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# The Rigid Analytical Regulator and $K_2$ of Drinfeld Modular Curves

by

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**Abstract**

We evaluate a rigid analytical analogue of the Beilinson–Bloch–Deligne regulator on certain explicit elements in the $K_2$ of Drinfeld modular curves, constructed from analogues of modular units, and relate its value to special values of $L$-series using the Rankin–Selberg method.

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## 1. Introduction

**Motivation 1.1.** In the paper [10] which is now classical B. Gross formulated a generalization of his original $p$-adic analogue of Stark's conjecture in a form which makes good sense both over number fields and function fields. This conjecture was proved by D. Hayes for function fields in [12]. In this paper Hayes gave an explicit rigid analytical construction of Stark units and expressed them in terms of special values of $L$-functions using this explicit construction. This paper is part of the project to formulate and prove results which generalize Hayes's theorem the same way as Beilinson's conjectures generalize Stark's. In a previous paper ([22]) we constructed a rigid analytical regulator analogous to the classical Beilinson–Bloch–Deligne regulator refining the tame regulator in case of Mumford curves. In our current work we express the value of this regulator on certain explicit elements of the $K_2$ group of Drinfeld modular curves, which are analogues of A. Beilinson's construction using modular units, in terms of special values of $L$-functions. Using

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the function field analogue of the Shimura–Taniyama–Weil conjecture we derive a formula for every elliptic curve defined over the rational function field of transcendence degree one over a finite field having split multiplicative reduction at the point at infinity analogous to the classical theorem of Beilinson on the $K_2$ of elliptic curves defined over the rational number field.

In the rest of this introductory chapter we first describe the rigid analytic regulator for Tate elliptic curves, then define the $\infty$-adic $L$-function of elliptic curves of the type mentioned above and formulate our main theorem.

**Notation 1.2.** Let $F_\infty$ be a field complete with respect to a discrete valuation and let $\mathcal{O}_\infty$ be its valuation ring. There is a canonical way to extend the absolute value of $F_\infty$ induced by its valuation to its algebraic closure. Let $\mathbb{C}_\infty$ denote the completion of the algebraic closure of $F_\infty$ with respect to this absolute value and let $|\cdot|$ denote the absolute value induced by the completion process. Let $|\mathbb{C}_\infty|$ denote the set of values of the latter. Let $\mathbb{P}^1$ denote the projective line over $\mathbb{C}_\infty$. We call a set $D \subset \mathbb{P}^1$ an *open disc* if it is the image of the set $\{z \in \mathbb{C}_\infty \mid |z| < 1\}$ under a Möbius transformation. Recall that a subset $U$ of $\mathbb{P}^1$ is a *connected rational subdomain* if it is non-empty and it is the complement of the union of finitely many pairwise disjoint open discs. Let $\partial U$ denote the set of these complementary open discs. Let $\mathcal{O}(U)$ and $\mathcal{O}^*(U)$ denote the algebra of holomorphic functions on $U$ and the group of invertible elements of this algebra, respectively. For each $f \in \mathcal{O}(U)$ let $\|f\|$ denote $\sup_{z\in U} |f(z)|$. This is a finite number, and makes $\mathcal{O}(U)$ a Banach algebra over $\mathbb{C}_\infty$. The latter is the closure of the subalgebra of restrictions of rational functions with respect to the supremum norm $\|\cdot\|$ by definition. For every real number $0 < \epsilon < 1$ we define the sets $\mathcal{O}_\epsilon(U) = \{f \in \mathcal{O}(U) \mid \|1-f\| \leq \epsilon\}$ and $\mathcal{U}_\epsilon = \{z \in \mathbb{C}_\infty \mid |1-z| \leq \epsilon\}$. Recall that a function $f : \mathbb{C}_\infty^* \to \mathbb{C}_\infty$ is holomorphic if its restriction $f|_U$ is holomorphic for every connected rational subdomain $U \subset \mathbb{C}_\infty^*$. For every $x \in \mathbb{P}^1$ and every pair of rational non-zero functions $f, g \in \mathbb{C}_\infty((t))$ on the projective line let $\{f, g\}_x$ denote the tame symbol of the pair $(f, g)$ at $x$. Let $\mathcal{M}(\mathbb{C}_\infty^*)$ denote the field of meromorphic functions of $\mathbb{C}_\infty^*$. For every field $L$ let $K_2(L)$ denote the Milnor $K_2$ of the field $L$. Finally for every $x \in \mathbb{C}_\infty$ and positive number $\rho \in |\mathbb{C}_\infty|$ let $D(x, \rho)$ denote the open disc $\{z \in \mathbb{C}_\infty \mid |z - x| < \rho\}$. The following result is an immediate consequence of the results of [22].

**Theorem 1.3.** *For every $0 < r \in |\mathbb{C}_\infty|$ there is a unique homomorphism:*

$$\{\cdot\}_r : K_2(\mathcal{M}(\mathbb{C}_\infty^*)) \to \mathbb{C}_\infty^*$$

*with the following properties:*

(i) *for every pair of rational functions* $f, g \in \mathcal{M}(\mathbb{C}_\infty^*)^*$ *we have:*

$$\{f \otimes g\}_r = \prod_{x \in D(0,r)} \{f, g\}_x,$$

(ii) *for every real number* $0 < \epsilon < 1$ *and functions* $f \in \mathcal{M}(\mathbb{C}_\infty^*) \cap \mathcal{O}_\epsilon(U)$ *and* $g \in \mathcal{M}(\mathbb{C}_\infty^*) \cap \mathcal{O}^*(U)$ *we have* $\{f, g\}_r \in \mathcal{U}_\epsilon$ *where* $U$ *is a connected rational subdomain in* $\mathbb{C}_\infty^*$ *such that* $D(0, r) \in \partial U$. □

**Notation 1.4.** For every field $K$, for any variety $V$ defined over $K$ and for any extension $L$ of $K$ let $V_L$ denote the base change of $V$ to $L$. For every field $K$ and regular irreducible projective curve $C$ defined over $K$ let $\mathcal{F}(C)$ denote the function field of the curve $C$ over $K$. For every closed point $x$ of $C$ there is a tame symbol at $x$ which is a homomorphism from $K_2(\mathcal{F}(C))$ into the multiplicative group of the residue field at $x$. We define the group $K_2(C)$ as the intersection of the kernels of all tame symbols. (In this paper we will sometimes use the somewhat incorrect notation $K_2(X)$ to denote $H^2_{\mathcal{M}}(X, \mathbb{Z}(2))$ for various types of spaces $X$ as the latter is rather awkward.) Let $E$ be an elliptic curve defined over $F_\infty$ which has a rigid-analytic Tate uniformization over $F_\infty$. The latter is equivalent to the property that the special fiber of the Néron model of $E$ over the spectrum of $\mathcal{O}_\infty$ is split multiplicative. Let $\theta : \mathbb{C}_\infty^* \to E(\mathbb{C}_\infty)$ be the Tate uniformization (over $\mathbb{C}_\infty$). It induces a homomorphism

$$\theta^* : \mathcal{F}(E_{F_\infty}) \to \mathcal{M}(\mathbb{C}_\infty^*)$$

by pull-back which in turn induces a homomorphism $K_2(\mathcal{F}(E_{F_\infty})) \to K_2(\mathcal{M}(\mathbb{C}_\infty^*))$ which will be denoted by the same symbol by slight abuse of notation.

**Proposition 1.5.** *For every* $k \in K_2(E_{F_\infty})$ *and* $0 < r \in |\mathbb{C}_\infty|$ *we have* $\{\theta^*(k)\}_r \in F_\infty^*$ *and the latter is independent of the choice of* $r$. □

Let $\{\cdot\} : K_2(E_{F_\infty}) \to F_\infty^*$ denote the homomorphism defined by the common value of the regulators $\{\theta^*(\cdot)\}_r$.

**Definition 1.6.** For every field $K$ let $\overline{K}$ denote its separable closure. Let $F$ denote the function field of $X$, where the latter is a geometrically connected smooth projective curve defined over the finite field $\mathbb{F}_q$ of characteristic $p$. Fix a closed point $\infty$ of the curve $X$ and let $E$ be an elliptic curve defined over $F$ which has split multiplicative reduction at $\infty$. For every closed point $x$ of $X$ let $\deg(x)$ and $L_x(E, t)$ denote the degree of $x$ and the local factor of the Hasse–Weil $L$-function of $E$ at $x$, respectively. The latter is an element of $\mathbb{Z}[[t]]$. Let $\psi_E^*(x^n) \in \mathbb{Z}$ denote the unique number such that

$$L_x(E,t)=\sum_{n=0}^{\infty}\psi_E^*(x^n)t^{n\deg(x)}.$$

Let $K$ be a number field and let $\Delta$ denote its ring of integers. Let $\chi:\mathrm{Gal}(\overline{F}|F)\to K^*$ be a $K$-valued one-dimensional Galois representation of $F$ which has finite image. Note that $\chi$ is automatically almost everywhere unramified and its image lies in $\Delta^*$. Let $\Gamma$ denote the quotient of $\mathrm{Gal}(\overline{F}|F)$ by the kernel of $\chi$. Assume that $\chi$ splits at $\infty$ and let $\mathfrak{m}$ be an effective divisor whose support does not contain $\infty$ and the conductor of $\chi$ and $E$ divides $\mathfrak{m}$ and $\mathfrak{m}\infty$, respectively. (Note that such an $\mathfrak{m}$ exists because we assumed that $E$ has split multiplicative reduction at $\infty$.) For every Galois group $G$ of a finite abelian extension of $F$ and for every closed point $x$ of $X$ where $G$ is unramified let $\phi_x^G$ denote the image of a geometric Frobenius at $x$ in $G$. The element $\phi_x^G\in G$ is well-defined as $G$ is abelian. Assume now that $G$ is the Galois group of a finite abelian extension of $F$ which only ramifies at $\infty$. We define the $L$-function $\mathcal{L}_{\mathfrak{m}}^G(E,\chi,t)$ as the Euler product:

$$\mathcal{L}_{\mathfrak{m}}^G(E,\chi,t)=\prod_{x\notin\mathrm{supp}(\mathfrak{m}\infty)}\Big(\sum_{n=0}^{\infty}\psi_E^*(x^n)\chi(\phi_x^\Gamma)(\phi_x^G)^n t^{n\deg(x)}\Big)\in\Delta[G][[t]],$$

where $\mathrm{supp}(\mathfrak{d})$ denotes the support of any effective divisor $\mathfrak{d}$ on $X$. The infinite product $\mathcal{L}_{\mathfrak{m}}^G(E,\chi,t)$ is well-defined, as the constant term of every factor appearing in the product is 1, and there are only finitely many factors with a term of degree less than $m$ for any positive integer $m$. Actually even more is true:

**Proposition 1.7.** *The power series $\mathcal{L}_{\mathfrak{m}}^G(E,\chi,t)$ is an element of $\Delta[G][t]$.*

**Definition 1.8.** An important consequence of the proposition above is that the polynomial $\mathcal{L}_{\mathfrak{m}}^G(E,\chi,t)$ can be evaluated at 1, i.e. the element $\mathcal{L}_{\mathfrak{m}}^G(E,\chi,1)\in\Delta[G]$ is well-defined. Let $G_\infty$ denote the Galois group of the maximal abelian extension of $F$ unramified at every closed point $x$ of $X$ different from $\infty$. It is a profinite group. Also note that if $H$ denotes the Galois group of the maximal abelian extension of $F$ unramified at every closed point $x$ of $X$ and totally split at $\infty$, then the kernel of the natural projection $G_\infty\to H$ is canonically isomorphic to the profinite completion of $F_\infty^*/\mathbb{F}_q^*$, the multiplicative group of the completion $F_\infty$ of $F$ with respect to the valuation at $\infty$ divided out by the multiplicative group of the constant field of $X$. (Note that this notation is compatible with what we have introduced in 1.2 and 1.4.) For any ring $R$ and abelian profinite group $M$ let $R[[M]]$ denote the $R$-dual of the ring of continuous functions $f:M\to R$, where $f$ is continuous with respect to the discrete topology on $R$ and the Krull topology on $M$. The ring $R[[M]]$ is also the projective limit of $R$-coefficient group rings of the finite quotients of $M$. The elements $\mathcal{L}^G_{\mathfrak{m}}(E,\chi,1)$ satisfy the obvious compatibility, so their limit defines an element $\mathcal{L}_{\mathfrak{m}}(E,\chi)$ in $\Delta[[G_\infty]]$, which we will call the $\infty$*-adic $L$-function of $E$ twisted with* $\chi$. For every $M$ as above let $I_M \triangleleft \Delta[[M]]$ denote the kernel of the natural augmentation map $\Delta[[M]] \to \Delta$. We will usually drop the subscript $M$ to ease notation. It is known that the group $I_M/I_M^2$ is naturally isomorphic to $M \otimes \Delta$. Finally let $\theta' \in M \otimes \Delta$ denote the class of any $\theta \in I_M$ in $I_M/I_M^2$.

**Proposition 1.9.** *We have $\mathcal{L}_{\mathfrak{m}}(E,\chi) \in I$ and $\mathcal{L}_{\mathfrak{m}}(E,\chi)' \in F^*_\infty/\mathbb{F}^*_q \otimes \Delta$.*

Let $L$ denote the Galois extension of $F$ whose Galois group is $\Gamma$. By our assumptions the field $L$ has an imbedding into $F_\infty$ which extends the canonical inclusion $F \subset F_\infty$. Fix once and for all such an imbedding. By slight abuse of notation let $\{\cdot\} : K_2(E_L) \otimes K \to F^*_\infty \otimes K$ denote also the composition of the homomorphism $K_2(E_L)\otimes K \to K_2(E_{F_\infty})\otimes K$ induced by the imbedding above and the unique $K$-linear extension of the homomorphism $\{\cdot\}$. Assume that $F = \mathbb{F}_q(T)$ is the rational function field of transcendence degree one over $\mathbb{F}_q$, where $T$ is an indeterminate, and $\infty$ is the point at infinity on $X = \mathbb{P}^1_{\mathbb{F}_q}$. Also assume that $\chi$ is non-trivial. Now we are able to state our main result:

**Theorem 1.10.** *There is an element $\kappa_E(\chi) \in K_2(E_L) \otimes K$ such that*

$$\{\kappa_E(\chi)\} = L(E,q^{-1})\mathcal{L}_{\mathfrak{m}}(E,\chi)' \quad \text{in } F^*_\infty \otimes K.$$

It is easy to deduce that the valuation of $\mathcal{L}_{\mathfrak{m}}(E,\chi)'$ with respect to $\infty \otimes \mathrm{id}_K$ is equal to $-L_{\mathfrak{m}}(E,\chi,1)$ from the interpolation property. (For the explanation of this notation see the next chapter.) Deligne's purity theorem implies that the latter is non-zero under mild, purely local conditions on $\chi$ and $\mathfrak{m}$. If the special value $L(E,q^{-1})$ also happens to be non-zero we get that the element $\kappa_E(\chi) \in K_2(E_L) \otimes K$ is not torsion hence our main result is non-vacuous.

**Contents 1.11.** In the next chapter we prove the basic properties of the $L$-function $\mathcal{L}_{\mathfrak{m}}(E,\chi)$ by simple cohomological means. We introduce our mail tool, which we call double Eisenstein series, in the third chapter. They are really analogous to the product of two Eisenstein series in the classical setting, but they cannot be written as such due to the lack of logarithm in positive characteristic. Here we also establish their basic properties, among them Proposition 3.5, which is analogous to analytic continuation. The link between double Eisenstein series and the rigid analytic regulator of elements in $K_2$ analogous to Beilinson's construction is provided by the Kronecker limit formula 4.10 of the fourth chapter.

The fifth chapter is somewhat technical: it identifies function field analogues of modular units with the rigid analytic functions appearing in the previous chapter and studies the action of the Hecke algebra on the source and target groups of the rigid analytic regulator. We execute the principal calculation of the paper in the sixth chapter. Perhaps the crucial reason why the Rankin–Selberg computation can be carried out is that the double Eisenstein series does become a product of two series after the first step of the calculation. In the seventh chapter we use the function field analogue of the Shimura–Taniyama–Weil conjecture as well as its explicit description due to Gekeler and Reversat to conclude the proof of our main result. The aim of the last chapter is to prove a useful lemma on the action of correspondences on motivic cohomology groups which is used in the fifth chapter.

## 2. The $\infty$-adic $L$-functions of elliptic curves

**Definition 2.1.** Note that for a finite group $G$ we have $\Delta[[G]] = \Delta[G]$ naturally. Let $M$ be an abelian profinite group, let $H$ be a finite quotient of $M$ and let $K$ denote the kernel of the quotient map $M \to H$. We let $I_H^M$ denote the ideal of the quotient map $\Delta[[M]] \to \Delta[H]$. It is obvious that the augmentation ideal $I = I_{\{1\}}^M$ and $I_H^M \subseteq I$ for any $H$.

**Lemma 2.2.** *We have $\theta' \in K \otimes \Delta$ for any $\theta \in I_H^M$.*

*Proof.* The same as the proof of Lemma 3.9 of [21] whose claim is just slightly different. □

**Notation 2.3.** Let $E$ be an elliptic curve defined over $F$ which has split multiplicative reduction at $\infty$ as in the introduction whose notation we are going to use without further notice. Let $G$ be the Galois group of a finite abelian extension of $F$ which only ramifies at $\infty$ and let $H(G)$ denote the maximal quotient of $G$ such that the corresponding abelian extension of $F$ is unramified at every closed point $x$ of $X$ and totally split at $\infty$.

