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On the Kernel and the Image of the Rigid Analytic Regulator in Positive Characteristic

by

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Abstract

We will formulate and prove a certain reciprocity law relating certain residues of the differential symbol $dlog^2$ from the K_2 of a Mumford curve to the rigid analytic regulator constructed by the author in a previous paper. We will use this result to deduce some consequences on the kernel and image of the rigid analytic regulator analogous to some old conjectures of Beilinson and Bloch on the complex analytic regulator. We also relate our construction to the symbol defined by Contou-Carrère and to Kato's residue homomorphism, and we show that Weil's reciprocity law directly implies the reciprocity law of Anderson and Romo.

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1. Introduction and announcement of results

Motivation 1.1. In the paper [17] the author introduced the concept of the rigid analytic regulator. This is a homomorphism from the motivic cohomology group $H^2_{\mathcal{M}}(X, \mathbb{Z}(2))$ of a Mumford curve X over a local field F into the F^*-valued harmonic cochains of the graph of components of the special fiber. It is defined through non-archimedean integration, hence it is elementary in nature and it is amenable to computation. In particular the author was able to compute its value on some explicit elements of the K_2 of Drinfeld modular curves constructed using modular units and relate it to special values of L-functions in the paper [18]. It is quite reasonable to consider this result as a function field analogue of Beilinson's classical theorem on the K_2 of elliptic modular curves as well as the rigid analytic regulator is a non-archimedean analogue of the complex analytic Beilinson–Bloch–Deligne regulator.

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An old conjecture of Bloch and Beilinson predicts that the complex analytic regulator is injective on the motivic cohomology group $H^2_{\mathcal{M}}(\mathfrak{X}, \mathbb{Q}(2))$ of a regular integral model $\mathfrak{X}$ of a smooth, projective curve X defined over a number field and the image of $H^2_{\mathcal{M}}(\mathfrak{X}, \mathbb{Z}(2))$ is a $\mathbb{Z}$-lattice of a conjecturally described rank. Hence it is desirable to understand the basic properties of the rigid analytic regulator such as its image and kernel, partially because any analogous result would be evidence towards the conjecture mentioned above. We offer the following result: let X be a Mumford curve which is the general fiber of a regular quasi-projective surface $\mathfrak{X}$ fibred over a smooth affine curve defined over a finite field. We show that the kernel of the rigid analytic regulator in $H^0(\mathfrak{X}, K_{2,\mathfrak{X}})$ is a p-divisible group and the image of $H^0(\mathfrak{X}, K_{2,\mathfrak{X}})$ is a finitely generated $\mathbb{Z}$-module whose closure in the p-completion of the target group is a $\mathbb{Z}_p$-module of the same rank which is at most as large as conjectured by Beilinson. In particular the kernel of this map is torsion if Parshin's conjecture holds. As a key ingredient of the proof we compute certain residues of the logarithmic differentials of elements of $H^2_{\mathcal{M}}(\mathfrak{X}, \mathbb{Z}(2))$ in terms of the rigid analytic regulator, generalising a formula of Osipov, proved for fields of zero characteristic in [16], to any characteristic. (A closely related result has been obtained by M. Asakura in a recent preprint [2] for certain two-dimensional local fields of zero characteristic using similar methods.) Of course the case of positive characteristic is the most intricate, due to the lack of logarithm. This formula can be considered as a relative of the explicit reciprocity law of Kato and Besser's theorem expressing the Coleman–de Shalit regulator in terms of the syntomic regulator (see [19] and [4]), although it is simpler to prove. Then our main theorems follow from the Bloch–Gabber–Kato theorem, Deligne's purity theorem and the degeneration of the slope spectral sequence. As an important intermediate step we also relate the rigid analytic regulator to Kato's residue homomorphism for higher local fields and the Contou-Carrère symbol. The symbols mentioned above are essentially just different formalizations of the same phenomenon which was discovered independently at least three times. As an easy application of our results we show that the latter is bilinear and satisfies the Steinberg relation. At the same stroke we show that Weil's reciprocity law directly implies the reciprocity law of Anderson and Romo.

Notation 1.2. By slightly extending the usual terminology we will call a scheme C defined over a field a *curve* if it is reduced, locally of finite type and of dimension one. A curve C is said to have *normal crossings* if every singular point of C is an ordinary double point in the usual sense. We say that a curve C over a field $\mathbf{f}$ is *totally degenerate* if it has normal crossings, every ordinary double point is defined over $\mathbf{f}$ and its irreducible components are projective lines over $\mathbf{f}$. For any curve C with normal crossings let $\widetilde{C}$ denote its normalization, and let $\widetilde{S}(C)$ denote the pre-image of the set $S(C)$ of singular points of C. We denote by $\Gamma(C)$ the oriented graph whose set of vertices is the set of irreducible components of $\widetilde{C}$, and its set of edges is the set $\widetilde{S}(C)$ such that the initial vertex of any edge $x \in \widetilde{S}(C)$ is the irreducible component of $\widetilde{C}$ which contains x and the terminal vertex of x is the irreducible component which contains the other element of $\widetilde{S}(C)$ which maps to the same singular point with respect to the normalization map as x. The normalization map identifies the irreducible components of C and $\widetilde{C}$ which we will use without further notice.

Definition 1.3. For any (oriented) graph G let $\mathcal{V}(G)$ and $\mathcal{E}(G)$ denote its set of vertices and edges, respectively. Let G be a locally finite oriented graph which is equipped with an involution $\bar{\cdot} : \mathcal{E}(G) \to \mathcal{E}(G)$ such that for each edge $e \in \mathcal{E}(G)$ the initial and terminal vertices of the edge $\overline{e} \in \mathcal{E}(G)$ are the terminal and initial vertices of e, respectively. The edge $\overline{e}$ is called the edge e *with reversed orientation.* Let R be a commutative group. A function $\phi : \mathcal{E}(G) \to R$ is called a *harmonic R-valued cochain* if it satisfies the following conditions:

(i) We have:
$$\phi(e) + \phi(\overline{e}) = 0 \qquad (\forall e \in \mathcal{E}(G)).$$

(ii) If for an edge e we introduce the notation $o(e)$ and $t(e)$ for its initial and terminal vertex respectively,
$$\sum_{\substack{e \in \mathcal{E}(G) \\ o(e)=v}} \phi(e) = 0 \qquad (\forall v \in \mathcal{V}(G)).$$

We denote by $\mathcal{H}(G, R)$ the group of R-valued harmonic cochains on G.

Notation 1.4. Let $\mathbf{k}$ be a perfect field and let B be a smooth irreducible projective curve over $\mathbf{k}$. Let ∞ be a closed point of B and let F denote the completion of the function field of B at ∞. Let $\mathcal{O}$ denote the valuation ring of F and let $\pi : \mathfrak{X} \to B$ be a regular irreducible projective surface fibred over B such that the fiber $\mathfrak{X}_\infty$ of $\mathfrak{X}$ over ∞ is totally degenerate. Then the base change X of $\mathfrak{X}$ to F is a Mumford curve over F which has a regular, semistable model over the spectrum of $\mathcal{O}$ whose special fiber is $\mathfrak{X}_\infty$. The rigid analytic regulator introduced in [17] is a homomorphism:
$$\{\cdot\} : H^2_{\mathcal{M}}(X, \mathbb{Z}(2)) \to \mathcal{H}(\Gamma(\mathfrak{X}_\infty), F^*).$$

Let $\mathfrak{U} \subset \mathfrak{X}$ be an open subvariety such that its complement is a normal crossings divisor D which is the preimage of a finite set of closed points of B containing ∞. Let the symbol $\{\cdot\}$ also denote the composition of the functorial map $H^0(\mathfrak{U}, K_{2,\mathfrak{U}}) \to H^2_{\mathcal{M}}(X, \mathbb{Z}(2))$ and the rigid analytic regulator.

The main result of this paper is the following:

Theorem 1.5. *Assume that* $\mathbf{k}$ *has characteristic* p. *Then the map:*

$$\{\cdot\} : H^0(\mathfrak{U}, K_{2,\mathfrak{U}})/p^n H^0(\mathfrak{U}, K_{2,\mathfrak{U}}) \to \mathcal{H}(\Gamma(\mathfrak{X}_\infty), F^*/(F^*)^{p^n})$$

induced by the regulator $\{\cdot\}$ *is injective for every natural number* n.

Notation 1.6. We call a $\mathbb{Z}$-submodule Λ of a Hausdorff topological group G which is the direct sum of a discrete group and a pro-p group *p-saturated* if it is finitely generated and the map $\Lambda \otimes \mathbb{Z}_p \to \widehat{G}$ is an injection, where $\widehat{G}$ is the p-completion of G. Note that every discrete, finitely generated $\mathbb{Z}$-submodule is p-saturated. Assume now that $\mathbf{k}$ is a finite field of characteristic p. The result above, Deligne's purity theorem and the degeneration of the slope spectral sequence imply the following:

Corollary 1.7. *The image of the regulator* $\{\cdot\} : H^0(\mathfrak{U}, K_{2,\mathfrak{U}}) \to \mathcal{H}(\Gamma(\mathfrak{X}_\infty), F^*)$ *is* p*-saturated and its rank is at most as large as the rank of the group* $\mathcal{H}(\Gamma(D), \mathbb{Z})$. *This lattice is discrete if* $D = \mathfrak{X}_\infty$. *The kernel of this regulator is a* p*-divisible group. In particular it is torsion if Parshin's conjecture holds for* $\mathfrak{X}$, *and it is finite if the Bass conjecture holds for* $\mathfrak{X}$.

Further results 1.8. Let A be a local Artinian ring with residue field $\mathbf{k}$ which is again allowed to be an arbitrary perfect field. In [7] a map:

$$\langle \cdot, \cdot \rangle : A((t))^* \times A((t))^* \to A^*$$

was defined, called the Contou-Carrère symbol in [1], where t is a variable. The Contou-Carrère symbol is equal to the tame symbol if A is a field. In [1] G. Anderson and F. P. Romo proved that the Contou-Carrère symbol is bilinear and proved a reciprocity law for it. Their proof is the generalization of the proof of the residue theorem by Tate and the Weil reciprocity law by Arabello, De Concini and Kac. They work directly over Artinian rings, so they have to develop an elaborate theory generalizing all concepts appearing in the proofs quoted above for Artinian rings. In this article we will present a different proof of the bilinearity of this symbol and of the reciprocity law. It is based on the observation that if A is the quotient of a discrete valuation ring, then the Contou-Carrère symbol $\langle f, g \rangle$ of any f and g in $A((t))^*$ is just the reduction of Kato's residue (see [12] and [13]) of some lifts of f and g. The existence and the basic properties of the latter follow at once from the properties of the rigid analytic regulator. Hence the aforementioned results follow immediately from some well-known facts such as the deformation theory of smooth projective curves is unobstructed and Weil's reciprocity law. One may even give a new proof of the Weil reciprocity law using the observed continuity of the tame symbol by degenerating the curve to a stable curve with rational components in its special fiber. Since the Anderson–Romo reciprocity law implies them, the reciprocity laws of Tate and Witt also follow from Weil's law.

Contents 1.9. The goal of the next chapter is to review the construction of the rigid analytic regulator and to list its basic properties for rational subdomains of the projective line without proofs, mainly for the sake of the reader. The relationship with Kato's residue homomorphism is established in the third chapter. The fourth chapter is concerned with the Contou-Carrère symbol and the Anderson–Romo reciprocity law. We review the general construction of the residue homomorphism for Kähler differentials in the fifth chapter; we relate that homomorphism to the Contou-Carrère symbol in the case of local Artinian rings, and to Kato's residue homomorphism in the case of local fields of dimension two. This reciprocity law is used to deduce Theorem 1.5 and Corollary 1.7 in chapter six.

2. Review of the rigid analytic regulator

Notation 2.1. In this chapter all claims are stated without proof. The interested reader is kindly asked to consult [17]. Let F be a local field and let $\mathbb{C}$ denote the completion of the algebraic closure of F with respect to the unique extension of the absolute value on F. Recall that $\mathbb{C}$ is an algebraically closed field complete with respect to an ultrametric absolute value which will be denoted by $|\cdot|$. Let $|\mathbb{C}|$ denote the set of values of the latter. Let $\mathbb{P}^1$ denote the projective line over $\mathbb{C}$. For any $x \in \mathbb{P}^1$ and any two rational non-zero functions $f, g \in \mathbb{C}((t))$ on the projective line let $\{f, g\}_x$ denote the tame symbol of the pair (f, g) at x. Recall that a subset U of $\mathbb{P}^1$ is a *connected rational subdomain* if it is non-empty and it is the complement of the union of finitely many pairwise disjoint open discs. Let ∂U denote the set of those complementary open discs. The elements of ∂U are called the *boundary components* of U, by slight abuse of language. Let $\mathcal{O}(U)$, $\mathcal{R}(U)$ denote the algebra of holomorphic functions on U and the subalgebra of restrictions of rational functions, respectively. Let $\mathcal{O}^*(U)$, $\mathcal{R}^*(U)$ denote the groups of invertible elements of these algebras. The group $\mathcal{R}^*(U)$ consists of rational functions which do not have poles or zeros lying in U. For each $f \in \mathcal{O}(U)$ let $\|f\|$ denote $\sup_{z\in U} |f(z)|$. This is a finite number, and makes $\mathcal{O}(U)$ a Banach algebra over $\mathbb{C}$. We say that the sequence $f_n \in \mathcal{O}(U)$ converges to $f \in \mathcal{O}(U)$, denoted by $f_n \to f$, if f_n converges to f with respect to the topology of this Banach algebra, i.e. $\lim_{n\to\infty} \|f - f_n\| = 0$. For every real number $0 < \epsilon < 1$ we define the sets $\mathcal{O}_\epsilon(U) = \{f \in \mathcal{O}(U) \mid \|1 - f\| \leq \epsilon\}$ and $U_\epsilon = \{z \in \mathbb{C} \mid |1 - z| \leq \epsilon\}$.

