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A Characterization of Stein Completion of 0-normal Coronae

by

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Abstract

We show that a 0-normal complex space X that is a corona admits a Stein completion if and only if $H^1(X, \mathcal{O})$ is separated.

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§1. Introduction

Let X be a complex space with countable topology. Let $\mathcal{F}$ be a coherent analytic sheaf on X. For each non-negative integer k the cohomology group $H^k(X, \mathcal{F})$ has a canonical structure of topological (complex) vector space, via the Čech cohomology, which, in general, is not separated.

There are some important particular cases when the separation of certain cohomology groups holds as shown in [1]. Here we mention that Bănică [3] states that this is true if: (i) $X = \widehat{X} \setminus K$, where $\widehat{X}$ is a Stein space and K is a holomorphically convex compact set in $\widehat{X}$ and (ii) $\mathcal{F}$ extends to a coherent analytic sheaf on $\widehat{X}$.

On the other hand, because a compact set L in a Stein space Y is holomorphically convex if, and only if, there exists a continuous plurisubharmonic (exhaustion) function $\varphi : Y \to [0, \infty)$ such that $L = \{\varphi = 0\}$ and φ is strictly plurisubharmonic on the set $\{\varphi > 0\}$, the space X from above carries a continuous, strictly plurisubharmonic, proper function $\psi : X \to (0, \infty)$; we refer to this property by saying that X is a *corona*.

In this paper we shall prove the following results (for definitions, see §2):

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Theorem 1. *Let X be a 0-normal complex space that is a corona. Then X admits a Stein completion if, and only if, the canonical topology of $H^1(X, \mathcal{O})$ is separated.*

Note. A complex space X is said to be *p-normal* (see [1]) if, for any point $x \in X$ and any open neighborhood U of x, the following holds: for any analytic subset A of U of dimension $\leq p$ and any holomorphic function f on $U \setminus A$, there is an open neighborhood W of x in U and a holomorphic function $\widehat{f}$ on W that extends $f|_{W \setminus A}$.

From [2] we recall the following facts:

(i) A complex space X is 0-normal if, and only if, prof $\mathcal{O}_X \geq 2$. Consequently, if X is 0-normal, then $\dim_x X \geq 2$ for every $x \in X$.

(ii) Let X be normal of pure dimension $n \geq 2$. Then X is p-normal for each non-negative integer $p \leq n-2$.

Proposition 1. *Let X be a normal Stein space of pure dimension $n \geq 2$ and A a finite set of points in X. Then the following statements hold true:*

(a) *Let $\mathcal{F}$ be a torsion free coherent sheaf on $X \setminus A$. Then $H^1(X \setminus A, \mathcal{F})$ is separated if and only if $\mathcal{F}$ extends to a coherent analytic sheaf on X.*

(b) *Let $n = 2$. Assume also that $H^2(X, \mathbb{Z}) = 0$ and A contains no singular point of X. Then, for a holomorphic line bundle L on $X \setminus A$, the cohomology group $H^1(X \setminus A, L)$ is separated if and only L is trivial.*

Remark 1. Statement (a) of our proposition can be regarded as an improvement of a particular case of the main result in [11], where it is required that, for **any** point $a \in A$ and for **any** Stein open neighborhood W of a, the canonical topology on the cohomology group $H^1(W \setminus A, \mathcal{F})$ is separated.

Corollary 1. *Let $\mathbb{B}^2$ be the open unit ball in $\mathbb{C}^2$. Then any non-trivial cohomology class ξ in $H^1(\mathbb{B}^2 \setminus \{0\}, \mathcal{O})$ defines via the exponential sequence a topologically trivial holomorphic line bundle L_ξ such that $H^1(\mathbb{B}^2 \setminus \{0\}, L_\xi)$ is not separated.*

Remark 2. This corollary points out that there are some gaps in [8] where it is claimed that for **any** smooth 1-corona M of dimension two and **any** locally free coherent analytic sheaf $\mathcal{F}$ on M the canonical topology on $H^1(M, \mathcal{F})$ is separated [8, Théorème 2, p. 934].