**Proposition 2.4.** *The following holds:*

(i) *the power series $\mathcal{L}_{\mathfrak{m}}^G(E,\chi,t)$ is an element of $\Delta[G][t]$,*
(ii) *we have $\mathcal{L}_{\mathfrak{m}}^G(E,\chi,1) \in I_{H(G)}^G$.*

*Proof.* Let $l$ be a prime different from $p$. The $\mathrm{Gal}(\overline{F}|F)$-module $H^1(E_{\overline{F}},\mathbb{Q}_l)$ is absolutely irreducible because the curve $E$ is not isotrivial. Let $\rho$ denote the corresponding $l$-adic Galois representation. For every character $\phi : G \to \overline{\mathbb{Q}}_l^*$ let the same symbol denote the corresponding homomorphism $\overline{\mathbb{Q}}_l[G][[t]] \to \overline{\mathbb{Q}}_l[[t]]$ and the corresponding $l$-adic Galois representation by the usual abuse of notation. For every $l$-adic Galois representation $\psi$ which is unramified at almost all closed points of $X$ we will use the same symbol to denote the constructible $l$-adic sheaf on $X$ which is the direct image of $\psi$ with respect to the generic point $\mathrm{Spec}(F) \to X$. Fix an imbedding of $K$ into $\overline{\mathbb{Q}}_l$. This way the Galois representation $\chi$ becomes an $l$-adic representation, too. The series $\mathcal{L}^G_{\mathfrak{m}}(E,\chi,t) \in \overline{\mathbb{Q}}_l[[t]]$ is characterized by the property:

$$\phi(\mathcal{L}^G_{\mathfrak{m}}(E,\chi,t)) = L(X(\mathfrak{m}\infty), \rho\otimes\chi\phi, t)$$

for every character $\phi : G \to \overline{\mathbb{Q}}_l^*$ where $X(\mathfrak{d})$ denotes the complement of the support of any effective divisor $\mathfrak{d}$ in $X$ and $L(U,\psi,t)$ denotes the Grothendieck $L$-function of any constructible $l$-adic sheaf $\psi$ on a variety $U$ over $\mathbb{F}_q$. The $l$-adic Galois representation $\rho\otimes\chi\phi$ is absolutely irreducible, therefore the twisted $L$-function $L(X(\mathfrak{m}\infty),\rho\otimes\chi\phi,t)$ is a polynomial for every character $\phi : G \to \overline{\mathbb{Q}}_l^*$ by the Grothendieck–Verdier formula. Hence so is $\mathcal{L}^G_{\mathfrak{m}}(E,\chi,t)$ as claim (i) says. For every character $\phi : H(G) \to \overline{\mathbb{Q}}_l^*$ let the same symbol denote the composition of the quotient map $G \to H(G)$ and the character $\phi$ as well. In this case the restriction of the $l$-adic Galois representation corresponding to $\phi$ to the decomposition group at $\infty$ is trivial. The same holds for $\chi$ by assumption. Moreover $E$ has split multiplicative reduction at $\infty$ so we have:

$$\phi(\mathcal{L}^G_{\mathfrak{m}}(E,\chi,t)) = (1-t^{\deg(\infty)})L(X(\mathfrak{m}),\rho\otimes\chi\phi,t)$$

for every such character. As the twisted $L$-function $L(X(\mathfrak{m}),\rho\otimes\chi\phi,t)$ is a polynomial by the Grothendieck–Verdier formula, we have $\phi(\mathcal{L}^G_{\mathfrak{m}}(E,\chi,1)) = 0$ for every such character as well. The latter is equivalent to the property that $\mathcal{L}^{H(G)}_{\mathfrak{m}}(E,\chi,1)$ is zero as claim (ii) says. $\square$

As we explained in Definition 1.8 part (i) of the proposition above implies that the object $\mathcal{L}_{\mathfrak{m}}(E,\chi)$ is well-defined. For every group $M$ let $\widehat{M}$ denote its profinite completion and let $\infty : \widehat{F^*_\infty/\mathbb{F}^*_q}\otimes\Delta \to \widehat{\Delta} = \widehat{\mathbb{Z}}\otimes\Delta$ denote the profinite completion of the valuation $\infty$ as well. The following proposition takes care of Proposition 1.9 and the remark after Theorem 1.10. For the sake of simple notation let $L_{\mathfrak{m}}(E,\chi,t)$ denote $L(X(\mathfrak{m}),\rho\otimes\chi,t)$.

**Proposition 2.5.** *The following holds:*

(i) *we have* $\mathcal{L}_{\mathfrak{m}}(E,\chi) \in I$ *and* $\mathcal{L}_{\mathfrak{m}}(E,\chi)' \in F^*_\infty/\mathbb{F}^*_q\otimes\Delta$,

(ii) *we have* $\infty(\mathcal{L}_{\mathfrak{m}}(E,\chi)') = -L_{\mathfrak{m}}(E,\chi,1)$.

*Proof.* The first half of claim (i) and the fact that $\mathcal{L}_{\mathfrak{m}}(E,\chi)' \in \widehat{F^*_\infty/\mathbb{F}^*_q}\otimes\Delta$ follow at once from claim (ii) of Proposition 2.4 and Lemma 2.2 by taking the limit.

On the other hand note that $F_\infty^*/\mathbb{F}_q^* \otimes \Delta$ is the pre-image of $\Delta$ with respect to $\infty$ in $\widehat{F_\infty^*/\mathbb{F}_q^*} \otimes \Delta$ hence the second half of claim (i) is an immediate consequence of claim (ii). Now we only have to show the latter. The profinite group $G_\infty$ surjects onto the Galois group of the maximal constant field extension of $F$ which is isomorphic to $\widehat{\mathbb{Z}}$. This induces a surjection $\Delta[[G_\infty]] \to \Delta[[\widehat{\mathbb{Z}}]]$. The choice of a topological generator of $\widehat{\mathbb{Z}}$, or equivalently the choice of a system of generators of the finite quotients of $\widehat{\mathbb{Z}}$ compatible with the projections furnishes an injection $\Delta[t] \to \Delta[[\widehat{\mathbb{Z}}]]$ such that the image of $t$ is the generator. In case of the natural choice of the global geometric Frobenius as a topological generator, the image $\phi_x$ of a geometric Frobenius at $x$ in $G_\infty$ maps to $t^{\deg(x)}$ for every closed point $x$ on $X$ under the map above. Hence the image of $\mathcal{L}_{\mathfrak{m}}(E,\chi)$ under this map is $\widetilde{L}_{\mathfrak{m}}(E,\chi,t) = (1-t^{\deg(\infty)})L_{\mathfrak{m}}(E,\chi,t)$ as we saw in the proof of Proposition 2.4. The ideal $I \triangleleft \Delta[[G_\infty]]$ maps into the augmentation ideal $J \triangleleft \Delta[[\widehat{\mathbb{Z}}]]$ corresponding to the trivial quotient of $\widehat{\mathbb{Z}}$, and the induced map $I/I^2 \to J/J^2$ is the tensor product of the surjection $G_\infty \to \widehat{\mathbb{Z}}$ introduced above and the identity of $\Delta$. Since the intersection $J \cap \Delta[t]$ is the ideal generated by $t-1$, the image of $\mathcal{L}_{\mathfrak{m}}(E,\chi)'$ under the map $I/I^2 \to J/J^2$ is just the derivative $\widetilde{L}(E,1)' \in \Delta \subset \widehat{\Delta}$. On the other hand the restriction of the surjection $G_\infty \to \widehat{\mathbb{Z}}$ to $F_\infty^*$ is $\deg(\infty)$ times the valuation map $\infty : F_\infty^* \to \mathbb{Z}$, so:

$$\deg(\infty)\infty(\mathcal{L}_{\mathfrak{m}}(E,\chi)') = ((1-t^{\deg(\infty)})L_{\mathfrak{m}}(E,\chi,t))'|_{t=1} = -\deg(\infty)L_{\mathfrak{m}}(E,\chi,1)$$

as we claimed. □

## 3. Double Eisenstein series

**Notation 3.1.** Let $|X|$, $\mathbb{A}$, $\mathcal{O}$ denote the set of closed points of $X$, the ring of adeles of $F$ and its maximal compact subring of $\mathbb{A}$, respectively. As in the introduction we will fix a closed point $\infty$ in the set $|X|$. For every divisor $\mathfrak{m}$ of $X$ let $\mathfrak{m}$ also denote the $\mathcal{O}$-module in the ring $\mathbb{A}$ generated by the ideles whose divisor is $\mathfrak{m}$, by abuse of notation. For every idele $m \in \mathbb{A}^*$ let the same symbol also denote the divisor of $m$ if this notation does not cause confusion. For any closed point $v$ in $|X|$ we will let $F_v$, $\mathbf{f}_v$ and $\mathcal{O}_v$ denote the corresponding completion of $F$, its constant field, and its discrete valuation ring, respectively. For every $v \in |X|$ let $v : F_v^* \to \mathbb{Z}$ denote the valuation normalized such that $v(\pi_v) = \deg(v)$ for every uniformizer $\pi_v \in F_v$. For any idele, adele, adele-valued matrix or function defined on the above which decomposes as an infinite product of functions defined on the individual components the subscript $v$ will denote the $v$-th component. Let $\mathbb{A}_f$, $\mathcal{O}_f$ denote the restricted direct product $\prod'_{x\neq\infty} F_x$ and the direct product $\prod_{x\neq\infty}\mathcal{O}_x$, respectively. The former is also called the ring of finite adeles of $F$ and the latter is its maximal compact subring. For every $g \in GL_2(\mathbb{A})$ (or $g \in \mathbb{A}$, etc.) let $g_f$ denote its finite component in $GL_2(\mathbb{A}_f)$. We will consider $\mathbb{A}_f^*$ as well as $F_v^*$ (for every place $v \in |X|$) as a subgroup of $\mathbb{A}^*$ in the natural way. Similarly we will consider $\mathbb{A}_f$ and $F_v$ as a subring of $\mathbb{A}$ and $GL_2(\mathbb{A})$ and $GL_2(F_v)$ as a subgroup of $GL_2(\mathbb{A})$. Let $|\cdot|$ denote the normalized absolute value on the ring $\mathbb{A}$ and for any idele or divisor $y$ let $\deg(y)$ denote its degree related to the normalized absolute value by the formula $|y| = q^{-\deg(y)}$. In accordance with our convention $|\cdot|$ will denote the absolute value with respect to $\infty$ if its argument is in $F_\infty$. For each $(u,v) \in F_\infty^2$ let $\|(u,v)\|$, $\infty(u,v)$ denote $\max(|u|,|v|)$ and $\min(\infty(u),\infty(v))$, respectively. Let $Z$ denote the center of the group scheme $GL_2$, let

$$\Gamma_\infty = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_\infty) \,\middle|\, \infty(c) > 0 \right\}$$

be the Iwahori subgroup of $GL_2(F_\infty)$ and let

$$\mathbb{K}(\mathfrak{m}) = \{ g \in GL_2(\mathcal{O}) \mid g \equiv I \bmod \mathfrak{m} \},$$

for every effective divisor $\mathfrak{m}$ where $I$ is the identity matrix. We will adopt the convention which assigns 0 or 1 as value to the empty sum or product, respectively.

**Definition 3.2.** Let $F_<^2$ denote the set $\{(a,b) \in F_\infty^2 \mid |a| < |b|\}$. Let $\mathfrak{m}$ be an effective divisor on $X$ whose support does not contain $\infty$. Let the same symbol also denote the ideal $\mathfrak{m} \cap \mathcal{O}_f$ by abuse of notation. For every $g \in GL_2(\mathbb{A})$, $(\alpha,\beta) \in (\mathcal{O}_f/\mathfrak{m})^2$, and integer $n$ let

$$\begin{aligned} W_\mathfrak{m}(\alpha,\beta,g,n) &= \{0 \neq f \in F^2 \mid fg_f \in (\alpha,\beta) + \mathfrak{m}\mathcal{O}_f^2,\ -n = \infty(fg_\infty)\}, \\ V_\mathfrak{m}(\alpha,\beta,g,n) &= \{ f \in W_\mathfrak{m}(\alpha,\beta,g,n) \mid fg_\infty \in F_<^2\}, \\ U_\mathfrak{m}(\alpha,\beta,g,n) &= W_\mathfrak{m}(\alpha,\beta,g,n) - V_\mathfrak{m}(\alpha,\beta,g,n). \end{aligned}$$

Also let

$$W_\mathfrak{m}(\alpha,\beta,g_f) = \bigcup_{n\in\mathbb{Z}} W_\mathfrak{m}(\alpha,\beta,g,n),$$

$$U_\mathfrak{m}(\alpha,\beta,g) = \bigcup_{n\in\mathbb{Z}} U_\mathfrak{m}(\alpha,\beta,g,n) \quad \text{and} \quad V_\mathfrak{m}(\alpha,\beta,g) = \bigcup_{n\in\mathbb{Z}} V_\mathfrak{m}(\alpha,\beta,g,n).$$

Obviously the first set is well-defined. For every finite quotient $G$ of $F^*\backslash\mathbb{A}^*/\mathcal{O}_f^*$ let $\cdot^G : \mathbb{A}^* \to G$ denote the quotient map. Let $E_\mathfrak{m}^G(\alpha,\beta,\gamma,\delta,g,x,y)$ denote the $\mathbb{Z}[G][[x,y]](x^{-1},y^{-1})$-valued function

$$E^G_{\mathfrak{m}}(\alpha,\beta,\gamma,\delta,g,x,y)$$
$$= \frac{\det(g_f^{-1})^G}{(xy)^{\deg(\det(g))}} \cdot \sum_{\substack{(a,b)\in U_{\mathfrak{m}}(\alpha,\beta,g)\\(c,d)\in V_{\mathfrak{m}}(\gamma,\delta,g)}} \det\begin{pmatrix} a & b\\ c & d\end{pmatrix}^G_\infty x^{2\infty((a,b)g_\infty)}y^{\infty(2(c,d)g_\infty)},$$

for every $g\in GL_2(\mathbb{A})$, variables $x$, $y$, and pairs $(\alpha,\beta)$ and $(\gamma,\delta)$ as above. In order to see that this function is indeed well-defined first note that

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix}\cdot\det(g_\infty)^{-1} = (a_1d_1-b_1c_1)\cdot\det(g_\infty)^{-1}$$

is non-zero where $(a_1,b_1)=(a,b)g_\infty$ and $(c_1,d_1)=(c,d)g_\infty$ because $|a_1|\geq|b_1|$ and $|c_1|<|d_1|$ by the definition of the sets $U_{\mathfrak{m}}(\alpha,\beta,g)$ and $V_{\mathfrak{m}}(\gamma,\delta,g)$ therefore

$$|a_1d_1-b_1c_1| = |a_1d_1|\neq 0.$$

Hence the terms of the infinite sum above are defined. The sum itself is well-defined and $\mathbb{Z}[G][[x,y]](x^{-1},y^{-1})$-valued as the cardinality of the sets $U_{\mathfrak{m}}(\alpha,\beta,g)$ and $V_{\mathfrak{m}}(\gamma,\delta,g)$ is finite for all $n$ and zero for $n$ sufficiently small.