Theorem 2.2. *There is a unique map $\{\cdot,\cdot\}_D : \mathcal{O}^*(U) \times \mathcal{O}^*(U) \to \mathbb{C}^*$ for every $D \in \partial U$, called the rigid analytic regulator, with the following properties:*

(i) *For any two $f, g \in \mathcal{R}^*(U)$ their regulator is:*

$$\{f,g\}_D = \prod_{x \in D} \{f,g\}_x,$$

(ii) *the regulator $\{\cdot,\cdot\}_D$ is bilinear in both variables,*

(iii) *the regulator $\{\cdot,\cdot\}_D$ is alternating: $\{f,g\}_D \cdot \{g,f\}_D = 1$,*

(iv) *if f, $1-f \in \mathcal{O}(U)^*$, then $\{f,1-f\}_D$ is 1,*

(v) *for each $f \in \mathcal{O}_\epsilon(U)$ and $g \in \mathcal{O}^*(U)$ we have $\{f,g\}_D \in U_\epsilon$.* $\square$

Remark 2.3. It is an immediate consequence of property (v) that the rigid analytic regulator is continuous with respect to the supremum norm topology. Explicitly, if f and g are elements of $\mathcal{O}^*(U)$, $D \in \partial U$ is a boundary component, and $f_n \in \mathcal{O}^*(U)$, $g_n \in \mathcal{O}^*(U)$ are sequences such that $f_n \to f$ and $g_n \to g$, then the limit

$$\lim_{n\to\infty} \{f_n, g_n\}_D$$

exists, and it is equal to $\{f,g\}_D$.

Let $\mathcal{M}(U)$ denote the field of meromorphic functions of U and let $\mathcal{M}^*(U)$ denote the multiplicative group of non-zero elements of $\mathcal{M}(U)$.

Theorem 2.4. *There is a unique set of homomorphisms $\deg_D : \mathcal{M}^*(U) \to \mathbb{Z}$ where U is any connected rational subdomain and $D \in \partial U$ is a boundary component with the following properties:*

(i) *the homomorphism $\deg_D$ is zero on $\mathcal{O}_1(U)$,*

(ii) *for every $f \in \mathcal{R}^*(U)$ the integer $\deg_D(f)$ is the number of zeros z of f with $z \in D$ counted with multiplicities minus the number of poles z of f with $z \in D$ counted with multiplicities,*

(iii) *for every $f \in \mathcal{M}^*(U)$ we have $\deg_D(f|_Y) = \deg_D(f)$ where $Y \subseteq U$ is any connected rational subdomain satisfying the property $D \in \partial Y$.* $\square$

Definition 2.5. If U is still a connected rational subdomain of $\mathbb{P}^1$, and f, g are two meromorphic functions on U, then for all $x \in U$ the functions f and g have a power series expansion around x, so in particular their tame symbol $\{f,g\}_x$ at x is defined. The tame symbol extends to a homomorphism $\{\cdot,\cdot\}_x : K_2(\mathcal{M}(U)) \to \mathbb{C}^*$. We define the group $K_2(U)$ as the kernel of the direct sum of the tame symbols:

$$\bigoplus_{x \in U} \{\cdot,\cdot\}_x : K_2(\mathcal{M}(U)) \to \bigoplus_{x \in U} \mathbb{C}^*.$$

Let $k = \sum_i f_i \otimes g_i \in K_2(U)$, where $f_i, g_i \in \mathcal{M}(U)$, and let $D \in \partial U$. Let moreover Y be a connected rational subdomain of U such that $f_i, g_i \in \mathcal{O}^*(Y)$ for all i and $\partial U \subseteq \partial Y$. Define the *rigid analytical regulator* $\{k\}_D$ by the formula:

$$\{k\}_D = \prod_i \{f_i|_Y, g_i|_Y\}_D.$$

Theorem 2.6. (i) *For each $k \in K_2(U)$ the rigid analytical regulator $\{k\}_D$ is well-defined, and it is a homomorphism $\{\cdot\}_D : K_2(U) \to \mathbb{C}^*$,*

(ii) *for any two functions $f, g \in \mathcal{O}^*(U)$ we have $\{f \otimes g\}_D = \{f, g\}_D$,*

(iii) *for every $k \in K_2(U)$ the product of all regulators on the boundary components of U is equal to 1:*

$$\prod_{D \in \partial U} \{k\}_D = 1. \qquad \square$$

Definition 2.7. For every connected rational subdomain $U \subset \mathbb{P}^1$ let $\mathbb{Z}\partial U$ denote the free abelian group with the elements of ∂U as free generators. Let $H_1(U)$ denote the quotient of $\mathbb{Z}\partial U$ by the $\mathbb{Z}$-module generated by $\sum_{D \in \partial U} D$. For every $D \in \partial U$ we let D denote the class of D in $H_1(U)$ as well. Let $\mathcal{A}b$ denote the category of abelian groups. Let $\mathcal{C}rs$ denote the category whose objects are connected rational subdomains of $\mathbb{P}^1$ and whose morphisms are holomorphic maps between them. Let $D(w, r)$ denote the open disc of radius r and center w, that is,

$$D(w, r) = \{z \in \mathbb{C} \mid |z - w| < r\}$$

where $0 < r \in |\mathbb{C}|$. Finally for every pair $a \leq b$ of numbers in $|\mathbb{C}|$ let $A(a, b)$ denote the closed annulus $\mathbb{P}^1 - D(0, a) - D(\infty, 1/b)$. Of course it is a connected rational subdomain.

Theorem 2.8. *There is a unique functor $H_1 : \mathcal{C}rs \to \mathcal{A}b$ with the following properties:*

(i) *for every connected rational subdomain $U \subset \mathbb{P}^1$, $H_1(U)$ is the group defined in 2.7,*

(ii) *for every map $U \to Y$ which is the restriction of a projective linear transformation f and every boundary component $D \in \partial U$ we have:*

$$H_1(f)(D) = f(D) \in H_1(Y),$$

(iii) *for every holomorphic map $f : U \to A(a, b)$ and boundary component $D \in \partial U$ we have:*

$$H_1(f)(D) = \deg_D(f) D(0, a) \in H_1(A(a, b)). \qquad \square$$

Definition 2.9. Let $U \subset \mathbb{P}^1$ be a connected rational subdomain. For every class $c \in H_1(U)$ and element $k \in K_2(U)$ we define the regulator $\{k\}_c$ as

$$\{k\}_c = \prod_{D \in \partial U} \{k\}_D^{c(D)},$$

where $\sum_{D \in \partial U} c(D)D$ is a lift of c in $\mathbb{Z}\partial U$. By claim (iii) of Theorem 2.6 this regulator is well-defined. For every holomorphic map $h : U \to Y$ between two connected rational subdomains let $h^* : K_2(\mathcal{M}(Y)) \to K_2(\mathcal{M}(U))$ be the pull-back homomorphism induced by h. By restriction it induces a homomorphism $K_2(Y) \to K_2(U)$.

Theorem 2.10. *For any $k \in K_2(Y)$ and $c \in H_1(U)$ we have:*

$$\{h^*(k)\}_c = \prod_{E \in \partial Y} \{k\}_{H_1(h)(c)}. \qquad \square$$

Definition 2.11. Let U be now a connected rational subdomain of $\mathbb{P}^1$ defined over F. This means that

$$U = \{z \in \mathbb{P}^1 \mid |f_i(z)| \leq 1 \ (\forall i = 1, \dots, n)\}$$

as a set for some natural number n and rational functions $f_1, \dots, f_n \in F(t)$. Let $\mathcal{O}_F(U)$, $\mathcal{R}_F(U)$, $\mathcal{O}_F^*(U)$, $\mathcal{R}_F^*(U)$ and $\mathcal{M}_F(U)$ denote the algebra of holomorphic functions, the subalgebra of restrictions of F-rational functions, the groups of invertible elements of these algebras and the field of meromorphic functions on the rigid analytic space U, respectively. Let U' denote the underlying rational subdomain over $\mathbb{C}$. Let $K_2(U)$ denote the largest subgroup of $K_2(\mathcal{M}_F(U))$ which maps into $K_2(U')$ under the restriction homomorphism $K_2(\mathcal{M}_F(U)) \to K_2(\mathcal{M}(U'))$. An *$F$-rational boundary component* of U is a set $D \in \partial U$ such that D is the image of the open disc of radius 1 and center 0 under an F-linear projective linear transformation of $\mathbb{P}^1$.

Proposition 2.12. *Let D be an F-rational boundary component of U, and let $k \in K_2(U)$. Then $\{k\}_D \in F^*$.* $\square$

3. Kato's residue homomorphism

Definition 3.1. In this chapter we will continue to use the notation of the second chapter. In this chapter we will also assume that the absolute value on F is induced by a discrete valuation. Let D denote the open disc $D(0,1)$. Let M be the field of fractions of $\mathcal{O}[[z]]$ and let $\widehat{M}$ denote the completion of M with respect to the discrete valuation of M defined by the prime ideal $\mathfrak{m}\mathcal{O}[[z]]$ of height one, where

$\mathfrak{m}$ is the unique maximal ideal of $\mathcal{O}$. The field $\widehat{M}$ is just the field of bidirectional formal Laurent series of the form $\sum_{n\in\mathbb{Z}} a_n z^n$ over F such that $|a_n|$ is bounded above and $\lim_{n\to-\infty} |a_n| = 0$. It is a local field equipped with the absolute value

$$\Big\|\sum_{n\in\mathbb{Z}} a_n z^n\Big\|_s = \max_{n\in\mathbb{Z}} |a_n|.$$

Every element of the formal Laurent series ring $\mathcal{O}[[z]]$ defines a holomorphic function on the rigid analytic space D, hence every element M gives a meromorphic function on D. By Weierstrass' preparation theorem each element of $\mathcal{O}[[z]]$ is the product of a polynomial and a unit of this ring, hence it has only a finite number of zeros in D. Therefore the limit

$$\{f,g\}_D = \lim_{\substack{\epsilon\to 0\\ 0<\epsilon<1\\ \epsilon\in|\mathbb{C}|}} \{f,g\}_{D(0,1-\epsilon)} = \prod_{x\in D} \{f,g\}_x$$

becomes stationary for any pair of elements $f, g \in M^*$ and defines an F^*-valued bilinear map satisfying the Steinberg relation by Theorem 2.2 and Proposition 2.12. Therefore it induces a homomorphism $\{\cdot\}_D : K_2(M) \to F^*$. Note that the rigid analytic regulator denoted by the same symbol has the same value as this pairing for those pairs of functions for which both of them are defined by Theorem 2.6. Hence our notation will not cause confusion.

Proposition 3.2. *There is a unique homomorphism $\{\cdot\}_D : K_2(\widehat{M}) \to F^*$, called Kato's residue homomorphism, such that*

(i) *the composition of the natural homomorphism $K_2(M) \to K_2(\widehat{M})$ and Kato's residue homomorphism is the homomorphism $\{\cdot\}_D$ defined above,*
(ii) *for each $f \in \widehat{M}^*$ and $g \in \mathcal{O}[[z]]$ with $\|1-g\|_s < \epsilon < 1$ we have $\{f,g\}_D \in U_\epsilon$.*

Proof. Clearly Kato's residue homomorphism is unique if it exists. We claim that for each $f, g \in M^*$ with $\|1-g\|_s < \epsilon < 1$ we have $\{f,g\}_D \in U_\epsilon$. We first show that this claim implies the proposition. In this case we may define $\{f,g\}_D$ for any two elements f and g of $\widehat{M}^*$ as the limit

$$\lim_{n\to\infty} \{f_n, g_n\}_D,$$

where $f_n \in M^*$, $g_n \in M^*$ are sequences such that $f_n \to f$ and $g_n \to g$. This limit exists because the sequence above is Cauchy by the claim above. Its value is non-zero as

$$1 = \lim_{n\to\infty} \{f_n, g_n\}_D \cdot \{g_n, f_n\}_D = \lim_{n\to\infty} \{f_n, g_n\}_D \cdot \lim_{n\to\infty} \{g_n, f_n\}_D.$$

It is also independent of the sequences chosen as any two sequences may be combed together to show that they give the same limit. The map $\{\cdot,\cdot\}_D$ defined this way is automatically a bilinear map satisfying claim (ii) and the Steinberg relation, hence the existence follows. For every $1 > \delta \in |\mathbb{C}|$ sufficiently close to 1 the holomorphic functions f and g are elements of $\mathcal{O}^*(A_\delta)$, where A_δ is the annulus $\{z \in \mathbb{C} \mid |z| = \delta\}$. Write $1-g = \sum_{n\in\mathbb{Z}} a_n z^n$ as an element of $\widehat{M}$. This power series will converge for all $z \in A_\delta$ when δ sufficiently close to 1, hence there is a number $0 < \rho < 1$ and a negative integer N such that $|a_n| \leq \epsilon\rho^{-n}$ for all $n < N$. For all $\delta \in |\mathbb{C}|$ such that $\rho < \delta < 1$ we have the following estimate for the supremum norm $\|1-g\|$ on the annulus A_δ:

$$\|1-g\| \leq \max\Big(\Big\|\sum_{n<N} a_n z^n\Big\|, \Big\|\sum_{n\geq N} a_n z^n\Big\|\Big) \leq \max(\epsilon, \epsilon\delta^N) = \epsilon\delta^N.$$

Therefore the limit inferior of the supremum norms $\|1-g\|$ on the annuli A_δ is at most $\|1-g\|_s$, so the claim is now clear by (v) of Theorem 2.2. □

Let $t \in \mathcal{O}[[z]]$ be a *uniformizer*, which here means that t is of the form $cz+z^2h$, where $c \in \mathcal{O}^*$ and $h \in \mathcal{O}[[z]]$. Then there is a unique $\mathcal{O}$-algebra automorphism $\phi : \mathcal{O}[[z]] \to \mathcal{O}[[z]]$ such that $\phi(z) = t$ and $\|\phi(h)\|_s = \|h\|_s$ for every $h \in \mathcal{O}[[z]]$. The automorphism ϕ extends uniquely to a norm-preserving automorphism $\phi : \widehat{M} \to \widehat{M}$. Let $\phi_* : K_2(\widehat{M}) \to K_2(\widehat{M})$ denote the induced automorphism.

