} \otimes \dots \otimes e_{i_k}\}_{i_1, \dots, i_k \in \{1, \dots, n\}}$ forms an orthonormal basis of $T^k\overline{E}$. To ease notation, let us write

$$e_{i_1, \dots, i_k}^{\otimes} = e_{i_1} \otimes \dots \otimes e_{i_k}.$$

For an element of the form $\sum_{\lambda \in I} \alpha_\lambda e_\lambda^{\otimes}$ with I a subset of $\{1, \dots, n\}^k$ and $\alpha_\lambda \in \mathbb{C}$, the square of its norm is given by $\sum_{\lambda \in I} \alpha_\lambda^2$.

Notation. If $\sigma \in \mathfrak{S}_p$ and $\lambda = (\lambda_1, \dots, \lambda_p) \in \{1, \dots, n\}^p$ we will write

$$\sigma(\lambda_1, \dots, \lambda_p) = (\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(p)}).$$

Renaming the indices, it is sufficient to compute the norms for an element of the form

$$\varphi_p(e) \quad \text{with} \quad e = e_{i_1} \dots e_{i_p} \otimes e_1 \wedge \dots \wedge e_{k-p}$$

where $i_1, \dots, i_p \in \{1, \dots, n\}$. Let m_i denote the number of times that e_i appears in $e_{i_1} \dots e_{i_p}$. Then

$$e = e_1^{m_1} \dots e_n^{m_n} \otimes e_1 \wedge \dots \wedge e_{k-p},$$

where $e_i^{m_i} = e_i \cdot \dots \cdot e_i$. Note that $m_1 + \dots + m_n = p$.

Then, by (4.9), we have

$$\varphi_p(e) = \sum_{j=1}^{k-p} (-1)^{j-1} e_{i_1} \dots e_{i_p} \cdot e_j \otimes e_1 \wedge \dots \wedge \widehat{e}_j \dots \wedge e_{k-p}.$$

The norm $\|\varphi_p(e)\|_i$ is computed as the norm of $\frac{(\iota_{p+1} \otimes j_{k-p-1})}{\sqrt{(p+1)!(k-p-1)!}}(\varphi_p(e))$ in $T^k\overline{E}$.

The term $(\iota_{p+1} \otimes j_{k-p-1})(\varphi_p(e))$ is

$$\sum_{\substack{\sigma \in \mathfrak{S}_{p+1} \\ \tau \in \mathfrak{S}_{k-p-1}}} (-1)^{|\tau|} \sum_{j=1}^{k-p} (-1)^{j-1} e_{\sigma(i_1, \dots, i_p, j), \tau(1, \dots, \widehat{j}, \dots, k-p)}^{\otimes}.$$

Its squared norm is given by the sum of the squares of the numbers of occurrences of each different summand. Given two different permutations $\tau, \tau' \in \mathfrak{S}_{k-p-1}$, we have

$$e_{\sigma(i_1, \dots, i_p, j), \tau(1, \dots, \widehat{j}, \dots, k-p)}^{\otimes} \neq e_{\sigma(i_1, \dots, i_p, j), \tau'(1, \dots, \widehat{j}, \dots, k-p)}^{\otimes}.$$

Moreover, if $j \neq j'$, the terms obtained are different as well. The only repetitions will come from the permutations $\sigma, \sigma' \in \mathfrak{S}_{p+1}$ satisfying $\sigma(i_1, \dots, i_p, j) = \sigma'(i_1, \dots, i_p, j)$. For fixed τ and j , there are $\frac{(p+1)!}{m_1! \dots (m_j+1)! \dots m_n!}$ different terms, each of them appearing $m_1! \dots (m_j+1)! \dots m_n!$ times (and with the same sign). Therefore,

$$\begin{aligned} \|\varphi_p(e)\|_i^2 &= \frac{(k-p-1)!}{(p+1)!(k-p-1)!} \sum_{j=1}^{k-p} \frac{(p+1)!(m_1! \dots (m_j+1)! \dots m_n!)^2}{m_1! \dots (m_j+1)! \dots m_n!} \\ &= \sum_{j=1}^{k-p} m_1! \dots (m_j+1)! \dots m_n! = m_1! \dots m_n! \sum_{j=1}^{k-p} (m_j+1) \\ &= m_1! \dots m_n! \left(k-p + \sum_{j=1}^{k-p} m_j \right). \end{aligned}$$

Let us proceed now to the computation of $\|\varphi_p(e)\|_q$. This norm is given by $\left\| \frac{1}{\sqrt{p!(k-p)!}} (\iota_p \otimes j_{k-p})(w) \right\|_{T^k \overline{E}}$, where $w \in (\ker \varphi_p)^\perp$ satisfies $\varphi_p(w) = \varphi_p(e)$. Let us see that $w = \psi_p \varphi_p(e)$. Since we have already seen that $\varphi_p \psi_p \varphi_p(e) = \varphi_p(e)$, it is enough to check that $\psi_p \varphi_p(e) \in (\ker \varphi_p)^\perp$.

By (4.10), we have

$$k\psi_p \varphi_p(e) = (k-p)e + \sum_{j=1}^{k-p} (-1)^{j-1} \sum_{t=1}^p e_{i_1} \dots \widehat{e_{i_t}} \dots e_{i_p} \cdot e_j \otimes e_{i_t} \wedge e_1 \wedge \dots \widehat{e_j} \dots \wedge e_{k-p}.$$

If $i_t \in \{1, \dots, k-p\}$ and $i_t \neq j$, we have $e_{i_t} \wedge e_1 \wedge \dots \widehat{e_j} \dots \wedge e_{k-p} = 0$. If $i_t = j$, then

$$e_{i_1} \dots \widehat{e_{i_t}} \dots e_{i_p} \cdot e_j \otimes e_{i_t} \wedge e_1 \wedge \dots \widehat{e_j} \dots \wedge e_{k-p} = (-1)^{j-1} e.$$

Hence,

$$\begin{aligned} k\psi_p \varphi_p(e) &= \left(k-p + \sum_{j=1}^{k-p} m_j \right) e \\ &+ \sum_{j=1}^{k-p} \sum_{\substack{t \in \{k-p+1, \dots, n\} \\ m_t \neq 0}} (-1)^{j-1} m_t e_1^{\cdot m_1} \dots e_j^{\cdot m_j+1} \dots e_t^{\cdot m_t-1} \dots e_n^{\cdot m_n} \\ &\quad \otimes e_t \wedge e_1 \wedge \dots \widehat{e_j} \dots \wedge e_{k-p}. \end{aligned}$$

Let $\Phi = \iota_p \otimes j_{k-p}$, $A = \frac{1}{k}(k-p + \sum_{j=1}^{k-p} m_j)\Phi(e)$ and $B = \Phi(\psi_p \varphi_p(e)) - A$. We have

$$\begin{aligned} A &= \frac{1}{k} \left(k-p + \sum_{j=1}^{k-p} m_j \right) \sum_{\substack{\sigma \in \mathfrak{S}_p \\ \tau \in \mathfrak{S}_{k-p}}} (-1)^{|\tau|} e_{\sigma(i_1, \dots, i_p), \tau(1, \dots, k-p)}^{\otimes}, \\ B &= \frac{1}{k} \sum_{j=1}^{k-p} \sum_{\substack{t \in \{k-p+1, \dots, n\} \\ m_t \neq 0}} (-1)^{j-1} \\ &\quad \cdot \sum_{\substack{\sigma \in \mathfrak{S}_p \\ \tau \in \mathfrak{S}_{k-p}}} (-1)^{|\tau|} m_t e_{\sigma(1^{m_1}, \dots, j^{m_j+1}, \dots, t^{m_t-1}, \dots, n^{m_n})}^{\otimes} \otimes e_{\tau(t, 1, \dots, \widehat{j}, \dots, k-p)}^{\otimes}. \end{aligned}$$

