Publ. RIMS Kyoto Univ. **46** (2010), 171–182
DOI 10.2977/PRIMS/4

Denseness of Norm-Attaining Mappings on Banach Spaces

by

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Abstract

Let X and Y be Banach spaces. Let $P(^nX : Y)$ be the space of all Y-valued continuous n-homogeneous polynomials on X. We show that the set of all norm-attaining elements is dense in $P(^nX : Y)$ when a set of u.s.e. points of the unit ball B_X is dense in the unit sphere S_X. Applying strong peak points instead of u.s.e. points, we generalize this result to a closed subspace of $C_b(M, Y)$, where M is a complete metric space. For complex Banach spaces X and Y, let $A_b(B_X : Y)$ be the Banach space of all bounded continuous Y-valued mappings f on B_X whose restrictions $f|_{B_X^\circ}$ to the open unit ball are holomorphic. It follows that the set of all norm-attaining elements is dense in $A_b(B_X : Y)$ if the set of all strong peak points in $A_b(B_X)$ is a norming subset for $A_b(B_X)$.

2010 Mathematics Subject Classification: 46B04, 46G20, 46G25, 46B22.
Keywords: homogeneous polynomial, norm-attaining element, uniformly strongly exposed point, strong peak point.

§1. Introduction

Let X, Y be Banach spaces over a scalar field $\mathbb{F}$, where $\mathbb{F}$ is the real or complex field. The celebrated Bishop–Phelps theorem [6] says that the set of all norm-attaining continuous linear functionals is dense in the dual space X^*. Motivated by the Bishop–Phelps theorem, Lindenstrauss [20] studied the denseness of the norm-attaining operators in the space $L(X, Y)$ of all continuous linear operators. Recall that an operator $T \in L(X, Y)$ is said to *attain its norm* if $\|T\| = \|T(x_0)\|$ for some $x_0 \in B_X$, where B_X is the unit ball of X. In [20, Proposition 1] he showed that if

Communicated by H. Okamoto. Received April 18, 2009. Revised August 10, 2009.

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B_X is the closed convex hull of a set of uniformly strongly exposed (u.s.e.) points, then X has property A, that is, for every Banach space Y, the set of norm-attaining elements is dense in $L(X,Y)$. Payá and Saleh [22] extended this result to n-linear forms on X, and showed that if B_X is the closed absolutely convex hull of a set of u.s.e. points, then the set of all norm-attaining elements is dense in $L({}^nX)$, the Banach space of all bounded n-linear forms on X. Notice that if B_X is the closed absolutely convex hull of a set E of u.s.e. points, then E is a norming set for $L(X,Y)$.

We shall study a similar question for the space $P({}^nX : Y)$ of all continuous n-homogeneous polynomials from X into Y. In particular, if a set of u.s.e. points on S_X is a norming set for $P({}^nX : Y)$, then the set of norm-attaining elements is dense in $P({}^nX : Y)$. Applying strong peak points instead of u.s.e. points, we shall generalize this result to a closed subspace of $C_b(M,Y)$, where M is a complete metric space and $C_b(M,Y)$ is the Banach space of all bounded continuous mappings from X to Y with the norm $\|f\| = \sup\{\|f(t)\| : t \in M\}$.

On the other hand, Lindenstrauss [20, Theorem 1] proved that the set of all bounded linear operators of X into Y whose second adjoints attain their norms is dense in $L(X,Y)$. In 1996 Acosta [2] extended this result to continuous bilinear forms, and in 2002 Aron, Garcia and Maestre [5] showed that this is also true for scalar-valued continuous 2-homogeneous polynomials. Recently, Acosta, Garcia and Maestre [4] extended it to n-linear mappings. We shall extend the result of [5] to the vector-valued case by modifying their proof, which is originally based on that of Lindenstrauss. Finally, we give a necessary condition for a complex Banach space to have property A.