§2. Preliminaries

For a topological vector space E we let E_0 be the closure of $\{0\}$ in E and $E_{\mathrm{s}} := E/E_0$ be the separated space associated to E. Note that E_0 equals the intersection of all open neighborhoods of 0 in E.

Now it is easily seen that:

(iii) E is separated if and only if $E_0 = \{0\}$.

(iv) E has the trivial topology precisely when $E_{\mathrm{s}} = \{0\}$.

The canonical map $S_E : E \to E_{\mathrm{s}}$, which is continuous and open, has certain functorial properties. For instance, taking the separated part commutes with projective limits. We also note that if $\varphi : E \to F$ is a continuous map of topological vector spaces, then $\varphi(E_0) \subset F_0$ so that φ induces continuous linear mappings $\varphi_0 : E_0 \to F_0$ and $\varphi_{\mathrm{s}} : E_{\mathrm{s}} \to F_{\mathrm{s}}$ that make the following diagram commutative:

$$\begin{tikzcd} 0 \arrow[r] & E_0 \arrow[r, "\iota"] \arrow[d, "\varphi_0"'] & E \arrow[r, "S_E"] \arrow[d, "\varphi"'] & E_{\mathrm{s}} \arrow[r] \arrow[d, "\varphi_{\mathrm{s}}"'] & 0 \\ 0 \arrow[r] & F_0 \arrow[r, "\iota"] & F \arrow[r, "S_{\mathrm{F}}"] & F_{\mathrm{s}} \arrow[r] & 0 \end{tikzcd}$$

Here by ι we denote the canonical inclusions. (We refer the interested reader to [4] for more properties.)

Remark 3. As a consequence of the five lemma, one has the following straightforward fact: If φ is surjective and φ_{s} is injective, then φ_0 is surjective.

Let X be a topological space. An *exhaustion* of X is a covering $\mathcal{X}$ of X made up of an increasing sequence of non-empty open subsets of X.

If $\mathcal{F}$ is a sheaf of abelian groups on X and k an integer, then we define the pre-sheaf $H^k(\mathcal{F})$ which associates to an open set V of X the cohomology group $H^k(V, \mathcal{F})$. Except for $k = 0$, $H^k(\mathcal{F})$ is, in general, not a sheaf!

Henceforth X denotes a complex space, reduced and satisfying the second axiom of countability (i.e. its topology has a countable base). Assume that $\mathcal{X}$ is an exhaustion of X by an increasing sequence $\{X_n\}_n$ of open sets. Let $\mathcal{F}$ be a coherent analytic sheaf on X. Then, for every non-negative integer q, the natural map

$$H^q(X, \mathcal{F}) \to \varprojlim H^q(X_n, \mathcal{F})$$

is surjective. The relation between the separated parts and the trivial parts of the cohomology groups in question is established by Cassa [5] whose main result we now quote.

Theorem 2. *Let X be a complex space that is exhausted by an increasing sequence $\{X_n\}_n$ of open sets. Then, for any coherent analytic sheaf $\mathcal{F}$ on X and for any integer $q \geq 1$ we have:*

(a) $H^q(X, \mathcal{F})_{\mathrm{s}} = \varprojlim H^q(X_n, \mathcal{F})_{\mathrm{s}}$.

(b) $H^q(X, \mathcal{F})_0 = \varprojlim H^q(X_n, \mathcal{F})_0 \oplus H^1(\mathcal{X}, H^{q-1}(\mathcal{F}))$.

(c) *If every* $H^q(X_n, \mathcal{F})$ *is separated, then* $H^q(X, \mathcal{F})$ *is separated if, and only if, the projective system* $\{H^{q-1}(X_n, \mathcal{F}), \rho_{m,n}\}$ *satisfies the topological Mittag-Leffler condition.*

Note. For the definition of "topological Mittag-Leffler condition" see [5]; for instance, this condition is satisfied if the restrictions

$$H^{q-1}(X_{n+1}, \mathcal{F})_{\mathrm{s}} \to H^{q-1}(X_n, \mathcal{F})_{\mathrm{s}}$$

have dense images.