We want to see that $A + B$ belongs to $\Phi(\ker \varphi_p)^\perp = \Phi(\text{im } \varphi_{p-1})^\perp$ in $T^k \overline{E}$. Let $v = \varphi_{p-1}(f)$, where $f = e_{r_1} \cdots e_{r_{p-1}} \otimes e_{s_1} \wedge \cdots \wedge e_{s_{k-p+1}}$ with $s_1 < \cdots < s_{k-p+1}$. Then

$$v = \sum_{t=1}^{k-p+1} (-1)^{t-1} e_{r_1} \cdots e_{r_{p-1}} \cdot e_{s_t} \otimes e_{s_1} \wedge \cdots \widehat{e_{s_t}} \cdots \wedge e_{s_{k-p+1}},$$

and so

$$\Phi(v) = \sum_{t=1}^{k-p+1} (-1)^{t-1} \sum_{\substack{\sigma \in \mathfrak{S}_p \\ \tau \in \mathfrak{S}_{k-p}}} (-1)^{|\tau|} e_{\sigma(r_1, \dots, r_{p-1}, s_t), \tau(s_1, \dots, \widehat{s_t}, \dots, s_{k-p+1})}^{\otimes}.$$

It is straightforward to see that the scalar product $\langle \Phi(v), A \rangle$ is 0 unless $f = e_{r_1} \cdots e_{r_{p-1}} \otimes e_1 \wedge \cdots \wedge e_{k-p} \wedge e_s$ with $\{i_1, \dots, i_p\} = \{r_1, \dots, r_{p-1}, s\}$ and $s \in \{k-p+1, \dots, n\}$. For B , we have two situations where it is not obvious that $\langle \Phi(v), B \rangle$ is 0. The first is the same as above. The second is when there exists j such that $\{1, \dots, \widehat{j}, \dots, k-p\} = \{s_1, \dots, s_{k-p-1}\}$, $s_{k-p}, s_{k-p+1} > k-p$ and $\{r_1, \dots, r_{p-1}\} = \{i_1, \dots, i_p, j\} \setminus \{s_{k-p}, s_{k-p+1}\}$.

In the first case ($f = e_{r_1} \cdots e_{r_{p-1}} \otimes e_1 \wedge \cdots \wedge e_{k-p} \wedge e_s$ with $\{i_1, \dots, i_p\} = \{r_1, \dots, r_{p-1}, s\}$ and $s \in \{k-p+1, \dots, n\}$), we have

$$\begin{aligned} \langle \Phi(v), A \rangle &= \alpha \cdot \sum_{\substack{\sigma, \sigma' \in \mathfrak{S}_p \\ \tau, \tau' \in \mathfrak{S}_{k-p}}} (-1)^{|\tau|+|\tau'|} \langle e_{\sigma(i_1, \dots, i_p), \tau(1, \dots, k-p)}^{\otimes}, e_{\sigma'(r_1, \dots, r_{p-1}, s), \tau'(1, \dots, k-p)}^{\otimes} \rangle \\ &= \alpha \cdot \sum_{\tau \in \mathfrak{S}_{k-p}} \sum_{\sigma, \sigma' \in \mathfrak{S}_p} \langle e_{\sigma(i_1, \dots, i_p), \tau(1, \dots, k-p)}^{\otimes}, e_{\sigma'(r_1, \dots, r_{p-1}, s), \tau(1, \dots, k-p)}^{\otimes} \rangle \end{aligned}$$

with $\alpha = \frac{(-1)^{k-p}}{k} (k-p + \sum_{j=1}^{k-p} m_j)$. Note that for every $\sigma \in \mathfrak{S}_p$, the number of $\sigma' \in \mathfrak{S}_p$ such that

$$\langle e_{\sigma(i_1, \dots, i_p), \tau(1, \dots, k-p)}^{\otimes}, e_{\sigma'(r_1, \dots, r_{p-1}, s), \tau(1, \dots, k-p)}^{\otimes} \rangle = 1$$

is the same, namely $\lambda = m_1! \dots m_n!$. Thus,

$$\begin{aligned} \langle \Phi(v), A \rangle &= \alpha \cdot \sum_{\tau \in \mathfrak{S}_{k-p}} \sum_{\sigma \in \mathfrak{S}_p} \lambda = \frac{(-1)^{k-p}}{k} \left(k-p + \sum_{j=1}^{k-p} m_j \right) (k-p)! p! \lambda \\ &= \frac{(-1)^{k-p}}{k} (k-p)! p! \left(k-p + \sum_{j=1}^{k-p} m_j \right) \lambda. \end{aligned}$$

We proceed in the same way for B :

$$\begin{aligned} &\langle \Phi(v), B \rangle \\ &= \frac{1}{k} \sum_{j=1}^{k-p} \sum_{\substack{t \in \{k-p+1, \dots, n\} \\ m_t \neq 0}} \sum_{\substack{\sigma, \sigma' \in \mathfrak{S}_p \\ \tau, \tau' \in \mathfrak{S}_{k-p}}} \sum_{l=1}^{k-p+1} (-1)^{j+l} (-1)^{|\tau|+|\tau'|} m_t \\ &\quad \cdot \langle e_{\sigma(1^{m_1}, \dots, j^{m_{j+1}}, \dots, t^{m_t-1}, \dots, n^{m_n}), \tau(1, \dots, \widehat{j}, \dots, k-p)}^{\otimes}, e_{\sigma'(r_1, \dots, r_{p-1}, l), \tau'(1, \dots, \widehat{l}, \dots, s)}^{\otimes} \rangle \\ &= \frac{1}{k} \sum_{j=1}^{k-p} \sum_{\substack{\sigma, \sigma' \in \mathfrak{S}_p \\ \tau \in \mathfrak{S}_{k-p}}} (-1)^{k-p+1} m_s \\ &\quad \cdot \langle e_{\sigma(1^{m_1}, \dots, j^{m_{j+1}}, \dots, s^{m_s-1}, \dots, n^{m_n}), \tau(1, \dots, \widehat{j}, \dots, k-p, s)}^{\otimes}, e_{\sigma'(r_1, \dots, r_{p-1}, j), \tau(1, \dots, \widehat{j}, \dots, s)}^{\otimes} \rangle \\ &= \frac{1}{k} (-1)^{k-p+1} m_s (k-p)! p! \sum_{j=1}^{k-p} m_1! \dots (m_j+1)! \dots (m_s-1)! \dots m_n! \\ &= \frac{(k-p)! p!}{k} (-1)^{k-p+1} \lambda \left(k-p + \sum_{j=1}^{k-p} m_j \right) = -\Phi(v) \cdot A. \end{aligned}$$

Therefore,

$$\langle \Phi(v), A + B \rangle = 0$$

in $T^k \overline{E}$ for all $v \in \text{im } \varphi_{p-1}$ of the first form.

Now assume there exists j such that $\{1, \dots, \widehat{j}, \dots, k-p\} = \{s_1, \dots, s_{k-p-1}\}$, $s_{k-p}, s_{k-p+1} > k-p$, and $\{r_1, \dots, r_{p-1}\} = \{i_1, \dots, i_p, j\} \setminus \{s_{k-p}, s_{k-p+1}\}$. Then $\langle \Phi(v), A \rangle = 0$ and we have (with β a constant)

$$\begin{aligned}
& \langle \Phi(v), B \rangle \\
&= \beta \sum_{\substack{t \in \{k-p+1, \dots, n\} \\ m_t \neq 0}} \sum_{l=1}^{k-p+1} \sum_{\substack{\sigma, \sigma' \in \mathfrak{S}_p \\ \tau \in \mathfrak{S}_{k-p}}} m_t \\
&\quad \cdot \langle e^{\otimes_{\sigma(1m_1, \dots, j^{m_j+1}, \dots, t^{m_t-1}, \dots, n^{m_n}), \tau(1, \dots, \widehat{j}, \dots, k-p, t)}, e^{\otimes_{\sigma'(r_1, \dots, r_{p-1}, s_l), \tau(s_1, \dots, \widehat{s_l}, \dots, s_{k-p+1})}} \rangle \\
&= \beta p!(k-p)! m_1! \dots (m_j+1)! \dots m_n! ((-1)^{k-p-1} + (-1)^{k-p}) = 0,
\end{aligned}$$

where we have used the fact that the scalar product is non-zero only for the pairs of indices $l = k-p$, $t = s_{k-p+1}$ and $l = k-p+1$, $t = s_{k-p}$.