For completeness we recall some terminology. A *continuous n-homogeneous polynomial* P from X to Y is a mapping $P : X \to Y$ defined by $P(x) = L(x,\dots,x)$, where L is a continuous n-linear mapping from $\overbrace{X \times \cdots \times X}^{n} \to Y$. The space of all continuous n-homogeneous polynomials from X to Y is denoted by $P({}^nX : Y)$ and it is a Banach space when equipped with the norm $\|P\| = \sup\{\|P(x)\| : x \in B_X\}$. An n-homogeneous polynomial P is said to attain its norm if $\|P\| = \|P(x_0)\|$ for some $x_0 \in B_X$. It is clear that $P({}^nX : Y)$ is a closed subspace of $C_b(B_X : Y)$. For the scalar-valued case we denote the former spaces simply by $P({}^nX)$ or $C_b(M)$. When A is a subspace of $C_b(M : Y)$, a subset Γ of M is said to be a *norming set* for A if $\|f\| = \sup\{\|f(t)\| : t \in \Gamma\}$ for each $f \in A$. It is easy to see that if $E \subset B_X$ is a norming set for $P({}^nX)$, then it is also a norming set for $P({}^nX : Y)$ for every Banach space Y.

An element $x \in B_X$ is said to be a *strongly exposed point* for B_X if there is a linear functional $f \in B_{X^*}$ such that $f(x) = 1$ and whenever there is a sequence $\{x_n\}_{n=1}^{\infty}$ in B_X satisfying $\lim_n \operatorname{Re} f(x_n) = 1$, we get $\lim_n \|x_n - x\| = 0$. A set

$\{x_\alpha\}_\alpha$ of points on S_X is called *uniformly strongly exposed* (u.s.e.) if there are a function $\delta(\epsilon)$ with $\delta(\epsilon) > 0$ for every $\epsilon > 0$, and a set $\{f_\alpha\}_\alpha$ of elements of norm 1 in X^* such that for every α, $f_\alpha(x_\alpha) = 1$, and for any x,

$$\|x\| \le 1 \text{ and Re } f_\alpha(x) \ge 1 - \delta(\epsilon) \quad \text{imply} \quad \|x - x_\alpha\| \le \epsilon.$$

In this case we say that $\{f_\alpha\}$ uniformly strongly exposes $\{x_\alpha\}$.

A nonzero function $f \in C_b(M : Y)$ is said to be a *strong peak function* at t if whenever there is a sequence $\{t_n\}_n$ in M with $\lim_n \|f(t_n)\| = \|f\|$, we get $\lim_n t_n = t$. Such a point t is said to be a *strong peak point* of f. When A is a subspace of $C_b(M : Y)$, we set $\rho A = \{t : t \text{ is a strong peak point of some } f \in A\}$.

§2. Main results

Theorem 2.1. *Let X and Y be Banach spaces and $n \in \mathbb{N}$. Suppose that a set E of u.s.e. points on S_X is a norming subset of $P(^nX)$. Then the set of all norm-attaining elements is dense in $P(^nX : Y)$.*

Proof. Suppose that a set E of u.s.e. points on S_X is a norming subset of $P(^nX)$. Let $P \in P(^nX : Y)$, $\|P\| = 1$, and $0 < \epsilon < 1/3$ be given. We first choose a decreasing sequence $\{\epsilon_k\}$ of positive numbers so that

$$(2.1)\qquad 4\sum_{i=1}^{\infty} \epsilon_i < \epsilon < \frac{1}{3}, \qquad 4\sum_{i=k+1}^{\infty} \epsilon_i < \epsilon_k^2, \qquad \epsilon_k < \frac{1}{10k}, \qquad k = 1, 2, \ldots.$$

Using induction, we next choose sequences $\{P_k\}_{k=1}^\infty$ in $\mathcal{P}(^nX : Y)$, $\{x_k\}_{k=1}^\infty$ in E and $\{x_k^*\}_{k=1}^\infty$ in S_{X^*} so that

$$(2.2)\qquad P_1 = P,$$

$$(2.3)\qquad \|P_k(x_k)\| \ge \|P_k\| - \epsilon_k^2 \quad \text{and} \quad \|x_k\| = 1, \quad x_k^*(x_k) = 1,$$

where $\{x_k^*\}$ uniformly strongly exposes $\{x_k\}$,

$$(2.4)\qquad P_{k+1}(x) = P_k(x) + \epsilon_k (x_k^*(x))^n P_k(x_k) \quad (x \in X).$$