Corollary 2. *Let* X *be a complex space that is exhausted by an increasing sequence* $\{X_n\}_n$ *of open sets and* $\mathcal{F}$ *be a coherent sheaf on* X*. Suppose the following conditions hold:*

(1) $H^q(X, \mathcal{F})$ *is separated.*
(2) *Each restriction* $H^q(X_{n+1}, \mathcal{F}) \to H^q(X_n, \mathcal{F})$ *is surjective and induces a bijection between the associated separated spaces.*

Then all $H^q(X_n, \mathcal{F})$ *are separated.*

Proof. On the one hand, we note that from hypothesis (1) and statement (b) of Theorem 2 we get

$$\varprojlim H^q(X_n, \mathcal{F})_0 = 0;$$

on the other hand, (2) gives, thanks to Remark 3, surjections

$$H^q(X_{n+1}, \mathcal{F})_0 \to H^q(X_n, \mathcal{F})_0.$$

Therefore, for all n, $H^q(X_n, \mathcal{F})_0 = 0$, whence the corollary. □

Definition 1. We say that a complex space X is a *corona* if there is a continuous, proper, strictly plurisubharmonic function $\varphi : X \to (a, b)$ with $a \in \{-\infty\} \cup \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$, $a < b$. If we may choose $a = -\infty$, then X is called a *hyperconcave corona.*

Remark 4. The terminology of $(1,1)$-*convex-concave* complex spaces is also used in [2] instead of coronae. Also hyperconcave coronae are called *hyperconcave ends* in [6].

Notation. Let X be a corona defined by a function $\varphi : X \to (a, b)$. For any c, d with $a \le c < d \le b$ we consider the subsets of X defined by

$$X_c^d = \{c < \varphi < d\}, \quad K^d = \{\varphi \le d\}, \quad X^d = \{\varphi < d\}.$$

Notice that X_c^d and X^d are coronae and, if $a < c < d < b$, X_c^d is relatively compact in X.

Definition 2. A complex space Y is called a *Stein completion* of X if

(1) X is an open subset of Y,
(2) Y is a Stein space,
(3) for any $d \in (a, b)$ the set $X^d \cup (Y \setminus X)$ is open and relatively compact.

Remark 5. We point out that the original definition in [2] requires "for any $d \in (a, b)$ the set $K^d \cup (Y \setminus X)$ is compact" instead of (3) above. However, it is not clear how this implies that the set $X^d \cup (Y \setminus X)$ is open, an assertion that is used throughout that paper (see [2, p. 241, l. 10]).

Now let X be a corona. If X admits a 0-normal Stein completion, then the 0-normal Stein completion is unique up to an isomorphism which is the identity on X (see [2]).

In general, if X admits a Stein completion, say Y, then $X = Y \setminus K$, where K is a holomorphically convex compact set in Y. This can be seen as follows. Let $\varphi : X \to (a, b)$ be the function displaying X as a corona. Clearly we may assume that $a \in \mathbb{R}$, otherwise we take $\exp \varphi$ instead of φ. We show that the trivial extension $\widetilde{\varphi}$ of φ to Y by setting $\widetilde{\varphi}(y) = a$ for $y \in Y \setminus X$ becomes continuous and plurisubharmonic. Indeed, to check continuity, we notice that the restriction of $\widetilde{\varphi}$ to each $Y^d := X^d \cup (Y \setminus X)$ is continuous and then use the obvious fact that the family $\{Y^d\}_{d \in (a,b)}$ of open subsets of Y exhausts Y. Now, for plurisubharmonicity, observe that the sequence $\{\psi_n\}_n$ of functions on Y, where $\psi_n := \max(\widetilde{\varphi}, a + 1/n)$, decreases pointwise to $\widetilde{\varphi}$ and each ψ_n is plurisubharmonic (because $\psi_n = a + 1/n$ on an open neighborhood of the compact set $Y \setminus X$, and $\psi_n = \max(\varphi, a + 1/n)$ on X). Consequently, the open sets $\{\widetilde{\varphi} < a + \epsilon\}$ for $\epsilon > 0$ are Runge in X thanks to [7]. Therefore $Y \setminus X$ is compact and holomorphically convex in Y, as desired.