We conclude that $\langle \Phi(v), A+B \rangle = 0$ in $T^k \overline{E}$ for all $v \in \text{im } \varphi_{p-1}$ as desired.

We proceed now to compute the norm $\|\varphi_p(e)\|_q$ which is given by

$$\left\| \frac{1}{\sqrt{p!(k-p)!}} (A+B) \right\|_{T^k \overline{E}}.$$

As above, we should group the summands of A and B that are the same. In order to do this, observe that the terms obtained for different permutations τ , different j or different t are not equal. Moreover, the summands in A are all different from the summands in B . Therefore, the only repetitions are obtained with the permutations $\sigma \in \mathfrak{S}_p$.

With these observations we obtain, as in the computation of $\|\cdot\|_i$,

$$\begin{aligned}
\frac{\|A\|_q^2}{p!(k-p)!} &= \frac{1}{k^2} \left(k-p + \sum_{j=1}^{k-p} m_j \right)^2 m_1! \dots m_n! \\
\frac{\|B\|_q^2}{p!(k-p)!} &= \frac{1}{k^2 p!(k-p)!} \sum_{j=1}^{k-p} \sum_{\substack{t \in \{k-p+1, \dots, n\} \\ m_t \neq 0}} \sum_{\tau \in \mathfrak{S}_{k-p}} \\
&\quad \frac{p!(m_t)^2 (m_1! \dots (m_j+1)! \dots (m_t-1)! \dots m_n!)^2}{m_1! \dots (m_j+1)! \dots (m_t-1)! \dots m_n!} \\
&= \frac{1}{k^2} \sum_{j=1}^{k-p} \sum_{\substack{t \in \{k-p+1, \dots, n\} \\ m_t \neq 0}} m_t m_1! \dots (m_j+1)! \dots m_n!.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\varphi_p(e)\|_q^2 &= \frac{m_1! \dots m_n!}{k^2} \left[\left(k-p + \sum_{j=1}^{k-p} m_j \right)^2 + \sum_{j=1}^{k-p} \sum_{t=k-p+1}^n m_t (m_j+1) \right] \\
&= \frac{m_1! \dots m_n!}{k^2} \left[\left(k-p + \sum_{j=1}^{k-p} m_j \right)^2 + \left(\sum_{j=1}^{k-p} (m_j+1) \right) \left(\sum_{t=k-p+1}^n m_t \right) \right].
\end{aligned}$$

Note that

$$\sum_{t=k-p+1}^n m_t = p - \sum_{j=1}^{k-p} m_j.$$

Hence, denoting $\beta = m_1! \dots m_n! / k^2$, we have

$$\begin{aligned} \|\varphi_p(e)\|_q^2 &= \beta \left[\left(k - p + \sum_{j=1}^{k-p} m_j \right)^2 + \left(k - p + \sum_{j=1}^{k-p} m_j \right) \left(p - \sum_{j=1}^{k-p} m_j \right) \right] \\ &= \beta k \left(k - p + \sum_{j=1}^{k-p} m_j \right) = \frac{m_1! \dots m_n!}{k} \left(k - p + \sum_{j=1}^{k-p} m_j \right). \end{aligned}$$

Therefore,

$$\frac{\|\varphi_p(e)\|_i^2}{\|\varphi_p(e)\|_q^2} = k.$$

It follows that if we define $\varphi'_p = \frac{1}{\sqrt{k}} \varphi_p$ we have

$$\|\varphi'_p(e)\|_i^2 = \frac{1}{k} \|\varphi_p(e)\|_i^2 = \|\varphi_p(e)\|_q^2 = \|\varphi'_p(e)\|_q^2.$$

The last equality follows from the fact that if $w \in (\ker \varphi_p)^\perp$ satisfies $\varphi_p(w) = \varphi_p(e)$ then $w \in (\ker \varphi'_p)^\perp$ and $\varphi'_p(w) = \varphi'_p(e)$. \square

Remark 4.4. Note that if we had defined from the very beginning the arrows φ_p of the Koszul complex Ψ^k to be $\frac{1}{\sqrt{k}} \varphi_p$, then we would not have the isometry (4.5) of chain complexes.

Corollary 4.1. *Let \bar{E} be a hermitian vector bundle. Then for all $0 \leq p \leq k-1$ we have*

$$d_{\mathcal{D}} \operatorname{ch}(\mu^p(\Psi^k(\bar{E})^*)) = 0.$$

Proof. Consider the commutative diagram of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker \varphi_p & \longrightarrow & S^p \bar{E} \otimes \wedge^{k-p} \bar{E} & \xrightarrow{\varphi_p} & \ker \varphi_{p+1} \longrightarrow 0 \\ & & \downarrow = & & \downarrow = & & \downarrow \frac{1}{\sqrt{k}} \\ 0 & \longrightarrow & \ker(\frac{1}{\sqrt{k}} \varphi_p) & \longrightarrow & S^p \bar{E} \otimes \wedge^{k-p} \bar{E} & \xrightarrow{\frac{1}{\sqrt{k}} \varphi_p} & \ker(\frac{1}{\sqrt{k}} \varphi_{p+1}) \longrightarrow 0 \end{array}$$

By the last proposition, $\operatorname{ch}(\mu^p(\lambda_k \Psi^k(\bar{E})^*)) = 0$. Hence

$$\begin{aligned} d_{\mathcal{D}} \operatorname{ch}(\mu^p(\Psi^k(\bar{E})^*)) &= d_{\mathcal{D}} \operatorname{ch}(\mu^p(\lambda_k \Psi^k(\bar{E})^*)) - d_{\mathcal{D}} \operatorname{ch}(id_{\ker \varphi_p}) \\ &\quad + d_{\mathcal{D}} \operatorname{ch}(id_{S^p \bar{E} \otimes \wedge^{k-p}}) - d_{\mathcal{D}} \operatorname{ch}(\ker \varphi_{p+1} \xrightarrow{\frac{1}{\sqrt{k}}} \ker \varphi_{p+1}) \\ &= 0. \end{aligned} \quad \square$$

Adams operations and the secondary Euler characteristic. Let E be a locally free sheaf of finite rank and let

$$\psi^k(E) = N_k(\lambda^1(E), \dots, \lambda^k(E)),$$

with N_k being the k -th Newton polynomial. These are the Adams operations associated to the lambda operations λ^k on locally free sheaves of finite rank. Let $\Psi^k(E)$ be the secondary Euler characteristic class of the Koszul complex of E . As shown by Grayson in [15, §3], the secondary Euler characteristic class of the k -th Koszul complex agrees, in the quotient group $K_0(X)$, with the usual k -th Adams operation. Therefore, in $K_0(X)$,

$$\psi^k(E) = \Psi^k(E).$$

This means that there exist short exact sequences s_1, \dots, s_r such that

$$\psi^k(E) - \Psi^k(E) = \sum_{i=1}^r d(s_i).$$

In the next proposition, we construct such a set of short exact sequences explicitly. For that, let $\Psi^k(E)^{t*}$ denote the Koszul complex obtained by replacing the p -th component $S^p E \otimes \bigwedge^{k-p} E$ with $\bigwedge^{k-p} E \otimes S^p E$ via the canonical isomorphism.

Proposition 4.5. *Let E be a locally free sheaf of finite rank. Then*

$$\psi^k(E) - \Psi^k(E) = \sum_{i=1}^r d(s_i),$$

where, for each i , either s_i is $\mu^p(\Psi^{k_i}(E)^{t*}) \otimes A_i$ with A_i some locally free sheaves of finite rank of the form $\bigwedge^{j_i} E$ or $T^{j_i} E$ and some indices k_i, p, j_i , or s_i is the canonical isomorphism $\bigwedge^{k-p} E \otimes S^p E \cong S^p E \otimes \bigwedge^{k-p} E$.