Having chosen these sequences, we see that the following hold:

$$(2.5)\qquad \|P_j - P_k\| \le \frac{4}{3}\sum_{i=j}^{k-1} \epsilon_i, \qquad \|P_k\| \le \frac{4}{3}, \qquad j < k,$$

$$(2.6)\qquad \|P_{k+1}\| \ge \|P_k\| + \epsilon_k\|P_k\| - \epsilon_k^2 - \epsilon_k^3,$$

$$(2.7)\qquad \|P_{k+1}\| \le \|P_k\| + \epsilon_k|x_k^*(x_l)|^n\|P_k\| + \epsilon_k^2 + 2 \cdot \frac{4}{3}\sum_{i=k+1}^{l-1} \epsilon_i, \qquad k+1 < l.$$

The assertion (2.5) can be easily proved by induction and (2.6) follows directly from (2.3) and (2.4). To see (2.7), for $k+1 < l$ we have

$$\begin{aligned}\|P_{k+1}\| &\le \|P_l\| + \|P_{k+1} - P_l\| \le \|P_l(x_l)\| + \epsilon_l^2 + \frac{4}{3}\sum_{i=k+1}^{l-1}\epsilon_i \\ &\le \|P_k(x_l)\| + \epsilon_k |x_k^*(x_l)|^n \|P_k\| + \epsilon_k^2 + 2\cdot\frac{4}{3}\sum_{i=k+1}^{l-1}\epsilon_i \\ &\le \|P_k\| + \epsilon_k |x_k^*(x_l)|^n \|P_k\| + \epsilon_k^2 + 2\cdot\frac{4}{3}\sum_{i=k+1}^{l-1}\epsilon_i.\end{aligned}$$

By (2.5), the sequence $\{P_k\}$ converges in the norm topology to $Q \in P({}^nX : Y)$ satisfying $\|P - Q\| < \epsilon$.

By (2.6) and (2.7) we have, for every $l > k+1$,

$$\epsilon_k\|P_k\| - \epsilon_k^2 - \epsilon_k^3 \le \epsilon_k|x_k^*(x_l)|^n\|P_k\| + 2\epsilon_k^2,$$

and hence $1 - 4\epsilon_k < |x_k^*(x_l)|^n$ since $\|P_k\| \ge 1$ for each $k \ge 1$.

Since E is uniformly strongly exposed, $\{x_k\}$ has a norm convergent subsequence to some $x_0 \in S_X$ by Lemma 6 in [1]. Then we can see that $\|Q(x_0)\| = \|Q\|$. □

Now we are going to generalize this result to a closed subspace of $C_b(M, Y)$ with strong peak points instead of u.s.e. points.

Theorem 2.2. *Let (M, d) be a complete metric space, Y a Banach space and A a closed subspace of $C_b(M : Y)$. Assume that there exist a norming subset $\{x_\alpha\}_\alpha \subset M$ for A and a family $\{\varphi_\alpha\}_\alpha$ of functions in $C_b(M)$ such that each φ_α is a strong peak function at x_α. Assume also that A contains $\varphi_\alpha^n \otimes y$ for each $y \in Y$ and $n \ge 1$. Then the set of norm-attaining elements is dense in A.*

Proof. We may assume that $\varphi_\alpha(x_\alpha) = 1$ for each α. Let $f \in A$ with $\|f\| = 1$ and ϵ with $0 < \epsilon < 1/3$ be given. We choose a decreasing sequence $\{\epsilon_k\}$ of positive numbers so that

$$(2.8) \qquad 2\sum_{i=1}^{\infty}\epsilon_i < \epsilon, \quad 2\sum_{i=k+1}^{\infty}\epsilon_i < \epsilon_k^2, \quad \epsilon_k < \frac{1}{10k}, \quad k = 1, 2, \ldots.$$

We next choose inductively sequences $\{f_k\}_{k=1}^{\infty}$, $\{x_{\alpha_k}\}_{k=1}^{\infty}$ satisfying

$$f_1 = f, \tag{2.9}$$

$$\|f_k(x_{\alpha_k})\| \geq \|f_k\| - \epsilon_k^2, \tag{2.10}$$

$$f_{k+1}(x) = f_k(x) + \epsilon_k \tilde{\varphi}_{\alpha_k}(x) \cdot f_k(x_{\alpha_k}) \quad (x \in M), \tag{2.11}$$

$$|\tilde{\varphi}_{\alpha_k}(x)| > 1 - 1/k \quad \text{implies} \quad d(x, x_{\alpha_k}) < 1/k, \tag{2.12}$$

where $\tilde{\varphi}_{\alpha_j}$ is $\varphi_{\alpha_j}^{n_j}$ for some positive integer n_j. Having chosen these sequences, we verify the following:

$$\|f_j - f_k\| \leq 2\sum_{i=j}^{k-1} \epsilon_i, \quad \|f_k\| \leq 4/3, \quad j < k,\ k = 2, 3, \ldots, \tag{2.13}$$

$$\|f_{k+1}\| \geq \|f_k\| + \epsilon_k \|f_k\| - 2\epsilon_k^2, \qquad k = 1, 2, \ldots, \tag{2.14}$$

$$\|f_k\| \geq \|f_j\| \geq 1, \qquad j < k,\ k = 2, 3, \ldots, \tag{2.15}$$

$$|\tilde{\varphi}_{\alpha_j}(x_{\alpha_k})| > 1 - 1/j, \qquad j < k,\ k = 2, 3, \ldots. \tag{2.16}$$

Assertion (2.13) is easy by using induction on k. By (2.10) and (2.11),

$$\begin{aligned}\|f_{k+1}\| \geq \|f_{k+1}(x_{\alpha_k})\| &= \|f_k(x_{\alpha_k})(1 + \epsilon_k \tilde{\varphi}_{\alpha_k}(x_{\alpha_k}))\| \\ &= \|f_k(x_{\alpha_k})\|(1 + \epsilon_k) \geq (\|f_k\| - \epsilon_k^2)(1 + \epsilon_k) \geq \|f_k\| + \epsilon_k\|f_k\| - 2\epsilon_k^2,\end{aligned}$$

so (2.14) is proved. Therefore (2.15) is an immediate consequence of (2.9) and (2.14). For $j < k$, by the triangle inequality, (2.8), (2.10), (2.13) and (2.15), we have

$$\begin{aligned}\|f_{j+1}(x_{\alpha_k})\| &\geq \|f_k(x_{\alpha_k})\| - \|f_k - f_{j+1}\| \\ &\geq \|f_k\| - \epsilon_k^2 - 2\sum_{i=j+1}^{k-1} \epsilon_i \geq \|f_{j+1}\| - 2\epsilon_j^2.\end{aligned}$$

Hence by (2.11) and (2.14),

$$\begin{aligned}\epsilon_j |\tilde{\varphi}_{\alpha_j}(x_{\alpha_k})| \cdot \|f_j\| + \|f_j\| \geq \|f_{j+1}(x_{\alpha_k})\| &\geq \|f_{j+1}\| - 2\epsilon_j^2 \\ &\geq \|f_j\| + \epsilon_j\|f_j\| - 4\epsilon_j^2,\end{aligned}$$

so that

$$|\tilde{\varphi}_{\alpha_j}(x_{\alpha_k})| \geq 1 - 4\epsilon_j > 1 - 1/j,$$

and this proves (2.16). Let $\hat{f} \in A$ be the limit of $\{f_k\}$ in the norm topology. By (2.8) and (2.13), $\|\hat{f} - f\| = \lim_n \|f_n - f_1\| \leq 2\sum_{i=1}^{\infty} \epsilon_i \leq \epsilon$. The relations (2.12) and (2.16) mean that the sequence $\{x_{\alpha_k}\}$ converges to some $\tilde{x}$, and by (2.10), we have $\|\hat{f}\| = \lim_n \|f_n\| = \lim_n \|f_n(x_{\alpha_n})\| = \|\hat{f}(\tilde{x})\|$. Hence $\hat{f}$ attains its norm. □

For complex Banach spaces X and Y, consider the following closed subspaces of $C_b(B_X:Y)$:

$$A_b(B_X:Y)=\{f\in C_b(B_X:Y): f|_{B_X^\circ} \text{ is holomorphic}\},$$
$$A_u(B_X:Y)=\{f\in A_b(B_X:Y): f \text{ is uniformly continuous on } B_X\}.$$

We denote by $A(B_X:Y)$ either $A_b(B_X:Y)$ or $A_u(B_X:Y)$ and write $A(B_X)$ in the scalar valued case.