Proposition 3.3. *The automorphism ϕ_* leaves Kato's residue homomorphism invariant.*

Proof. By continuity we only have to show that the equation $\{\phi_*(k)\}_D = \{k\}_D$ holds for any $k \in K_2(M)$. Note that the power series t as a holomorphic function $t : D \to D$ leaves the annulus A_δ invariant for any positive rational $\delta < 1$ where we continue to use the notation of the proof above. In fact for any $z \in A_\delta$ we have

$$|t(z) - cz| \leq |z^2| = \delta^2 < \delta = |cz|.$$

The inequality above also implies that $\deg_{D(0,\delta)}(t) = 1$ by claim (iii) of Theorem 2.8, hence the claim follows at once from Theorem 2.10. □

Lemma 3.4. *For every pair of positive integers n and m the following identities hold:*

(i) $\{1-at^{-n}, 1-bt^{-m}\}_D = 1$, *if* $|a| < 1$ *and* $|b| < 1$,

(ii) $\{1-at^{n}, 1-bt^{m}\}_D = 1$, *if* $|a| \leq 1$ *and* $|b| \leq 1$,

(iii) $\{1-at^{n}, 1-bt^{-m}\}_D = (1-a^{m/(n,m)}b^{n/(n,m)})^{(n,m)}$, *if* $|a| \leq 1$ *and* $|b| < 1$,

(iv) $\{1-at^{-n}, 1-bt^{m}\}_D = (1-a^{m/(n,m)}b^{n/(n,m)})^{-(n,m)}$, *if* $|a| < 1$ *and* $|b| \leq 1$.

Proof. Note that the equations (iii) and (iv) are equivalent, because we can get the latter from the former by reversing the roles of the symbols a and b, and using the antisymmetry of the rigid regulator. Hence we only have to show (i), (ii) and (iii). First assume that both m and n are equal to 1. We may assume that both a and b are non-zero. In case (i) the two linear expressions in t^{-1} each have one zero, which are a and b, respectively. They also have a pole, which is the point zero. These points all lie in D, so Weil's reciprocity law implies:

$$\{1-at^{-1},1-bt^{-1}\}_D=\prod_{x\notin D}\{1-at^{-1},1-bt^{-1}\}_x^{-1}=1.$$

In case (ii) the zeros of the two linear polynomials are $1/a$ and $1/b$, respectively, which do not lie in D. Hence the equation holds in this case. In case (iii) the expression $1-at$ does not have a zero or a pole in D, but $1-bt^{-1}$ does, hence:

$$\{1-at,1-bt^{-1}\}_D=\{1-at,1-bt^{-1}\}_0\cdot\{1-at,1-bt^{-1}\}_b=1-ab.$$

Now assume that n and m are relatively prime and none of them is divisible by the characteristic of $\mathbb{C}$. Let ϵ_1, ϵ_2 be a primitive n-th and a primitive m-th root of unity, respectively. Let α and β be an n-th root of a and an m-th root of b, respectively. Since $|\epsilon_1^i\alpha|^n=|a|$ and $|\epsilon_2^j\beta|^m=|b|$, the conditions of claim (iii) hold for these values, so we get:

$$\{1-at^n,1-bt^{-m}\}_D=\prod_{i=1}^{n}\prod_{j=1}^{m}\{1-\epsilon_1^i\alpha t,1-\epsilon_2^j\beta t^{-1}\}_D=\prod_{i=1}^{n}\prod_{j=1}^{m}(1-\epsilon_1^i\alpha\epsilon_2^j\beta)=1-ab.$$

The other two claims follow similarly. Now assume that n and m are still relatively prime, but one of them, for example n, is divisible by p, the characteristic of $\mathbb{C}$. In this case $1-at^{\pm n}=(1-\alpha t^{\pm n/p})^p$, where $\alpha^p=a$, so the claims follow from what we have proved already, by induction on the exponent of p in the primary factorization of n. In the general case we have:

$$\{1-at^{\pm n},1-bt^{\pm m}\}_D=\{1-at^{\pm n/(n,m)},1-bt^{\pm m/(n,m)}\}_D^{(n,m)},$$

which follows from applying Theorem 2.10 to the map $t\mapsto t^{(n,m)}$. □

Lemma 3.5. *For every pair of integers n and m the following identities hold:*

(i) $\{at^n,bt^m\}_D=(-1)^{nm}a^mb^{-n}$, *if both a and b are non-zero,*

(ii) $\{at^n,1-bt^m\}_D=1$, *if $a\neq0$, $|b|\leq1$ and m is positive, or $a\neq0$, $|b|<1$ and m is negative.*

Proof. In case (i) both expressions have at most one singularity on the disc D which is the point zero. Therefore

$$\{at^n, bt^m\}_D = \{at^n, bt^m\}_0 = (-1)^{nm} a^m b^{-n}.$$

In case (ii) we may immediately reduce to the case $b \neq 0$ and $|m| = 1$ using the same arguments as the proof above. If $m = 1$ then the linear expression $1 - bt$ has no singularity on the disc D, hence

$$\{at^n, 1 - bt\}_D = \{at^n, 1 - bt\}_0 = 1.$$

In the other case the expression $1 - bt^{-1}$ has two singularities on the disc D: a pole at 0 and a zero at b. Therefore

$$\{at^n, 1 - bt^{-1}\}_D = \{at^n, 1 - bt^{-1}\}_0 \{at^n, 1 - bt^{-1}\}_b = 1. \qquad \square$$

Definition 3.6. Fix a uniformizer $\pi \in F$ and let $\mathcal{R}$ denote the valuation ring of $\widehat{M}$. For every $u \in \mathcal{R}$ let $\overline{u} \in \mathcal{R}/\pi\mathcal{R}$ denote the reduction of u modulo the proper maximal ideal of $\mathcal{R}$. Note that $\mathcal{R}/\pi\mathcal{R}$ is a local field since it is canonically isomorphic to $\mathbf{f}((\overline{z}))$ where $\mathbf{f}$ is the residue field of F. Let ν denote the valuation of $\mathcal{R}/\pi\mathcal{R}$ normalised such that $\nu(\overline{z}) = 1$. Every element $u \in \widehat{M}^*$ can be written uniquely in the form $\pi^n v$ for some $n \in \mathbb{Z}$ and $v \in \mathcal{R}^*$. We define $\deg(u)$ as $\nu(\overline{v})$.

Lemma 3.7. *We have $\{c, u\}_D = c^{\deg(u)}$ for every $c \in F^*$ and $u \in \widehat{M}^*$.*

Proof. By the continuity and the bilinearity of Kato's residue homomorphism we only have to show that the equation in the claim above is true when $u = dv$ where $d \in F^*$ and $v \in \mathcal{O}[[z]]$. Because $\{c, d\} = 1$ by definition we may assume that $u = v$. In this case the number of zeros of the convergent power series u on D counted with multiplicities is exactly $\deg(u)$ so the claim holds. $\square$

4. The Contou-Carrère symbol and the Anderson–Romo reciprocity law

Notation 4.1. Let $\mathbf{k}$ be a perfect field and let $\mathcal{C}$ denote the category of local Artinian rings with residue field $\mathbf{k}$. By slight abuse of notation we will let $\mathcal{C}$ denote the class of objects of this category as well.

Lemma 4.2. *Assume that $\mathbf{k}$ has characteristic zero. Then for every object A in $\mathcal{C}$ there is a homomorphism $i : \mathbf{k} \to A$ such that the composition of the reduction map $A \to \mathbf{k}$ modulo the maximal ideal of A and i is the identity map.*

Proof. This is a special case of Proposition 6 of [21] on pages 33–34. $\square$

Lemma 4.3. *Assume that* $\mathbf{k}$ *has positive characteristic* p. *Then for every object* A *in* $\mathcal{C}$ *there is a homomorphism* $i : \mathbb{W}(\mathbf{k}) \to A$ *of local rings, where* $\mathbb{W}(\mathbf{k})$ *is the ring of Witt vectors of* $\mathbf{k}$ *of infinite length, such that the map induced by* i *on the residue fields is the identity.*

Proof. By Theorem 8 of [21] on page 43 the ring $\mathbb{W}(\mathbf{k})$ is strict hence the claim follows from Proposition 10 of [21] on pages 38–39. □

Proposition 4.4. *Assume that* $\mathbf{k}$ *is algebraically closed and let* $\mathcal{D}$ *be a subclass of the class of objects of* $\mathcal{C}$ *such that the following conditions hold:*

(i) *if* $A \in \mathcal{C}$ *is the quotient of a discrete valuation ring with residue field* $\mathbf{k}$, *then* $A \in \mathcal{D}$,

(ii) *if* $A \in \mathcal{C}$ *is the quotient of an element of* $\mathcal{D}$, *then* $A \in \mathcal{D}$,

(iii) *if* $A \in \mathcal{C}$ *and for every* $x \in A^*$ *different from* 1 *there is a* $B \in \mathcal{D}$ *and a homomorphism* $\phi : A \to B$ *such that* $\phi(x) \neq 1$, *then* $A \in \mathcal{D}$.

In this case $\mathcal{D}$ *is the whole class of objects of* $\mathcal{C}$.

Proof. For every pair of natural numbers n and m let $A_{n,m}$ denote the local Artinian algebra:

$$A_{n,m} = \mathbf{k}[[x_1, \ldots, x_n]]/\Big(\Big\{\prod_{j=1}^{n} x_j^{J(j)} \;\Big|\; J : \{1, 2, \ldots, n\} \to \mathbb{N}, \sum_{j=1}^{n} J(j) = m+1\Big\}\Big).$$

First assume that $\mathbf{k}$ has characteristic zero. In this case for every $A \in \mathcal{C}$ there is a surjective local homomorphism $A_{n,m} \to A$ for some n and m. On the other hand $A_{1,m} \in \mathcal{D}$ by condition (i). Therefore it will be enough to show that for every $x \in A^*_{n,m}$ with $x \neq 1$ there is a homomorphism $\phi : A_{n,m} \to A_{1,m}$ such that $\phi(x) \neq 1$ by condition (iii). There is a positive integer $k \leq m$ such that $x \equiv 1 \bmod \mathfrak{m}^{k-1}$, but $x \not\equiv 1 \bmod \mathfrak{m}^k$. Every local homomorphism $\phi : A_{n,m} \to A_{1,m}$ induces a $\mathbf{k}$-linear homomorphism $T^l_\phi : \mathfrak{m}^l/\mathfrak{m}^{l+1} \to \mathfrak{n}^l/\mathfrak{n}^{l+1}$ for every positive $l \leq m$, where $\mathfrak{n}$ is the maximal proper ideal of $A_{1,m}$. For every vector space V over $\mathbf{k}$ let $Sym^l(V)$ denote the l-th symmetric power of V and for every $\mathbf{k}$-linear map $h : V \to W$ between vector spaces over $\mathbf{k}$ let $Sym^l(h) : Sym^l(V) \to Sym^l(W)$ denote the l-th symmetric power of this homomorphism. The multiplication induces a natural isomorphism between $Sym^k(\mathfrak{m}/\mathfrak{m}^2)$, $Sym^k(\mathfrak{n}/\mathfrak{n}^2)$ and $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ and $\mathfrak{n}^k/\mathfrak{n}^{k+1}$, respectively, and under these identifications we have $T^k_\phi = Sym^k(T^1_\phi)$. Since any $\mathbf{k}$-linear map $h : \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \cong \mathbf{k}$ is induced by a local homomorphism $\phi : A_{n,m} \to A_{1,m}$, it will be sufficient to prove the following lemma.

Lemma 4.5. *For every* $0 \neq v \in Sym^k(\mathbf{k}^n)$ *there is a* $\mathbf{k}$-*linear map* $\phi : \mathbf{k}^n \to \mathbf{k}$ *such that* $Sym^k(\phi)(v) \neq 0$.

Proof. We are going to prove the claim by induction on n. The case $n = 1$ is obvious. Let $x_1, x_2, \dots, x_n$ be a basis of $\mathbf{k}^n$. Write v as

$$v = \sum_{j=0}^{k} p_j(x_1, x_2, \dots, x_{n-1}) x_n^j,$$

where $p_j \in Sym^{k-j}(\mathbf{k}^{n-1})$. For a $0 \leq j \leq k$ the polynomial p_j is not zero, therefore there is a $\mathbf{k}$-linear map $\phi_1 : \mathbf{k}^{n-1} \to \mathbf{k}$ by induction, where $\mathbf{k}^{n-1}$ is spanned by $x_1, x_2, \dots, x_{n-1}$, such that $Sym^{k-j}(\phi_1)(p_j) \neq 0$. The polynomial:

$$p(t) = \sum_{j=0}^{k} Sym^{k-j}(\phi_1)(p_j) t^j$$

is not identically zero, hence it has finitely many roots. The field $\mathbf{k}$ is assumed to be algebraically closed, in particular it is not finite. Hence there is a $\beta \in \mathbf{k}$ which is not a root of the polynomial above. Let $\phi : \mathbf{k}^n \to \mathbf{k}$ be the unique $\mathbf{k}$-linear extension of ϕ_1 with $\phi(x_n) = \beta$. In this case $Sym^k(v) = p(\beta) \neq 0$, so the claim is proved. $\square$

Now assume that $\mathbf{k}$ has characteristic $p > 0$ and let $\mathbb{W}(\mathbf{k})$ denote the ring of Witt vectors of $\mathbf{k}$ of infinite length. For every pair of natural numbers n and m let $B_{n,m}$ denote the local Artinian algebra:

$$B_{n,m} = \mathbb{W}(\mathbf{k})[[x_1, \dots, x_n]] / \Big(\Big\{ p^{J(0)} \cdot \prod_{j=1}^{n} x_j^{J(j)} \;\Big|\; J : \{0, 1, \dots, n\} \to \mathbb{N}, \sum_{j=0}^{n} J(j) > m \Big\} \Big).$$