Examples. 1. Let Y be a Stein space and K a holomorphically convex compact set. Then $Y \setminus K$ is a corona whose Stein completion is Y.

In order to see this we let $\psi : Y \to (0, \infty)$ be a continuous strictly plurisubharmonic exhaustion function. Let $r > \max_K \psi$. Since K is holomorphically convex, there is a sequence $\{f_n\}_n$ of holomorphic functions on Y such that $|f_n| \leq 1$ on K for all n and for any $y_0 \in Y \setminus K$ there is an index n_0 with $|f_{n_0}(y_0)| \geq \sqrt{1 + r}$. Select $\rho : [0, \infty) \to [0, \infty)$ continuous and convex such that $\{\rho = 0\} = [0, 1 + r]$ and ρ is strictly increasing on $[1 + r, \infty)$. Then we define $\varphi : Y \to [0, \infty)$ by setting

$$\varphi(y) := \sum \epsilon_n \rho(|f_n(y)|^2 + \psi(y)), \qquad y \in Y,$$

where $\{\epsilon_\nu\}_\nu$ is a sequence of positive numbers that decreases fast enough to zero. This φ has the required properties.

2. Let Γ be the unit circle in $\mathbb{C}$. Then $\mathbb{C} \setminus \Gamma$ is a corona which is not hyperconcave. (Note that Γ is not a pluripolar set!)

3. Let X be a compact complex space with only isolated singularities. Then X_{reg}, the set of regular points of X, is a hyperconcave corona.

4. Let X be a hyperconcave corona. Then, for any discrete set A in X, $X \setminus A$ is a hyperconcave corona.

Indeed, if $\varphi : X \to (-\infty, \infty)$ displays X as a hyperconcave corona and $A = \{x_i : i \in I\}$, where I is an at most countable set of indices, then it is easy to find a continuous, plurisubharmonic, proper function $\widetilde{\varphi} : X \setminus A \to (-\infty, \infty)$ of the form $\widetilde{\varphi} := \varphi + \sum \epsilon_i \psi_i$, where $\epsilon_i > 0$ are small enough and on suitable neighborhoods of x_i, ψ_i are psh with $\{\psi_i = -\infty\} = \{x_i\}$.

Remark 6. By [9], [2] any corona X with $\operatorname{prof} \mathcal{O}_X \geq 3$ admits a Stein completion. Moreover, this property may fail for two-dimensional (even non-singular) coronae. However, two-dimensional non-singular hyperconcave coronae do have Stein completions; see [6].

In this circle of ideas, the following question seems to be open.

Let X be a 0-normal complex space of pure dimension ≥ 3. If X is a corona, does it follow that X admits a Stein completion?

§3. The proofs

First let us collect some useful lemmata.

Lemma 1. *Let X be a complex space and $\alpha : \mathcal{F} \to \mathcal{G}$ be a sheaf morphism between coherent analytic sheaves on X. Let q be a positive integer and $\alpha^q : H^q(X, \mathcal{F}) \to H^q(X, \mathcal{G})$ the canonical induced morphism. If $H^q(X, \mathcal{G})$ is separated and* $\operatorname{Ker} \alpha^q$ *has finite dimension, then $H^q(X, \mathcal{F})$ is separated.*