Recall that a Banach space is said to be *locally uniformly convex* if for each $x\in B_X$ and for each sequence $\{x_n\}_n$ in B_X with $\lim_n\|x+x_n\|=2$, we have $\lim_n\|x_n-x\|=0$. Let A be the closed linear span of the constant 1 and X^* as a subspace of $C_b(B_X)$. Notice that if X is locally uniformly convex, then every element of S_X is a strong peak point for A, and clearly the set $\rho A(B_X:Y)$ of all strong peak points of $A(B_X:Y)$ is also S_X, hence a norming subset for $A(B_X:Y)$ for every complex Banach space Y. Indeed, if $x\in S_X$, choose $x^*\in S_{X^*}$ so that $x^*(x)=1$. Set $f(y)=(x^*(y)+1)/2$ for $y\in B_X$. Then $f\in A$ and $f(x)=1$. If $\lim_n|f(x_n)|=1$ for some sequence $\{x_n\}$ in B_X, then $\lim_n x^*(x_n)=1$. Since $|x^*(x_n)+x^*(x)|\le\|x_n+x\|\le 2$ for every n, $\|x_n+x\|\to 2$ and $\|x_n-x\|\to 0$ as $n\to\infty$. It is also clear that every strongly exposed point for B_X is a strong peak point for A.

Following [13, 15], a point $x\in S_X$ is said to be a *complex extreme point* of B_X if for any nonzero $y\in X$, $\int_0^{2\pi}\|x+e^{i\theta}y\|^2\,\frac{d\theta}{2\pi}>1$. A point $x\in S_X$ is called a *strong complex extreme point* of B_X if for each $\epsilon>0$, there is $\delta>0$ such that

$$\inf\left\{\int_0^{2\pi}\|x+e^{i\theta}y\|^2\,\frac{d\theta}{2\pi}: y\in X,\ \|y\|\ge\epsilon\right\}\ge 1+\delta.$$

The set of all complex extreme points of B_X is denoted by $\mathrm{ext}_{\mathbb{C}}(B_X)$, and a complex Banach space X is said to be *strictly complex convex* if $\mathrm{ext}_{\mathbb{C}}(B_X)=S_X$. When every point of S_X is a strong complex extreme point of B_X, the Banach space X is called *locally uniformly c-convex.*

It was shown in [7] that if a Banach sequence space X is locally uniformly c-convex and order continuous, then $\rho A(B_X)$ is dense in S_X. Therefore, $\rho A(B_X)$ is a norming subset for $A(B_X:Y)$ for every complex Banach space Y. For the definition of a Banach sequence space and order continuity, see [7, 16, 21]. We also notice that if $E\subset B_X$ is a norming set for $A(B_X)$, then it is also a norming set for $A(B_X:Y)$ for every Banach space Y. By the remarks above, we get the following.

Corollary 2.1. *Suppose that X and Y are complex Banach spaces and $\rho A(B_X)$ is a norming subset for $A(B_X:Y)$. Then the set of norm-attaining elements is* *dense in $A(B_X : Y)$. In particular, if X is locally uniformly convex, or if it is a locally uniformly c-convex, order continuous Banach sequence space, then the set of norm-attaining elements is dense in $A(B_X : Y)$.*

It is shown in [3] that if X has the Radon–Nikodým property, then the set of norm-attaining elements is dense in $A(B_X : Y)$. However, the complex Banach space c_0 renormed by Day's norm is locally uniformly convex [11, 12], but it does not have the Radon-Nikodým property. In addition, it is a locally uniformly c-convex and order continuous Banach sequence space.

It is also worth remarking that it is shown in [9] that $\rho A(B_X)$ is a norming subset for $A(B_X)$ if X has the Radon–Nikodým property. Further, very recently, it has been shown in [18] that the set of all strong peak functions is dense in $A(B_X : Y)$ if $\rho A(B_X)$ is a norming subset for $A(B_X)$.

Example 1. Let $\varphi : \mathbb{R} \to [0, \infty]$ be an even, convex continuous function vanishing only at zero and let $w = \{w(n)\}$ be a nonincreasing sequence of positive real numbers satisfying $\sum_{n=1}^{\infty} w(n) = \infty$. For a sequence $x = \{x(n)\}_{n=1}^{\infty}$, x^* is the decreasing rearrangement of $|x| = \{|x(n)|\}_{n=1}^{\infty}$.