For every $A \in \mathcal{C}$ there is a surjective local homomorphism $B_{n,m} \to A$ for some n and m. By repeating the argument above we may reduce the proof of the proposition to showing that there is a homomorphism $\phi : B_{n,m} \to B_{0,m}$ such that $\phi(x) \neq 1$ for every $x \in B^*_{n,m}$ with $x \equiv 1 \mod \mathfrak{m}^{k-1}$, but $x \not\equiv 1 \mod \mathfrak{m}^k$ for some positive integer $k \leq m$, where $\mathfrak{m}$ is the maximal ideal of $B_{n,m}$. Every local homomorphism $\phi : B_{n,m} \to B_{0,m}$ induces a $\mathbf{k}$-linear homomorphism $T^l_\phi : \mathfrak{m}^l/\mathfrak{m}^{l+1} \to \mathfrak{n}^l/\mathfrak{n}^{l+1}$ for any positive $l \leq m$, where $\mathfrak{n}$ is the maximal proper ideal of $B_{0,m}$. Let T_p, $T^\perp$ denote the $\mathbf{k}$-linear subspace of $\mathfrak{m}/\mathfrak{m}^2$ generated by p and the elements $x_1, x_2, \dots, x_n$, respectively. The multiplication induces a natural isomorphism:

$$\mathfrak{m}^k/\mathfrak{m}^{k+1} = \bigoplus_{j=0}^{k} Sym^j(T_p) \otimes Sym^{k-j}(T^\perp)$$

and another between $Sym^k(\mathfrak{n}/\mathfrak{n}^2)$ and $\mathfrak{n}^k/\mathfrak{n}^{k+1}$. Moreover there is a canonical isomorphism $\iota : T_p \to \mathfrak{n}/\mathfrak{n}^2$ between these one-dimensional vector spaces. Under these identifications we have

$$T^k_\phi = \bigoplus_{j=0}^{k} Sym^j(\iota) \otimes Sym^{k-j}(T^1_\phi|_{T^\perp}).$$

Since every $\mathbf{k}$-linear map $h : T^\perp \to \mathfrak{n}/\mathfrak{n}^2 \cong \mathbf{k}$ is induced by a local homomorphism $\phi : B_{n,m} \to B_{0,m}$, the proposition follows from the lemma above. □

Definition 4.6. Let $A \in \mathcal{C}$ be a local Artinian ring with maximal ideal $\mathfrak{m}$ and let f be any element of $A((t))^*$. Then there is an integer $w(f) \in \mathbb{Z}$, and a sequence of elements $a_i \in A$ indexed by the integers such that $a_0 \in A^*$, $a_{-i} \in \mathfrak{m}$ for $i > 0$, $a_{-i} = 0$ for i sufficiently large, and

$$f = a_0 \cdot t^{w(f)} \cdot \prod_{i=1}^{\infty}(1 - a_i t^i) \cdot \prod_{i=1}^{\infty}(1 - a_{-i} t^{-i}),$$

and $w(f)$ and a_i are uniquely determined by f. The integer $w(f)$ is called the *winding number* of f and the elements a_i are called the *Witt coordinates* of f. Let $f, g \in A((t))^*$ be arbitrary with winding numbers $w(f)$, $w(g)$ and Witt coordinates a_i, b_j, respectively. By definition the *Contou-Carrère symbol* $\langle f, g\rangle$ is:

$$\langle f, g\rangle = (-1)^{w(f)w(g)} \frac{a_0^{w(g)} \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} (1 - a_i^{j/(i,j)} b_{-j}^{i/(i,j)})^{(i,j)}}{b_0^{w(f)} \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} (1 - a_{-i}^{j/(i,j)} b_i^{i/(i,j)})^{(i,j)}} \in A^*.$$

Obviously all but finitely many terms are equal to one in the infinite products above, hence the Contou-Carrère is a well-defined alternating map:

$$\langle \cdot, \cdot \rangle : A((t))^* \times A((t))^* \to A^*.$$

It is also clear from the formula that the Contou-Carrère symbol is equal to the tame symbol if A is a field.

Proposition 4.7. *The Contou-Carrère symbol is a bilinear map satisfying the Steinberg relation.*

Proof. Because $\mathbf{k}$ is perfect for every object A of $\mathcal{C}$ there is a local Artinian ring B with residue field $\overline{\mathbf{k}}$, where the latter is the algebraic closure of $\mathbf{k}$, and an injective local homomorphism $i : A \to B$. Indeed the algebra $B = A \otimes_{\mathbf{k}} \overline{\mathbf{k}}$ and $B = A \otimes_{\mathbb{W}(\mathbf{k})} \mathbb{W}(\overline{\mathbf{k}})$ will do, when $\mathbf{k}$ has characteristic zero or positive characteristic, respectively, using the fact has A can be equipped with the structure of a $\mathbf{k}$-algebra or $\mathbb{W}(\mathbf{k})$-algebra, respectively, by Lemmas 4.2 and 4.3. Hence we may assume that $\mathbf{k}$ is algebraically closed. Let $\mathcal{D}$ denote the subclass of those local Artinian rings with residue field $\mathbf{k}$ which satisfy the claim of the proposition above. Clearly we only have to show that this subclass satisfies the conditions of Proposition 4.4. If $A \in \mathcal{C}$ is the quotient of a discrete valuation ring R with residue field $\mathbf{k}$, we may assume that R is complete with respect to its valuation. Let K be the quotient field of R and let $\mathbb{C}$ be the completion of the algebraic closure of K. The latter is an algebraically closed field complete with respect to an ultrametric absolute value. For every $f \in A((t))^*$ there is a lift $\tilde{f} \in R((t))^*$ whose image is f under the functorial map $R((t)) \to A((t))$. By Lemmas 3.4 and 3.5 the Contou-Carrère symbol of f and $1-f$ is the reduction of the rigid analytic regulator $\{\tilde{f}, 1-\tilde{f}\}_D \in R$ modulo the maximal ideal of R, hence the Contou-Carrère symbol satisfies the Steinberg relation. A similar argument shows that it is also bilinear, therefore (i) of Proposition 4.4 holds for $\mathcal{D}$. Property (ii) also follows from same reasoning, because every $f \in A((t))^*$ has a lift $\tilde{f} \in B((t))^*$ if the map $B \to A$ is surjective. Finally let $A \in \mathcal{C}$ be an algebra which satisfies the condition in (iii) of Proposition 4.4. Assume that there is an $1 \neq f \in A((t))^*$ such that $\langle f, 1-f\rangle \neq 1$. Then there is a $B \in \mathcal{D}$ and a homomorphism $\phi : A \to B$ such that $1 \neq \phi(\langle f, 1-f\rangle) = \langle \phi_*(f), 1-\phi_*(f)\rangle = 1$, where $\phi_* : A((t)) \to B((t))$ is the functorial map induced by ϕ, which is a contradiction. A similar argument shows that the Contou-Carrère symbol is bilinear over A, therefore property (iii) also holds for $\mathcal{D}$. □

Let $x \in A[[t]]$ be a uniformizer, which means that x is of the form $ct + t^2h$, where $c \in A^*$ and $h \in A[[t]]$. In this case there is a unique A-algebra automorphism $\phi : A[[t]] \to A[[t]]$ such that $\phi(t) = x$. On the other hand every A-algebra automorphism of $A[[t]]$ is of this form. The automorphism ϕ extends uniquely to an automorphism $\phi : A((t)) \to A((t))$ by localizing at the maximal ideal.

Proposition 4.8. *The automorphism ϕ leaves the Contou-Carrère symbol invariant.*

Proof. As in the proof above we may assume that $\mathbf{k}$ is algebraically closed. Let $\mathcal{D}$ again denote the subclass of those local Artinian rings with residue field $\mathbf{k}$ which satisfy the claim of the proposition above. We need to show only that this subclass satisfies the conditions of Proposition 4.4. Let $\psi : B \to A$ be a surjective homomorphism of local algebras with residue field $\mathbf{k}$ and let $\psi_* : B((t)) \to A((t))$ be the functorial map induced by ψ. If B is Artinian or a discrete valuation ring then there is a B-algebra automorphism $\phi_B : B((t)) \to B((t))$ such that $\psi_* \circ \phi_B = \phi \circ \psi_*$ which is of type described before Proposition 3.3 if B is a discrete valuation ring. Hence (i) and (ii) of Proposition 4.4 hold for $\mathcal{D}$ by Proposition 3.3. A similar argument as above shows that condition (iii) also holds for $\mathcal{D}$. □

Notation 4.9. Let $A \in \mathcal{C}$ be a local Artinian ring and let $\pi : X \to \mathrm{Spec}(A)$ be a projective flat morphism whose fiber X_0 over the unique closed point of $\mathrm{Spec}(A)$ is a reduced, connected, regular curve over $\mathbf{k}$. Let S be a finite set of closed points of X (or, equivalently, of X_0) and let f and g be two rational functions on X which are invertible on the complement of S. For every $s \in S$ choose an A-algebra isomorphism ϕ_s between the completion $\widehat{\mathcal{O}}_{s,X}$ of the stalk $\mathcal{O}_{s,X}$ of the structure sheaf of X at s and $A[[t]]$. The latter induces an isomorphism between the localization of $\widehat{\mathcal{O}}_{s,X}$ by the semigroup of those elements whose image under the canonical map $\widehat{\mathcal{O}}_{s,X} \to \widehat{\mathcal{O}}_{s,X_0}$ is non-zero, where $\widehat{\mathcal{O}}_{s,X_0}$ is the the completion of the stalk $\mathcal{O}_{s,X_0}$, and $A((t))$, which will be denoted by ϕ_s, by the usual abuse of notation. Let $\langle f, g\rangle_s$ denote the Contou-Carrère symbol of the image of f and g under ϕ_s for every s in S. By Proposition 4.8 the value of $\langle f, g\rangle_s$ is independent of the choice of the isomorphism ϕ_s, so the symbol $\langle f, g\rangle_s$ is well-defined. The following result is the reciprocity law of Anderson and Romo (see [1]).

Proposition 4.10. *The product of all the Contou-Carrère symbols of f and g is equal to* 1*:*

$$\prod_{s\in S} \langle f, g\rangle_s = 1.$$

Proof. We are going to use the same strategy for proof as we used before: in particular we assume that $\mathbf{k}$ is algebraically closed. Let $\mathcal{D}$ denote the subclass of those local Artinian rings with residue field $\mathbf{k}$ which satisfy the claim of the proposition above. We will show that this subclass satisfies the conditions of Proposition 4.4. If $A \in \mathcal{C}$ is the quotient of a discrete valuation ring R with residue field $\mathbf{k}$, we may again assume that R is complete with respect to the valuation. Let K denote the quotient field of R and let $\mathbb{C}$ denote the completion of the algebraic closure of K as above. Because the deformation theory of regular projective curves is unobstructed, there is a formal scheme $\mathfrak{X}$ over the formal spectrum of R whose fiber over $\mathrm{Spec}(A)$ is X. By the algebraicity theorem of Grothendieck $\mathfrak{X}$ is actually the formal completion of a smooth curve over $\mathrm{Spec}(R)$ which will be denoted by $\mathfrak{X}$ by abuse of notation. By flatness there are rational functions $\tilde{f}$ and $\tilde{g}$ on $\mathfrak{X}$ whose restrictions to the fiber over $\mathrm{Spec}(A)$ are f and g, respectively. The rigid analytic domain D_s of $\mathbb{C}$-valued points of $\mathfrak{X}$ which reduces to s is isomorphic to the open disc D by the formal inverse function theorem. By Lemmas 3.4 and 3.5 the Contou-Carrère symbol of f and g is the reduction of the product of the tame symbols $\{\tilde{f}, \tilde{g}\}_x$ modulo the maximal ideal of R where x is running through the $\mathbb{C}$-valued points of the set D_s. The rational functions $\tilde{f}$ and $\tilde{g}$ have only poles or zeros in the union of the sets D_s hence the reciprocity law of Anderson and Romo holds by Weil's reciprocity law.

Property (ii) also follows from the same reasoning. If the map $B \to A$ is surjective and X, f and g are as above, then there is a similar triple $\tilde{X}$, $\tilde{f}$ and $\tilde{g}$ over $\mathrm{Spec}(B)$ such that the fiber of $\tilde{X}$ over $\mathrm{Spec}(A)$ is X and the restrictions of $\tilde{f}$ and $\tilde{g}$ to X are f and g, respectively, because the deformation theory of X is unobstructed. If $B \in \mathcal{D}$ then the claim holds for the triple $\tilde{X}, \tilde{f}, \tilde{g}$, so it must hold for the triple X, f, g as well. Finally let $A \in \mathcal{C}$ be an algebra which satisfies the condition in (iii) of Proposition 4.4. Assume that there are rational functions f and g as above such that $\prod_{s\in S}\langle f, g\rangle_s \neq 1$. Then there is a $B \in \mathcal{D}$ and a homomorphism $\phi : A \to B$ such that $1 \neq \phi(\prod_{s\in S}\langle f, g\rangle_s) = \prod_{s\in\phi^*(S)}\langle \phi^*(f), \phi^*(g)\rangle_s$, where $\phi^*(f)$, $\phi^*(g)$ and $\phi^*(S)$ are the base changes of the corresponding objects on the curve $\phi^*(X)$ which is the base change of X with respect to the map $\phi^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$. This is a contradiction, therefore property (iii) also holds for $\mathcal{D}$. □

Remark 4.11. It is possible to push the methods of this paper a bit further to actually give a proof of Weil's reciprocity law itself by reducing it to the case of Mumford curves, when it follows from (iii) of Theorem 2.6 at once. We will only sketch this argument because it uses a considerable amount of machinery compared to the relatively elementary nature of Weil's reciprocity law. For any scheme S and any stable curve $\pi : C \to S$ of genus g let $\omega_{C/S}$ denote the relative dualizing sheaf. By Theorem 1.2 of [8], page 77 the functor which assigns to each scheme S the set of stable curves $\pi : C \to S$, and an isomorphism $\mathbb{P}(\pi_*(\omega_{C/S}^{\otimes 3})) \cong \mathbb{P}_S^{5g-6}$ (modulo isomorphism) is represented by a fine moduli scheme $\mathfrak{H}_g$. By Corollary 1.7 of [8], page 83 and the main result of [8], pages 92–96, the scheme $\mathfrak{H}_g$ is smooth over the spectrum of $\mathbb{Z}$ and the base change $(\mathfrak{H}_g)_{\mathrm{Spec}(\mathbf{k})}$ is irreducible for any algebraically closed field $\mathbf{k}$. Let X be a smooth, projective curve over $\mathbf{k}$ and let f and g be two non-zero rational functions on X. We may assume that the genus g of X is at least two by taking a cover of X and proving the reciprocity law for the pull-back of f and g instead. Let x be a $\mathbf{k}$-valued point of $\mathfrak{H}_g$ such that the underlying curve is X and let y be another $\mathbf{k}$-valued point such that the underlying curve is totally degenerate. Since $(\mathfrak{H}_g)_{\mathrm{Spec}(\mathbf{k})}$ is an irreducible, smooth quasi-projective variety, repeated application of Bertini's theorem shows that there is a smooth, irreducible curve S mapping to $(\mathfrak{H}_g)_{\mathrm{Spec}(\mathbf{k})}$ whose image contains both x and y. Let $\pi : C \to S$ be the pull-back of the universal family. There are rational functions $\tilde{f}$ and $\tilde{g}$ on C whose restrictions to the fiber over x, which is X, are f and g, respectively. Since the base change of C to the spectrum of the local field of S at y is a Mumford curve, Weil's reciprocity law holds for $\tilde{f}$ and $\tilde{g}$, hence holds for f and g, too. One may say that this proof is close in spirit to the classical proof of the reciprocity law over the complex numbers using triangulation, since it decomposes the curve to small pieces in a suitable topology.