Proof. Consider the polynomial relating the lambda and Adams operations:

$$(4.11) \quad \psi^k = \psi^{k-1}\lambda^1 - \psi^{k-2}\lambda^2 + \dots + (-1)^{k-1}k\lambda^k.$$

For two linear combinations A, B of locally free sheaves of finite rank, write $A \simeq B$ if there exist short exact sequences s_1, \dots, s_l , of the form as in the statement of the proposition, such that

$$A - B = \sum_{i=1}^l d(s_i).$$

Let $L_k = \{(i_1, \dots, i_l) \mid i_j \in \{1, \dots, k\}, i_1 + \dots + i_l = k\}$ be the set of partitions of k and let S^k denote the symmetric product. We first show, by induction on k ,

that

$$S^k \simeq \sum_{(i_1, \dots, i_l) \in L_k} (-1)^{l+k} \lambda^{i_1} \dots \lambda^{i_l}.$$

For $k = 1$, this is obvious since $S^1 = \lambda^1 = \text{id}$. Assume that the result is true up to k . Considering the $(k+1)$ -th Koszul complex $\Psi^{k+1}(E)^{t*}$, we have

$$S^{k+1} \simeq \sum_{i=1}^{k+1} (-1)^{i+1} \lambda^i \otimes S^{k+1-i}.$$

By induction hypothesis,

$$\begin{aligned} S^{k+1} &\simeq \sum_{i=1}^{k+1} (-1)^{i+1} \lambda^i \otimes \left(\sum_{(i_1, \dots, i_l) \in L_{k+1-i}} (-1)^{l+k+1-i} \lambda^{i_1} \dots \lambda^{i_l} \right) \\ &\simeq \sum_{(i, i_1, \dots, i_l) \in L_{k+1}} (-1)^{(l+1)+(k+1)} \lambda^i \lambda^{i_1} \dots \lambda^{i_l} \end{aligned}$$

as desired.

Next, observe that by definition $\Psi^1 = \psi^1$. Hence, if we show that Ψ^k satisfies the recursive formula (4.11), up to short exact sequences of the desired form, we are done. By definition,

$$\begin{aligned} \Psi^k(E) &= \sum_{p \geq 0} (-1)^{k-p+1} (k-p) S^p(E) \otimes \wedge^{k-p} E \\ &\simeq \sum_{p \geq 0} (-1)^{k-p+1} (k-p) \lambda^{k-p} E \otimes S^p E. \end{aligned}$$

Hence, using the previous relation for S^k , we have (omitting E)

$$\begin{aligned} \Psi^k &\simeq \sum_{p \geq 0} (-1)^{k-p+1} (k-p) \lambda^{k-p} \otimes \left(\sum_{(i_1, \dots, i_l) \in L_p} (-1)^{l+p} \lambda^{i_1} \dots \lambda^{i_l} \right) \\ &\simeq \sum_{p \geq 0} \sum_{(i_1, \dots, i_l) \in L_p} (-1)^{k+l+1} (k-p) \lambda^{k-p} \lambda^{i_1} \dots \lambda^{i_l} \\ &\simeq \sum_{(i_1, \dots, i_{l+1}) \in L_k} (-1)^{k+l+1} i_1 \lambda^{i_1} \dots \lambda^{i_{l+1}} \\ &\simeq (-1)^{k+1} k \lambda^k + \sum_{s=1}^{k-1} (-1)^s \left(\sum_{(i_1, \dots, i_l) \in L_{k-s}} (-1)^{k+l+1-s} i_1 \lambda^{i_1} \dots \lambda^{i_l} \right) \lambda^s \\ &\simeq (-1)^{k+1} k \lambda^k - \sum_{s=1}^{k-1} (-1)^s \Psi^{k-s} \lambda^s. \end{aligned}$$

This finishes the proof of the proposition. \square

Remark 4.5. Note that last proposition applies to any suitable exact category where Grayson's Adams operations are defined, that is, where the correct notion of symmetric, exterior and tensor product is available (see [15]).

Let X be a proper arithmetic variety over \mathbb{Z} . Let

$$\Psi^k : \mathcal{D}^{2p-*}(X, p) \rightarrow \mathcal{D}^{2p-*}(X, p)$$

be the morphism that maps α to $k^p\alpha$. That is, we endow $\bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p)$ with the canonical λ -ring structure corresponding to the graduation given by p .

Proposition 4.6. *Let X be a proper arithmetic variety and let \bar{E} be a hermitian vector bundle over X . Then*

$$\Psi^k(\text{ch}(\bar{E})) = \text{ch}(\Psi^k(\bar{E}))$$

in the group $\bigoplus_{p \geq 0} \mathcal{D}^{2p}(X, p)$.

Proof. In [12, Lemma 7.3.3] Gillet and Soulé proved that $\lambda^k \text{ch} = \text{ch} \lambda^k$, from which it follows that

$$\psi^k(\text{ch}(\bar{E})) = \text{ch}(\psi^k(\bar{E})).$$

Observe that by definition,

$$\Psi^k(\text{ch}(\bar{E})) = \psi^k(\text{ch}(\bar{E})).$$

By Proposition 4.5, there are short exact sequences \bar{s}_i such that for each i , either \bar{s}_i is $\mu^p(\Psi^{k_i}(\bar{E})^{t*}) \otimes \bar{A}_i$ with \bar{A}_i some locally free sheaves of the form $\bigwedge^{j_i} \bar{E}$ or $T^{j_i} \bar{E}$ and some indices k_i, p, j_i , or \bar{s}_i is the canonical isomorphism $\bigwedge^{k-p} \bar{E} \otimes S^p \bar{E} \cong S^p \bar{E} \otimes \bigwedge^{k-p} \bar{E}$, and such that

$$\psi^k(\bar{E}) - \Psi^k(\bar{E}) = \sum_{i=1}^r d(\bar{s}_i).$$

The sequences \bar{s}_i are endowed with the hermitian metric induced by the hermitian metrics $h_{\bigwedge^* \bar{E}}, h_{T^* \bar{E}}$ and $h_{\Psi^*(\bar{E})^t}$. By Corollary 4.1 it follows that

$$\text{ch}(\psi^k(\bar{E})) - \text{ch}(\Psi^k(\bar{E})) = \sum_{i=1}^r d_{\mathcal{D}} \text{ch}(\bar{s}_i) = 0$$

and hence

$$\text{ch}(\Psi^k(\bar{E})) = \text{ch}(\psi^k(\bar{E})),$$

and the proposition is proved. \square

The Bott–Chern form of the Koszul complex. For any acyclic cochain complex of hermitian vector bundles, $\bar{A} : 0 \rightarrow \bar{A}^0 \xrightarrow{f^0} \dots \xrightarrow{f^{j-1}} \bar{A}^j \xrightarrow{f^j} \dots \xrightarrow{f^{k-1}} \bar{A}^k \rightarrow 0$, the Bott–Chern form of \bar{A} is defined by

$$\text{ch}(\bar{A}) = \text{ch}(\mu(\bar{A})) = \sum_{j \geq 0} (-1)^{j-1} \text{ch}(\mu^j(\bar{A})).$$

Proposition 4.7. *Let X be a smooth complex variety and let \bar{E} be a hermitian vector bundle. With the metrics on the Koszul complex $\Psi^k(\bar{E})^*$ induced by $h_{\wedge^* \bar{E}}, h_{T^* \bar{E}}$ and $h_{S^* (\bar{E})}$, we have*

$$\text{ch}(\Psi^k(\bar{E})^*) = \frac{(-1)^{k+1} \log(k)}{2} \Psi^k(\text{ch}(\bar{E}))$$

in $\bigoplus_{p \geq 0} \mathcal{D}^{2p-1}(X, p)$.