**Proposition 3.3.** *The following holds:*

(i) *the function $E^G_{\mathfrak{m}}(\alpha,\beta,\gamma,\delta,g,x,y)$ is left-invariant with respect to $GL_2(F)$ and right-invariant with respect to $\mathbb{K}(\mathfrak{m}\infty)\Gamma_\infty Z(F_\infty)$,*

(ii) *the $\mathbb{C}[G]$-valued infinite sum $E^G_{\mathfrak{m}}(\alpha,\beta,\gamma,\delta,g,q^{-s},q^{-t})$ converges absolutely, if $\mathrm{Re}(s)>1$ and $\mathrm{Re}(t)>1$, for every $g$.*

*Proof.* We are going to prove claim (i) first. Since for every $\rho\in GL_2(F)$ and $n\in\mathbb{Z}$ we have:

$$U_{\mathfrak{m}}(\alpha,\beta,\rho g,n) = U_{\mathfrak{m}}(\alpha,\beta,g,n)\rho^{-1}\quad\text{and}\quad V_{\mathfrak{m}}(\gamma,\delta,\rho g,n) = V_{\mathfrak{m}}(\gamma,\delta,g,n)\rho^{-1},$$

we get that

$$E^G_{\mathfrak{m}}(\alpha,\beta,\gamma,\delta,\rho g,x,y) = \frac{\det(\rho_f^{-1})^G\det(g_f^{-1})^G}{(xy)^{\deg(\det(\rho))+\deg(\det(g))}}$$
$$\cdot\sum_{\substack{(a,b)\in U_{\mathfrak{m}}(\alpha,\beta,g)\\(c,d)\in V_{\mathfrak{m}}(\gamma,\delta,g)}} \det\begin{pmatrix} a & b\\ c & d\end{pmatrix}^G_\infty \det(\rho_\infty^{-1})^G x^{2\infty((a,b)\rho^{-1}\rho g_\infty)}y^{2\infty((c,d)\rho^{-1}\rho g_\infty)}$$
$$= E^G_{\mathfrak{m}}(\alpha,\beta,\gamma,\delta,g,x,y),$$

because $\det(\rho^{-1})\in F^*$ and $\deg(\det(\rho))=0$ as the degree of every principal divisor is zero. On the other hand for every $\lambda\in GL_2(F_\infty)$ the set $\{f\in F^2_\infty\mid f\lambda\in F^2_<\}$ is obviously left-invariant by $\Gamma_\infty Z(F_\infty)$ hence

$$U_{\mathfrak{m}}(\alpha,\beta,g\rho)=U_{\mathfrak{m}}(\alpha,\beta,g)\quad\text{and}\quad V_{\mathfrak{m}}(\gamma,\delta,g\rho)=V_{\mathfrak{m}}(\gamma,\delta,g)$$

for every $\rho\in\mathbb{K}(\mathfrak{m}\infty)\Gamma_\infty Z(F_\infty)$ and $g\in GL_2(\mathbb{A})$. Therefore

$$\begin{aligned}E^G_{\mathfrak{m}}(\alpha,\beta,\gamma,\delta,g\rho,x,y)&=\frac{\det(\rho_f^{-1})^G\det(g_f^{-1})^G}{(xy)^{\deg(\det(z))+\deg(\det(g))}}\\&\quad\cdot\sum_{\substack{(a,b)\in U_{\mathfrak{m}}(\alpha,\beta,g)\\(c,d)\in V_{\mathfrak{m}}(\gamma,\delta,g)}}\det\begin{pmatrix}a&b\\c&d\end{pmatrix}^G_\infty x^{2\infty((a,b)g_\infty)+\infty(\det(z))}y^{2\infty((c,d)g_\infty)+\infty(\det(z))}\\&=E^G_{\mathfrak{m}}(\alpha,\beta,\gamma,\delta,g,x,y),\end{aligned}$$

where $\rho=\kappa z$ with $\kappa\in\mathbb{K}(\mathfrak{m}\infty)\Gamma_\infty$ and $z\in Z(F_\infty)$ because $\kappa_\infty$ is an isometry with respect to the norm $\|\cdot\|$, $\deg(\det(\kappa))=0$ and $\det(\kappa_f)^G=1$ by definition. Our proof of claim (ii) is the same as the argument that may be found in [20]. The coefficient of each element of $G$ in the series $E^G_{\mathfrak{m}}(\alpha,\beta,\gamma,\delta,g,q^{-s},q^{-t})$ is majorized by the product $E(g,s)E(g,t)$ where:

$$E(g,s)=|\det(g)|^s\sum_{\substack{f\in F^2-\{0\}\\fg\in\mathcal{O}_f^2}}\|(fg)_\infty\|^{-2s},$$

so it will be sufficient to prove that $E(g,s)$ converges absolutely for each $g\in GL_2(\mathbb{A})$ if $\mathrm{Re}(s)>1$. For every $g\in GL_2(\mathbb{A})$ let $\mathcal{E}(g)$ denote the sheaf on $X$ whose group of sections for every open subset $U\subseteq X$ is

$$\mathcal{E}(g)(U)=\{f\in F^2\mid fg\in\mathcal{O}_v^2,\ \forall v\in|U|\},$$

where we denote the set of closed points of $U$ by $|U|$. The sheaf $\mathcal{E}(g)$ is a coherent locally free sheaf of rank two. If $\mathcal{F}_n$ denotes the sheaf $\mathcal{F}\otimes\mathcal{O}_X(\infty)^n$ for every coherent sheaf $\mathcal{F}$ on $X$ and integer $n$, then for every $g\in GL_2(\mathbb{A})$ and $s\in\mathbb{C}$ the series above can be rewritten as

$$E(g,s)=\sum_{n\in\mathbb{Z}}|H^0(X,\mathcal{E}(g)_n)-H^0(X,\mathcal{E}(g)_{n-1})|q^{-s\deg(\mathcal{E}(g)_n)}.$$

By the Riemann–Roch theorem for curves:

$$\dim H^0(X,\mathcal{F})-\dim H^0(X,K_X\otimes\mathcal{F}^\vee)=2-2g(X)+\deg(\mathcal{F})$$

for any coherent locally free sheaf $\mathcal{F}$ of rank two on $X$, where $K_X$, $\mathcal{F}^\vee$ and $g(X)$ is the canonical bundle on $X$, the dual of $\mathcal{F}$, and the genus of $X$, respectively.

Because $\dim H^0(X,\mathcal{F}_{-n}) = 0$ for $n$ sufficiently large depending on $\mathcal{F}$, we have that

$$|H^0(X,\mathcal{E}(g)_n)| = q^{2-2g(X)+\deg(\mathcal{E}(g))+2n\deg(\infty)} \quad \text{and} \quad |H^0(X,\mathcal{E}(g)_{-n})| = 1,$$

if $n$ is a sufficiently large positive number. Hence

$$E(g,s) = p(q^{-s}) + q^{2-2g(X)+(1-s)\deg(\mathcal{E}(g))}(1-q^{-\deg(\infty)})\sum_{n=0}^{\infty} q^{2n(1-s)\deg(\infty)},$$

where $p$ is a polynomial. The claim now follows from the convergence of the geometric series. □

**Definition 3.4.** For every abelian group $M$ and for every finite set $S$ let $M[S]$ and $M[S]_0$ denote the group of functions $f : S \to M$ and its subgroup consisting of functions $f \in M[S]$ with the property

$$\sum_{\alpha\in S} f(\alpha) = 0,$$

respectively. Let $\mathcal{V}_{\mathfrak{m}}$ denote the set $(\mathcal{O}_f/\mathfrak{m})^2 - \{0,0\}$ and for every $C \in R[\mathcal{V}_{\mathfrak{m}}]$ and $D \in R[\mathcal{V}_{\mathfrak{m}}]$ let $E^G_{\mathfrak{m}}(C,D,g,x,y)$ denote the function:

$$E^G_{\mathfrak{m}}(C,D,g,x,y) = \sum_{\substack{(\alpha,\beta)\in\mathcal{V}_{\mathfrak{m}}\\(\gamma,\delta)\in\mathcal{V}_{\mathfrak{m}}}} C(\alpha,\beta)D(\gamma,\delta)E^G_{\mathfrak{m}}(\alpha,\beta,\gamma,\delta,g,x,y),$$

where $R \supseteq \mathbb{Z}$ is an arbitrary commutative ring.

**Proposition 3.5.** *For every $C,D \in R[\mathcal{V}_{\mathfrak{m}}]_0$ the function $E^G_{\mathfrak{m}}(C,D,g,x,y)$ takes values in $R[G][x,y,x^{-1},y^{-1}]$.*

*Proof.* We may assume by bilinearity that $C = (\alpha,\beta)-(\gamma,\delta)$ and $D = (\epsilon,\iota)-(\kappa,\lambda)$ for some pairs $(\alpha,\beta),(\gamma,\delta),(\epsilon,\iota),(\kappa,\lambda) \in \mathcal{V}_{\mathfrak{m}}$. Now pick two elements $(r,s) \in U_{\mathfrak{m}}(\alpha-\gamma,\beta-\delta,g)$ and $(u,v) \in V_{\mathfrak{m}}(\epsilon-\kappa,\iota-\lambda,g)$. Then for every sufficiently large natural number $n$ we have:

$$\begin{aligned} U_{\mathfrak{m}}(\alpha,\beta,g,n) &= \{(a+r,b+s) \mid (a,b) \in U_{\mathfrak{m}}(\gamma,\delta,g,n)\},\\ V_{\mathfrak{m}}(\epsilon,\iota,g,n) &= \{(a+u,b+v) \mid (a,b) \in V_{\mathfrak{m}}(\kappa,\lambda,g,n)\}.\end{aligned}$$

Therefore

$$\begin{aligned} E^G_{\mathfrak{m}}(C,D,g,x,y) &= P(x,y)\\ &\quad + \frac{\det(g_f^{-1})^G}{(xy)^{\deg(\det(g))}} \sum_{\substack{(a,b)\in U_{\mathfrak{m}}(\gamma,\delta,g_f)\\(c,d)\in V_{\mathfrak{m}}(\kappa,\lambda,g)}} \begin{bmatrix} a & b\\ c & d\end{bmatrix} x^{2\infty((a,b)g_\infty)} y^{2\infty((c,d)g_\infty)},\end{aligned}$$

where $P(x,y) \in \mathbb{Z}[G][x,y,x^{-1},y^{-1}]$ and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det\begin{pmatrix} a+r & b+s \\ c+u & d+v \end{pmatrix}_\infty^G - \det\begin{pmatrix} a+r & b+s \\ c & d \end{pmatrix}_\infty^G$$
$$- \det\begin{pmatrix} a & b \\ c+u & d+v \end{pmatrix}_\infty^G + \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}_\infty^G.$$

In order to finish the proof it is enough to show that the determinants in the expression above can be paired in such a way that in every pair the determinants have different signs and they represent the same element in $G$ if $\max(\|(a,b)g_\infty\|, \|(c,d)g_\infty\|)$ is sufficiently large. This follows from the lemma below or its pair which we get by switching the rows of the matrices depending on whether $\|(a,b)g_\infty\|$ or $\|(c,d)g_\infty\|$ is the larger one among the two, respectively. $\square$

**Lemma 3.6.** *For each $k \in GL_2(F_\infty)$ let $\|k\|$ denote the maximum of the absolute values of the entries of $k$. Then for every $g \in GL_2(F_\infty)$ and $(r,s),(a,b),(c,d) \in F_\infty^2$ such that $(a,b)g \notin F_<^2$ and $(c,d)g \in F_<^2$ we have:*

$$\left|1 - \det\left(\begin{pmatrix} a+r & b+s \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\right)\right| \leq \frac{\|(r,s)\|\,\|g^{-1}\|\,|\det(g)|}{\|(a,b)g\|}.$$

*Proof.* Using Cramer's rule we get that

$$\begin{pmatrix} a+r & b+s \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 + \frac{rd-sc}{ad-bc} & \frac{-rb+sa}{ad-bc} \\ 0 & 1 \end{pmatrix},$$

so

$$\left|1 - \det\left(\begin{pmatrix} a+r & b+s \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\right)\right| = \left|\frac{rd-sc}{ad-bc}\right| \leq \frac{\|(r,s)\| \cdot \|(c,d)\|}{|ad-bc|}.$$

On the other hand let $(a_1,b_1) = (a,b)g$ and $(c_1,d_1) = (c,d)g$. Then $|a_1| \geq |b_1|$ and $|c_1| < |d_1|$ similarly as we noted at the end of Definition 3.2 therefore

$$\begin{aligned} |ad-bc| &= \left|\det\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \cdot \det(g^{-1})\right| = |a_1d_1 - b_1c_1| \cdot |\det(g)|^{-1} \\ &= |a_1d_1| \cdot |\det(g)|^{-1} = \|(a_1,b_1)\| \cdot \|(c_1,d_1)\| \cdot |\det(g)|^{-1} \\ &\geq \|(a,b)g\| \cdot \|(c,d)\| \cdot \|g^{-1}\|^{-1} \cdot |\det(g)|^{-1}. \end{aligned}$$ $\square$

**Definition 3.7.** As a consequence of Proposition 3.5 the function $E_{\mathfrak{m}}^G(C,D,g,x,y)$ can be evaluated at $x=y=1$. Let

$$E_{\mathfrak{m}}^G(C,D,g) = E_{\mathfrak{m}}^G(C,D,g,1,1) \in R[G]$$

for every $g \in GL_2(\mathbb{A})$. In accordance with the previously introduced notation for every finite abelian group $G$ we let $I_G \triangleleft \mathbb{Z}[G]$ denote the augmentation ideal of $\mathbb{Z}[G]$, that is, the kernel of the augmentation map $\mathbb{Z}[G] \to \mathbb{Z}$. There is an isomorphism $I_G/I_G^2 = G$ induced by the map given by the rule $g \mapsto 1 - g \in I_G$ for every $g \in G$.

**Proposition 3.8.** *Assume that $R = \mathbb{Z}$. Then we have $E^G_{\mathfrak{m}}(C, D, g) \in I_G$ for every $g \in GL_2(\mathbb{A})$.*

*Proof.* It will be sufficient to prove that $E^{\{1\}}_{\mathfrak{m}}(C, D, g) = 0$ where $\{1\}$ is the trivial group. We may assume again by bilinearity that $C = (\alpha, \beta) - (\gamma, \delta)$ and $D = (\epsilon, \iota) - (\kappa, \lambda)$ for some pairs $(\alpha, \beta), (\gamma, \delta), (\epsilon, \iota), (\kappa, \lambda) \in \mathcal{V}_{\mathfrak{m}}$. Pick again two elements $(r, s) \in U_{\mathfrak{m}}(\alpha - \gamma, \beta - \delta, g)$ and $(u, v) \in V_{\mathfrak{m}}(\epsilon - \kappa, \iota - \lambda, g)$. Then for every sufficiently large natural number $n$ we have:

$$\bigcup_{m \le n} U_{\mathfrak{m}}(\alpha, \beta, g, m) = \bigcup_{m \le n} \{(a + r, b + s) \mid (a, b) \in U_{\mathfrak{m}}(\gamma, \delta, g, m)\}$$

and

$$\bigcup_{m \le n} V_{\mathfrak{m}}(\epsilon, \iota, g, m) = \bigcup_{m \le n} \{(a + u, b + v) \mid (a, b) \in V_{\mathfrak{m}}(\kappa, \lambda, g, m)\}.$$

Hence we have:

$$\Big|\bigcup_{m \le n} U_{\mathfrak{m}}(\alpha, \beta, g, m)\Big| = \Big|\bigcup_{m \le n} U_{\mathfrak{m}}(\gamma, \delta, g, m)\Big|, \qquad |U_{\mathfrak{m}}(\alpha, \beta, g, n)| = |U_{\mathfrak{m}}(\gamma, \delta, g, n)|,$$
$$\Big|\bigcup_{m \le n} V_{\mathfrak{m}}(\epsilon, \iota, g, m)\Big| = \Big|\bigcup_{m \le n} V_{\mathfrak{m}}(\kappa, \lambda, g, m)\Big|, \qquad |V_{\mathfrak{m}}(\epsilon, \iota, g, n)| = |V_{\mathfrak{m}}(\kappa, \lambda, g, n)|$$

for every sufficiently large natural number $n$. Therefore

$$\begin{aligned} E^{\{1\}}_{\mathfrak{m}}(C, D, g) = & \sum_{m, n \in \mathbb{Z}} \big((|U_{\mathfrak{m}}(\alpha, \beta, g, m)| - |U_{\mathfrak{m}}(\gamma, \delta, g, m)|) \\ & \cdot (|V_{\mathfrak{m}}(\epsilon, \iota, g, n)| - |V_{\mathfrak{m}}(\kappa, \lambda, g, n)|)\big) \\ = & \lim_{n \to \infty} \Big(\Big(\Big|\bigcup_{m \le n} U_{\mathfrak{m}}(\alpha, \beta, g, m)\Big| - \Big|\bigcup_{m \le n} U_{\mathfrak{m}}(\gamma, \delta, g, m)\Big|\Big) \\ & \cdot \Big(\Big|\bigcup_{k \le n} V_{\mathfrak{m}}(\epsilon, \iota, g, k)\Big| - \Big|\bigcup_{k \le n} V_{\mathfrak{m}}(\kappa, \lambda, g, k)\Big|\Big)\Big) \\ = & \ 0. \end{aligned}$$

$\square$

**Definition 3.9.** In accordance with the notation we introduced in Definition 1.8 let $\theta' \in G$ denote the class of any $\theta \in I_G$. For every $C, D \in \mathbb{Z}[\mathcal{V}_{\mathfrak{m}}]_0$ and $N \in \mathbb{Z}$ let

$\mathcal{E}_{\mathfrak{m}}(C,D,g,n)$ denote the $F^*_\infty$-valued function:

$$\mathcal{E}_{\mathfrak{m}}(C,D,g,N)=\prod_{\substack{m,n\leq n\\(\alpha,\beta)\in\mathcal{V}_{\mathfrak{m}}\\(\gamma,\delta)\in\mathcal{V}_{\mathfrak{m}}}}\prod_{\substack{(a,b)\in U_{\mathfrak{m}}(\alpha,\beta,g,m)\\(c,d)\in V_{\mathfrak{m}}(\gamma,\delta,g,n)}}\det\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{C(\alpha,\beta)D(\gamma,\delta)}.$$

Finally let $\mathcal{E}_{\mathfrak{m}}(C,D,g)$ denote the limit

$$\mathcal{E}_{\mathfrak{m}}(C,D,g)=\lim_{N\to\infty}\mathcal{E}_{\mathfrak{m}}(C,D,g,N)$$

if the latter exists. The following claim is an immediate corollary to Lemma 3.6 and Proposition 3.8 using the same argument we used in the proof of Proposition 3.5.

**Proposition 3.10.** *The limit above exists and*

$$E^G_{\mathfrak{m}}(C,D,g)'=\mathcal{E}_{\mathfrak{m}}(C,D,g)^G. \qquad \square$$

## 4. The Kronecker limit formula

**Notation 4.1.** We are going to use the notation we introduced in 1.2. For every connected rational subdomain $U$ of $\mathbb{P}^1$ the elements of $\partial U$ are called the *boundary components* of $U$, by slight abuse of language. Let $\mathcal{R}(U)\subset\mathcal{O}(U)$ denote the subalgebra of restrictions of rational functions holomorphic on $U$ and $\mathcal{R}^*(U)$ denote the group of invertible elements of this algebra. The group $\mathcal{R}^*(U)$ consists of rational functions which do not have poles or zeros lying in $U$.