An *Orlicz–Lorentz sequence space* $\lambda_{\varphi,w}$ consists of all sequences $x = \{x(n)\}$ such that for some $\lambda > 0$,

$$\varrho_\varphi(\lambda x) = \sum_{n=1}^{\infty} \varphi(\lambda x^*(n))w(n) < \infty,$$

and has the norm $\|x\| = \inf\{\lambda > 0 : \varrho_\varphi(x/\lambda) \leq 1\}$. Then $\lambda_{\varphi,w}$ is a Banach sequence space. We say that the function φ satisfies *condition* δ_2 ($\varphi \in \delta_2$) if there exist $K > 0$ and $u_0 > 0$ such that

$$\varphi(2u) \leq K\varphi(u) \quad \text{for } u \in [0, u_0].$$

If $\varphi \in \delta_2$, then $\lambda_{\varphi,w}$ is locally uniformly c-convex [7] and order continuous [16]. Notice that if $\varphi(t) = |t|^p$ for $p \geq 1$ and $w = 1$, then $\lambda_{\varphi,w} = \ell_p$. A characterization of the local uniform convexity of an Orlicz–Lorentz function space is given in [16, 17] and a characterization of the local uniform c-convexity of a complex function space is given in [19].

We now extend the result of [5] to the vector-valued case, that is, show that the set of all elements whose Aron–Berner extensions attain their norms is dense in $P(^2X : Y)$.

A continuous n-homogeneous polynomial $P \in P(^nX : Y)$ has an extension $\overline{P} \in P(^nX^{**}, Y^{**})$ to the bidual X^{**} of X, which is called the *Aron–Berner extension* of P. In fact, $\overline{P}$ is defined in the following way. Let $X_1, \ldots, X_n$ be an arbitrary collection of Banach spaces and let $\mathcal{L}(^n(X_1 \times \cdots \times X_n))$ denote the space of bounded n-linear forms. Given $z_i \in X_i^{**}$, $1 \le i \le n$, define $\overline{z}_i$ from $\mathcal{L}(^n(X_1 \times \cdots \times X_i \times X_{i+1}^{**} \times \cdots \times X_n^{**}))$ to $\mathcal{L}(^{n-1}(X_1 \times \cdots \times X_{i-1} \times X_{i+1}^{**} \times \cdots \times X_n^{**}))$ by

$$\overline{z}_i(T)(x_1, \dots, x_{i-1}, x_{i+1}^{**}, \dots, x_n^{**}) = \langle z_i, T(x_1, \dots, x_{i-1}, \cdot, x_{i+1}^{**}, \dots, x_n^{**})\rangle,$$

where $T(x_1, \dots, x_{i-1}, \cdot, x_{i+1}^{**}, \dots, x_n^{**})$ is the linear functional on X_i defined by $\cdot \mapsto T(x_1, \dots, x_{i-1}, \cdot, x_{i+1}^{**}, \dots, x_n^{**})$ and $\langle z, x^*\rangle$ is the duality between X_i^{**} and X_i^*. The map $\overline{z}_i$ is a bounded operator with norm $\|z_i\|$. Now, given $T \in \mathcal{L}(^n(X_1 \times \cdots \times X_n))$, define the extended n-linear form $\overline{T} \in \mathcal{L}(^n(X_1^{**} \times \cdots \times X_n^{**}))$ by

$$\overline{T}(z_1, \dots, z_n) := \overline{z}_1 \circ \cdots \circ \overline{z}_n(T).$$

For a vector-valued n-linear mapping $L \in \mathcal{L}(^n(X_1 \times \cdots \times X_n), Y)$, define

$$\overline{L}(x_1^{**}, \dots, x_n^{**})(y^*) = \overline{y^* \circ L}(x_1^{**}, \dots, x_n^{**}),$$

where $x_i^{**} \in X_i^{**}$, $1 \le i \le n$, and $y^* \in Y^*$. Then $\overline{L} \in \mathcal{L}(^n(X_1^{**} \times \cdots \times X_n^{**}), Y^{**})$ has the same norm as L. Let $S \in \mathcal{L}_s(^nX : Y)$ be the symmetric n-linear mapping corresponding to P. Then S can be extended to an n-linear mapping $\overline{S} \in \mathcal{L}(^nX^{**}, Y^{**})$ as described above. The restriction

$$\overline{P}(z) = \overline{S}(z, \dots, z)$$

is called the Aron–Berner extension of P. Given $z \in X^{**}$ and $w \in Y^*$, we have

$$\overline{P}(z)(w) = \overline{w \circ P}(z).$$

Actually this equality is often used as the definition of the vector-valued Aron–Berner extension based upon the scalar-valued Aron–Berner extension. Davie and Gamelin [10, Theorem 8] proved that $\|P\| = \|\overline{P}\|$. It is also worth noting that $\overline{S}$ is not symmetric in general.