Proof. Consider the commutative diagram of acyclic chain complexes (we omit the hermitian vector bundle \bar{E}):

$$\begin{array}{ccccccccccccccc} 0 & \rightarrow & \wedge^k & \xrightarrow{\varphi_0} & S^1 \otimes \wedge^{k-1} & \xrightarrow{\varphi_1} & \dots & \rightarrow & S^p \otimes \wedge^{k-p} & \xrightarrow{\varphi_p} & \dots & \rightarrow & S^{k-1} \otimes \wedge^1 & \xrightarrow{\varphi_{k-1}} & S^k & \rightarrow & 0 \\ & & \downarrow \sqrt{k} \text{ Id} & & \downarrow \sqrt{k}^{k-1} \text{ Id} & & & & \downarrow \sqrt{k}^{k-p} \text{ Id} & & & & \downarrow \sqrt{k} \text{ Id} & & \downarrow \text{ Id} & & \\ 0 & \rightarrow & \wedge^k & \xrightarrow{\frac{\varphi_0}{\sqrt{k}}} & S^1 \otimes \wedge^{k-1} & \xrightarrow{\frac{\varphi_1}{\sqrt{k}}} & \dots & \rightarrow & S^p \otimes \wedge^{k-p} & \xrightarrow{\frac{\varphi_p}{\sqrt{k}}} & \dots & \rightarrow & S^{k-1} \otimes \wedge^1 & \xrightarrow{\frac{\varphi_{k-1}}{\sqrt{k}}} & S^k & \rightarrow & 0 \end{array}$$

with the first row at degree 1 and the second at degree 2.

Let $\text{ch}(\sqrt{k}^{k-p} \text{ Id}_{S^p \bar{E} \otimes \wedge^{k-p} \bar{E}})$ denote $\text{ch}(0 \rightarrow S^p \bar{E} \otimes \wedge^{k-p} \bar{E} \xrightarrow{\sqrt{k}^{k-p} \text{ Id}} S^p \bar{E} \otimes \wedge^{k-p} \bar{E})$.

For $p = 0, \dots, k - 1$, let Λ_p be the diagram

$$\begin{array}{ccccc} \ker \varphi_p & \longrightarrow & S^p \bar{E} \otimes \wedge^{k-p} \bar{E} & \xrightarrow{\varphi_p} & \ker \varphi_{p+1} \\ \downarrow \sqrt{k}^{k-p} & & \downarrow \sqrt{k}^{k-p} & & \downarrow \sqrt{k}^{k-p-1} \\ \ker \varphi_p & \longrightarrow & S^p \bar{E} \otimes \wedge^{k-p} \bar{E} & \xrightarrow{\frac{\varphi_p}{\sqrt{k}}} & \ker \varphi_{p+1} \end{array}$$

(here $\varphi_k = 0$). Then we have

$$\begin{aligned} \text{ch}(\Psi^k(\bar{E})^*) &= \text{ch}(\lambda_k(\Psi^k(\bar{E})^*)) + \sum_{p=0}^k (-1)^{p+1} \text{ch}(\sqrt{k}^{k-p} \text{ Id}_{S^p \bar{E} \otimes \wedge^{k-p}}) \\ &\quad + \sum_{p=0}^{k-1} (-1)^p d_{\mathcal{D}} \text{ch}(\Lambda_p). \end{aligned}$$

Let us see that $d_{\mathcal{D}} \text{ch}(\Lambda_p) = 0$ for all p . The lower sequence of Λ_p is split exact. Therefore, Λ_p is isometric to the following diagram, denoted by Λ'_p :

$$\begin{array}{ccccc} \ker \varphi_p & \xrightarrow{i} & \ker \varphi_p \oplus \ker \varphi_{p+1} & \xrightarrow{\sqrt{k}\pi} & \ker \varphi_{p+1} \\ \downarrow \sqrt{k}^{k-p} & & \downarrow \sqrt{k}^{k-p} & & \downarrow \sqrt{k}^{k-p-1} \\ \ker \varphi_p & \xrightarrow{i} & \ker \varphi_p \oplus \ker \varphi_{p+1} & \xrightarrow{\pi} & \ker \varphi_{p+1} \end{array}$$

(i denotes the canonical inclusion and π the canonical projection). Since ch behaves additively for direct sums of cubes, we have $\text{ch}(\Lambda'_p) = \text{ch}(A) + \text{ch}(B)$ with A, B the following two diagrams:

$$\begin{array}{ccccccc} \ker \varphi_p & \xrightarrow{\text{Id}} & \ker \varphi_p & \longrightarrow & 0 & 0 & \longrightarrow & \ker \varphi_{p+1} & \xrightarrow{\sqrt{k}} & \ker \varphi_{p+1} \\ \downarrow \sqrt{k}^{k-p} & & \downarrow \sqrt{k}^{k-p} & & & & & \downarrow \sqrt{k}^{k-p} & & \downarrow \sqrt{k}^{k-p-1} \\ \ker \varphi_p & \xrightarrow{\text{Id}} & \ker \varphi_p & \longrightarrow & 0 & 0 & \longrightarrow & \ker \varphi_{p+1} & \xrightarrow{\text{Id}} & \ker \varphi_{p+1} \end{array}$$

By Lemma 4.3 below, $\text{ch}(a \text{Id}_{\overline{F}}) = -\log(a) \text{ch}(\overline{F})$ for any hermitian vector bundle \overline{F} and $a > 0$ a real number. Using this lemma, we obtain

$$\begin{aligned} d_{\mathcal{D}} \text{ch}(B) &= \text{ch}(\sqrt{k} \text{Id}_{\ker \varphi_{p+1}}) - \text{ch}(\text{Id}_{\ker \varphi_{p+1}}) \\ &\quad - \text{ch}(\sqrt{k}^{k-p} \text{Id}_{\ker \varphi_{p+1}}) + \text{ch}(\sqrt{k}^{k-p-1} \text{Id}_{\ker \varphi_{p+1}}) \\ &= (-\log(\sqrt{k}) + \log(\sqrt{k}^{k-p}) - \log(\sqrt{k}^{k-p-1})) \text{ch}(\ker \varphi_{p+1}) = 0. \end{aligned}$$

Similarly, or using that A is a degenerate cube, we obtain the fact $d_{\mathcal{D}} \text{ch}(A) = 0$ as well. Therefore, $d_{\mathcal{D}} \text{ch}(\Lambda_p) = d_{\mathcal{D}} \text{ch}(\Lambda'_p) = 0$.

By Proposition 4.4, we know that $\text{ch}(\lambda_k(\Psi^k(\overline{E})^*)) = 0$. Therefore, using Lemma 4.3 again we get

$$\begin{aligned} \text{ch}(\Psi^k(\overline{E})^*) &= \sum_{p=0}^k (-1)^p \log(\sqrt{k}^{k-p}) \text{ch}(S^p \overline{E} \otimes \wedge^{k-p} \overline{E}) \\ &= \sum_{p=0}^k (-1)^p (k-p) \log(\sqrt{k}) \text{ch}(S^p \overline{E} \otimes \wedge^{k-p} \overline{E}) \\ &= (-1)^{k+1} \log(\sqrt{k}) \sum_{p=0}^k (-1)^{k-p+1} (k-p) \text{ch}(S^p \overline{E} \otimes \wedge^{k-p} \overline{E}) \\ &= (-1)^{k+1} \log(\sqrt{k}) \text{ch}(\Psi^k(\overline{E})) = \frac{(-1)^{k+1} \log(k)}{2} \Psi^k(\text{ch}(\overline{E})), \end{aligned}$$

where the last equality follows from Proposition 4.6. \square

Lemma 4.3. *Let X be a smooth complex variety, let \bar{F} be a hermitian vector bundle over X and let a be a non-zero complex number. Then*

$$\text{ch}(0 \rightarrow \bar{F} \xrightarrow{a \text{Id}} \bar{F}) = -\log(\|a\|) \text{ch}(\bar{F})$$

in $\bigoplus_{p \geq 0} \mathcal{D}^{2p-1}(X, p)$.