**Theorem 4.2.** *There is a unique map $\{\cdot,\cdot\}_D:\mathcal{O}^*(U)\times\mathcal{O}^*(U)\to\mathbb{C}^*_\infty$ for every $D\in\partial U$, called the rigid analytic regulator, with the following properties:*

(i) *For any two $f,g\in\mathcal{R}^*(U)$ their regulator is:*

$$\{f,g\}_D=\prod_{x\in D}\{f,g\}_x,$$

(ii) *the regulator $\{\cdot,\cdot\}_D$ is bilinear in both variables,*
(iii) *the regulator $\{\cdot,\cdot\}_D$ is alternating: $\{f,g\}_D\cdot\{g,f\}_D=1$,*
(iv) *if $f,1-f\in\mathcal{O}(U)^*$, then $\{f,1-f\}_D$ is $1$,*
(v) *for each $f\in\mathcal{O}_\epsilon(U)$ and $g\in\mathcal{O}^*(U)$ we have $\{f,g\}_D\in\mathcal{U}_\epsilon$.*

*Proof.* This is Theorem 2.2 of [22]. $\square$

**Definition 4.3.** If $U$ is still a connected rational subdomain of $\mathbb{P}^1$, and $f,g$ are two meromorphic functions on $U$, then for all $x\in U$ the functions $f$ and $g$ have a power series expansion around $x$, so in particular their tame symbol $\{f, g\}_x$ at $x$ is defined. Let $\mathcal{M}(U)$ denote the field of meromorphic functions on $U$. The tame symbol extends to a homomorphism $\{\cdot,\cdot\}_x : K_2(\mathcal{M}(U)) \to \mathbb{C}_\infty^*$. We define the group $K_2(U)$ as the kernel of the direct sum of the tame symbols:

$$\bigoplus_{x\in U}\{\cdot,\cdot\}_x : K_2(\mathcal{M}(U)) \to \bigoplus_{x\in U}\mathbb{C}_\infty^*.$$

Let $k = \sum_i f_i \otimes g_i \in K_2(U)$, where $f_i, g_i \in \mathcal{M}(U)$, and let $D \in \partial U$. Let moreover $Y$ be a connected rational subdomain of $U$ such that $f_i, g_i \in \mathcal{O}^*(Y)$ for all $i$ and $\partial U \subseteq \partial Y$. Define the *rigid analytical regulator* $\{k\}_D$ by the formula:

$$\{k\}_D = \prod_i \{f_i|_Y, g_i|_Y\}_D.$$

Theorem 1.3 is based on the previous result and the following theorem:

**Theorem 4.4.** (i) *For each $k \in K_2(U)$ the rigid analytical regulator $\{k\}_D$ is well-defined, and it is a homomorphism $\{\cdot\}_D : K_2(U) \to \mathbb{C}_\infty^*$,*

(ii) *for any two functions $f, g \in \mathcal{O}^*(U)$ we have $\{f \otimes g\}_D = \{f, g\}_D$,*

(iii) *for every $k \in K_2(U)$ the product of all regulators on the boundary components of $U$ is equal to 1:*

$$\prod_{D\in\partial U}\{k\}_D = 1,$$

(iv) *for every connected subdomain $Y \subseteq U$, boundary component $D \in \partial Y \cap \partial U$ and $k \in K_2(\mathcal{M}(U))$ we have:*

$$\{k|_Y\}_D = \{k\}_D.$$

*Proof.* This is Theorem 3.2 of [22]. □

**Definition 4.5.** For every $\rho \in GL_2(F_\infty)$ and $z \in \mathbb{P}^1$ let $\rho(z)$ denote the image of $z$ under the Möbius transformation corresponding to $\rho$. Let moreover $D(\rho)$ denote the open disc

$$D(\rho) = \{z \in \mathbb{P}^1(\mathbb{C}_\infty) \mid 1 < |\rho^{-1}(z)|\}.$$

Let $\mathcal{D}$ denote the set of open discs of the form $D(\rho)$ where $\rho \in GL_2(F_\infty)$. For each $D \in \mathcal{D}$ let $D(F_\infty)$ denote $D \cap \mathbb{P}^1(F_\infty)$. Let $\mathcal{P}$ denote those subsets $S$ of $\mathcal{D}$ such that the sets $D(F_\infty)$, $D \in S$ form a pairwise disjoint partition of $\mathbb{P}^1(F_\infty)$. For each $S \in \mathcal{P}$ let $\Omega(S)$ denote the unique connected rational subdomain defined over $F_\infty$ with the property $\partial\Omega(S) = S$. Let $\Omega$ denote the rigid analytic upper half plane, or Drinfeld's upper half plane over $F_\infty$. The set of points of $\Omega$ is $\mathbb{C}_\infty - F_\infty$, denoted also by $\Omega$ by abuse of notation. Recall that a function $f : \Omega \to \mathbb{C}_\infty$ is holomorphic if the restriction of $f$ onto $\Omega(S)$ is holomorphic for every $S \in \mathcal{P}$. Let $\mathcal{O}(\Omega)$ and $\mathcal{M}(\Omega)$ denote the $\mathbb{C}_\infty$-algebra of holomorphic functions and the field of meromorphic functions on $\Omega$, respectively. The latter is of course the quotient field of the former. We define $K_2(\Omega)$ as the intersection of the kernels of all the tame symbols $\{\cdot,\cdot\}_x$ inside $K_2(\mathcal{M}(\Omega))$ where $x$ runs through the set $\Omega$. By part (iv) of Theorem 4.4 for each $k \in K_2(\Omega)$ the value $\{k\}(\rho) = \{k|_{\Omega(S)}\}_{D(\rho)}$, where $\rho \in GL_2(F_\infty)$ and $D(\rho) \in S \in \mathcal{P}$, is independent of the choice of $S$. We define the regulator $\{k\} : GL_2(F_\infty) \to \mathbb{C}^*_\infty$ of $k$ as the function given by this rule.

**Lemma 4.6.** *Let $\rho = \left(\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix}\right)$ where $x \in F^*_\infty$ and $y \in F_\infty$. Then for every $0 \neq (a,b) \in F^2_\infty$ and $0 \neq (c,d) \in F^2_\infty$ the following holds:*

(i) *if $(a,b)\rho \in F^2_<$ and $(c,d)\rho \in F^2_<$ then*

$$\{(az+b)\otimes(cz+d)\}_{D(\rho)} = 1,$$

(ii) *if $(a,b)\rho \notin F^2_<$ and $(c,d)\rho \notin F^2_<$ then $a \neq 0$, $b \neq 0$ and*

$$\{(az+b)\otimes(cz+d)\}_{D(\rho)} = b/a,$$

(iii) *if $(a,b)\rho \notin F^2_<$ and $(c,d)\rho \in F^2_<$ then $a \neq 0$ and*

$$\{(az+b)\otimes(cz+d)\}_{D(\rho)} = \frac{1}{a}\det\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

(iv) *if $(a,b)\rho \in F^2_<$ and $(c,d)\rho \notin F^2_<$ then $b \neq 0$ and*

$$\{(az+b)\otimes(cz+d)\}_{D(\rho)} = c\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}.$$

*Proof.* Let $D(\rho)^c$ denote the complement of $D(\rho)$ in $\mathbb{P}^1$. Obviously

$$D(\rho)^c = \{z \in \mathbb{C}_\infty \mid |z-y| \leq |x|\}.$$

Hence $(a,b)\rho \in F^2_<$ if and only if the polynomial $az+b$ has no zeros in $D(\rho)^c$ and $(a,b)\rho \notin F^2_<$ if and only if $a \neq 0$ and the polynomial $az+b$ does have a zero in $D(\rho)^c$. By Weil's reciprocity law:

$$\{(az+b)\otimes(cz+d)\}^{-1}_{D(\rho)} = \prod_{t \in D(\rho)^{-1}} \{az+b, cz+d\}_t.$$

Since the tame symbol of $az+b$ and $cz+d$ at $t \in \mathbb{C}_\infty$ is 1 if neither $az+b$ nor $cz+d$ has a zero at $t$ claim (i) is clear. In the second case $az+b$ and $cz+d$ each have a single pole at $\infty$ but no zero in $D(\rho)$ therefore

$$\{(az+b)\otimes(cz+d)\}_{D(\rho)} = \{az+b, cz+d\}_\infty = b/a.$$

Claim (iv) follows from claim (iii) by the antisymmetry of the regulator. In the latter case $az+b$ has a single zero in $D(\rho)^c$ and $cz+d$ has no zeros in $D(\rho)^c$ hence

$$\{(az+b)\otimes(cz+d)\}_{D(\rho)} = \{az+b, cz+d\}^{-1}_{-b/a} = \frac{1}{a}\det\begin{pmatrix} a & b \\ c & d\end{pmatrix}. \qquad \square$$

**Definition 4.7.** We are going to need a mild extension of the regulator we introduced in Definition 4.5. Let $K_2(GL_2(\mathbb{A}_f)\times\Omega)$ denote the set of functions $k : GL_2(\mathbb{A}_f)\to K_2(\Omega)$. We define the regulator of an element $k\in K_2(GL_2(\mathbb{A}_f)\times\Omega)$ as the function $\{k\} : GL_2(\mathbb{A})\to\mathbb{C}^*_\infty$ given by the rule $\{k\}(g) = \{k(g_f)\}(g_\infty)$ for every $g\in GL_2(\mathbb{A})$. Since the set $K_2(GL_2(\mathbb{A}_f)\times\Omega)$ consists of functions taking values in the group $K_2(\Omega)$ it is equipped with a group structure whose operation will be denoted by addition. Let $\mathcal{O}^*(GL_2(\mathbb{A}_f)\times\Omega)$ denote the set of functions $u : GL_2(\mathbb{A}_f)\times\Omega\to\mathbb{C}^*_\infty$ which are holomorphic in the second variable. Then there is a bilinear map:

$$\otimes : \mathcal{O}^*(GL_2(\mathbb{A}_f)\times\Omega)\times\mathcal{O}^*(GL_2(\mathbb{A}_f)\times\Omega)\to K_2(GL_2(\mathbb{A}_f)\times\Omega)$$

given by the rule $(u\otimes v)(g) = u|_{g\times\{\cdot\}}\otimes v|_{g\times\{\cdot\}}$ for every $g\in GL_2(\mathbb{A}_f)$. For each $(\alpha,\beta)\in(\mathcal{O}_f/\mathfrak{m})^2$, and $N$ a positive integer let $\epsilon_{\mathfrak{m}}(\alpha,\beta,N)(g,z)$ denote the function:

$$\epsilon_{\mathfrak{m}}(\alpha,\beta,N)(g,z) = \prod_{n\le N}\Big(\prod_{(a,b)\in W_{\mathfrak{m}}(\alpha,\beta,g,n)}(az+b)\cdot\prod_{(c,d)\in W_{\mathfrak{m}}(0,0,g,n)}(cz+d)^{-1}\Big).$$

on the set $GL_2(\mathbb{A}_f)\times\Omega$. The latter is clearly holomorphic in the second variable.

**Lemma 4.8.** *The limit*

$$\epsilon_{\mathfrak{m}}(\alpha,\beta)(g,z) = \lim_{N\to\infty}\epsilon_{\mathfrak{m}}(\alpha,\beta,N)(g,z)$$

*converges uniformly in $z$ on every admissible open subdomain of $\Omega$ for every fixed $g$ and defines a function holomorphic in the second variable.*

*Proof.* See Lemma 4.5 of [20] on pages 145–146. $\square$

**Definition 4.9.** For every $C\in\mathbb{Z}[\mathcal{V}_{\mathfrak{m}}]_0$ let $\epsilon_{\mathfrak{m}}(C,g,z)$ denote the function:

$$\prod_{(\alpha,\beta)\in\mathcal{V}_{\mathfrak{m}}}\epsilon_{\mathfrak{m}}(\alpha,\beta)(g,z)^{C(\alpha,\beta)}$$

on the set $GL_2(\mathbb{A}_f)\times\Omega$. For every $C,D\in\mathbb{Z}[\mathcal{V}_{\mathfrak{m}}]_0$ let $\kappa_{\mathfrak{m}}(C,D)$ denote the element:

$$\epsilon_{\mathfrak{m}}(C,g,z)\otimes\epsilon_{\mathfrak{m}}(D,g,z)$$

of the set $K_2(GL_2(\mathbb{A}_f)\times\Omega)$.

**Kronecker Limit Formula 4.10.** *For all $g \in GL_2(\mathbb{A})$ we have:*

$$\{\kappa_{\mathfrak{m}}(C,D)\}(g)^G = (E^G_{\mathfrak{m}}(C,D,g) - E^G_{\mathfrak{m}}(D,C,g))'.$$

*Proof.* Assume first that $g_\infty = \left(\begin{smallmatrix} x & y \\ 0 & 1\end{smallmatrix}\right)$ for some $x \in F^*_\infty$ and $y \in F_\infty$. By Proposition 3.10 it will be sufficient to prove that

$$\Big\{ \prod_{(\alpha,\beta)\in\mathcal{V}_{\mathfrak{m}}} \epsilon_{\mathfrak{m}}(\alpha,\beta,N)(g,z)^{C(\alpha,\beta)} \otimes \prod_{(\gamma,\delta)\in\mathcal{V}_{\mathfrak{m}}} \epsilon_{\mathfrak{m}}(\gamma,\delta,N)(g,z)^{D(\gamma,\delta)} \Big\}_{D(g_\infty)}$$

is equal to

$$\mathcal{E}_{\mathfrak{m}}(C,D,g,N)\cdot\mathcal{E}_{\mathfrak{m}}(D,C,g,N)^{-1}$$

for every sufficiently large $N$. By bilinearity and Lemma 4.6 the regulator in the left hand side of the equation that we wish to prove is equal to

$$\prod_{\substack{m,n\leq N\\(\alpha,\beta)\in\mathcal{V}_{\mathfrak{m}}\\(\gamma,\delta)\in\mathcal{V}_{\mathfrak{m}}}} \Bigg( \prod_{\substack{(a_1,b_1)\in U_{\mathfrak{m}}(\alpha,\beta,g,m)\\(c_1,d_1)\in V_{\mathfrak{m}}(\gamma,\delta,g,n)}} a_1^{-1}\det\begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix}$$

$$\cdot \prod_{\substack{(a_2,b_2)\in V_{\mathfrak{m}}(\alpha,\beta,g,m)\\(c_2,d_2)\in U_{\mathfrak{m}}(\gamma,\delta,g,n)}} c_2\det\begin{pmatrix} a_2 & b_2\\ c_2 & d_2\end{pmatrix}^{-1} \cdot \prod_{\substack{(a_3,b_3)\in U_{\mathfrak{m}}(\alpha,\beta,g,m)\\(c_3,d_3)\in U_{\mathfrak{m}}(\gamma,\delta,g,n)}} a_3^{-1}c_3\Bigg)^{C(\alpha,\beta)D(\gamma,\delta)}.$$

Therefore what we need to show is:

$$\prod_{\substack{m,n\leq N\\(\alpha,\beta)\in\mathcal{V}_{\mathfrak{m}}\\(\gamma,\delta)\in\mathcal{V}_{\mathfrak{m}}}} \Bigg( \prod_{(a,b)\in U_{\mathfrak{m}}(\alpha,\beta,g,m)} a^{-|W_{\mathfrak{m}}(\gamma,\delta,g,n)|}$$

$$\cdot \prod_{(c,d)\in U_{\mathfrak{m}}(\gamma,\delta,g,n)} c^{|W_{\mathfrak{m}}(\alpha,\beta,g,m)|}\Bigg)^{C(\alpha,\beta)D(\gamma,\delta)} = 1.$$

The latter follows from the fact that for every $C \in \mathbb{Z}[\mathcal{V}_{\mathfrak{m}}]_0$ and for every sufficiently large $N$ the equation:

$$\sum_{\substack{n\leq N\\(\alpha,\beta)\in\mathcal{V}_{\mathfrak{m}}}} C(\alpha,\beta)|W_{\mathfrak{m}}(\alpha,\beta,g,n)| = 0$$

holds. On the other hand the latter has already been shown in the course of the proof of Proposition 3.8 (at least in the special case when $C = (\alpha,\beta)-(\gamma,\delta)$ for some $(\alpha,\beta),(\gamma,\delta)\in\mathcal{V}_{\mathfrak{m}}$ but the general case follows at once from this one by linearity). Let us consider now the general case. First note that both sides of the equation in the theorem above are right-invariant with respect to $Z(F_\infty)\Gamma_\infty$ hence if the claim is true for $g$ then it is true for $gz$ as well for every $z \in Z(F_\infty)\Gamma_\infty$. Let $\Pi \in GL_2(F_\infty)$ be the matrix whose diagonal entries are zero, and its lower left and upper right entries are $\pi$ and 1, respectively, where $\pi$ is a uniformizer of $F_\infty$. Then for every $\rho \in GL_2(F_\infty)$ the open discs $D(\rho)$ and $D(\rho\Pi)$ are complementary in $\mathbb{P}^1$ hence claim (iii) of Theorem 4.4 (Weil's reciprocity law) implies that

$$\{\kappa_{\mathfrak{m}}(C,D)\}(g)^G \cdot \{\kappa_{\mathfrak{m}}(C,D)\}(g\Pi)^G = 1$$

for every $g \in GL_2(\mathbb{A})$. A matrix $\rho \in GL_2(F_\infty)$ can be written as a product $\rho = \left(\begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix}\right) z$ where $x \in F_\infty^*$, $y \in F_\infty$ and $z \in Z(F_\infty)\Gamma_\infty$ if and only if $\infty \in D(\rho)$. Since either $D(\rho)$ or $D(\rho\Pi)$ contains the point $\infty$ it will be sufficient to prove that the identity above holds for $E^G_{\mathfrak{m}}(C,D,g)'/E^G_{\mathfrak{m}}(D,C,g)'$ as well. But

$$U_{\mathfrak{m}}(\alpha,\beta,g\Pi) = V_{\mathfrak{m}}(\alpha,\beta,g) \quad \text{and} \quad V_{\mathfrak{m}}(\alpha,\beta,g\Pi) = U_{\mathfrak{m}}(\alpha,\beta,g),$$

therefore $E^G_{\mathfrak{m}}(C,D,g) = (-1)^G E^G_{\mathfrak{m}}(D,C,g\Pi)$ for any $g \in GL_2(\mathbb{A})$, so the latter is obvious. □

## 5. Modular units and Hecke operators

**Notation 5.1.** Let $A = \mathcal{O}_f \cap F$: it is a Dedekind domain. The ideals of $A$ and the effective divisors on $X$ with support away from $\infty$ are in a bijective correspondence. These two sets will be identified in all that follows. For any non-zero ideal $\mathfrak{m} \triangleleft A$ let $Y(\mathfrak{m})$ denote the coarse moduli for parameterizing Drinfeld modules of rank two over $A$ of general characteristic with full level $\mathfrak{m}$-structure. It is an affine algebraic curve defined over $F$. For every Drinfeld module $\phi : A \to C\{\tau\}$ of rank two equipped with a full level $\mathfrak{m}$-structure $\iota : (A/\mathfrak{m})^2 \to C$, where $C$ is an $F$-algebra, let $u_{\phi,\iota} : Y(\mathfrak{m}) \to \mathrm{Spec}(C)$ be the universal map.