5. The differential reciprocity law

Definition 5.1. We will continue to use the notation of the previous chapter. For every $\mathbf{k}$-algebra A let $\Omega^{\cdot}_A$ denote the graded differential algebra of $\mathbf{k}$-linear Kähler differential forms of A and for every $\mathbf{k}$-algebra homomorphism $h : A \to B$ let $\Omega^k(h) : \Omega^{\cdot}_A \to \Omega^{\cdot}_B$ induced by h by functoriality. Every $\omega \in \Omega^k_{A((t))}$ can be written uniquely in the form:

$$\omega = \sum_{i=1}^{m} \beta_i \frac{dt}{t^i} + \omega_0$$

where m is a natural number, $\beta_i \in \Omega^{k-1}_A$ and $\omega_0 \in \Omega^k_{A[[t]]} + A((t))\Omega^k_A$. Let $\mathrm{Res}^k(\omega) \in \Omega^{k-1}_A$ denote the element β_1. We get a map $\mathrm{Res}^k : \Omega^k_{A((t))} \to \Omega^{k-1}_A$ which is called the *residue*.

Proposition 5.2. *The following holds:*

(i) *we have* $\mathrm{Res}^{k+i}(\alpha\omega) = \alpha\mathrm{Res}^k(\omega)$ *for every* $\alpha \in \Omega^i_A$ *and* $\omega \in \Omega^k_{A((t))}$,

(ii) *we have* $\Omega^{k-1}(h) \circ \mathrm{Res}^k = \mathrm{Res}^k \circ \Omega^k(h')$ *where* $h : A \to B$ *is a* $\mathbf{k}$*-algebra homomorphism and* $h' : A((t)) \to B((t))$ *is the corresponding* $\mathbf{k}$*-algebra homomorphism induced by functoriality,*

(iii) *we have* $\mathrm{Res}^k(\omega) = 0$ *for every* $\omega \in \Omega^k_{A[[t]]}$ *and for every* $\omega \in \Omega^k_{A[\frac{1}{t}]}$,

(iv) *the map* Res^k *does not depend on the choice of the uniformizer* t.

Proof. Our method of proving the first two claims is the same. Using the notation of Definition 5.1 we have:

$$\alpha\omega = \sum_{i=1}^{m} \alpha\beta_i \frac{dt}{t^i} + \alpha\omega_0.$$

Because $\alpha\beta_i \in \Omega^{k+i-1}_A$ and $\alpha\omega_0 \in \Omega^{k+i}_{A[[t]]} + A((t))\Omega^{k+i}_A$ we have $\mathrm{Res}^{k+i}(\alpha\omega) = \alpha\beta_1$ by definition so claim (i) is true. On the other hand:

$$\Omega^k(h')(\omega) = \sum_{i=1}^{m} \Omega^{k-1}(h)(\beta_i) \frac{dt}{t^i} + \Omega^k(h')(\omega_0)$$

where $\Omega^{k-1}(h)(\beta_i) \in \Omega^{k-1}_B$ and $\Omega^k(h')(\omega_0) \in \Omega^k_{B[[t]]} + B((t))\Omega^k_B$. Therefore we have $\mathrm{Res}^k(\Omega^k(h')(\omega)) = \Omega^{k-1}(h)(\beta_1)$ as claim (ii) says. The first half of claim (iii) is immediate from the definition of the residue. In order to prove the second half we only need to show the identity $\mathrm{Res}^1(dt^{-n}) = 0$ for all $n \geq 1$ by the $\Omega^{\cdot}_A$-linearity of the residue spelled out in claim (i). But the latter is obvious. Claim (iv) means the following: let $x \in A[[t]]$ be a uniformizer, which means that x is of the form tu, where $u \in A[[t]]^*$. In this case there is a unique A-algebra automorphism $\phi : A[[t]] \to A[[t]]$ such that $\phi(t) = x$ as we already saw when we prepared to formulate Proposition 4.8. The automorphism ϕ extends uniquely to an automorphism $\phi : A((t)) \to A((t))$ by localizing at the maximal ideal. Claim (iv) means that the equation $\mathrm{Res}^k \circ \Omega^k(\phi) = \mathrm{Res}^k$ holds. Because the homomorphism $H^k(\phi)$ maps $\Omega^k_{A[[t]]}$ and $A((t))\Omega^k_A$ into itself we only need to show that

$$\mathrm{Res}^1\left(\frac{dx}{x}\right) = 1 \quad \text{and} \quad \mathrm{Res}^1\left(\frac{dx}{x^n}\right) = 0 \qquad \text{for all } n \geq 2$$

by $\Omega^{\cdot}_A$-linearity. These identities follow at once from the same type of identity in Proposition 5′ in [20] on pages 20–21 by the principle of prolongation of algebraic identities quoted in the proof of the proposition just mentioned above. (Or one may use the functorial argument explained in the remark following the proof of Proposition 5′ in [20] instead.) □

Notation 5.3. For every **k**-algebra B let $dlog : B^* \to \Omega^1_B$ denote the logarithmic differential given by the rule $dlog(u) = u^{-1}du$ for every $u \in B^*$. Let moreover $dlog^2$ denote the $\mathbb{Z}$-bilinear pairing:

$$dlog^2 : B^* \otimes B^* \to \Omega^2_B$$

given by the rule:

$$dlog^2(a, b) = dlog(a)dlog(b) \quad (\forall a \in B^*, \forall b \in B^*).$$

Recall that for every object A of $\mathcal{C}$ the symbol $\langle\cdot,\cdot\rangle$ denotes the Contou-Carrère symbol.

Proposition 5.4. *The following diagram commutes:*

$$\begin{array}{ccc} A((t))^* \otimes A((t))^* & \xrightarrow{dlog^2} & \Omega^2_{A((t))} \\ \downarrow{\scriptstyle\langle\cdot,\cdot\rangle} & & \downarrow{\scriptstyle \mathrm{Res}^2} \\ A^* & \xrightarrow{dlog} & \Omega^1_A \end{array}$$

for every object A of $\mathcal{C}$.

Proof. By bilinearity and antisymmetry of the Contou-Carrère symbol and the map $dlog^2$ it will be sufficient to prove for every pair of integers n, m and elements $a, b \in A$ the following identities:

(i) $\mathrm{Res}^2(dlog^2(1 - at^n, 1 - bt^m)) = 0$, if $n, m > 0$,
(ii) $\mathrm{Res}^2(dlog^2(1 - at^{-n}, 1 - bt^{-m})) = 0$, if $a, b \in \mathfrak{m}$ and $n, m > 0$,

(iii) $\mathrm{Res}^2(dlog^2(at^n, 1-bt^m)) = 0$, if $a \in A^*$ and $m > 0$,

(iv) $\mathrm{Res}^2(dlog^2(at^n, 1-bt^m)) = 0$, if $a \in A^*$, $b \in \mathfrak{m}$ and $m < 0$,

(v) $\mathrm{Res}^2(dlog^2(at^n, bt^m)) = m(da/a) - n(db/b)$, if $a, b \in A^*$,

(vi) $\mathrm{Res}^2(dlog^2(1-at^n, 1-bt^{-m})) = dlog((1-a^{m/(n,m)}b^{n/(n,m)})^{(n,m)})$, if $b \in \mathfrak{m}$ and $n, m > 0$,

(vii) $\mathrm{Res}^2(dlog^2(1-ft^{Mn+1}, 1-bt^{-n})) = 0$, if $f \in A[[t]]$, $b \in \mathfrak{m}$ and $n > 0$,

where $\mathfrak{m}$ is the maximal proper ideal of A and M is a positive integer such that $\mathfrak{m}^M = 0$. Note that

$$dlog(1-at^n) = -(dat^n + nat^{n-1}dt)(1 + at^n + a^2t^{2n} + \cdots)$$

lies in $\Omega^1_{A[[t]]}$, if $a \in A$ and $n > 0$, and lies in $\Omega^1_{A[\frac{1}{t}]}$, if $a \in \mathfrak{m}$ and $n < 0$. Hence the first two identities follow from claim (iii) of Proposition 5.2. For every $a \in A^*$ and $b \in A$ we have:

$$\begin{aligned} dlog^2(at^n, 1-bt^m) &= -(1 + bt^m + b^2t^{2m} + \cdots)\left(\frac{da}{a} + n\frac{dt}{t}\right)(dbt^m + mbt^{m-1}dt) \\ &= \left(ndb - \frac{mb}{a}da\right)(t^{m-1}dt + bt^{2m-1}dt + b^2t^{3m-1}dt + \cdots) + \omega_0 \end{aligned}$$

where $\omega_0 \in A((t))\Omega^2_A$ when either $m > 0$ or when $b \in \mathfrak{m}$ and $m < 0$. The first summand in the second line lies in $\Omega^2_{A[[t]]}$, if $m > 0$, and lies in $\Omega^2_{A[\frac{1}{t}]}$, if $b \in \mathfrak{m}$ and $m < 0$. Hence its residue is zero so identities (iii) and (iv) are true. For every $a, b \in A^*$ we have:

$$dlog^2(at^n, bt^m) = \left(\frac{da}{a} + n\frac{dt}{t}\right)\left(\frac{db}{b} + m\frac{dt}{t}\right) = \left(m\frac{da}{a} - n\frac{db}{b}\right)\frac{dt}{t} + \omega_0$$

where $\omega_0 \in \Omega^2_A$ so identity (v) is clear. By definition:

$$\begin{aligned} &dlog((1-a^{m/(n,m)}b^{n/(n,m)})^{(n,m)}) \\ &\qquad = -\frac{ma^{m/(n,m)}b^{n/(n,m)}\frac{da}{a} + na^{m/(n,m)}b^{n/(n,m)}\frac{db}{b}}{1-a^{m/(n,m)}b^{n/(n,m)}} \end{aligned}$$

for every $a \in A$, $b \in \mathfrak{m}$ and $n, m > 0$. We also have:

$$(1-at^n)^{-1}(1-b^{-m})^{-1} = \sum_{k\in\mathbb{Z}} \sum_{\substack{i,j\in\mathbb{N} \\ in-jm=k}} a^ib^jt^k$$

for such a and b. Because $in - jm = m - n$ for any $i, j \in \mathbb{N}$ if and only if $i + 1 = lm/(n,m)$ and $j + 1 = ln(n,m)$ for some $l \in \mathbb{N}$ we have:

$$(1-at^n)^{-1}(1-b^{-m})^{-1} = (ab)^{-1}\Big(\sum_{l=1}^{\infty} a^{lm/(n,m)}b^{ln/(n,m)}\Big)t^{m-n} + (r+s)t^{m-n}$$

for some $r \in \frac{1}{t}A[\frac{1}{t}]$ and $s \in tA[[t]]$. Hence we have:

$$\begin{aligned} dlog^2(1-at^n, 1-bt^{-m}) &= \frac{(dat^n + nat^{n-1}dt)(dbt^{-m} - mbt^{-m-1}dt)}{(1-at^n)(1-bt^{-m})} \\ &= \frac{-(mbda + nadb)t^{n-m-1}dt}{(1-at^n)(1-bt^{-m})} + \omega_0 \\ &= -\frac{a^{m/(n,m)-1}b^{n/(n,m)-1}(mbda + nadb)t^{-1}dt}{1 - a^{m/(n,m)}b^{n/(n,m)}} \\ &\quad + \omega_0 + \omega_1 \end{aligned}$$

where $\omega_0 \in A((t))\Omega^2_A$ and $\omega_1 \in \Omega^2_{A[\frac{1}{t}]} + \Omega^2_{A[[t]]}$. Identity (vi) is now obvious. Finally consider the last identity. Note that

$$dlog(1-bt^{-n}) = -(dbt^{-n} - nbt^{-n-1}dt)(1 + bt^{-n} + b^2t^{-2n} + \cdots + b^{M-1}t^{(1-M)n})$$

because $b^M = 0$ by assumption, and

$$dlog(1-ft^{Mn+1}) = -(dft^{Mn+1} + (Mn+1)ft^{Mn}dt)(1 + ft^{Mn+1} + f^2t^{2Mn+2} + \cdots)$$

hence

$$\begin{aligned} dlog^2(1-ft^{Mn+1}, 1-bt^{-n}) &= (dft + (Mn+1)fdt)(db - nbt^{-1}dt)g \\ &= (tdfdb - nbdfdt + (Mn+1)fdtdb)g \end{aligned}$$

where $g \in A[[t]]$. The claim is now clear. □

Definition 5.5. Let L be a field complete with respect to a discrete valuation and let $\mathcal{R}$, $\mathfrak{m}$ denote its discrete valuation ring and the maximal ideal of $\mathcal{R}$, respectively. Assume that the residue field of L is $\mathbf{k}$ and the quotient map $\mathcal{R} \to \mathbf{k}$ has a section which is a ring homomorphism. The latter equips L and $\mathcal{R}$ with a $\mathbf{k}$-algebra structure. Let $\widehat{\Omega}^{\cdot}_L$ denote the graded differential algebra which is the quotient of the complex $\Omega^{\cdot}_L$ by the homogeneous ideal generated by $\bigcap_{n\geq 1}\mathfrak{m}^n\Omega^{\cdot}_{\mathcal{R}}$ and let $\widehat{\Omega}^k_{\mathcal{R}}$ denote the image of $\Omega^k_{\mathcal{R}}$ in $\widehat{\Omega}^d_L$ under the quotient map. For every natural number n let $\mathcal{R}_n$ denote the truncated ring $\mathcal{R}/\mathfrak{m}^{n+1}$ and for every pair $m \leq n$ of natural numbers let $\pi_n : \mathcal{R} \to \mathcal{R}_n$ and $\pi_{n,m} : \mathcal{R}_n \to \mathcal{R}_m$ denote the canonical projections. The system of modules $\{\Omega^k_{\mathcal{R}_n}\}_{n\in\mathbb{N}}$ forms a compatible system with respect to the morphisms $\Omega^k(\pi_{n,m})$ $(m \leq n)$ hence it has a projective limit $\varprojlim_{n\to\infty}(\Omega^k_{\mathcal{R}_n})$. The maps $\Omega^k(\pi_n) : \Omega^k_{\mathcal{R}} \to \Omega^k_{\mathcal{R}_n}$ factor through $\widehat{\Omega}^k_{\mathcal{R}}$ and their limit induces an identification: $\widehat{\Omega}^k_{\mathcal{R}} \cong \varprojlim_{n\to\infty}(\Omega^k_{\mathcal{R}_n})$ which we will use without further notice. Let $dlog : L^* \to \widehat{\Omega}^1_L$ and $dlog^2 : K_2(L) \to \widehat{\Omega}^2_L$ also denote the composition of $dlog$, $dlog^2$ and the quotient map $\Omega^1_L \to \widehat{\Omega}^1_L$, $\Omega^2_L \to \widehat{\Omega}^1_L$, respectively.