Proof. Let $\text{ch}(a \text{Id})$ denote $\text{ch}(0 \rightarrow \bar{F} \xrightarrow{a \text{Id}} \bar{F})$. Let $t = x/y$ be the local coordinates on \mathbb{P}^1 . By definition,

$$\text{ch}(a \text{Id}) = \frac{1}{2\pi i} \int_{\mathbb{P}^1} \text{ch}(\text{tr}_1(a \text{Id})) \wedge \left(\frac{1}{2} \log t\bar{t} \right).$$

Let h denote the hermitian metric of \bar{F} . Let ξ be a local frame for F on an open set U . By [4], the resulting local frame on $\text{tr}_1(a \text{Id}) \cong p_0^* \bar{F}$ has metric given by the matrix

$$\frac{y\bar{y}h(\xi) + x\bar{x}\|a\|^2 h(\xi)}{y\bar{y} + x\bar{x}} = \frac{y\bar{y} + x\bar{x}\|a\|^2}{y\bar{y} + x\bar{x}} h(\xi) = \frac{1 + \|a\|^2 t\bar{t}}{1 + t\bar{t}} h(\xi)$$

using local projective coordinates. Let $h_a = \frac{1 + \|a\|^2 t\bar{t}}{1 + t\bar{t}}$.

Let $p_0, p_1 : X \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ be the projections onto the first and second coordinates respectively. Since the hermitian metric on $\text{tr}_1(a \text{Id})$ is expressed as the product of a hermitian metric on the line bundle $\mathcal{O}(1)$ and the hermitian metric h of \bar{F} , we have

$$\text{ch}(\text{tr}_1(a \text{Id})) = \text{ch}(p_1^* \mathcal{O}(1), h_a) \wedge \text{ch}(p_0^* \bar{F}).$$

Hence,

$$\begin{aligned} \text{ch}(a \text{Id}) &= \frac{1}{2\pi i} \left(\int_{\mathbb{P}^1} \text{ch}(p_1^* \mathcal{O}(1), h_a) \wedge \left(\frac{1}{2} \log t\bar{t} \right) \right) \wedge \text{ch}(\bar{F}) \\ &= \frac{1}{2\pi i} \left(\int_{\mathbb{P}^1} c_1(p_1^* \mathcal{O}(1), h_a) \wedge \left(\frac{1}{2} \log t\bar{t} \right) \right) \wedge \text{ch}(\bar{F}) \\ &= \frac{-\log(\|a\|^2)}{2} \text{ch}(\bar{F}) = -\log(\|a\|) \text{ch}(\bar{F}). \quad \square \end{aligned}$$

§4.3. Adams operations and the Beilinson regulator

We will define the Adams operations on the rational higher arithmetic K -groups of X from a commutative diagram of the form

$$(4.12) \quad \begin{array}{ccc} N\widehat{C}_*(X) & \xrightarrow{\text{ch}} & \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p) \\ \Psi^k \downarrow & & \downarrow \Psi^k \\ \widetilde{Z}\widehat{C}_*(X) & \xrightarrow{\text{ch}} & \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p) \end{array}$$

We proceed as follows:

- (1) We first define the bottom arrow $\text{ch} : \widetilde{\mathbb{Z}}\widehat{C}_*^{\mathbb{P}}(X) \rightarrow \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p)$.
- (2) We show that there are isomorphisms

$$H_n(s(\widehat{\text{ch}}), \mathbb{Q}) \cong \widehat{K}_n(X)_{\mathbb{Q}}, \quad \widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}_*^{\mathbb{P}}(X), \text{ch})_{\mathbb{Q}} \cong \widehat{K}_n^T(X)_{\mathbb{Q}},$$

with $\widehat{\text{ch}}$ the composition

$$\widetilde{\mathbb{Z}}\widehat{C}_*^{\mathbb{P}}(X) \xrightarrow{\text{ch}} \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p) \rightarrow \bigoplus_{p \geq 0} \sigma_{>0} \mathcal{D}^{2p-*}(X, p).$$

- (3) We prove that the diagram (4.12) is commutative.

Let $\overline{E} \in \widehat{C}_n(X \times (\mathbb{P}^1)^m)$ be a hermitian n -cube on $X \times (\mathbb{P}^1)^m$. We define

$$\text{ch}_{n,m}(\overline{E}) := \frac{(-1)^{n(m+1)}}{(2\pi i)^{n+m}} \int_{(\mathbb{P}^1)^{n+m}} \text{ch}(\text{tr}_n(\lambda(\overline{E}))) \bullet W_{n+m} \in \bigoplus_{p \geq 0} \mathcal{D}^{2p-n-m}(X, p).$$

Proposition 4.8. *There is a chain morphism*

$$\text{ch} : \widetilde{\mathbb{Z}}\widehat{C}_*^{\mathbb{P}}(X) \rightarrow \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p),$$

which maps $\overline{E} \in \widehat{C}_n(X \times (\mathbb{P}^1)^m)$ to $\text{ch}_{n,m}(\overline{E})$. The composition

$$K_n(X) \rightarrow H_n(\widetilde{\mathbb{Z}}\widehat{C}_*^{\mathbb{P}}(X), \mathbb{Q}) \xrightarrow{\text{ch}} \bigoplus_{p \geq 0} H_{\mathcal{D}}^{2p-n}(X, p)$$

is the Beilinson regulator.

Proof. First of all, observe that the map ch is well defined. Indeed, if $\overline{E} = p_i^* \overline{\mathcal{O}(1)} \otimes \overline{F}$, then $\text{ch}(\overline{E}) \in \sigma_i \mathcal{D}^{2p-*}(X \times (\mathbb{P}^1)^{n-1}, p-1) + \omega_i \wedge \sigma_i \mathcal{D}^{2p-*2}(X \times (\mathbb{P}^1)^{n-1}, p-1)$ and hence $\text{ch}(\overline{E}) = 0$.

In order to prove that ch is a chain morphism, observe that it factors as

$$\widetilde{\mathbb{Z}}\widehat{C}_{n,m}^{\mathbb{P}}(X) \xrightarrow{\overline{\text{ch}}} \bigoplus_{p \geq 0} \widetilde{\mathcal{D}}_{\mathbb{P}}^{2p, -n-m}(X, p) \xrightarrow{\varphi} \bigoplus_{p \geq 0} \mathcal{D}^{2p-n-m}(X, \mathbb{R}(p)),$$

where φ is the quasi-isomorphism of Proposition 1.2 and $\overline{\text{ch}}(\overline{E}) = \overline{\text{ch}}_{n,m}(\overline{E})$ is defined by

$$\overline{\text{ch}}_{n,m}(\overline{E}) = (-1)^{nm} \text{ch}(\text{tr}_n(\lambda(\overline{E}))) \in \bigoplus_{p \geq 0} \widetilde{\mathcal{D}}^{2p}(X \times (\mathbb{P}^1)^{m+n}, p)$$

for any $\overline{E} \in \widehat{C}_n(X \times (\mathbb{P}^1)^m)$. Hence, it is enough to see that $\overline{\text{ch}}$ is a chain morphism.

Let $\bar{E} \in \widetilde{\mathbb{Z}}\widehat{C}_{n,m}(X)$. Since ch is a closed differential form, we have

$$\begin{aligned} d_s(\overline{\text{ch}}_{n,m}(\bar{E})) &= (-1)^{nm} \delta \text{ch}(\text{tr}_n(\lambda(\bar{E}))) \\ &= \sum_{i=1}^m (-1)^{i+nm} \text{ch}(\text{tr}_n(\lambda(\delta_i^1 \bar{E} - \delta_i^0 \bar{E}))) \\ &\quad + \sum_{i=m+1}^{n+m} \sum_{j=0}^2 (-1)^{i+j+nm} \text{ch}(\text{tr}_n(\lambda(\partial_{i-m}^j \bar{E}))) \\ &= (-1)^n \overline{\text{ch}}_{n,m-1}(\delta \bar{E}) + \overline{\text{ch}}_{n-1,m}(d\bar{E}), \end{aligned}$$

as desired.