**Lemma 5.2.** *There is a unique element $\epsilon_{\mathfrak{m}}(D) \in \Gamma(Y(\mathfrak{m}), \mathcal{O}^*)$ for every $D \in \mathbb{Z}[\mathcal{V}_{\mathfrak{m}}]_0$ such that*

$$u^*_{\phi,\iota}(\epsilon_{\mathfrak{m}}(D)) = \prod_{(\alpha,\beta)\in\mathcal{V}_{\mathfrak{m}}} \iota(\alpha,\beta)^{D(\alpha,\beta)} \in C$$

*for every $C$, $\phi$ and $\iota$ as above.*

*Proof.* We may assume that $\mathfrak{m}$ is a proper ideal without loss of generality. For any non-zero ideal $\mathfrak{m} \triangleleft A$ let $H(\mathfrak{m})$ denote $\Gamma(Y(\mathfrak{m}), \mathcal{O})$. We may assume by linearity that $D = (\alpha,\beta) - (\gamma,\delta)$ for some $(\alpha,\beta),(\gamma,\delta) \in \mathcal{V}_{\mathfrak{m}}$. Let $(\phi,\iota)$ and $(\psi,\kappa)$ be ordered pairs of two Drinfeld modules $\phi$ and $\psi$ of rank two over $C$ equipped with a full level $\mathfrak{m}$-structure $\iota$ and $\kappa$, respectively. Recall that $(\phi,\iota)$ and $(\psi,\kappa)$ are isomorphic if there is an isomorphism $j : \mathbb{G}_a \to \mathbb{G}_a$ between $\phi$ and $\psi$ such that the composition $j \circ \iota$ is equal to $\kappa$. As $j$ is just multiplication by a scalar we get that the element $\iota(\alpha,\beta)/\iota(\gamma,\delta)$ depends only on the isomorphism class of the pair $(\phi,\iota)$. In particular the claim is obvious when the moduli scheme $Y(\mathfrak{m})$ is fine because we have $\epsilon_{\mathfrak{m}}(D) = \iota_{\mathfrak{m}}(\alpha,\beta))/\iota_{\mathfrak{m}}(\gamma,\delta)$ in this case where the map $\iota_{\mathfrak{m}} : (A/\mathfrak{m})^2 \to H(\mathfrak{m})$ is the universal full level $\mathfrak{m}$-structure for the universal Drinfeld module $\phi_{\mathfrak{m}} : A \to H(\mathfrak{m})\{\tau\}$ over $Y(\mathfrak{m})$. The latter holds if $\mathfrak{m}$ has at least two prime divisors. In general the universal map $H(\mathfrak{m}) \to \bigoplus_{\mathfrak{p}\nmid\mathfrak{m}} H(\mathfrak{m}\mathfrak{p})$ is an étale injection, so it is faithfully flat. Therefore the sequence

$$0 \to H(\mathfrak{m}) \to \bigoplus_{\mathfrak{p}\nmid\mathfrak{m}} H(\mathfrak{m}\mathfrak{p}) \rightrightarrows \Big(\bigoplus_{\mathfrak{p}\nmid\mathfrak{m}} H(\mathfrak{m}\mathfrak{p})\Big) \otimes_{H(\mathfrak{m})} \Big(\bigoplus_{\mathfrak{p}\nmid\mathfrak{m}} H(\mathfrak{m}\mathfrak{p})\Big)$$

is exact by Proposition 2.18 of [18], pages 16–17. For every $(\kappa,\lambda) \in \mathcal{V}_{\mathfrak{m}}$ and prime ideal $\mathfrak{p} \nmid \mathfrak{m}$ let $(\kappa,\lambda,\mathfrak{p})$ denote the unique element of $\mathcal{V}_{\mathfrak{m}p}$ such that $(\kappa,\lambda,\mathfrak{p}) \equiv (\kappa,\lambda) \bmod \mathfrak{m}$ and $(\kappa,\lambda,\mathfrak{p}) \equiv (0,0) \bmod \mathfrak{p}$. Moreover for every prime ideal $\mathfrak{p} \nmid \mathfrak{m}$ let $D(\mathfrak{p}) \in \mathbb{Z}[\mathcal{V}_{\mathfrak{m}p}]_0$ denote the element $(\alpha,\beta,\mathfrak{p}) - (\gamma,\beta,\mathfrak{p})$. Then the element

$$\bigoplus_{\mathfrak{p}\nmid\mathfrak{m}} \epsilon_{\mathfrak{m}p}(D(\mathfrak{p})) \in \bigoplus_{\mathfrak{p}\nmid\mathfrak{m}} H(\mathfrak{m}p)$$

is in the kernel of the second map in the exact sequence above, therefore it is the image of a unique element $\epsilon_{\mathfrak{m}}(D) \in H(\mathfrak{m})$ which satisfies the required property. □

**Definition 5.3.** The group $GL_2(F)$ acts on the product $GL_2(\mathbb{A}_f) \times \Omega$ on the left by acting on the first factor via the natural embedding and on Drinfeld's upper half plane via Möbius transformations. The group $\mathbb{K}_f(\mathfrak{m}) = \mathbb{K}(\mathfrak{m}) \cap GL_2(\mathcal{O}_f)$ acts on the right of this product by acting on the first factor via the regular action. Since the quotient set $GL_2(F)\backslash GL_2(\mathbb{A}_f)/\mathbb{K}_f(\mathfrak{m})$ is finite, the set

$$GL_2(F)\backslash GL_2(\mathbb{A}_f) \times \Omega/\mathbb{K}_f(\mathfrak{m})$$

is the disjoint union of finitely many sets of the form $\Gamma\backslash\Omega$, where $\Gamma$ is a subgroup of $GL_2(F)$ of the form $GL_2(F) \cap g\mathbb{K}_f(\mathfrak{m})g^{-1}$ for some $g \in GL_2(\mathbb{A}_f)$. As these groups act on $\Omega$ discretely, the set above naturally has the structure of a rigid analytic curve. Let $Y(\mathfrak{m})_{F_\infty}$ also denote the underlying rigid analytical space of the base change of $Y(\mathfrak{m})$ to $F_\infty$ by abuse of notation.

**Theorem 5.4.** *There is a rigid-analytical isomorphism:*

$$Y(\mathfrak{m})_{F_\infty} \cong GL_2(F)\backslash GL_2(\mathbb{A}_f) \times \Omega/\mathbb{K}_f(\mathfrak{m}).$$

*Proof.* See [4], Theorem 6.6. □

**Proposition 5.5.** *For every $D \in \mathbb{Z}[\mathcal{V}_{\mathfrak{m}}]_0$ the function corresponding to $\epsilon_{\mathfrak{m}}(D)$ under the isomorphism of Theorem* 5.4 *above is the function $\epsilon_{\mathfrak{m}}(D, g, z)$ introduced in Definition* 4.9.

*Proof.* First we are going to recall the map underlying the isomorphism of Theorem 5.4 on $\mathbb{C}_\infty$-valued points. For every $(g, z) \in GL_2(\mathbb{A}_f) \times \Omega$ let $e_{(g,z)}(w)$ denote the corresponding exponential function:

$$e_{(g,z)}(w) = z \prod_{(a,b)\in W_{\mathfrak{m}}(0,0,g_f)} \left(1 - \frac{w}{az+b}\right).$$

The infinite product above is converging absolutely and defines an entire function. The exponential $e_{(g,z)}$ uniformizes a Drinfeld module $\phi_{(g,z)}$ over $\mathbb{C}_\infty$ which is equipped with a full level $\mathfrak{m}$-structure $\iota$ given by the formula:

$$\iota(\alpha, \beta) = e_{(g,z)}(az + b) \quad \text{ where } (a, b) \in W_{\mathfrak{m}}(\alpha, \beta, g_f)$$

for every $(\alpha, \beta) \in (\mathcal{O}_f/\mathfrak{m})^2$ independent of the choice of $(a, b)$. Since obviously we have $\iota(\alpha, \beta) = \epsilon_{\mathfrak{m}}(\alpha, \beta, g, z)$ the claim is now clear. □

**Definition 5.6.** Let $M$ be an abelian group and let $\mathfrak{n}$ be any effective divisor on $X$. By an $M$-valued *automorphic form* over $F$ of level $\mathfrak{n}$ (and trivial central character) we mean a locally constant function $\phi : GL_2(\mathbb{A}) \to M$ satisfying $\phi(\gamma gkz) = \phi(g)$ for all $\gamma \in GL_2(F)$, $z \in Z(\mathbb{A})$, and $k \in \mathbb{K}_0(\mathfrak{n})$, where

$$\mathbb{K}_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}) \;\middle|\; c \equiv 0 \bmod \mathfrak{n} \right\}.$$

Let $\mathcal{A}(\mathfrak{n}, M)$ denote the $\mathbb{Z}$-module of $M$-valued automorphic forms of level $\mathfrak{n}$. Now let $\mathfrak{n}$ be an effective divisor on $X$ whose support does not contain $\infty$. Let $\mathcal{H}(\mathfrak{n}, M)$ denote the $\mathbb{Z}$-module of automorphic forms $\phi$ of level $\mathfrak{n}\infty$ satisfying the following two identities:

$$\phi\left(g\begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}\right) = -\phi(g) \quad (\forall g \in GL_2(\mathbb{A})),$$

and

$$\phi\left(g\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) + \sum_{\epsilon\in\mathbf{f}_\infty} \phi\left(g\begin{pmatrix} 1 & 0 \\ \epsilon & 1 \end{pmatrix}\right) = 0 \quad (\forall g \in GL_2(\mathbb{A})),$$

where $\pi$ is a uniformizer in $F_\infty$ and we consider $GL_2(F_\infty)$ as a subgroup of $GL_2(\mathbb{A})$ and we understand the product of their elements accordingly. Such automorphic forms are called *harmonic*.

**Definition 5.7.** Let $\mathfrak{m}$, $\mathfrak{n}$ be effective divisors of $X$. Define the set:

$$H(\mathfrak{m},\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{A}) \;\middle|\; a,b,c,d \in \mathcal{O},\ (ad-cb)=\mathfrak{m},\ \mathfrak{n}\supseteq (c),\ (d)+\mathfrak{n}=\mathcal{O} \right\}.$$

The set $H(\mathfrak{m},\mathfrak{n})$ is compact and it is a double $\mathbb{K}_0(\mathfrak{n})$-coset, so it is a disjoint union of finitely many right $\mathbb{K}_0(\mathfrak{n})$-cosets. Let $R(\mathfrak{m},\mathfrak{n})$ be a set of representatives of these cosets. For any $\phi \in \mathcal{A}(\mathfrak{n},R)$ define the function $T_{\mathfrak{m}}(\phi)$ by the formula:

$$T_{\mathfrak{m}}(\phi)(g) = \sum_{h\in R(\mathfrak{m},\mathfrak{n})} \phi(gh).$$

It is easy to check that $T_{\mathfrak{m}}(\phi)$ is independent of the choice of $R(\mathfrak{m},\mathfrak{n})$ and $T_{\mathfrak{m}}(\phi)\in \mathcal{A}(\mathfrak{n},M)$ as well. So we have a $\mathbb{Z}$-linear operator $T_{\mathfrak{m}} : \mathcal{A}(\mathfrak{n},M)\to\mathcal{A}(\mathfrak{n},M)$.

**Definition 5.8.** Let $\mathfrak{X}$ be a Hausdorff topological space. For any commutative ring $R$ let $\mathcal{M}(\mathfrak{X},R)$ denote the set of $R$-valued finitely additive measures on the open and compact subsets of $\mathfrak{X}$. For every abelian group $M$ let $\mathcal{C}_0(\mathfrak{X},M)$ denote the group of compactly supported locally constant functions $f:\mathfrak{X}\to M$. For every $f\in\mathcal{C}_0(\mathfrak{X},M)$ and $\mu\in\mathcal{M}(\mathfrak{X},R)$ we define the *modulus* $\mu(f)$ of $f$ with respect to $\mu$ as the $\mathbb{Z}$-submodule of $R$ generated by the elements $\mu(f^{-1}(g))$, where $0\neq g\in M$. We also define the integral of $f$ on $\mathfrak{X}$ with respect to $\mu$ as the sum:

$$\int_{\mathfrak{X}} f(x)\,d\mu(x) = \sum_{g\in M} g\otimes\mu(f^{-1}(g)) \in M\otimes\mu(f).$$

Of course this definition is nothing more than a convenient formalism. For its elementary properties see Lemma 5.2 of [22]. Let $M$ be a $\mathbb{Q}$-vector space and let $\phi$ be an element of $\mathcal{A}(\mathfrak{n},M)$. If for all $g\in GL_2(\mathbb{A})$:

$$\int_{F\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} g\right) d\mu(x) = 0,$$

where $\mu$ is the normalized Haar measure on $F\backslash\mathbb{A}$ such that $\mu(F\backslash\mathbb{A})=1$ we call $\phi$ a *cusp form.* Let $\mathcal{A}_0(\mathfrak{n},M)$ (respectively $\mathcal{H}_0(\mathfrak{n},M)$) denote the $\mathbb{Q}$-module of $M$-valued cuspidal automorphic forms (respectively cuspidal harmonic forms) of level $\mathfrak{n}$ (resp. of level $\mathfrak{n}\infty$).

**Notation 5.9.** For any ideal $\mathfrak{n}\triangleleft A$ let $Y_0(\mathfrak{n})$ denote the coarse moduli scheme for rank two Drinfeld modules of general characteristic equipped with a Hecke level-$\mathfrak{n}$ structure. It is an affine algebraic curve defined over $F$. Let $X_0(\mathfrak{n})$ denote the unique irreducible smooth projective curve over $F$ which contains $Y_0(\mathfrak{n})$ as an open subvariety. For every proper ideal $\mathfrak{m} \triangleleft A$ there is the $\mathfrak{m}$-th Hecke correspondence on the Drinfeld modular curve $X_0(\mathfrak{n})$ which in turn induces an endomorphism of the Jacobian $J_0(\mathfrak{n})$ of the curve $X_0(\mathfrak{n})$, called the *Hecke operator* $T_\mathfrak{m}$ (for a detailed description see for example [6] or [7]). The $\mathfrak{m}$-th Hecke correspondence also induces a pair of compatible homomorphisms:

$$T_\mathfrak{m} : H^2_\mathcal{M}(X_0(\mathfrak{n})_L, K(2)) \to H^2_\mathcal{M}(X_0(\mathfrak{n})_L, K(2))$$

and

$$T_\mathfrak{m} : H^2_\mathcal{M}(Y_0(\mathfrak{n})_L, K(2)) \to H^2_\mathcal{M}(Y_0(\mathfrak{n})_L, K(2))$$

for every number field $K$ and for every extension $L \supseteq F$. These operators are denoted by the same symbol we use for the operators introduced in Definition 5.7, but this will not cause confusion as we will see. For the moment it is sufficient to remark that they act on different objects.

**Notation 5.10.** Let $\pi(\mathfrak{n}) : Y(\mathfrak{n}) \to Y_0(\mathfrak{n})$ be the map induced by the forgetful functor which assigns to every Drinfeld module $\phi : A \to C\{\tau\}$ of rank two equipped with a full level $\mathfrak{m}$-structure $\iota : (A/\mathfrak{m})^2 \to C$, where $C$ is an $F$-algebra, the Drinfeld module $\phi$ equipped with the Hecke level-$\mathfrak{n}$ structure generated by $\iota(0,1)$. Hence $Y_0(\mathfrak{n})$ also has a rigid analytic uniformization of the kind described in Theorem 5.4 where the role of the group $\mathbb{K}(\mathfrak{n})$ is played by $\mathbb{K}_0(\mathfrak{n})$. Hence we may evaluate the regulator introduced in Definition 4.7 on the pull-back of the elements of $H^2_\mathcal{M}(Y_0(\mathfrak{n})_{F_\infty}, \mathbb{Z}(2))$ with respect to this uniformization. Let $\{\cdot\}$ denote also the unique $K$-linear extension to $H^2_\mathcal{M}(Y_0(\mathfrak{n})_{F_\infty}, K(2))$ of this regulator for every number field $K$ by abuse of notation.