Theorem 2.3. *Let X and Y be Banach spaces. The subset of $P(^2X : Y)$ each of whose elements has the norm-attaining Aron–Berner extension is dense in $P(^2X : Y)$.*

Proof. Let $P \in \mathcal{P}(^2X : Y)$, $\|P\| = 1$, and let S be the symmetric bilinear mapping corresponding to P. Let ϵ with $0 < \epsilon < 1/4$ be given. We first choose a decreasing sequence $\{\epsilon_k\}$ of positive numbers which satisfies the following conditions:

$$(2.17) \qquad 8\sum_{i=1}^{\infty} \epsilon_i < \epsilon < \frac{1}{4}, \quad 8\sum_{i=k+1}^{\infty} \epsilon_i < \epsilon_k^2, \quad \epsilon_k < \frac{1}{10k}, \quad k = 1, 2, \dots.$$

Using induction, we next choose sequences $\{P_k\}_{k=1}^{\infty}$ in $P(^2X : Y)$, $\{x_k\}_{k=1}^{\infty}$ in S_X and $\{f_k\}_{k=1}^{\infty}$ in S_{Y^*} so that

$$P_1 = P, \quad \|P\| = 1, \tag{2.18}$$

$$f_k(P_k(x_k)) = \|P_k(x_k)\| \geq \|P_k\| - \epsilon_k^2, \tag{2.19}$$

$$P_{k+1}(x) = P_k(x) + \epsilon_k (f_k(S_k(x_k, x)))^2 P_k(x_k) \quad (x \in X), \tag{2.20}$$

where each S_k is the symmetric bilinear mapping corresponding to P_k. Having chosen these sequences, we see that the following hold:

$$\|P_j - P_k\| \leq 4\left(\frac{5}{4}\right)^3 \sum_{i=j}^{k-1} \epsilon_i, \quad \|P_k\| \leq \frac{5}{4}, \quad j < k, \tag{2.21}$$

$$\|P_{k+1}\| \geq \|P_k\| + \epsilon_k \|P_k\|^3 - 4\epsilon_k^2, \tag{2.22}$$

$$\|P_{j+1}(x_k)\| > \|P_{j+1}\| - 2\epsilon_j^2, \quad j < k, \tag{2.23}$$

$$|f_j(S_j(x_j, x_k))|^2 \geq \|P_j\|^2 - 6\epsilon_j, \quad j < k. \tag{2.24}$$

By (2.21) and the polarization formula [14], the sequences $\{P_k\}$ and $\{S_k\}$ converge in the norm topology to Q and T, say, respectively. Clearly T is the symmetric bilinear mapping corresponding to Q, and $\|P - Q\| < \epsilon$.

Let $\eta > 0$ be given. Then there exists $j_0 \in \mathbb{N}$ such that

$$\|Q - P_j\| \leq \|T - S_j\| < \eta \quad \text{for all } j \geq j_0,$$

hence $\|P_j\| \geq \|Q\| - \eta$ for all $j \geq j_0$.

From

$$\|T - S_j\| \geq |f_j(T(x_j, x_k)) - f_j(S_j(x_j, x_k))|$$

and (2.24), we have

$$\begin{aligned} |f_j(T(x_j, x_k))| &\geq |f_j(S_j(x_j, x_k))| - \|T - S_j\| \\ &\geq \sqrt{\|P_j\|^2 - 6\epsilon_j} - \eta \geq \sqrt{(\|Q\| - \eta)^2 - 6\epsilon_j} - \eta \end{aligned}$$

for all $k > j \geq j_0$. Let $z \in X^{**}$ be a weak-$*$ limit point of the sequence $\{x_k\}$. Then for all $j \geq j_0$,

$$\|\overline{T}(x_j, z)\| \geq \sqrt{(\|Q\| - \eta)^2 - 6\epsilon_j} - \eta.$$

Hence $\|\overline{T}(z, z)\| \geq \|Q\| - 2\eta$. Since $\eta > 0$ is arbitrary, we have

$$\|\overline{Q}(z)\| = \|\overline{T}(z, z)\| \geq \|Q\| = \|\overline{Q}\|. \qquad \square$$

We finally investigate a version of Theorem 2 in [20] related to complex convexity.