Finally, since by definition there is a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathbb{Z}}\widehat{C}_*(X) & \xrightarrow{\text{ch}} & \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p), \\ \downarrow i \cong & & \uparrow \\ \widetilde{\mathbb{Z}}\widehat{C}_*^{\mathbb{P}}(X) & \xrightarrow{\text{ch}} & \end{array}$$

the morphism ch induces the Beilinson regulator. □

We have therefore constructed the bottom arrow of diagram (4.12). For the next proposition, let $\widehat{\text{ch}} : \widetilde{\mathbb{Z}}\widehat{C}_*^{\mathbb{P}}(X) \rightarrow \bigoplus_{p \geq 0} \sigma_{>0} \mathcal{D}^{2p-*}(X, p)$ be the composition of the morphism defined in Proposition 4.8 with the natural projection $\bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p) \rightarrow \bigoplus_{p \geq 0} \sigma_{>0} \mathcal{D}^{2p-*}(X, p)$.

Proposition 4.9. *There are isomorphisms*

$$\widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}_*^{\mathbb{P}}(X), \text{ch})_{\mathbb{Q}} \cong \widehat{K}_n^T(X)_{\mathbb{Q}}, \quad H_n(s(\widehat{\text{ch}}), \mathbb{Q}) \cong \widehat{K}_n(X)_{\mathbb{Q}},$$

induced by the isomorphism $H_n(\widetilde{\mathbb{Z}}\widehat{C}_*^{\mathbb{P}}(X), \mathbb{Q}) \cong K_n(X)_{\mathbb{Q}}$ of Proposition 4.2.

Proof. Both isomorphisms are a consequence of Proposition 4.2, and the five lemma using the exact sequences of Lemma 3.1 and Proposition 2.1. □

At this point, all that remains to see is that the diagram (4.12) is commutative. This will be a consequence of the next series of lemmas and propositions.

The next lemma tells us that the morphism ch maps the split exact sequences to zero in the complex $\bigoplus_{p \geq 0} \widetilde{\mathcal{D}}_{\mathbb{P}}^{2p-*}(X, p)$.

Lemma 4.4. *Let X be a smooth proper complex variety. Consider a split exact sequence*

$$\bar{E} : 0 \rightarrow \bar{E}^0 \rightarrow \bar{E}^0 \oplus \bar{E}^1 \rightarrow \bar{E}^1 \rightarrow 0$$

of hermitian vector bundles over X . Then we have $\text{ch}(\overline{E}) = 0$ in the complex $\bigoplus_{p \geq 0} \widetilde{\mathcal{D}}_{\mathbb{P}}^{2p-*}(X, p)$.

Proof. Clearly, the exact sequence \overline{E} already has canonical kernels. Let us compute $\text{ch}(\text{tr}_1(\overline{E}))$. By definition, $\text{tr}_1(\overline{E})$ is the kernel of the morphism

$$\overline{E}^0(1) \oplus \overline{E}^1(1) \oplus \overline{E}^1(1) \rightarrow \overline{E}^1(2), \quad (a, b, c) \mapsto b \otimes x - c \otimes y.$$

For every locally free sheaf B , there is a short exact sequence

$$0 \rightarrow B \xrightarrow{f} B(1) \oplus B(1) \xrightarrow{g} B(2) \rightarrow 0$$

where f sends b to $(b \otimes y, b \otimes x)$ and g sends (b, c) to $b \otimes x - c \otimes y$. Moreover, if \overline{B} is a hermitian vector bundle, then the monomorphism f preserves the hermitian metric. It follows that the hermitian vector bundle $\text{tr}_1(\overline{E})$ is $\overline{E}^0(1) \oplus \overline{E}^1$ and therefore

$$\text{ch}(\text{tr}_1(\overline{E})) = \text{ch}(\overline{E}^0(1) \oplus \overline{E}^1) = \text{ch}(\overline{E}^0(1)) + \text{ch}(\overline{E}^1).$$

Since $\text{ch}(\overline{\mathcal{O}}(1)) = 1 + \omega \in D_1^2 + \mathcal{W}_1^2$, the differential form $\text{ch}(\overline{E}^0(1)) + \text{ch}(\overline{E}^1)$ is zero in the complex $\bigoplus_{p \geq 0} \widetilde{\mathcal{D}}_{\mathbb{P}}^{2p-*}(X, p)$. \square

Lemma 4.5. *Let $n > 0$ and let $\overline{E} \in \mathbb{Z}\widehat{\mathcal{C}}_n(X \times (\mathbb{P}^1)^m)$ be a hermitian n -cube which is split in the last direction, that is, for every $\mathbf{j} \in \{0, 1, 2\}^{n-1}$, the 1-cube*

$$(\partial_n^0 \overline{E})^{\mathbf{j}} \rightarrow (\partial_n^1 \overline{E})^{\mathbf{j}} \rightarrow (\partial_n^2 \overline{E})^{\mathbf{j}}$$

is hermitian split. Then $\text{ch}_{n,m}(\overline{E}) = 0$ in $\bigoplus_{p \geq 0} \mathcal{D}^{2p-n-m}(X, p)$.

Proof. Recall that if \overline{E} is a hermitian n -cube then

$$\text{tr}_n(\overline{E}) = \text{tr}_1 \text{tr}_{n-1}(\overline{E}) = \text{tr}_1(\text{tr}_{n-1}(\partial_n^0 \overline{E}) \rightarrow \text{tr}_{n-1}(\partial_n^1 \overline{E}) \rightarrow \text{tr}_{n-1}(\partial_n^2 \overline{E})).$$

Thus, if \overline{E} is split in the last direction, the 1-cube

$$\text{tr}_{n-1}(\partial_n^0 \overline{E}) \rightarrow \text{tr}_{n-1}(\partial_n^1 \overline{E}) \rightarrow \text{tr}_{n-1}(\partial_n^2 \overline{E})$$

is orthogonally split. Now the result follows from Lemma 4.4. \square

Corollary 4.2. *Let $n > 0$ and let $\overline{E} \in \mathbb{Z}\widehat{\mathcal{C}}_n(X \times (\mathbb{P}^1)^m)$ be a hermitian n -cube which is split in any direction. Then $\text{ch}_{n,m}(\overline{E}) = 0$ in the group $\bigoplus_{p \geq 0} \mathcal{D}^{2p-n-m}(X, p)$.* \square

Theorem 4.1. *Let X be a proper arithmetic variety over \mathbb{Z} . The diagram*

$$\begin{array}{ccc} N\widehat{C}_*(X) & \xrightarrow{\text{ch}} & \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p) \\ \Psi^k \downarrow & & \downarrow \Psi^k \\ \widetilde{\mathbb{Z}}\widehat{C}_*^{\widetilde{\mathbb{P}}}(X) & \xrightarrow{\text{ch}} & \bigoplus_{p \geq 0} \mathcal{D}^{2p-*}(X, p) \end{array}$$

is commutative.

Proof. Let \overline{E} be a hermitian n -cube. If $n \geq 2$, then by Proposition 4.1, all the cubes in the image of Ψ^k are hermitian split in at least one direction. Therefore, by Corollary 4.2, they vanish after applying ch . The same reasoning applies to some of the summands of the image of Ψ^k when $n = 1$. The rest of the terms are hermitian 1-cubes of the form $\mu(\overline{A})$ with $\overline{A} = \lambda_l(\Psi^l(\overline{F})^*) \otimes \overline{G}$ or $\overline{A} = \overline{F} \otimes \lambda_l(\Psi^l(\overline{G})^*)$ for some hermitian bundles $\overline{F}, \overline{G}$. By Proposition 4.4, these terms vanish as well after applying ch .