For the rest of the paper we assume that $F = \mathbb{F}_q(T)$ is the rational function field of transcendence degree one over $\mathbb{F}_q$, where $T$ is an indeterminate, and $\infty$ is the point at infinity on $X = \mathbb{P}^1_{\mathbb{F}_q}$.

**Proposition 5.11.** *For every $k \in H^2_\mathcal{M}(Y_0(\mathfrak{n})_{F_\infty}, K(2))$ the following holds:*

(i) *we have $\{k\} \in \mathcal{H}(\mathfrak{n}, F^*_\infty \otimes K)$,*
(ii) *we have $\{k\} \in \mathcal{H}_0(\mathfrak{n}, F^*_\infty \otimes K)$ when $k \in H^2_\mathcal{M}(X_0(\mathfrak{n})_{F_\infty}, K(2))$,*
(iii) *we have $\{T_\mathfrak{m}(k)\} = T_\mathfrak{m}\{k\}$ for every proper ideal $\mathfrak{m} \triangleleft A$.*

*Proof.* By definition and the invariance theorem of [22] the regulator $\{k\}$ is left $GL_2(F)$-invariant and right $\mathbb{K}_0(\mathfrak{n}\infty)Z(F_\infty)$-invariant. By our assumptions on $F$ and $\infty$ we have $F^*\mathcal{O}^*_f = \mathbb{A}^*_f$ hence $\{k\}$ is also $Z(\mathbb{A})$-invariant. Therefore it is an element of $\mathcal{A}(\mathfrak{n}\infty, F^*_\infty \otimes K)$. By claim (iii) of Theorem 4.4 the additional conditions of Definition 5.6 also hold for $\{k\}$ as the following two sets of discs:

$$D(\rho), D\left(\rho\begin{pmatrix}0 & 1\\ \pi & 0\end{pmatrix}\right) \quad \text{and} \quad D\left(\rho\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right), \left\{D\left(\rho\begin{pmatrix}1 & 0\\ \epsilon & 1\end{pmatrix}\right) \,\middle|\, \epsilon \in \mathbf{f}_\infty\right\}$$

each give a pairwise disjoint covering of the set $\mathbb{P}^1(F_\infty)$ for every $\rho \in GL_2(F_\infty)$. Claim (i) is proved. The second claim is an immediate consequence of Theorem 6.3 of [22]. Finally let us concern ourselves with the proof of claim (iii). For every $h \in GL_2(\mathbb{A}_f)$ let $h : GL_2(\mathbb{A}_f) \times \Omega \to GL_2(\mathbb{A}_f) \times \Omega$ simply denote the map given by the rule $(g, z) \mapsto (gh, z)$ for every $g \in GL_2(\mathbb{A}_f)$ and $z \in \Omega$. By slight abuse of notation let the same symbol denote the unique map $h : Y(\mathfrak{n})_{F_\infty} \to Y(\mathfrak{n})_{F_\infty}$ which satisfies the relation $h \circ \pi(\mathfrak{n}) = \pi(\mathfrak{n}) \circ h$. Let $R(\mathfrak{m}, \mathfrak{n}) \subset H(\mathfrak{m}, \mathfrak{n})$ be a set of representatives which is also a subset of $GL_2(\mathbb{A}_f)$. Then we have:

$$\pi(\mathfrak{n})^*(T_\mathfrak{m}(k)) = \sum_{h \in R(\mathfrak{m},\mathfrak{n})} h^*(k)$$

in $H^2_\mathcal{M}(Y(\mathfrak{n})_{F_\infty}, K(2))$. Hence by the invariance theorem (Theorem 3.11 of [22]) for every $g \in GL_2(\mathbb{A})$ we have:

$$\{T_\mathfrak{m}(k)\}(g) = \sum_{h \in R(\mathfrak{m},\mathfrak{n})} \{h^*(k)\}(g) = \sum_{h \in R(\mathfrak{m},\mathfrak{n})} \{k\}(gh) = T_\mathfrak{m}\{k\}(g)$$

as we claimed. □

Let $L \subset F_\infty$ be a finite extension of $F$ and let

$$\{\cdot\} : H^2_\mathcal{M}(X_0(\mathfrak{n})_L, K(2)) \to F^*_\infty \otimes K$$

denote also the composition of the homomorphism

$$H^2_\mathcal{M}(X_0(\mathfrak{n})_L, K(2)) \to H^2_\mathcal{M}(X_0(\mathfrak{n})_{F_\infty}, K(2))$$

induced by the functoriality of motivic cohomology and the homomorphism $\{\cdot\}$. Let $C \subset X_0(\mathfrak{n}) \times X_0(\mathfrak{n})$ be a correspondence and let

$$C_* : H^2_\mathcal{M}(X_0(\mathfrak{n})_L, K(2)) \to H^2_\mathcal{M}(X_0(\mathfrak{n})_L, K(2))$$

denote the homomorphism induced by $C$.

**Lemma 5.12.** *We have $\{C_*(k)\} = 0$ for every $k \in H^2_\mathcal{M}(X_0(\mathfrak{n})_L, K(2))$ if the endomorphism $J(C) : J_0(\mathfrak{n}) \to J_0(\mathfrak{n})$ induced by $C$ is zero.*

*Proof.* We may assume without loss of generality that $k \in H^2_\mathcal{M}(X_0(\mathfrak{n})_L, \mathbb{Z}(2))$. Let $Y$ be a smooth, projective curve over $\mathbb{F}_q$ whose function field is $L$ and let $\mathfrak{X}$ be a regular flat projective model of $X_0(\mathfrak{n})_L$ over $Y$. By passing to a finite extension of $L$, if it is necessary, we may also assume that $\mathfrak{X}$ is semi-stable. By Proposition 8.6 there is a positive integer $j$ such that the element $jC_*(k) \in H^2_{\mathcal{M}}(X_0(\mathfrak{n})_L, \mathbb{Z}(2))$ lies in the image of the natural map $H^2_{\mathcal{M}}(\mathfrak{X}, \mathbb{Z}(2)) \to H^2_{\mathcal{M}}(X_0(\mathfrak{n})_L, \mathbb{Z}(2))$. By Proposition 6.5 of [23] the group $H^2_{\mathcal{M}}(\mathfrak{X}, \mathbb{Z}(2))$ is the extension of a torsion group by a $p$-divisible subgroup. The image of the restriction of the regulator of Notation 5.10 to $H^2_{\mathcal{M}}(X_0(\mathfrak{n})_L, \mathbb{Z}(2))$ lies in $\mathcal{H}(\mathfrak{n}, F^*_\infty)$ so its image has a torsion $p$-divisible part. Therefore the image of the restriction of this regulator to $H^2_{\mathcal{M}}(\mathfrak{X}, \mathbb{Z}(2))$ is torsion. The claim is now clear. □

**Remarks 5.13.** Let $\mathbb{T}(\mathfrak{n})$ denote the algebra with unity generated by the endomorphisms $T_\mathfrak{m}$ of the Jacobian $J_0(\mathfrak{n})$, where $\mathfrak{m} \lhd A$ is any proper ideal. The algebra $\mathbb{T}(\mathfrak{n})$ is known to be commutative. By claim (iii) of Proposition 5.11 and the lemma above the algebra of correspondences generated by the Hecke correspondences leaves the kernel of the regulator of Notation 5.10 restricted to $H^2_{\mathcal{M}}(X_0(\mathfrak{n})_L, K(2))$ invariant and its action on the image of this homomorphism factors through the Hecke algebra $\mathbb{T}(\mathfrak{n}) \otimes \mathbb{Q}$. Moreover the Hecke operator $T_\mathfrak{m}$ acts on this image via the operator $T_\mathfrak{m}$ given by the formula in Definition 5.7 by claim (iii) of Proposition 5.11.

**Definition 5.14.** Let $\mu_G$ be the unique left-invariant Haar measure on the locally compact abelian topological group $GL_2(\mathbb{A})/Z(\mathbb{A})$ such that $\mu_G(GL_2(\mathcal{O})/Z(\mathcal{O}))$ is equal to 1. Since this measure is left-invariant with respect to the discrete action of the group $GL_2(F)/Z(F)$, it induces a measure on $Z(\mathbb{A})GL_2(F)\backslash GL_2(\mathbb{A})$ which will be denoted by the same symbol by abuse of notation. Let $V$, $W$ be vector spaces over $\mathbb{Q}$, and let $\phi$, $\psi$ be a $V$-valued and a $W$-valued, locally constant function on $Z(\mathbb{A})GL_2(F)\backslash GL_2(\mathbb{A})$, respectively. Also assume that $\psi$ has compact support, for example $\psi \in \mathcal{H}_0(\mathfrak{n}, W)$. Then the integral

$$\int_{Z(\mathbb{A})GL_2(F)\backslash GL_2(\mathbb{A})} \phi(g) \otimes \psi(g)\, d\mu_G(g) \in V \otimes_\mathbb{Q} W$$

is well-defined. It will be denoted by $\langle \phi, \psi \rangle$, and will be called the *Petersson product* of $\phi$ and $\psi$.

**Lemma 5.15.** *For every $k \in H^2_{\mathcal{M}}(Y_0(\mathfrak{n})_L, K(2))$ there is a $k' \in H^2_{\mathcal{M}}(X_0(\mathfrak{n})_L, K(2))$ such that $\langle \{k\}, \psi \rangle = \langle \{k'\}, \psi \rangle$ for every $\psi \in \mathcal{H}_0(\mathfrak{n}, \mathbb{Q})$.*

*Proof.* Let $\mathcal{U}$ denote the group $H^0(Y_0(\mathfrak{n})_L, \mathcal{O}^*)$ and let $\mathcal{V}$ denote the $K$-vector subspace of $H^2_{\mathcal{M}}(Y_0(\mathfrak{n})_L, K(2))$ generated by the product $L^* \otimes \mathcal{U} \subseteq H^2_{\mathcal{M}}(Y_0(\mathfrak{n})_L, \mathbb{Z}(2))$. By our assumptions on $F$ and $\infty$ the curve $X_0(\mathfrak{n})$ is geometrically irreducible. Moreover the group generated by the linear equivalence class of degree zero divisors defined over $L$ supported on the complement of $Y_0(\mathfrak{n})_L$ in the Jacobian of $X_0(\mathfrak{n})_L$ is finite by the main theorem of [8]. Hence there is a $u \in \mathcal{V}$ such that

$\{k\}_x = \{u\}_x$ for every closed point $x$ in the complement of $Y_0(\mathfrak{n})_L$ where $\{\cdot\}_x$ denotes the $K$-linear extension of the tame symbol at $x$ (this fact is referred to as Bloch's lemma in [24]). Therefore $k' = k - u$ lies in $H^2_{\mathcal{M}}(X_0(\mathfrak{n})_L, K(2))$ by the localization sequence. Hence it will be sufficient to prove that $\langle\{u\},\psi\rangle = 0$ for every $u \in \mathcal{V}$ and for every $\psi \in \mathcal{H}_0(\mathfrak{n},\mathbb{Q})$. In fact we will show the same claim for every $\psi \in \mathcal{H}_0(\mathfrak{n},\overline{\mathbb{Q}})$. The operators $T_{\mathfrak{m}}$ act semisimply on the finite-dimensional vector space $\mathcal{H}_0(\mathfrak{n},\overline{\mathbb{Q}})$ therefore the latter decomposes as the direct sum of Hecke eigenspaces. Hence we may assume that $\psi$ above is a Hecke eigenform by linearity. By the projection formula for the norm map in Milnor $K$-theory we have $T_{\mathfrak{m}}(u_1 \otimes u_2) = u_1 \otimes T_{\mathfrak{m}}(u_2)$ for every $u_1 \in L^*$, $u_2 \in \mathcal{U}$ and Hecke correspondence $T_{\mathfrak{m}}$. Let $\mathfrak{q} \nmid \mathfrak{n}$ be a non-zero prime ideal which has a generator $\pi \in A$ such that $\pi \equiv 1 \bmod \mathfrak{n}$. Then the Hecke correspondence $T_{\mathfrak{q}}$ maps the cusps, the geometric points in the complement of $Y_0(\mathfrak{n})$, into themselves with multiplicity $1 + q^{\deg(\mathfrak{q})}$ according to the proof of Proposition 3.1 of [8] on page 365. (Strictly speaking this claim is proved for the cusps of the Drinfeld modular curve $X(\mathfrak{n})$ there but the former claim immediately follows from the latter.) Hence we have:

$$(1 + q^{\deg(\mathfrak{q})})\langle\{u\},\psi\rangle = \langle T_{\mathfrak{q}}\{u\},\psi\rangle = \langle\{u\}, T_{\mathfrak{q}}(\psi)\rangle = \psi^*(\mathfrak{q})\langle\{u\},\psi\rangle$$

using the self-adjointness of the operator $T_{\mathfrak{q}}$ with respect to the Petersson product where $\psi^*(\mathfrak{q}) \in \overline{\mathbb{Q}}$ is the $\mathfrak{q}$-th Hecke eigenvalue of $\psi$. By the Ramanujan–Petersson conjecture (proved in [4] first in this case) $\psi^*(\mathfrak{q})$ is not equal to $1 + q^{\deg(\mathfrak{q})}$ when $\deg(\mathfrak{q})$ is sufficiently large hence $\langle\{u\},\psi\rangle$ must be zero. $\square$

## 6. The Rankin–Selberg method

**Notation 6.1.** Let $\mathfrak{m} \triangleleft A$ be a proper ideal. Recall that a Dirichlet character of conductor $\mathfrak{m}$ is a continuous homomorphism $\chi : \mathbb{A}_f^* \to \mathbb{C}^*$ which is trivial on $F^*\mathcal{O}_{\mathfrak{m}}$ where $\mathcal{O}_{\mathfrak{m}} = \{u \in \mathcal{O}_f^* \mid u \equiv 1 \bmod \mathfrak{m}\}$. Then there is a unique homomorphism from $(\mathcal{O}_f/\mathfrak{m})^*$ into $\mathbb{C}^*$, which will be denoted by $\chi$ as well by abuse of notation, such that $\chi$ is trivial on the class of constants $\mathbb{F}_q^* \subset \mathcal{O}_f^*$ and we have $\chi(z) = \chi(\overline{z})$ for every $z \in \mathcal{O}_f^*$ where $\overline{z}$ denotes the class of $z$ in the quotient group $(\mathcal{O}_f/\mathfrak{m})^*$. Moreover we let $\chi$ denote also the unique extensions of these two homomorphisms onto $\mathbb{A}_f$ and $\mathcal{O}_f/\mathfrak{m}$ which are zero on the complement of $\mathbb{A}_f^*$ and $(\mathcal{O}_f/\mathfrak{m})^*$, respectively. We are going to assume that the homomorphism $\chi$ is non-trivial. In this case $\sum_{\alpha \in \mathcal{O}_f/\mathfrak{m}} \chi(\alpha)$ is zero. Let $\chi_1, \chi_2 \in \mathbb{C}[\mathcal{V}_{\mathfrak{m}}]_0$ denote the functions given by the rules:

$$\chi_1(\alpha,\beta) = \chi(\alpha) \quad (\forall(\alpha,\beta) \in \mathcal{V}_{\mathfrak{m}}), \qquad \chi_2(\alpha,\beta) = \begin{cases} \chi(\beta) & \text{when } \alpha = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $E^G_{\mathfrak{m}}(\chi, g, x, y)$ denote the function $\chi(\det(g_f))^{-1}E^G_{\mathfrak{m}}(\chi_1, \chi_2, g, x, y)$ for every finite quotient $G$ of $F^*\backslash\mathbb{A}^*/\mathcal{O}^*_f$.