Theorem 2.4. *Let X be a complex Banach space with property A.*

(1) *If X is isomorphic to a strictly c-convex space, then B_X is the closed convex hull of its complex extreme points.*

(2) *If X is isomorphic to a locally uniformly c-convex space, then B_X is the closed convex hull of its strong complex extreme points.*

Proof. We prove only (2). Let C be the closed convex hull of the strong complex extreme points of B_X. Suppose that $C \neq B_X$. Then there are $f \in X^*$ with $\|f\| = 1$ and δ, $0 < \delta < 1$, such that $|f(x)| < 1-\delta$ for $x \in C$. Let $|\!|\!|\cdot|\!|\!|$ be a locally uniformly c-convex norm on X, equivalent to the given norm $\|\cdot\|$, such that $|\!|\!|x|\!|\!| \leq \|x\|$ for $x \in X$. Let Y be the space $X \oplus_2 \mathbb{C}$ with the norm $\|(x,c)\| = (|\!|\!|x|\!|\!|^2 + |c|^2)^{1/2}$. Then Y is locally uniformly c-convex. Indeed, otherwise there exist $(x,c) \in S_{X\oplus_2\mathbb{C}}$, $\epsilon > 0$ and a sequence $\{(x_n, c_n)\}$ such that for every $n \geq 1$, $\|(x_n, c_n)\| \geq \epsilon$ and

$$\lim_n \int_0^{2\pi} \|(x,c) + e^{i\theta}(x_n, c_n)\|^2 \, \frac{d\theta}{2\pi} = 1.$$

Since the norm is plurisubharmonic,

$$\begin{aligned} 1 = |\!|\!|x|\!|\!|^2 + |c|^2 &\leq \int_0^{2\pi} \|(x,c) + e^{i\theta}(x_n, c_n)\|^2 \, \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} |\!|\!|x + e^{i\theta}x_n|\!|\!|^2 \, \frac{d\theta}{2\pi} + \int_0^{2\pi} |c + e^{i\theta}c_n|^2 \, \frac{d\theta}{2\pi} \to 1. \end{aligned}$$

So

$$\lim_{n\to\infty} \int_0^{2\pi} |\!|\!|x + e^{i\theta}x_n|\!|\!|^2 \, \frac{d\theta}{2\pi} = |\!|\!|x|\!|\!|^2 \quad \text{and} \quad \lim_{n\to\infty} \int_0^{2\pi} |c + e^{i\theta}c_n|^2 \, \frac{d\theta}{2\pi} = |c|^2.$$

Since both $(X, |\!|\!|\cdot|\!|\!|)$ and $\mathbb{C}$ are locally uniformly c-convex, we get $\lim_n |\!|\!|x_n|\!|\!| = \lim |c_n| = 0$, which contradicts $\inf_n \|(x_n, c_n)\| \geq \epsilon$.

Let V be the operator from X into Y defined by $Vx = (x, Mf(x))$, where $M > 2/\delta$. Then V is an (into) isomorphism and the same is true for every operator sufficiently close to V. We have

$$\|V\| \geq M, \qquad \|Vx\| \leq (1 + (M-2)^2)^{1/2} \qquad \text{for } x \in C.$$

It follows that operators sufficiently close to V cannot attain their norm at a point belonging to C. To conclude the proof we only have to show that if T is an (into) isomorphism which attains its norm at a point x and if the range of T is locally uniformly c-convex, then x is a strong complex extreme point of B_X.

We may assume that $\|Tx\| = \|T\| = 1$. If x is not a strong complex extreme point, then there are $\epsilon > 0$ and a sequence $\{y_n\} \subset X$ such that $\|y_n\| \geq \epsilon$ for every n and

$$\lim_n \int_0^{2\pi} \|x + e^{i\theta} y_n\|^2 \ \frac{d\theta}{2\pi} = 1.$$

Then

$$1 \le \int_0^{2\pi} \|Tx + e^{i\theta} Ty_n\|^2 \ \frac{d\theta}{2\pi} \le \int_0^{2\pi} \|x + e^{i\theta} y_n\|^2 \ \frac{d\theta}{2\pi}$$

shows that $\{Ty_n\}$ converges to 0, because the range of T is locally uniformly c-convex. Therefore, $\{y_n\}$ converges to 0, which is a contradiction. □

Acknowledgments

The first named author is supported by POSTECH BSRI research fund-2007. The second named author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2009-0069100).

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