Definition 5.6. Let π be a uniformizer of L and let $\widehat{\Omega}^k_L(log)$ denote the subgroup $\pi^{-1}\widehat{\Omega}^k_{\mathcal{R}}$ of $\widehat{\Omega}^k_L$. Clearly the group $\widehat{\Omega}^k_L(log)$ is independent of the choice of the uniformizer π. Let $\mathcal{O}$ denote the discrete valuation ring $\mathbf{k}[[x]]$ and let F denote its quotient field. Let $\widehat{M}$ denote the field attached to F which was introduced in Definition 3.1 and let $\mathcal{R}$ denote the valuation ring of $\widehat{M}$. The uniformizer x of F is also a uniformizer in $\widehat{M}$. There is a natural isomorphism $\mathcal{R}_n \cong \mathcal{O}_n((\overline{z}))$ for every $n \in \mathbb{N}$, where $\overline{z}$ denotes also the reduction of z in $\mathcal{R}_n$ for every n by slightly extending the notation introduced in Definition 3.6, therefore for every $\omega \in \Omega^k_{\mathcal{R}_n}$ the residue $\mathrm{Res}^k(\omega) \in \Omega^{k-1}_{\mathcal{O}_n}$ is well-defined. For every $\omega \in \widehat{\Omega}^k_{\widehat{M}}(log)$ let $\mathrm{Res}^k(\omega) \in \widehat{\Omega}^{k-1}_F$ be given by the rule:

$$\mathrm{Res}^k(\omega) = \frac{1}{x} \varprojlim_{n\to\infty} (\mathrm{Res}^k(\widehat{\Omega}^k(\pi_n)(x\omega)))$$

where the map $\widehat{\Omega}^k(\pi_n) : \widehat{\Omega}^k_{\mathcal{R}} \to \Omega^k_{\mathcal{R}_n}$ is induced by $\Omega^k(\pi_n)$. The system:

$$\{\mathrm{Res}^k(\widehat{\Omega}^k(\pi_n)(x\omega))\}_{n\in\mathbb{N}}$$

satisfies the compatibility described above by claim (ii) of Proposition 5.2 hence $\mathrm{Res}^k(\omega)$ is well-defined. Because of the $\mathcal{O}_n$-linearity of the residue it is obvious that $\mathrm{Res}^k(\omega)$ is independent of the choice of x as the notation indicates.

Remark 5.7. Let $\phi : \widehat{M} \to \widehat{M}$ be a valuation-preserving F-algebra automorphism. Then there is a unique map $\widehat{\Omega}^k(\phi) : \widehat{\Omega}^k_{\widehat{M}} \to \widehat{\Omega}^k_{\widehat{M}}$ such that $\widehat{\Omega}^k(\phi) \circ q_k = q_k \circ \Omega^k(\phi)$ where $q_k : \Omega^k_{\widehat{M}} \to \widehat{\Omega}^k_{\widehat{M}}$ is the quotient map. The automorphism $\widehat{\Omega}^k(\phi)$ of $\widehat{\Omega}^k_{\widehat{M}}$ preserves the subgroup $\widehat{\Omega}^k_{\widehat{M}}(log)$ and it commutes with the residue map Res^k by claim (iv) of Proposition 5.2.

Theorem 5.8. *We have $dlog^2(k) \in \widehat{\Omega}^2_{\widehat{M}}$ for every $k \in K_2(\widehat{M})$ and the diagram:*

$$\begin{array}{ccc} K_2(\widehat{M}) & \xrightarrow{dlog^2} & \widehat{\Omega}^2_{\widehat{M}}(log) \\ \Big\downarrow{\scriptstyle \{\cdot,\cdot\}_D} & & \Big\downarrow{\scriptstyle \mathrm{Res}^2} \\ F^* & \xrightarrow{dlog} & \widehat{\Omega}^1_F \end{array}$$

is commutative where $\{\cdot,\cdot\}_D$ denotes Kato's residue homomorphism.

Proof. By the linearity of the $dlog^2$ map we only have to verify the first claim of the theorem as well as the identity expressed by the commutative diagram above for the elements of any set of generators of $K_2(\widehat{M})$. Hence we may assume that $k = u \otimes v$ where either $u, v \in \mathcal{R}^*$ or $u = x$ and v is arbitrary. In the first case we have $dlog^2(k) \in \widehat{\Omega}^k_{\mathcal{R}}$ obviously and the identity holds by Proposition 5.4. In the second case we may write v in the form $v = x^n w$ for some $n \in \mathbb{Z}$ and $w \in \mathcal{R}^*$. Because $\{x, x\}_D = 1$ and $dlog^2(x \otimes x) = 0$ by definition we may assume that $v = w$. The first claim is now obvious. Moreover in this case we may write v in the form $v = z^{\deg(v)} t$ for some $t \in \mathcal{R}^*$ such that the reduction t_k of t modulo $x^k\mathcal{R}$ lies in $\mathcal{O}_k[[\overline{z}]]^* \subset \mathcal{R}_k$ for every $k \in \mathbb{N}$. Therefore $dlog(t_k) \in \Omega^1_{\mathcal{O}_k[[\overline{z}]]}$ and we have:

$$\begin{aligned}\mathrm{Res}^2\left(\Omega^2(\pi_k)\left(dx\frac{dv}{v}\right)\right) &= \mathrm{Res}^2\left(\deg(v)\Omega^2(\pi_k)\left(dx\frac{dz}{z}\right)\right) + \mathrm{Res}^2\left(\Omega^2(\pi_k)(dx)\frac{dt_n}{t_n}\right)\\ &= \deg(v)\Omega^1(\pi_k)(dx)\end{aligned}$$

for every $k \in \mathbb{N}$. The claim now follows from Lemma 3.7. □

6. The image and kernel of the rigid analytic regulator in positive characteristic

Notation 6.1. For every scheme X let $K_{2,X}$ denote the sheaf on X associated to the presheaf $U \mapsto K_2(H^0(U, \mathcal{O}_X))$ for the Zariski topology where $K_2(A)$ denotes Milnor's K-group of any ring A. Let $\mathbb{W}_n\Omega^*_X$ denote the de Rham–Witt pro-complex of any ringed topos X of $\mathbb{F}_p$-algebras. Moreover we let F denote the Frobenius morphism of the de Rham–Witt pro-complex. Recall that the logarithmic differential $dlog^1 : \mathcal{O}^*_X \to \mathbb{W}_n\Omega^1_X$ is defined as the composition of the Teichmüller lift $\mathcal{O}^*_X \to \mathbb{W}_n\Omega^0_X$ and the differential $d : \mathbb{W}_n\Omega^0_X \to \mathbb{W}_n\Omega^1_X$, where X is the same as above. The bilinear map of sheaves:

$$dlog^2 : \mathcal{O}^*_X \times \mathcal{O}^*_X \to \mathbb{W}_n\Omega^2_X$$

given by the formula:

$$dlog^2(f \otimes g) = dlog^1(f)dlog^1(g)$$

also satisfies the Steinberg relation $dlog^2(f \otimes (1-f)) = 0$ for all $f \in \mathcal{O}^*_X$ with $1 - f \in \mathcal{O}^*_X$, hence it induces a map $dlog^2 : K_{2,X} \to \mathbb{W}_n\Omega^2_X$. Moreover let $\nu_n(k)$ denote the kernel of $1 - F$ on the degree k term $\mathbb{W}_n\Omega^k_X$ of the de Rham–Witt pro-complex on the topos X. Let $\mathbb{W}_n\Omega^i_{X,log}$ denote the abelian sub-sheaf generated by the image of $dlog^i$, where $i = 1, 2$. It is easy to see using the defining relations of the de Rham–Witt pro-complex that $\mathbb{W}_n\Omega^i_{X,\mathrm{log}}$ lies in $\nu_n(i)$.

We will need the following result which is a special case of the celebrated theorem in [6] due to Bloch, Gabber and Kato.

Theorem 6.2. *Let F be a field of characteristic p. Then the map*

$$K_2(F)/p^nK_2(F) \xrightarrow{dlog^2} H^0(F_{et}, \mathbb{W}_n\Omega^2_{F_{et},log})$$

is an isomorphism, where $H^0(F_{et}, \mathbb{W}_n\Omega^2_{F_{et},log})$ denotes the group of global sections of the sheaf $\mathbb{W}_n\Omega^2_{F_{et},log}$ on the étale site of the spectrum of F.

Proof. The map is well-defined as $\mathbb{W}_n\Omega^*$ is annihilated by p^n. The map is an isomorphism by Corollary 2.8 of [6], pages 117–118. □

Notation 6.3. Let $\mathbf{k}$ be a perfect field as in the previous two chapters. For every $\mathbf{k}$-scheme X let $\Omega^{\cdot}_X$ denote the complex of graded differential $\mathcal{O}_X$-algebras of $\mathbf{k}$-linear Kähler differential forms on X. Note that the complex $\Omega^{\cdot}_X$ is canonically isomorphic to the complex $\mathbb{W}_1\Omega^{\cdot}_X$. In particular there is a map $dlog^2 : K_{2,X} \to \Omega^{\cdot}_X$. For every $k \in \mathbb{N}$ and for every Cartier divisor D on X let $\Omega^k_X(D)$ denote the sheaf $\Omega^k_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$. Let $i : X - D \to X$ denote the open immersion of the complement of the support of D into X. Then the pull-back $i^*\Omega^k_X(D)$ is canonically isomorphic to Ω^k_{X-D}. Let

$$i^* : H^0(X, \Omega^k_X(D)) \to H^0(X - D, \Omega^k_{X-D})$$

denote the composition of the pull-back and this identification, too.

Lemma 6.4. *Assume that X is a smooth surface over $\mathbf{k}$ and D is a normal crossings divisor. Then the image of the map:*

$$dlog^2 : H^0(X - D, K_{2,X}) \to H^0(X - D, \Omega^2_{X-D})$$

lies in the image of the map $i^* : H^0(X, \Omega^k_X(D)) \to H^0(X - D, \Omega^k_{X-D})$ introduced above.

Proof. The claim is clearly local on X with respect to the Zariski topology hence we may assume that X is the spectrum of an integral regular $\mathbf{k}$-algebra A. We may also assume that D has at most one singular point and its branches are the zeros of some elements of A. The localization sequence for K-theory induces a complex:

$$H^0(X, K_{2,X}) \to H^0(X - D, K_{2,X}) \xrightarrow{T} \bigoplus_{C \in \mathcal{V}(\Gamma(D))} H^0(X - S(D), \mathcal{O}^*_C) \to \bigoplus_{e \in S(D)} \mathbb{Z}e$$

which is exact at the term $H^0(X - D, K_{2,S})$, where the second map is the direct sum of tame symbols along the irreducible components and the third map is the sum of the maps which assign to every element of $H^0(C - S(D), \mathcal{O}^*_C)$ its divisor considered as a zero cycle supported on $S(D)$ for every $C \in \mathcal{V}(\Gamma(D))$. Let k be an arbitrary element of $H^0(X - D, K_{2,X})$. Assume first that D is irreducible and let $t \in A$ be an element whose zero scheme is D. Pick an element $u \in A$ whose pull-back to D is equal to the tame symbol $T(k)$. By shrinking X further we may assume that $u \in A^*$. Then $T(k) = T(t \otimes u)$ hence $dlog^2(k - t \otimes u)$ is the pull-back of a differential form on X by the localization sequence. On the other hand $dlog^2(t \otimes u)$ clearly lies in the image of the map i^*. Assume now that D has one ordinary double point s and let $t_1, t_2 \in A$ be two elements whose zeros are the two branches of D. According to the complex above there is an $n \in \mathbb{Z}$ such that the valuations of the restrictions of $T(k)$ onto the zero scheme of t_1 and t_2 at s are n and $-n$, respectively. Hence by shrinking X further we may assume that there are $u_1, u_2 \in A^*$ such that the restrictions of $T(k)$ onto the zero scheme of t_1 and t_2 are the restrictions of $u_1 t_2^n$ and $u_2 t_1^{-n}$, respectively. Then we have:

$$T(k) = T(t_1 \otimes t_2)^n \cdot T(t_1 \otimes u_1) \cdot T(t_2 \otimes u_2)$$

and we may argue as above to conclude the proof. □

The lemma above has the following important corollary: because $X - D$ is Zariski-dense in X the map $i^* : H^0(X, \Omega^k_X(D)) \to H^0(X - D, \Omega^k_{X-D})$ is injective. Hence the map $dlog^2$ has a unique lift:

$$dlog^2 : H^0(X - D, K_{2,X}) \to H^0(X, \Omega^2_X(D))$$

which will be denoted by the same symbol by the usual abuse of notation.