Proof. This results immediately from [12, Lemma 2, p. 359]. □

Corollary 3. *Let X be a complex space and $\mathcal{F}$ a coherent analytic sheaf on X such that $H^q(X, \mathcal{F})$ is separated for some integer $q \geq 1$. Let $\mathcal{I}$ be a coherent ideal subsheaf of $\mathcal{O}_X$ such that* $\operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$ *is a finite set. Then $H^q(X, \mathcal{I}\mathcal{F})$ is separated.*

Proof. This results immediately from the above lemma and the long cohomology sequence associated to the short exact sequence

$$0 \to \mathcal{I}\mathcal{F} \to \mathcal{F} \to \mathcal{F}/\mathcal{I}\mathcal{F} \to 0. \qquad \square$$

Lemma 2. *Let X be a complex space which is the union of two open sets Y and U such that U and $Y \cap U$ are Stein and $Y \cap U$ is Runge in U. Let $\mathcal{F}$ be a coherent analytic sheaf on X. Then the restriction morphism $H^1(X, \mathcal{F}) \to H^1(Y, \mathcal{F})$ induces a bijection between the associated separated parts. Moreover, if $H^1(X, \mathcal{F})$ is separated, then the restriction morphism $H^0(X, \mathcal{F}) \to H^0(Y, \mathcal{F})$ has dense image.*

Proof. Consider the exact portion of the Mayer–Vietoris sequence with coefficients in $\mathcal{F}$ (which we omit for practical purposes) associated to $X = U \cup Y$:

$$H^0(X) \to H^0(Y) \oplus H^0(U) \to H^0(Y \cap U) \to H^1(X) \to H^1(Y) \to 0,$$

where we used Theorem B for vanishing of cohomology of coherent sheaves on Stein spaces. It is known that in the above sequence the canonical maps are continuous for the natural topologies.

Let $\mathcal{W} = \{W_m\}_{m=0,1,\ldots}$ be a Stein open covering of X with $W_0 = U$ and $W_m \subset Y$ for $m > 0$. Let $\mathcal{V} = \{V_m\}_m$, where $V_m := U_m \cap Y$ for $m \geq 0$. Clearly $\mathcal{V}$ is a Stein covering of Y.

Since $W_k \cap W_m = V_k \cap V_m$ for $k \neq m$ and $\mathcal{F}(U) \to \mathcal{F}(U \cap Y)$ has dense image, it follows that $C^i(\mathcal{W}, \mathcal{F}) = C^i(\mathcal{V}, \mathcal{F})$ for $i > 0$ and the canonical map $C^0(\mathcal{W}, \mathcal{F}) \to C^0(\mathcal{V}, \mathcal{F})$ has dense image. The lemma follows readily using the Čech definition of cohomology with alternate cycles.

Now, the "moreover" statement results in the following way. Because the restriction map $H^0(U, \mathcal{F}) \to H^0(Y \cap U, \mathcal{F})$ has dense image it follows that the natural map $u : H^0(Y, \mathcal{F}) \oplus H^0(U, \mathcal{F}) \to H^0(Y \cap U, \mathcal{F})$ has dense image, too. But $\operatorname{Im} u$ is the kernel of the continuous map $H^0(Y \cap U, \mathcal{F}) \to H^1(X, \mathcal{F})$ which is closed since $\{0\}$ is closed in $H^1(X, \mathcal{F})$. Therefore, the map u is surjective and the proof finishes in a standard way by diagram chasing. $\square$

Putting these together we obtain:

Proposition 2. *Let X be a complex space and $\mathcal{F}$ a coherent analytic sheaf on X. Assume that X admits an exhaustion by an increasing sequence $\{X_n\}_n$ of open sets with $X_{n+1} = X_n \cup U_{n+1}$, where U_{n+1} and $U_{n+1} \cap X_n$ are Stein open sets and $U_{n+1} \cap X_n$ is Runge in U_{n+1}. Then the following statements hold true:*

(a) *Each restriction $H^1(X, \mathcal{F}) \to H^1(X_n, \mathcal{F})$ induces a bijection between their separated spaces.*