**Lemma 6.2.** *The function $E^G_{\mathfrak{m}}(\chi, g, x, y)$ is left-invariant with respect to $GL_2(F)$ and right-invariant with respect to $\mathbb{K}(\mathfrak{m}\infty)\Gamma_\infty Z(\mathbb{A})$.*

*Proof.* By claim (i) of Proposition 3.3 we only need to show that $E^G_{\mathfrak{m}}(\chi, g, x, y)$ is right-invariant with respect to $Z(\mathbb{A}_f)$. But $Z(\mathbb{A}_f) = Z(F)Z(\mathcal{O}_f)$ hence we only have to show that $E^G_{\mathfrak{m}}(\chi, g, x, y)$ is right-invariant with respect to $Z(\mathcal{O}_f)$. In order to do so we will introduce some convenient notation which we will also use later on without further notice. By our usual abuse of notation for $i = 1, 2$ let $\chi_i : \mathcal{O}^2_f \to \mathbb{C}$ denote the function such that $\chi_i(f) = \chi_i(\overline{f})$ for every $f \in \mathcal{O}^2_f$ where $\overline{f}$ denotes the class of $f$ in the quotient group $(\mathcal{O}_f/\mathfrak{m})^2$. For every $g \in GL_2(\mathbb{A})$ let

$$W(g) = \{0 \neq f \in F^2 \mid fg_f \in \mathcal{O}^2_f\}, \quad V(g) = \{f \in W(g) \mid fg_\infty \in F^2_<\}$$

and $U(g) = W(g) - V(g)$. For every $g \in GL_2(\mathbb{A})$ and $z \in Z(\mathcal{O}_f) = \mathcal{O}^*_f$ we have $U(gz) = U(g)$ and $V(gz) = V(g)$. Moreover we have $\chi_i(fz) = \chi_i(f)\chi(z)$ for every $f \in \mathcal{O}^2_f$ and $i = 1, 2$. Therefore

$$\begin{aligned} E^G_{\mathfrak{m}}(\chi, gz, x, y) = \chi(z)^{-2}\chi(\det(g_f))^{-1}&\frac{\det(z^{-1})^G \det(g_f^{-1})^G}{(xy)^{\deg(\det(z))+\deg(\det(g))}} \\ &\cdot \sum_{\substack{(a,b)\in U(g)\\(c,d)\in V(g)}} \Bigg(\chi_1((a,b)g_f z)\chi_2((c,d)g_f z) \\ &\cdot \det\begin{pmatrix} a & b \\ c & d\end{pmatrix}^G_\infty x^{2\infty((a,b)g_\infty)} y^{2\infty((c,d)g_\infty)}\Bigg) \\ = E^G_{\mathfrak{m}}(\chi, g, x, y),& \end{aligned}$$

because $\deg(\det(z)) = 0$ and $\det(z)^G = 1$ by definition. □

**Definition 6.3.** Let $\chi_0 : \mathcal{O}_f \to \mathbb{C}$ denote the function such that $\chi_0(u) = \chi(\overline{u})$ for every $u \in \mathcal{O}_f$ where $\overline{u}$ denotes again the class of $u$ in the quotient group $\mathcal{O}_f/\mathfrak{m}$. Let $G$ be a finite quotient group of $F^*\backslash\mathbb{A}^*/\mathcal{O}^*_f$. Note that for every non-zero $\mathfrak{q} \triangleleft A$ the value $y^G \in G$ depends only on $\mathfrak{q}$ for every $y \in \mathbb{A}^*_f$ where the divisor of $y$ is $\mathfrak{q}$. Let $\mathfrak{q}^G$ denote this common value. Similarly note that for every non-zero ideal $\mathfrak{q} \triangleleft A$ relatively prime to $\mathfrak{m}$ the value $\chi_0(a)$ depends only on $\mathfrak{q}$ for every $a \in A$ which generates the ideal $\mathfrak{q}$. We let $\chi(\mathfrak{q})$ denote this common value. For every $G$ as above let $L^G_{\mathfrak{m}}(\chi, x)$ be the infinite series:

$$\sum_{(\mathfrak{q},\mathfrak{m})=1} \chi(\mathfrak{q})(\mathfrak{q}^G)^{-1}x^{\deg(\mathfrak{q})} \in \mathbb{C}[G][[x]].$$

Note that for every complex number $s$ the $\mathbb{C}[G]$-valued series $L^G_{\mathfrak{m}}(\chi, q^{-s})$ is absolutely convergent when $\mathrm{Re}(s) > 1$.

For every $z \in \mathbb{A}^*_f$ let $L^G_{\mathfrak{m}}(\chi, z, s)$ denote the $\mathbb{C}[G]$-valued series:

$$L^G_{\mathfrak{m}}(\chi, z, s) = (z^{-1})^G |z|^s \sum_{\substack{u \in F^* \\ uz \in \mathcal{O}_f}} \chi_0(uz) u^G_\infty |u|^{-s}_\infty$$

if the latter is absolutely convergent.

**Lemma 6.4.** *For every $z \in \mathbb{A}^*_f$ we have $L^G_{\mathfrak{m}}(\chi, z, s) = \chi(z)(q-1)L^G_{\mathfrak{m}}(\chi, q^{-s})$.*

*Proof.* First we are going to show that the function $\overline{\chi(z)} L^G_{\mathfrak{m}}(\chi, s)$ is invariant with respect to $\mathcal{O}^*_f$. For every $v \in \mathcal{O}^*_f$ we have

$$\begin{aligned}\overline{\chi(zv)} L^G_{\mathfrak{m}}(\chi, \eta zv, s) &= \overline{\chi(v)}(v^{-1})^G |v|^s \overline{\chi(z)}(z^{-1})^G |z|^s \cdot \sum_{\substack{u \in F^* \\ uzv \in \mathcal{O}_f}} \chi_0(uz)\chi_0(v) u^G_\infty |u|^{-s}_\infty \\ &= L^G_{\mathfrak{m}}(\chi, z, s),\end{aligned}$$

because $\chi(v) = \chi_0(v)$, $v^G = 1$ and $|v| = 1$. Now we may assume that $z \in F^*$ because $\mathbb{A}^*_F = F^* \mathcal{O}^*_f$. In this case

$$\begin{aligned}\overline{\chi(z)} L^G_{\mathfrak{m}}(\chi, z, s) = L^G_{\mathfrak{m}}(\chi, z, s) &= \sum_{0 \neq u \in A} \chi_0(u)(u^G_f)^{-1} |u|^{-s}_\infty \\ &= (q-1) L^G_{\mathfrak{m}}(\chi, q^{-s})\end{aligned}$$

because we have $(u^G_f)^{-1} = u^G_\infty$ for every $u \in F^*$, and because $\chi(z) = 1$ and $|z_f| = |z|^{-1}_\infty$ as the degree of every principal divisor is zero. □

**Definition 6.5.** Let $B$ denote the group scheme of invertible upper triangular two by two matrices. For every finite quotient $G$ as above and $g \in B(\mathbb{A})$ let $K^G_{\mathfrak{m}}(\chi, g, s)$ denote the $\mathbb{C}[G]$-valued function:

$$K^G_{\mathfrak{m}}(\chi, g, s) = \overline{\chi((xz)_f)}((xz)^{-1}_f)^G |xz^2|^s \sum_{(v,w) \in U(g)} \chi_0((vxz)_f) v^G_\infty \, |(vxz)_\infty|^{-2s}$$

where $g = \left(\begin{smallmatrix} xz & yz \\ 0 & z \end{smallmatrix}\right) \in B(\mathbb{A})$ and $s$ is a complex number when this infinite sum is absolutely convergent. The latter holds if $\mathrm{Re}(s) > 1$ because the series above is majorized by the series $E(g, s)$. Finally for every pair of complex numbers $s$, $t$ with $\mathrm{Re}(s) > 1$, $\mathrm{Re}(t) > 1$ we let $H^G_{\mathfrak{m}}(\chi, g, s, t)$ denote the $\mathbb{C}[G]$-valued function on $GL_2(\mathbb{A})$ given by the formula:

$$H^G_{\mathfrak{m}}(\chi, g, s, t) = L^G_{\mathfrak{m}}(\chi, q^{-2t}) |x|^t K^G_{\mathfrak{m}}\left(\chi, \begin{pmatrix} xz & yz \\ 0 & z \end{pmatrix}, s\right)$$

if

$$g = \begin{pmatrix} xz & yz \\ 0 & z \end{pmatrix} k \quad \text{where } k \in \mathbb{K}(\mathfrak{m}\infty)\Gamma_\infty,$$

and $H^G_{\mathfrak{m}}(\chi, g, s, t) = 0$, otherwise.

**Lemma 6.6.** *The following holds:*

(i) *the function $K^G_{\mathfrak{m}}(\chi, g, s)$ is left-invariant with respect to $B(F)$ and right-invariant with respect to $(B(\mathcal{O}) \cap \mathbb{K}(\mathfrak{m}))Z(\mathbb{A})$,*

(ii) *the function $H^G_{\mathfrak{m}}(\chi, g, s, t)$ is well-defined and it is left-invariant with respect to $B(F)$ and right-invariant with respect to $\mathbb{K}(\mathfrak{m}\infty)\Gamma_\infty Z(\mathbb{A})$.*

*Proof.* The proof of the first claim is the same as the proofs of claim (i) of Proposition 3.3 and Lemma 6.4. In order to prove that $H^G_{\mathfrak{m}}(\chi, g, s, t)$ is well-defined we need to show that $|x|^t = |a|^t$ and

$$K^G_{\mathfrak{m}}\left(\chi, \begin{pmatrix} xz & yz \\ 0 & z \end{pmatrix}, s\right) = K^G_{\mathfrak{m}}\left(\chi, \begin{pmatrix} ac & bc \\ 0 & c \end{pmatrix}, s\right)$$

where

$$\begin{pmatrix} xz & yz \\ 0 & z \end{pmatrix}, \begin{pmatrix} ac & bc \\ 0 & c \end{pmatrix} \in B(\mathbb{A}) \quad \text{and} \quad \begin{pmatrix} xz & yz \\ 0 & z \end{pmatrix} \cdot \begin{pmatrix} ac & bc \\ 0 & c \end{pmatrix}^{-1} \in \mathbb{K}(\mathfrak{m}\infty)\Gamma_\infty.$$

The latter is an immediate consequence of claim (i). Similarly the invariance properties of $H^G_{\mathfrak{m}}(\chi, g, s, t)$ claimed above are obvious from claim (i) and the definition. □

**Proposition 6.7.** *For every $g \in GL_2(\mathbb{A})$ the sum on the right hand side below is absolutely convergent and we have:*

$$E^G_{\mathfrak{m}}(\chi, g, q^{-s}, q^{-t}) = (q-1) \sum_{\rho \in B(F)\backslash GL_2(F)} H^G_{\mathfrak{m}}(\chi, \rho g, s, t),$$

*when* $\mathrm{Re}(s) > 1$ *and* $\mathrm{Re}(t) > 1$.

*Proof.* For every $\rho \in GL_2(F)$ the value of $H^G_{\mathfrak{m}}(\chi, \rho g, s, t)$ depends only on the left $B(F)$-coset of $\rho$ because $H^G_{\mathfrak{m}}(\chi, g, s, t)$ is left-invariant with respect to $B(F)$. Hence the infinite sum on the right hand side above is well-defined. By grouping the terms of the absolutely convergent series on the left hand side we get:

$$E^G_{\mathfrak{m}}(\chi, g, q^{-s}, q^{-t}) = \sum_{\rho \in B(F)\backslash GL_2(F)} \chi(\det((\rho g)_f))^{-1} \det((\rho g)_f^{-1})^G |\det(\rho g)|^{s+t}$$

$$\cdot\Big( \sum_{\substack{(v,w) \in U(\rho g) \\ u \in F^*, (0,u) \in V(\rho g)}} \chi_1((v,w)\rho g_f)\chi_2((0,u)\rho g_f)(vu)^G_\infty \|(v,w)\rho g_\infty\|^{-2s} \|(0,u)\rho g_\infty\|^{-2t} \Big).$$

By the Iwasawa decomposition we may write $g$ as

$$g = pk, \quad \text{where} \quad p = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in B(\mathbb{A}) \quad \text{and} \quad k = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \in GL_2(\mathcal{O}).$$

Because $k_\infty$ is an isometry we only have to show that

$$H^G_{\mathfrak{m}}(\chi, g, s, t) = \Big( \overline{\chi(c_f)} (c_f^{-1})^G |ac|^t \sum_{\substack{u \in F^* \\ (0,u) \in V(g)}} \chi_2((0,u)g_f) u^G_\infty \|(0,u)p_\infty\|^{-2t} \Big) \cdot \Big( \overline{\chi(a_f)} (a_f^{-1})^G |ac|^s \sum_{(v,w) \in U(g)} \chi_1((v,w)g_f) v^G_\infty \|(v,w)p_\infty\|)^{-2s} \Big)$$

by the above. The first infinite sum is zero unless there is a $d \in \mathbb{A}_f$ such that $d(k_{21}, k_{22})_f \in \mathcal{O}_f^2$ and the latter is congruent to $(0, \alpha)$ modulo $\mathfrak{m}\mathcal{O}_f$ for some $\alpha \in (\mathcal{O}_f/\mathfrak{m})^*$. The latter is possible exactly when $k_f$ is in $\mathbb{K}_0(\mathfrak{m})$. We may even assume that $k_f$ is in $\mathbb{K}(\mathfrak{m})$ by changing $p$, if necessary. By the definition of the set $V(g)$ we also need that $|(k_{22})_\infty| > |(k_{21})_\infty|$ for the first sum to be non-zero. Since $(k_{22})_\infty \in \mathcal{O}_\infty$ we have $\infty((k_{21})_\infty) > 0$ so $k_\infty \in \Gamma_\infty$. In this case we have $(0,u)g_\infty \in F^2_<$ for every $u \in F^*$ automatically so

$$\{u \in F^* \mid (0,u) \in V(g)\} = \{u \in F^* \mid uc \in \mathcal{O}_f\}.$$

Hence the first term of the product above is $|a/c|^t \overline{\chi(c_f)} L^G(\overline{\chi}, c_f, 2t)$. By Lemma 6.4 we know that the latter is $|a/c|^t (q-1) L(\overline{\chi}, q^{-2t})$. On the other hand the second term is visibly $K^G_{\mathfrak{m}}(\chi, g, s)$ because $U(g) = U(p)$ and $\chi_1(fk_f) = \chi_1(f)$ for every $f \in \mathcal{O}_f^2$ since $k_f \in \mathbb{K}(\mathfrak{m})$. □

**Definition 6.8.** Let $\mu_B$ be the unique left-invariant Haar measures on the locally compact abelian topological groups $Z(\mathbb{A})\backslash B(\mathbb{A})$ such that $\mu_B(Z(\mathcal{O})\backslash B(\mathcal{O}))$ is equal to 1. Since this measure is left-invariant with respect to the discrete action of the group $Z(F)\backslash B(F)$, it induces a measure on $Z(\mathbb{A})B(F)\backslash B(\mathbb{A})$, which will be denoted by the same symbol by abuse of notation. The measure $\mu_B$ has a simple description. Let $\mu$ and $\mu^*$ be the unique Haar measures on the locally compact abelian topological groups $\mathbb{A}$ and $\mathbb{A}^*$, respectively, such that $\mu(\mathcal{O})$ and $\mu^*(\mathcal{O}^*)$ are both equal to 1. Since the measures $\mu$ and $\mu^*$ are left-invariant with respect to the discrete subgroups $F \subset \mathbb{A}$ and $F^* \subset \mathbb{A}^*$, respectively, by definition, they induce a measure on $F\backslash\mathbb{A}$ and $F^*\backslash\mathbb{A}^*$, respectively, which will be denoted by the same letter by abuse of notation. Then we have

$$\int_{Z(\mathbb{A})B(F)\backslash B(\mathbb{A})} f\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} d\mu_B\left(\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\right) = \int_{F^*\backslash\mathbb{A}^*} d\mu^*(x) \int_{F\backslash\mathbb{A}} f\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} d\frac{\mu(y)}{|x|}$$

for every Lebesgue-measurable function $f : Z(\mathbb{A})B(F)\backslash B(\mathbb{A}) \to \mathbb{C}$.

**Lemma 6.9.** *For every $\psi \in \mathcal{A}_0(\mathfrak{m}\infty, \mathbb{C})$ the integrands of the two integrals below are absolutely Lebesgue-integrable and*

$$\int_{Z(\mathbb{A})GL_2(F)\backslash GL_2(\mathbb{A})} E^G_{\mathfrak{m}}(\chi, g, q^{-s}, q^{-t})\overline{\psi(g)}\, d\mu_G(g) = \mu(\mathfrak{m}) \int_{Z(\mathbb{A})B(F)\backslash B(\mathbb{A})} H^G_{\mathfrak{m}}(\chi, b, s, t)\overline{\psi(b)}\, d\mu_B(b)$$

*where* $\mu(\mathfrak{m}) = (q-1)\mu_G(Z(\mathcal{O})\backslash\mathbb{K}(\mathfrak{m}\infty)\Gamma_\infty Z(\mathcal{O}))$ *when* $\mathrm{Re}(s) > 1$ *and* $\mathrm{Re}(t) > 1$.