Proposition 6.5. *Assume that $\mathfrak{X}$ is a smooth irreducible projective surface over $\mathbf{k}$ and the field $\mathbf{k}$ is finite of characteristic p. Then the group $H^0(\mathfrak{X}, K_{2,\mathfrak{X}})$ is the extension of a torsion group by its maximal p-divisible subgroup.*

Proof. Using the notation of [15] on pages 307 and 309 let $H^2(\mathfrak{X}, \mathbb{Z}(2))$ denote the projective limit $\varprojlim(H^0(\mathfrak{X}, \nu_n(2)))$. The logarithmic differentials

$$dlog^2 : K_2(\mathfrak{X}) \to H^0(\mathfrak{X}, \nu_n(2))$$

satisfy the obvious compatibility hence they induce a map

$$dlog^2 : K_2(\mathfrak{X}) \to H^2(\mathfrak{X}, \mathbb{Z}(2)).$$

Let $\mathcal{F}(\mathfrak{U})$ denote the function field of $\mathfrak{X}$. Let $\mathcal{P}$ denote the set of prime divisors of $\mathfrak{X}$ and for every $P \in \mathcal{P}$ let $\mathbf{f}_P$ denote the function field of the irreducible curve P. The localization sequence for K-theory furnishes an exact sequence:

$$0 \to H^0(\mathfrak{X}, K_{2,\mathfrak{X}}) \to K_2(\mathcal{F}(\mathfrak{X})) \xrightarrow{T} \bigoplus_{P \in \mathcal{P}} \mathbf{f}^*_P$$

where the second map is the direct sum of the tame symbols along the irreducible components. Every element $k \in H^0(\mathfrak{X}, K_{2,\mathfrak{X}})$ of the kernel of $dlog^2$ lies in $M(\mathcal{F}(\mathfrak{X})) = \bigcap_{n \in \mathbb{N}} p^n K_2(\mathcal{F}(\mathfrak{X}))$ by the Bloch–Gabber–Kato theorem. Since $K_2(\mathcal{F}(\mathfrak{X}))$ has no p-torsion by Theorem 1.10 of [23] on page 10, the group $M(\mathcal{F}(\mathfrak{X}))$ is the maximal p-divisible subgroup of $K_2(\mathcal{F}(\mathfrak{X}))$. If the element l is in $M(\mathcal{F}(\mathfrak{X})) \cap H^0(\mathfrak{X}, K_{2,\mathfrak{X}})$ and $k \in M(\mathcal{F}(\mathfrak{X}))$ is its unique p^n-th root then $T(k)$ is p^n-torsion by the localization sequence. But the group $\mathbf{f}^*_P$ has no non-zero p-torsion so

k lies in the image of $H^0(\mathfrak{X}, K_{2,\mathfrak{X}})$. Therefore we get that $M(\mathcal{F}(\mathfrak{X}))\cap H^0(\mathfrak{X}, K_{2,\mathfrak{X}})$, the kernel of the map $dlog^2$ in $H^0(\mathfrak{X}, K_{2,\mathfrak{X}})$, is p-divisible. Hence it will be sufficient to show that the group $H^2(\mathfrak{X}, \mathbb{Z}(2))$ is torsion. This is proved in [15] (the claim itself can be found on page 335) although the proof is somewhat dispersed over the article. It is an immediate consequence of Proposition 5.4 of the paper cited above on pages 330–331, the validity of Weil's conjectures for crystalline cohomology (Remark 5.5 of [15] on page 331), and the exact sequence on page 335 of the same paper. $\square$

Notation 6.6. Let $A \in \mathcal{C}$ be a local Artinian $\mathbf{k}$-algebra and let $\pi : \mathbb{P}^1_A \to \mathrm{Spec}(A)$ be the projective line over A. Let S be a finite set of sections $s : \mathrm{Spec}(A) \to \mathbb{P}^1_A$ and for every $s \in S$ let s_0 denote the $\mathbf{k}$-valued point $s_0 : \mathrm{Spec}(\mathbf{k}) \to \mathbb{P}^1_{\mathbf{k}}$ we get from s via base change. Assume that s_0 is different from t_0 for every pair $s, t \in S$ of different sections. For every $s \in S$ choose an A-algebra isomorphism ϕ_s between the completion $\widehat{\mathcal{O}}_{s_0,\mathbb{P}^1_A}$ of the stalk $\mathcal{O}_{s_0,\mathbb{P}^1_A}$ of the structure sheaf of $\mathbb{P}^1_A$ at s_0 and $A[[t]]$. The latter induces an isomorphism between the localization $\mathcal{L}_s$ of $\widehat{\mathcal{O}}_{s_0,\mathbb{P}^1_A}$ by the semigroup of those elements whose image under the canonical map $\widehat{\mathcal{O}}_{s_0,\mathbb{P}^1_A} \to \widehat{\mathcal{O}}_{s_0,\mathbb{P}^1_{\mathbf{k}}}$ is non-zero, where $\widehat{\mathcal{O}}_{s_0,\mathbb{P}^1_{\mathbf{k}}}$ is the completion of the stalk $\mathcal{O}_{s_0,\mathbb{P}^1_{\mathbf{k}}}$, and $A((t))$, which will be denoted by ϕ_s as well. The image of s is a locally principal closed subscheme of codimension one in $\mathbb{P}^1_A$ for every element s of S. Let S also denote the Cartier divisor which is the sum of these divisors by slight abuse of notation. For every $s \in S$ let Res^k_s denote the composition of the map:

$$H^k_s : H^0(\mathbb{P}^1_A - S, \Omega^k_{\mathbb{P}^1_A - S}) \to \Omega^k_{\mathcal{L}_s} \xrightarrow{\Omega^k(\phi_s)} \Omega^k_{A((t))},$$

where the first arrow is induced by the tautological map $\mathbb{P}^1_A - S \to \mathrm{Spec}(\mathcal{L}_s)$, and the residue $\mathrm{Res}^k : \Omega^k_{A((t))} \to \Omega^{k-1}_A$. By claim (iv) of Proposition 5.2 the map:

$$\mathrm{Res}^k_s : H^0(\mathbb{P}^1_A - S, \Omega^k_{\mathbb{P}^1_A - S}) \to \Omega^{k-1}_A$$

is independent of the choice of ϕ_s. Recall that there is a canonical inclusion $H^0(\mathbb{P}^1_A, \Omega^2_{\mathbb{P}^1_A}(S)) \subset H^0(\mathbb{P}^1_A - S, \Omega^k_{\mathbb{P}^1_A - S})$.

Proposition 6.7. *The sequence:*

$$0 \to \Omega^2_A \xrightarrow{\pi^*} H^0(\mathbb{P}^1_A, \Omega^2_{\mathbb{P}^1_A}(S)) \xrightarrow{\oplus_{s\in S}\mathrm{Res}^2_s} \bigoplus_{s\in S} \Omega^1_A \xrightarrow{\sum_{s\in S}(\cdot)} \Omega^1_A \to 0$$

is exact where π^* is the pull-back with respect to the map $\pi : \mathbb{P}^1_A \to \mathrm{Spec}(A)$.

Proof. By base change we may assume that $\mathbf{k}$ is algebraically closed, which implies that it is infinite. Let $R \supseteq S$ be any finite set. Note that for every $\omega \in \frac{1}{t}\Omega^2_{A[[t]]}$ we have $\omega \in \Omega^2_{A[[t]]}$ if and only if $\mathrm{Res}^2(\omega) = 0$. Therefore

$$H^0(\mathbb{P}^1_A, \Omega^2_{\mathbb{P}^1_A}(S)) = \{\omega \in H^0(\mathbb{P}^1_A, \Omega^2_{\mathbb{P}^1_A}(R)) \mid \mathrm{Res}^2_s(\omega) = 0 \ (\forall s \in R - S)\}.$$

Hence it is sufficient to prove the proposition above for R instead of S. In particular we may assume that the point at infinity $\infty \in \mathbb{P}^1_A$ lies in S after a suitable automorphism of the A-scheme $\mathbb{P}^1_A$. Let x be the coordinate function of the affine line $\mathbb{A}^1_A = \mathbb{P}^1_A - \infty$. For every $\infty \neq s \in S$ let the same letter denote the unique element of A such that the image of the section s is the zero scheme of $x - s \in A[x]$. Every $\omega \in H^0(\mathbb{P}^1_A - S, \Omega^2_{\mathbb{P}^1_A - S})$ can be written uniquely in the form:

$$\omega = \omega_0 + \sum_{s \in S - \infty} \sum_{k=1}^{n(s)} \frac{\omega_{s,k}}{(x-s)^k} dx + \sum_{j=0}^{n(\infty)} \omega_{\infty,j} x^j dx$$

where $\omega_0 \in \Omega^2_A$, $n(s), n(\infty) \in \mathbb{N}$ and $\omega_{s,k}, \omega_{\infty,j} \in \Omega^1_A$. For every $\infty \neq s \in S$ we may assume that $x - s$ maps to t with respect to ϕ_s. Then it is obvious that

$$H^2_s(\eta x^n dx), H^2_s(\eta (x-r)^{-n} dx) \in \Omega^2_{A[[t]]}$$

for every $\eta \in \Omega^1_A$, $n \in \mathbb{N}$ and $s \neq r \in S - \infty$. Therefore we have $\omega_{s,k} = 0$ for every $k > 1$ when $\omega \in H^0(\mathbb{P}^1_A, \Omega^2_{\mathbb{P}^1_A}(S))$. We may assume also that x^{-1} maps to t with respect to ϕ_∞. In this case it is obvious that

$$H^2_\infty\left(\omega_{s,1} \frac{dx}{x-s}\right) = -\omega_{s,1}\frac{dt}{t} + \eta_s$$

for some $\eta_s \in \Omega^2_{A[[t]]}$ for every $s \in S - \infty$ but

$$H^2_\infty(\omega_{\infty,j} x^j dx) = -\omega_{\infty,j} t^{-j-2} dt$$

for every $j = 0, 1, \ldots, n(\infty)$ so we must have:

$$\omega = \omega_0 + \sum_{s \in S - \infty} \frac{\omega_{s,1}}{x-s} dx.$$

By the above $\mathrm{Res}^2_s(\omega) = \omega_{s,1}$ for every $s \in S - \infty$ and $\mathrm{Res}^2_\infty(\omega) = -\sum_{s \in S - \infty} \omega_{s,1}$ so the claim is now obvious. $\square$

Notation 6.8. Now we are going to consider the same situation that we looked at in the introduction. Let B be a smooth irreducible projective curve over $\mathbf{k}$ and let $\pi : \mathfrak{X} \to B$ be a regular irreducible projective surface fibred over B such that the fiber $\mathfrak{X}_\infty$ of $\mathfrak{X}$ over the closed point ∞ of B is totally degenerate. Then the base change X of $\mathfrak{X}$ to the completion F of the function field of B with respect to the valuation corresponding to ∞ is a Mumford curve over F. Let $\mathfrak{U} \subset \mathfrak{X}$ be an open subvariety such that its complement is a normal crossings divisor D which is the pre-image of a finite set of closed points of B containing ∞. The base change of $\mathfrak{X}$ to the valuation ring of F is a semi-stable model of X whose fiber is $\mathfrak{X}_\infty$ hence the rigid analytic regulator $\{\cdot\}$ introduced in Definition 5.12 of [17] supplies a diagram:

$$H^0(\mathfrak{U},, K_{2,\mathfrak{U}}) \to H^2_{\mathcal{M}}(X,\mathbb{Z}(2)) \xrightarrow{\{\cdot\}} \mathcal{H}(\Gamma(\mathfrak{X}_\infty), F^*),$$

where the first homomorphism is induced by functoriality. This composition will be denoted by the symbol $\{\cdot\}$ as well.

Definition 6.9. For every $\omega \in H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}}(D))$ we are going to define a function $\mathrm{Res}(\omega) : \mathcal{E}(\Gamma(\mathfrak{X}_\infty)) \to \widehat{\Omega}^1_F$ as follows. Fix an edge $e \in \mathcal{E}(\Gamma(\mathfrak{X}_\infty))$ and let $s \in S(\mathfrak{X}_\infty)$ denote the image of e under the normalization map. Let C be the irreducible component of $\mathfrak{X}_\infty$ which corresponds to the initial vertex of e under the identification of Notation 1.2. Let $\widehat{\mathcal{O}}_{s,\mathfrak{X}}$ denote the completion of the stalk $\mathcal{O}_{s,\mathfrak{X}}$ of the structure sheaf of $\mathfrak{X}$ at s and let $t \in \mathcal{O}_{s,\mathfrak{X}}$ be an element whose zero scheme is the germ of the curve C. Because t generates a prime ideal in $\widehat{\mathcal{O}}_{s,\mathfrak{X}}$ the latter gives rise to a discrete valuation on the quotient field M_e of $\widehat{\mathcal{O}}_{s,\mathfrak{X}}$. Let $\widehat{M_e}$ denote the completion of M_e with respect to this valuation and let $i_e : \mathrm{Spec}(\widehat{M_e}) \to \mathfrak{X}$ denote the tautological map. Note that the closure of the image of the stalk $\mathcal{O}_{\infty,B}$ of the structure sheaf of B at ∞ in $\mathcal{O}_{s,\mathfrak{X}}$ with respect to the map induced by $\pi : \mathfrak{X} \to B$ in $\widehat{\mathcal{O}}_{s,\mathfrak{X}}$ is canonically isomorphic to the valuation ring $\mathcal{O}$ of F. Hence $\widehat{M_e}$ is canonically equipped with the structure of an F-algebra. Let $\phi : \widehat{M_e} \to \widehat{M}$ be the unique valuation-preserving F-algebra homomorphism such that $\phi(t) = x$ where we continue to use the notation of the previous chapter. Note that $q_2(\Omega(\phi)(i_e^*(\omega))) \in \widehat{\Omega}^k_{\widehat{M}}(log)$ where $q_k : \Omega^k_{\widehat{M}} \to \widehat{\Omega}^k_{\widehat{M}}$ is the quotient map. Hence the value:

$$\mathrm{Res}(\omega)(e) = \mathrm{Res}_2(q_2(\Omega(\phi)(i_e^*(\omega)))) \in \widehat{\Omega}^1_F$$

is well-defined and it is independent of the choice of the element t by Remark 5.7.

For every oriented graph G and commutative group R let $\mathcal{F}(G,R)$ denote the group of functions $f : \mathcal{E}(G) \to R$.