(b) *If $H^1(X, \mathcal{F})$ is separated, then $H^1(X_n, \mathcal{F})$ is separated for all n and each restriction morphism $H^1(X, \mathcal{F}) \to H^1(X_n, \mathcal{F})$ is bijective. Moreover, each restriction $H^0(X, \mathcal{F}) \to H^0(X_n, \mathcal{F})$ has dense image.*

Proof. This follows readily by standard arguments from Theorem 2, Corollary 2 and Lemma 2. □

Corollary 4. *Let X be a corona defined by a function φ from X into (a,b). Let $\mathcal{F}$ be a coherent analytic sheaf on X such that $H^1(X,\mathcal{F})$ is separated. Then, for every $d \in (a,b)$, $H^1(X^d,\mathcal{F})$ is separated.*

Proof. First notice that the bumping lemma of Andreotti and Grauert [1] allows us to produce an exhaustion of X by an increasing sequence starting with X^d that satisfies the hypothesis of Proposition 2. Then the assertion follows readily from Proposition 2. □

Proof of Theorem 1. Let $\varphi : X \to (a,b)$ be the function displaying X as a corona.

Assume first that X admits a Stein completion, say Y. Then from Remark 5 it follows that $Y \setminus X$ is a holomorphically convex compact set in Y so that, by [3], the cohomology group $H^1(X,\mathcal{O})$ has separated canonical topology.

For the "if" part, we follow the recipe of [2] (see the proof of Prop. 3.2, pp. 243–245 there). Therefore it suffices to prove the following three assertions:

(1) $\mathcal{O}(X)$ separates the points of X.
(2) $\mathcal{O}(X)$ gives local coordinates.
(3) For every sequence $\{x_k\}_k$ in X such that $\{\varphi(x_k)\}_k$ tends to b, there is a holomorphic function f on X such that $\{|f(x_k)|\}_k$ tends to infinity.

In order to prove this, we make the following

Claim. *Let $\mathcal{I}$ be an ideal subsheaf of $\mathcal{O}$ such that the support of $\mathcal{O}/\mathcal{I}$, say S, is a discrete set and $c_0 := \inf_S \varphi > a$. (For instance one may take $\mathcal{I}$ to be defined by the sequence $\{x_k\}$.) Then the natural map $H^0(X,\mathcal{O}) \to H^0(X,\mathcal{O}/\mathcal{I})$ is surjective.*

In order to check this, let $\{D_n\}$ be an exhaustion of X by increasing open subsets obtained by the bumping method in [1] such that $D_0 = \{\varphi < c\}$ with $c \in (a,c_0)$. These $\{D_n\}$ are as in the hypothesis of Proposition 2. Moreover, it is easily seen that, for all n, $S \cap D_n$ is a finite set of points (possibly empty).

Now, from Proposition 2 it follows that, for any n, $H^1(D_n,\mathcal{O})$ is separated, the restrictions $H^1(D_{n+1},\mathcal{O}) \to H^1(D_n,\mathcal{O})$ are bijective and $H^0(D_{n+1},\mathcal{O}) \to H^0(D_n,\mathcal{O})$ have dense images. Therefore $H^1(X,\mathcal{O}) \to H^1(D_0,\mathcal{O})$ is bijective.

From Corollary 3, all cohomological vector spaces $H^1(D_n,\mathcal{I})$ are separated. Thanks to Lemma 2, the restrictions $H^1(D_{n+1},\mathcal{I}) \to H^1(D_n,\mathcal{I})$ are bijective and $H^0(D_{n+1},\mathcal{I}) \to H^0(D_n,\mathcal{I})$ have dense images. Thus $H^1(X,\mathcal{I}) \to H^1(D_0,\mathcal{I})$ is also bijective.