Finally, if $n = 0$, by Proposition 4.6 and the definition of Ψ^k on differential forms, we have

$$\begin{aligned} \text{ch}(\Psi^k(\overline{E})) &= \frac{(-1)^n}{(2\pi i)^n} \int_{(\mathbb{P}^1)^n} \text{ch}(\Psi^k(\text{tr}_n(\lambda(\overline{E})))) \wedge W_n \\ &= \frac{(-1)^n}{(2\pi i)^n} \int_{(\mathbb{P}^1)^n} \Psi^k(\text{ch}(\text{tr}_n(\lambda(\overline{E})))) \wedge W_n = \Psi^k(\text{ch}(\overline{E})). \quad \square \end{aligned}$$

§4.4. Adams operations on higher arithmetic K -theory

Let X be a proper arithmetic variety over \mathbb{Z} . Proposition 4.9 and Theorem 4.1 enable us to define, for every $k \geq 0$, the *Adams operation on higher arithmetic K -groups*:

- ▶ Since the simple complex associated to a morphism is a functorial construction, for every k there is an Adams operation morphism on the Deligne–Soulé higher arithmetic K -groups:

$$\Psi^k : \widehat{K}_n(X)_{\mathbb{Q}} \rightarrow \widehat{K}_n(X)_{\mathbb{Q}}, \quad n \geq 0.$$

- ▶ By Proposition 3.1 for every k there is an Adams operation morphism on the Takeda higher arithmetic K -groups:

$$\Psi^k : \widehat{K}_n^T(X)_{\mathbb{Q}} \rightarrow \widehat{K}_n^T(X)_{\mathbb{Q}}, \quad n \geq 0.$$

We have proved the following theorems.

Theorem 4.2 (Adams operations). *Let X be a proper arithmetic variety over \mathbb{Z} and let $\widehat{K}_n(X)$ be the n -th Deligne–Soulé arithmetic K -group. There are Adams operations*

$$\Psi^k : \widehat{K}_n(X)_{\mathbb{Q}} \rightarrow \widehat{K}_n(X)_{\mathbb{Q}},$$

compatible with the Adams operations in $K_n(X)_{\mathbb{Q}}$ and $\bigoplus_{p \geq 0} H_{\mathcal{D}}^{2p-n}(X, \mathbb{R}(p))$, by means of the morphisms a and ζ . \square

Theorem 4.3 (Adams operations). *Let X be a proper arithmetic variety over \mathbb{Z} and let $\widehat{K}_n^T(X)$ be the n -th arithmetic K -group defined by Takeda in [21]. Then, for every $k \geq 0$ there exists an Adams operation morphism $\Psi^k : \widehat{K}_n^T(X)_{\mathbb{Q}} \rightarrow \widehat{K}_n^T(X)_{\mathbb{Q}}$ such that the following diagram is commutative:*

$$\begin{array}{ccccccccc} K_{n+1}(X)_{\mathbb{Q}} & \xrightarrow{\text{ch}} & \bigoplus_{p \geq 0} \widetilde{\mathcal{D}}^{2p-n-1}(X, p) & \xrightarrow{a} & \widehat{K}_n^T(X)_{\mathbb{Q}} & \xrightarrow{\zeta} & K_n(X)_{\mathbb{Q}} & \longrightarrow & 0 \\ \Psi^k \downarrow & & \Psi^k \downarrow & & \Psi^k \downarrow & & \Psi^k \downarrow & & \\ K_{n+1}(X)_{\mathbb{Q}} & \xrightarrow{\text{ch}} & \bigoplus_{p \geq 0} \widetilde{\mathcal{D}}^{2p-n-1}(X, p) & \xrightarrow{a} & \widehat{K}_n^T(X)_{\mathbb{Q}} & \xrightarrow{\zeta} & K_n(X)_{\mathbb{Q}} & \longrightarrow & 0 \end{array}$$

Moreover, the diagram

$$\begin{array}{ccc} \widehat{K}_n^T(X)_{\mathbb{Q}} & \xrightarrow{\text{ch}} & \bigoplus_{p \geq 0} \mathcal{D}^{2p-n}(X, p) \\ \Psi^k \downarrow & & \downarrow \Psi^k \\ \widehat{K}_n^T(X)_{\mathbb{Q}} & \xrightarrow{\text{ch}} & \bigoplus_{p \geq 0} \mathcal{D}^{2p-n}(X, p) \end{array}$$

is commutative. \square

Lambda operations. Let X be a proper arithmetic variety over \mathbb{Z} . Consider the product structure on $\widehat{K}_*(X)_{\mathbb{Q}}$ defined before Lemma 3.2. Then, by the relation between the Adams and λ operations in a λ -ring (which is a \mathbb{Q} -algebra), there are induced λ -operations

$$\lambda^k : \widehat{K}_n(X)_{\mathbb{Q}} \rightarrow \widehat{K}_n(X)_{\mathbb{Q}}.$$

Corollary 4.3 (Pre- λ -ring). *Let X be a proper arithmetic variety over \mathbb{Z} . Then $\widehat{K}_*(X)_{\mathbb{Q}}$ is a pre- λ -ring. Moreover, there is a commutative square*

$$\begin{array}{ccc} \widehat{K}_n(X)_{\mathbb{Q}} & \xrightarrow{\zeta} & K_n(X)_{\mathbb{Q}} \\ \lambda^k \downarrow & & \downarrow \lambda^k \\ \widehat{K}_n(X)_{\mathbb{Q}} & \xrightarrow{\zeta} & K_n(X)_{\mathbb{Q}} \end{array}$$

Proof. The diagram is commutative since the Adams and lambda operations in $K_*(X)$ are related under the product structure on $K_*(X)$ which is zero in $\bigoplus_{n \geq 1} K_n(X)$. \square

Proposition 4.10. *Let X be a proper arithmetic variety over \mathbb{Z} . The Adams operations given here for $\widehat{K}_0(X)_{\mathbb{Q}}$ agree with the ones given by Gillet and Soulé in [12].*

Proof. This follows from the definition. \square

Consider the product structure in $\bigoplus_{n \geq 0} \widehat{K}_n^T(X)_{\mathbb{Q}}$ having $\bigoplus_{n \geq 1} \widehat{K}_n^T(X)_{\mathbb{Q}}$ as a zero square ideal and agreeing with the product defined by Takeda in [21] otherwise.

Corollary 4.4 (Pre- λ -ring). *Let X be a proper arithmetic variety over \mathbb{Z} . Then $\widehat{K}_*^T(X)_{\mathbb{Q}}$ is a pre- λ -ring. Moreover, there is a commutative square*

$$\begin{array}{ccc} \widehat{K}_n^T(X)_{\mathbb{Q}} & \xrightarrow{\zeta} & K_n(X)_{\mathbb{Q}} \\ \lambda^k \downarrow & & \downarrow \lambda^k \\ \widehat{K}_n^T(X)_{\mathbb{Q}} & \xrightarrow{\zeta} & K_n(X)_{\mathbb{Q}} \end{array}$$

Proof. The proof is analogous to the proof of Corollary 4.3. \square

Remark 4.6. One way to prove that the pre- λ -ring structure on $\widehat{K}_*(X)_{\mathbb{Q}}$ given here is actually a λ -ring structure, is to find precise exact sequences relating, at the level of vector bundles, the equalities in $K_0(X)$

$$\Psi^k(E \otimes F) = \Psi^k(E) \otimes \Psi^k(F), \quad \Psi^k(\Psi^l(E)) = \Psi^{kl}(E).$$

This implies finding formulas for

$$\wedge^k(E \otimes F), \quad \wedge^k(\wedge^l(E)),$$

in terms of tensor and exterior products. The theory of Schur functors gives a formula for the first term. However, the second formula is an open problem. Nevertheless, even for the first equality, when we try to apply the formulas to our concrete situation, the combinatorics becomes really complicated.

It would be desirable and interesting to find a non-direct approach to these relations. One attempt could be to go through the ‘‘Arakelov’’ representation ring on the linear group scheme over \mathbb{Z} , $\widehat{R}_{\mathbb{Z}}(GL_n \times GL_m)$, introduced by Rössler in [19].

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