Theorem 6.10. *We have* $\mathrm{Res}(dlog^2(k)) \in \mathcal{H}(\Gamma(\mathfrak{X}_0), \widehat{\Omega}^1_F)$ *for every* $k \in H^0(\mathfrak{U}, K_{2,\mathfrak{U}})$ *and the diagram:*

$$\begin{array}{ccc} H^0(\mathfrak{U}, K_{2,\mathfrak{U}}) & \xrightarrow{\{\cdot\}} & \mathcal{H}(\Gamma(\mathfrak{X}_\infty), F^*) \\ \Big\downarrow{\scriptstyle dlog^2} & & \Big\downarrow{\scriptstyle dlog} \\ H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}}(D)) & \xrightarrow{\mathrm{Res}} & \mathcal{F}(\Gamma(\mathfrak{X}_\infty), \widehat{\Omega}^1_F) \end{array}$$

is commutative.

Proof. We are going to show that $dlog(\{k\}(e)) = \mathrm{Res}(dlog^2(k))(e)$ for every edge $e \in \mathcal{E}(\Gamma(\mathfrak{X}_\infty))$. Then the theorem will follow immediately because $\{k\}$ is a harmonic cochain. The identity above follows immediately from Theorem 5.8 and the following alternate description of the rigid analytic regulator. The pull-back of k with respect to i_e is an element $i_e^*(k) \in K_2(\widehat{M}_e)$. Let $\phi_* : K_2(\widehat{M}_e) \to K_2(\widehat{M})$ be the homomorphism induced by ϕ. Then we have $\{k\}(e) = \{\phi_*(i_e^*(k))\}_D$. □

Proposition 6.11. *Let k be an element of $H^0(\mathfrak{U}, K_{2,\mathfrak{U}})$ such that* $\mathrm{Res}(dlog^2(k)) = 0$. *Then $dlog^2(k) = 0$, too.*

Proof. Let $x \in F$ be a uniformizer. The closed subscheme of B defined by the n-th power of the defining sheaf of ideals of the closed subscheme ∞ is canonically isomorphic to $\mathrm{Spec}(\mathcal{O}_n)$ where $\mathcal{O}_n = \mathcal{O}/x^n\mathcal{O}$ as in chapter 5. Let $i_n : \mathrm{Spec}(\mathcal{O}_n) \to B$ be the closed immersion corresponding to this isomorphism. For every irreducible component $C \in \mathcal{V}(\Gamma(\mathfrak{X}_\infty))$ let C_n denote the closed subscheme of $\mathfrak{X}$ defined by the n-th power of the defining sheaf of ideals of the closed subscheme C. Let $c_n : C_n \to \mathfrak{X}$ be the closed immersion. Then there is a unique morphism $p_n : C_n \to \mathrm{Spec}(\mathcal{O}_n)$ such that $c_n \circ \pi = p_n \circ i_n$. As an $\mathcal{O}_n$-scheme C_n is isomorphic to the projective line over $\mathrm{Spec}(\mathcal{O}_n)$. Let S denote the Cartier divisor on C_n which is the pull-back of the divisor on $\mathfrak{X}$ that is the sum of those irreducible components of $\mathfrak{X}_\infty$ which are intersecting C with respect to the map c_n and are different from C. Then S is the sum of images of sections of the map p_n. Let C_0 be the divisor of the element $x \in \mathcal{O}_n \subset H^0(C_n, \mathcal{O}_{C_n})$. Multiplication by x induces a map $\mathcal{O}(S + C_0) \to \mathcal{O}(S)$. By our assumptions the residues of $xc_n^*(dlog^2(k)) \in H^0(C_n, \Omega^2_{C_n}(S))$ introduced in Definition 6.6 are all zero. Hence $xc_n^*(dlog^2(k)) \in \Omega^2_{\mathcal{O}_n}$ by Proposition 6.7. But $\Omega^2_{\mathcal{O}_n} = 0$ hence we get that the formal completion of $dlog^2(k)$ along the closed scheme $\mathfrak{X}_\infty$ must be zero. The claim is now clear. □

Assume now that $\mathbf{k}$ is a field of characteristic p.

Corollary 6.12. *The map:*

$$\{\cdot\} : H^0(\mathfrak{U}, K_{2,\mathfrak{U}})/p^n H^0(\mathfrak{U}, K_{2,\mathfrak{U}}) \to H^0(\Gamma(\mathfrak{X}_\infty), F^*/(F^*)^{p^n})$$

induced by the regulator $\{\cdot\}$ is injective for every natural number n.

Proof. We are going to prove the claim by induction on n. Let $\mathcal{F}(\mathfrak{U})$ denote the function field of $\mathfrak{U}$. Assume first that $n = 1$ and let $k \in H^0(\mathfrak{U}, K_{2,\mathfrak{U}})$ be an element such that $\{k\} \in \mathcal{H}(\Gamma(\mathfrak{X}_\infty), (F^*)^p)$. By Theorem 6.10 we have:

$$\mathrm{Res}(dlog^2(k)) = dlog \circ \{k\} = 0,$$

hence $dlog^2(k) = 0$ by Proposition 6.11. Therefore $k = pl$ for some $l \in K_2(\mathcal{F}(\mathfrak{U}))$ by the Bloch–Gabber–Kato theorem. Let $\mathcal{P}$ denote the set of prime divisors of $\mathfrak{U}$ and for every $P \in \mathcal{P}$ let $\mathbf{f}_P$ denote the function field of the irreducible curve P. The localization sequence for K-theory furnishes an exact sequence:

$$0 \to H^0(\mathfrak{U}, K_{2,\mathfrak{U}}) \to K_2(\mathcal{F}(\mathfrak{U})) \to \bigoplus_{P \in \mathcal{P}} \mathbf{f}_P^*$$

where the second map is the direct sum of tame symbols along the irreducible components. The image of l with respect to the second map is p-torsion. But the group $\mathbf{f}_P^*$ has no non-zero p-torsion so l is the image of an element of $H^0(\mathfrak{U}, K_{2,\mathfrak{U}})$. Assume now that the claim is proved for $n-1$ and let $k \in H^0(\mathfrak{U}, K_{2,\mathfrak{U}})$ be an element such that $\{k\} \in \mathcal{H}(\Gamma(\mathfrak{X}_\infty), (F^*)^{p^n})$. By the induction hypothesis there is an element $l \in H^0(\mathfrak{U}, K_{2,\mathfrak{U}})$ such that $k = p^{n-1}l$. Because the group F^* has no p-torsion we have $\{l\} \in \mathcal{H}(\Gamma(\mathfrak{X}_\infty), (F^*)^p)$ therefore $l \in pH^0(\mathfrak{U}, K_{2,\mathfrak{U}})$ by the above. Hence $k \in p^n H^0(\mathfrak{U}, K_{2,\mathfrak{U}})$ as we wished to prove. □

Assume now that $\mathbf{k}$ is a finite field of characteristic p.

Theorem 6.13. *The following holds:*

(i) *the quotient group* $H^0(\mathfrak{U}, K_{2,\mathfrak{U}})/H^0(\mathfrak{X}, K_{2,\mathfrak{X}})$ *is a finitely generated abelian group whose rank is at most as large as the rank of the group* $\mathcal{H}(\Gamma(D), \mathbb{Z})$.

(ii) *the kernel* $\mathrm{Ker}(\{\cdot\})$ *of the regulator* $\{\cdot\} : H^0(\mathfrak{U}, K_{2,\mathfrak{U}}) \to \mathcal{H}(\Gamma(\mathfrak{X}_\infty), F^*)$ *has a subgroup of finite index which lies in* $H^0(\mathfrak{X}, K_{2,\mathfrak{X}})$,

(iii) *the kernel* $\mathrm{Ker}(\{\cdot\})$ *above is p-divisible. It is torsion if Parshin's conjecture holds, and it is finite if the Bass conjecture holds,*

(iv) *the image* $\mathrm{Im}(\{\cdot\})$ *of the regulator* $\{\cdot\} : H^0(\mathfrak{U}, K_{2,\mathfrak{U}}) \to \mathcal{H}(\Gamma(\mathfrak{X}_\infty), F^*)$ *is p-saturated,*

(v) *the rank of* $\mathrm{Im}(\{\cdot\})$ *is at most as large as the rank of the group* $\mathcal{H}(\Gamma(D), \mathbb{Z})$,

(vi) *the image* $\mathrm{Im}(\{\cdot\})$ *is discrete if* $D = \mathfrak{X}_\infty$.

Proof. Let us recall that $S(D)$, $\mathcal{V}(\Gamma(D))$ denote the set of singular points and the set of irreducible components of the curve D, respectively. The localization sequence for K-theory induces a complex:

$$H^0(\mathfrak{X}, K_{2,\mathfrak{X}}) \to H^0(\mathfrak{U}, K_{2,\mathfrak{U}}) \xrightarrow{T} \bigoplus_{C \in \mathcal{V}(\Gamma(D))} H^0(C - S(D), \mathcal{O}_C^*) \to \bigoplus_{e \in S(D)} \mathbb{Z}e$$

which is exact at the term $H^0(\mathfrak{U}, K_{2,\mathfrak{U}})$, where the second map is the direct sum of tame symbols along the irreducible components and the third map is the sum of the maps which for every $C \in \mathcal{V}(\Gamma(D))$ assign to every element of $H^0(C - S(D), \mathcal{O}_C^*)$

its divisor considered as a zero cycle supported on $S(D)$. The kernel of the latter is a finitely generated abelian group of rank $\mathcal{H}(\Gamma(D), \mathbb{Z})$ hence claim (i) is clear. By Corollary 6.9 the kernel $\mathrm{Ker}(\{\cdot\})$ of the map:

$$\{\cdot\} : H^0(\mathfrak{U}, K_{2,\mathfrak{U}}) \to \mathcal{H}(\Gamma(\mathfrak{X}_\infty), F^*)$$

is p-divisible. Therefore its image with respect to the map T above is finite because the maximal p-divisible subgroup of a finitely generated abelian group is finite. Hence the kernel of T in $\mathrm{Ker}(\{\cdot\})$ is a subgroup of finite index which lies in $H^0(\mathfrak{X}, K_{2,\mathfrak{X}})$. Therefore claim (ii) holds.

According to Parshin's conjecture the group $H^0(\mathfrak{X}, K_{2,\mathfrak{X}})$ should be torsion. Then the same is true for $\mathrm{Ker}(\{\cdot\}) \cap H^0(\mathfrak{X}, K_{2,\mathfrak{X}})$ and therefore $\mathrm{Ker}(\{\cdot\})$ is torsion as well by claim (ii). The Bass conjecture states that $H^0(\mathfrak{X}, K_{2,\mathfrak{X}})$ should be a finitely generated abelian group. Hence the same is true for its subgroup $\mathrm{Ker}(\{\cdot\}) \cap H^0(\mathfrak{X}, K_{2,\mathfrak{X}})$. Note that this group is also p-divisible: every element of $\mathrm{Ker}(\{\cdot\}) \cap H^0(\mathfrak{X}, K_{2,\mathfrak{X}})$ has a p-th root in $\mathrm{Ker}(\{\cdot\}) \subseteq H^0(\mathfrak{U}, K_{2,\mathfrak{U}})$. On the other hand if $px \in H^0(\mathfrak{X}, K_{2,\mathfrak{X}})$ for some $x \in H^0(\mathfrak{U}, K_{2,\mathfrak{U}})$ then $x \in H^0(\mathfrak{X}, K_{2,\mathfrak{X}})$ using the localization sequence the same way we did in the proof of Corollary 6.12 already. Hence $\mathrm{Ker}(\{\cdot\}) \cap H^0(\mathfrak{X}, K_{2,\mathfrak{X}})$ is a finite group whose order is not divisible by p so $\mathrm{Ker}(\{\cdot\})$ is finite as well by claim (ii). Claim (iii) is now proved.

Because the maximal p-divisible subgroup of $\mathcal{H}(\Gamma(\mathfrak{X}_\infty), F^*)$ is finite the image of $H^0(\mathfrak{X}, K_{2,\mathfrak{X}})$ with respect to the rigid analytic regulator is torsion by Proposition 6.5. But the torsion of $\mathcal{H}(\Gamma(\mathfrak{X}_\infty), F^*)$ is finite so $\mathrm{Im}(\{\cdot\})$ is finitely generated and claim (v) is true by claim (i). On the other hand note that a finitely generated subgroup $\Lambda \subset \mathcal{H}(\Gamma(\mathfrak{X}_\infty), F^*)$ is p-saturated if and only if

$$p^n\Lambda = \Lambda \cap p^n\mathcal{H}(\Gamma(\mathfrak{X}_\infty), F^*)$$

for every $n \in \mathbb{N}$. The latter holds for $\mathrm{Im}(\{\cdot\})$ by Corollary 6.12 so claim (iv) is true. Let $\mathrm{Reg} : H^0(\mathfrak{U}, K_{2,\mathfrak{U}}) \to \mathcal{H}(\Gamma(\mathfrak{X}_\infty), \mathbb{Z})$ denote the tame regulator which is defined as follows. For every $k \in H^0(\mathfrak{U}, K_{2,\mathfrak{U}})$ and for every edge $e \in \mathcal{E}(\Gamma(\mathfrak{X}_\infty))$ we define $\mathrm{Reg}(k)(e)$ as the valuation of the tame symbol of k along the irreducible component $o(e)$ of $\mathfrak{X}_\infty$ with respect to the valuation corresponding to the closed point which is the image of e with respect to the normalization map. By Theorem 5.6 of [17] the diagram:

$$\begin{tikzcd} H^0(\mathfrak{U}, K_{2,\mathfrak{U}}) \arrow[r, "\{\cdot\}"] \arrow[d, "\mathrm{Reg}"] & \mathcal{H}(\Gamma(\mathfrak{X}_\infty), F^*) \arrow[d, "v"] \\ \mathcal{H}(\Gamma(\mathfrak{X}_\infty), \mathbb{Z}) \arrow[r, equal] & \mathcal{H}(\Gamma(\mathfrak{X}_\infty), \mathbb{Z}) \end{tikzcd}$$

commutes where v is the map induced by the normalized valuation on F. If $D = \mathfrak{X}_\infty$ then the kernel of Reg contains $H^0(\mathfrak{X}, K_{2,\mathfrak{X}})$ as a subgroup of finite index according to the complex we wrote down above. Since $H^0(\mathfrak{X}, K_{2,\mathfrak{X}})$ is p-divisible its image with respect to the regulator $\{\cdot\}$ is finite. Hence the kernel of the map v in $\mathrm{Im}(\{\cdot\})$ is finite, too. Therefore $\mathrm{Im}(\{\cdot\})$ must be discrete as claim (vi) says. □

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