Consider now the following canonical commutative diagram:

$$\begin{tikzcd}
H^0(X,\mathcal{O}) \arrow[r] & H^0(X,\mathcal{O}/\mathcal{I}) \arrow[r] & H^1(X,\mathcal{I}) \arrow[r, "t"] \arrow[d, "u"'] & H^1(X,\mathcal{O}) \arrow[d, "v"] \\
 & & H^1(D_0,\mathcal{I}) \arrow[r, "w"] & H^1(D_0,\mathcal{O})
\end{tikzcd}$$

where the mappings u and v are bijective by the above discussion. Now w is obviously bijective. Therefore t is bijective too. Hence the restriction $H^0(X,\mathcal{O}) \to H^0(X,\mathcal{O}/\mathcal{I})$ is surjective, whence the claim.

Notice that the claim may be applied, thanks to Corollary 4, to any X^d, which is a corona. Finally, we may choose appropriate ideal subsheaves $\mathcal{I}$ of $\mathcal{O}_X$ to conclude the proof of the three assertions, whence the proof of Theorem 1. □

Proof of Proposition 1. First notice that by the example in §2 the space $X \setminus A$ is a corona. (As a matter of fact, it is a hyperconcave corona, but we shall not need this here.)

Now we deal with statement (a). Since the "if" part is obvious, let us assume that $H^1(X\setminus A,\mathcal{F})$ is separated. Since $X\setminus A$ is a corona, as in the proof of Theorem 1 and using Nakayama's lemma, we deduce that, given any point $a \in X$, $a \notin A$, the stalk $\mathcal{F}_a$ is generated by $H^0(X \setminus A,\mathcal{F})$, which completes the proof by [10].

Now consider (b). Again the "if" implication is obvious, so let us suppose that $H^1(X \setminus A, L)$ is separated. From (a) it follows that the sheaf of germs of sections of L extends to a coherent analytic sheaf $\widehat{\mathcal{L}}$ on X and this extension is reflexive; see [10, p. 372]. Hence $\widehat{\mathcal{L}}$ is locally free of rank one as A consists only of regular points and the ambient dimension is two. Finally, as X is Stein and $H^2(X,\mathbb{Z}) = 0$, the exponential sequence implies that $H^1(X,\mathcal{O}^\star) = 0$, completing the proof. □

Proof of Corollary 1. We consider the setting as in statement (b) of Proposition 1 with $A = \{x_0\}$. Clearly this may be applied for $X = \mathbb{B}^2$.

In order to construct a holomorphic line bundle L on $X \setminus \{x_0\}$ such that $H^1(X \setminus \{x_0\}, L)$ is non-separated, observe that if B is a ball around x_0 (in some local complex coordinates), then, using the Mayer–Vietoris sequence for $X = (X \setminus \{x_0\}) \cup B$, it follows that, for any coherent analytic sheaf $\mathcal{F}$ on X, the restriction map $\rho : H^1(X \setminus \{x_0\},\mathcal{F}) \to H^1(B \setminus \{x_0\},\mathcal{F})$ is bijective. Since topologically $B \setminus \{x_0\}$ is a real three-sphere, the exponential sequence shows that the natural map $\epsilon : H^1(B \setminus \{x_0\},\mathcal{O}) \to H^1(B \setminus \{x_0\},\mathcal{O}^\star)$ is also bijective. Pictorially we have the following commutative diagram:

$$\begin{tikzcd} H^1(X \setminus \{x_0\}, \mathcal{O}) \arrow[r, "u"] \arrow[d, "\rho"'] & H^1(X \setminus \{x_0\}, \mathcal{O}^\star) \arrow[d, "\alpha"] \\ H^1(B \setminus \{x_0\}, \mathcal{O}) \arrow[r, "\epsilon"] & H^1(B \setminus \{x_0\}, \mathcal{O}^\star) \end{tikzcd}$$

from which we easily obtain topologically trivial holomorphic line bundles L on $X \setminus \{x_0\}$ with $H^1(X \setminus \{x_0\}, L)$ non-separated; e.g. L given by $u(\rho^{-1}(\xi))$, for any non-trivial cohomology class ξ in $H^1(B \setminus \{x_0\}, \mathcal{O})$. $\square$

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