*Proof.* We may talk about the Lebesgue-integrability of the integrands above because they are $\mathbb{C}[G]$-valued functions. By Theorem 2.2.1 in [11], pages 255–256, we know that any cuspidal automorphic form which is invariant with respect to $Z(\mathbb{A})$ has compact support as a function on $Z(\mathbb{A})GL_2(F)\backslash GL_2(\mathbb{A})$. Hence the integral on the left in the equation above is absolutely convergent and we may interchange the integration and the summation in Proposition 6.7 to get that

$$\int_{Z(\mathbb{A})GL_2(F)\backslash GL_2(\mathbb{A})} E^G_{\mathfrak{m}}(\chi, g, q^{-s}, q^{-t})\overline{\psi(g)}\, d\mu_G(g) = (q-1) \int_{Z(\mathbb{A})B(F)\backslash GL_2(\mathbb{A})} H^G_{\mathfrak{m}}(\chi, b, s, t)\overline{\psi(b)}\, d\mu_G(b)$$

where the measure on $Z(\mathbb{A})B(F)\backslash GL_2(\mathbb{A})$ induced by $\mu_G$ will be denoted by the same symbol by the usual abuse of notation. The map:

$$\pi : Z(\mathbb{A})B(F)\backslash B(\mathbb{A}) \times Z(\mathcal{O})\backslash\mathbb{K}(\mathfrak{m}\infty)\Gamma_\infty Z(\mathcal{O}) \to Z(\mathbb{A})B(F)\backslash GL_2(\mathbb{A})$$

given by the rule $(b,k) \mapsto bk$ is continuous, hence for every Borel-measurable set $\mathfrak{B} \subseteq Z(\mathbb{A})B(F)\backslash GL_2(\mathbb{A})$ the pre-image $\pi^{-1}(\mathfrak{B})$ is also Borel-measurable. Let $\mu_B \times \mu_G$ denote the direct product of the measures $\mu_B$ and $\mu_G$ on the direct product $Z(\mathbb{A})B(F)\backslash B(\mathbb{A}) \times Z(\mathcal{O})\backslash\mathbb{K}(\mathfrak{m}\infty)\Gamma_\infty Z(\mathcal{O})$. Then we have $\mu_B \times \mu_G(\pi^{-1}(\mathfrak{B})) = \mu_G(\mathfrak{B})$ for every $\mathfrak{B}$ above. Moreover the map $\pi$ maps surjectively onto the support of $H^G_{\mathfrak{m}}(\chi, b, s, t)$ as a function on $GL_2(\mathbb{A})$ as we saw in Definition 6.4 so the integral above is equal to:

$$(q-1) \int_{Z(\mathbb{A})B(F)\backslash B(\mathbb{A})} d\mu_B(b) \int_{Z(\mathcal{O})\backslash\mathbb{K}(\mathfrak{m}\infty)\Gamma_\infty Z(\mathcal{O})} H^G_{\mathfrak{m}}(\chi, bk, s, t)\overline{\psi(bk)}\, d\mu_G(k)$$

by Fubini's theorem. By definition the integrand of the interior integral is constant on the domain of integration. The claim is now obvious. $\square$

**Definition 6.10.** Let $\tau : F\backslash\mathbb{A} \to \mathbb{C}^*$ be a non-trivial continuous character and let $\mathfrak{d}$ be an idele such that $\mathcal{D} = \mathfrak{d}\mathcal{O}$, where $\mathcal{D}$ is the $\mathcal{O}$-module defined as

$$\mathcal{D} = \{x \in \mathbb{A} \mid \tau(x\mathcal{O}) = 1\}.$$

It is well-known that the linear equivalence class of the divisor of $\mathfrak{d}$ is the anticanonical class. Moreover for every $\eta \in F^*$ the map $x \mapsto \tau(\eta x)$ is another non-trivial continuous homomorphism. Therefore by choosing an appropriate character $\tau$, we may assume that $\mathfrak{d}$ is any idele of degree two, as every such divisor is linearly equivalent to the anticanonical class. In particular we may assume that $\mathfrak{d} = \pi^2$ where $\pi \in F_\infty$ is a uniformizer. For every non-zero ideal $\mathfrak{r} \triangleleft A$ let $S(\mathfrak{m}, \mathfrak{r})$ denote the set:

$$S(\mathfrak{m}, \mathfrak{r}) = \{0 \neq \mathfrak{q} \triangleleft A \mid (\mathfrak{m}, \mathfrak{q}) = 1,\ \mathfrak{q} \mid \mathfrak{r}\}.$$

Moreover for every $G$ as in Definition 6.3 let $\sigma^G_{\mathfrak{m}}(\chi, \mathfrak{r}, x) \in \mathbb{C}[G][x]$ denote the polynomial given by the formula:

$$\sigma^G_{\mathfrak{m}}(\chi, \mathfrak{r}, x) = \sum_{\mathfrak{q} \in S(\mathfrak{m}, \mathfrak{r})} \chi(\mathfrak{q})(\mathfrak{q}^G)^{-1} x^{\deg(\mathfrak{q})}.$$

**Proposition 6.11.** *For each complex $s$ with* $\mathrm{Re}(s) > 1$ *we have:*

$$\int_{F\backslash\mathbb{A}} K^G_{\mathfrak{m}}\left(\chi, \begin{pmatrix} x\mathfrak{d} & y \\ 0 & 1 \end{pmatrix}, s\right) \tau(-y)\, d\mu(y) = (q-1)|x\mathfrak{d}|^{1-s} \sigma^G_{\mathfrak{m}}(\chi, x_f, q^{1-2s}),$$

*if the divisor of $x$ is effective, and the integral is zero, otherwise.*

*Proof.* The integral above is well-defined because the integrand is $F$-invariant by claim (i) of Lemma 6.6. Note that we have $u \neq 0$ for every $0 \neq (u, v) \in F^2$ such that $\chi_0((ux\mathfrak{d})_f) \neq 0$. Therefore by grouping the terms in the infinite sum of Definition 6.5 we get the following identity:

$$\begin{aligned} &K^G_{\mathfrak{m}}\left(\chi, \begin{pmatrix} x\mathfrak{d} & y \\ 0 & 1 \end{pmatrix}, s\right) \\ &\quad = \overline{\chi((x\mathfrak{d})_f)}((x\mathfrak{d})_f^{-1})^G |x\mathfrak{d}|^s \sum_{v \in F} \sum_{\substack{u \in F^* \\ (u,0) \in U\left(\begin{pmatrix} x\mathfrak{d} & (y+v) \\ 0 & 1 \end{pmatrix}\right)}} \chi_0((ux\mathfrak{d})_f) u^G_\infty |(ux\mathfrak{d})_\infty|^{-2s}. \end{aligned}$$

Hence

$$\begin{aligned} &\int_{F\backslash\mathbb{A}} K^G_{\mathfrak{m}}\left(\chi, \begin{pmatrix} x\mathfrak{d} & y \\ 0 & 1 \end{pmatrix}, s\right) \tau(-y)\, d\mu(y) \\ &\qquad = \overline{\chi((x\mathfrak{d})_f)}((x\mathfrak{d})_f^{-1})^G \\ &\qquad\quad \cdot |x\mathfrak{d}|^s \sum_{\substack{u \in F^* \\ u(x\mathfrak{d})_f \in \mathcal{O}_f}} \chi_0((ux\mathfrak{d})_f) u^G_\infty |ux\mathfrak{d}|_\infty^{-2s} \int_{\substack{uy_f \in \mathcal{O}_f \\ |y_\infty| \leq |x\mathfrak{d}|_\infty}} \tau(-y)\, d\mu(y) \end{aligned}$$

by interchanging summation and integration. For every $u \in F^*$ the domain of integration of the integral above is a direct product of the sets $u_f^{-1}\mathcal{O}_f \subset \mathbb{A}_f$ and $x\mathfrak{d}\mathcal{O}_\infty \subset F_\infty$. The integral itself is non-zero if and only if the product set above lies in the kernel of $\tau$. The latter is equivalent to the conditions $(u\mathfrak{d})_f^{-1} = u_f^{-1} \in \mathcal{O}_f$ and $\infty(x) \geq 0$. In this case the integral is equal to:

$$\mu(u_f^{-1}\mathcal{O}_f \times x\mathfrak{d}\mathcal{O}_\infty) = |u|^{-1}\mu(\mathcal{O}_f \times ux\mathfrak{d}\mathcal{O}_\infty) = |ux\mathfrak{d}|_\infty.$$

Let $T(\mathfrak{m}, x)$ denote the set:

$$T(\mathfrak{m}, x) = \{u \in F^* \mid (ux)_f \in \mathcal{O}_f,\, u_f^{-1} \in \mathcal{O}_f\}.$$

By the above the left hand side of the equation in the claim above is equal to:

$$\overline{\chi(x_f)}|x\mathfrak{d}|^{1-s} \sum_{u \in T(\mathfrak{m},x)} \chi_0((ux)_f)((ux)_f^{-1})^G |(ux)_f|^{2s-1}$$

when $\infty(x) \geq 0$, and it is zero, otherwise. The set $T(\mathfrak{m}, x)$ is empty when $x_f$ is not an element of $\mathcal{O}_f$. Therefore the expression above is zero unless the divisor of $x$ is effective. Note that in the latter case for every $u \in T(\mathfrak{m}, x)$ the number $\chi_0((ux)_f)$ is zero unless the divisor of $(ux)_f$ is an element of $S(\mathfrak{m}, x_f)$. On the other hand every element of $S(\mathfrak{m}, x_f)$ is the divisor of an idele of the form $(ux)_f$ for some $u \in T(\mathfrak{m}, x)$ and $u$ is unique up to a factor in $\mathbb{F}_q^*$. Note that the sum above is invariant in the variable $x$ with respect to the action of $\mathcal{O}_f^*$. Hence we may assume that $x_f = \eta_f$ for some $\eta \in F^*$. In this case we have $\chi(x_f) = 1$ and $\chi(\mathfrak{q}) = \chi_0((ux)_f)$ for every $u \in T(\mathfrak{m}, x)$ where $\mathfrak{q}$ is the divisor of $(ux)_f$. The claim is now clear. $\square$

**Notation 6.12.** Recall that we call two divisors $\mathfrak{r}$ and $\mathfrak{s}$ on $X$ *relatively prime* if their supports are disjoint. For every $\psi \in \mathcal{A}_0(\mathfrak{m}, \mathbb{C})$ let $\psi^* : \mathrm{Div}(X) \to \mathbb{C}$ denote the Fourier coefficients of $\psi$ whose existence was established in Proposition 1 of Chapter III in [25], page 21, proved on pages 19–20 of [25]. Recall that a function $f : \mathrm{Div}(X) \to R$ is called *multiplicative*, where $R$ is a commutative ring with unity, if it is zero on non-effective divisors, $f(1) = 1$ and for every pair of relatively prime divisors $\mathfrak{r}$ and $\mathfrak{s}$ we have $f(\mathfrak{r}\mathfrak{s}) = f(\mathfrak{r})f(\mathfrak{s})$. (Similarly an $R$-valued function on the set of non-zero ideals of $A$ is called multiplicative if it satisfies the last two properties of the previous definition.) Let us recall the situation considered in the introduction. Let $E$ be an elliptic curve defined over $F$ which has split multiplicative reduction at $\infty$. By assumption the conductor of $E$ is of the form $\mathfrak{n}\infty$ where $\mathfrak{n}$ is an effective divisor which is supported in the complement of $\infty$ in $X$. Let $\psi_E^*$ denote the unique multiplicative function into the multiplicative semigroup of $\mathbb{Q}$ such that $\psi_E^*(x^n)$ is the same as in 1.6 for each natural number $n$ and each closed point $x$ on $X$. A cuspidal harmonic form $\phi_E \in \mathcal{H}_0(\mathfrak{n}, \mathbb{Q})$ is called a *normalized Hecke eigenform* attached to $E$ if its Fourier coefficient $\phi_E^*(\mathfrak{q})$ is equal to $|\mathfrak{q}|\psi_E^*(\mathfrak{q})$ for every effective divisor $\mathfrak{q}$.

The following proposition is an easy consequence of the Langlands correspondence:

**Proposition 6.13.** *There is a unique normalized Hecke eigenform attached to $E$.*

*Proof.* The only not entirely obvious fact is that the normalized Hecke eigenform has values in $\mathbb{Q}$, see for example the proof of Proposition 3.3 in [21]. □

**Theorem 6.14.** *Assume that $\mathfrak{n}$ divides $\mathfrak{m}$. Then for every $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(t) > 1$ we have:*

$$\int_{Z(\mathbb{A})GL_2(F)\backslash GL_2(\mathbb{A})} E^G_{\mathfrak{m}}(\chi, g, q^{-s}, q^{-t})\phi_E(g)\, d\mu_G(g)$$
$$= (q-1)\mu(\mathfrak{m})L^G_{\mathfrak{m}}(\chi, q^{-2t})\frac{|\mathfrak{d}|^{t-s}}{1-q^{s-t-1}}\sum_{0\neq\mathfrak{r}\triangleleft A}|\mathfrak{r}|^{1+t-s}\sigma^G_{\mathfrak{m}}(\chi, \mathfrak{r}, q^{1-2s})\psi^*_E(\mathfrak{r}).$$

*Proof.* By Lemma 6.9 and the description of the measure $\mu_B$ as a double integral at the end of Definition 6.8 we know that the integral on the left hand side of the equation above is equal to:

$$\mu(\mathfrak{m})L^G_{\mathfrak{m}}(\chi, q^{-2t})\int_{F^*\backslash\mathbb{A}^*} d\mu^*(x)\int_{F\backslash\mathbb{A}}|x|^t K^G_{\mathfrak{m}}\left(\chi, \begin{pmatrix} x & y \\ 0 & 1\end{pmatrix}, s\right)\phi_E\left(\begin{pmatrix} x & y \\ 0 & 1\end{pmatrix}\right)d\frac{\mu(y)}{|x|}. \tag{6.14.1}$$

Using the Fourier expansion of $\phi_E = \overline{\phi_E}$ we get from Proposition 6.11 that

$$\int_{F\backslash\mathbb{A}} K^G_{\mathfrak{m}}\left(\chi, \begin{pmatrix} x\mathfrak{d} & y \\ 0 & 1\end{pmatrix}, s\right)\phi_E\left(\begin{pmatrix} x\mathfrak{d} & y \\ 0 & 1\end{pmatrix}\right)d\mu(y)$$
$$= (q-1)\sum_{\eta\in F^*}|\eta x\mathfrak{d}|^{1-s}\sigma^G_{\mathfrak{m}}(\chi, (\eta x)_f, q^{1-2s})\phi^*_E(\eta x).$$

By plugging the equation above into the double integral in (6.14.1) we get that the latter is equal to:

$$(q-1)\int_{\mathbb{A}^*}|x\mathfrak{d}|^{t-s}\sigma^G_{\mathfrak{m}}(\chi, x_f, q^{1-2s})\phi^*_E(x)\, d\mu^*(x)$$

if we also interchange the summation in the index $\eta$ and the integration. The integrand above is constant on the cosets of the subgroup $\mathcal{O}^* \subset \mathbb{A}^*$ hence the integral is equal to the infinite sum:

$$|\mathfrak{d}|^{t-s}\sum_{\substack{0\neq\mathfrak{r}\triangleleft A\\ k\in\mathbb{N}}}|\mathfrak{r}\infty^k|^{t-s}\sigma^G_{\mathfrak{m}}(\chi, \mathfrak{r}, q^{1-2s})\phi^*_E(\mathfrak{r}\infty^k)$$
$$= \frac{|\mathfrak{d}|^{t-s}}{1-q^{s-t-1}}\sum_{0\neq\mathfrak{r}\triangleleft A}|\mathfrak{r}|^{1+t-s}\sigma^G_{\mathfrak{m}}(\chi, \mathfrak{r}, q^{1-2s})\psi^*_E(\mathfrak{r}),$$

where we also used that the function $\psi^*_E$ is multiplicative and $\psi^*_E(\infty^k) = 1$ for every $k \in \mathbb{N}$. □

## 7. An $\infty$-adic analogue of Beilinson's theorem

Let $R$ be an arbitrary commutative ring with unity. Let $*: R[[t]] \times R[[t]] \to R[[t]]$ denote the map given by the rule:

$$\Big(\sum_{n\in\mathbb{N}} a_n t^n\Big) * \Big(\sum_{n\in\mathbb{N}} b_n t^n\Big) = \sum_{n\in\mathbb{N}} a_n b_n t^n.$$

**Lemma 7.1.** *We have:*

$$\frac{1}{(1-\alpha_1 t)(1-\beta_1 t)} * \frac{1}{(1-\alpha_2 t)(1-\beta_2 t)} = \frac{1-\alpha_1\beta_1\alpha_2\beta_2 t^2}{(1-\alpha_1\alpha_2 t)(1-\alpha_1\beta_2 t)(1-\beta_1\alpha_2 t)(1-\beta_1\beta_2 t)}$$

*for every $\alpha_1$, $\beta_1 \in R$ and $\alpha_2, \beta_2 \in R$.*

*Proof.* We may assume that $R = \mathbb{Z}[x_1, y_1, x_2, y_2]$ and $\alpha_i = x_i$, $\beta_i = y_i$ for $i = 1, 2$ without loss of generality. Note that

$$\frac{\alpha_i - \beta_i}{(1-\alpha_i t)(1-\beta_i t)} = \frac{\alpha_i}{1-\alpha_i t} - \frac{\beta_i}{1-\beta_i t}$$

for $i = 1$ and $i = 2$. Also note that

$$\frac{1}{1-\gamma_1 t} * \frac{1}{1-\gamma_2 t} = \frac{1}{1-\gamma_1\gamma_2 t}$$

for every $\gamma_1, \gamma_2 \in R$ by definition. Because the map $*$ is $R$-bilinear we have

$$\begin{aligned}(\alpha_1-\beta_1)(\alpha_2-\beta_2)&\frac{1}{(1-\alpha_1 t)(1-\beta_1 t)} * \frac{1}{(1-\alpha_2 t)(1-\beta_2 t)}\\ &= \left(\frac{\alpha_1}{1-\alpha_1 t} - \frac{\beta_1}{1-\beta_1 t}\right) * \left(\frac{\alpha_2}{1-\alpha_2 t} - \frac{\beta_2}{1-\beta_2 t}\right)\\ &= \frac{(\alpha_1-\beta_1)(\alpha_2-\beta_2)(1-\alpha_1\beta_1\alpha_2\beta_2 t^2)}{(1-\alpha_1\alpha_2 t)(1-\alpha_1\beta_2 t)(1-\beta_1\alpha_2 t)(1-\beta_1\beta_2 t)}\end{aligned}$$

by the above. Since our assumption above implies that $\alpha_i - \beta_i$ is not a zero divisor in $R$ for $i = 1, 2$ the claim is now clear. $\square$

**Notation 7.2.** Let us consider the situation described in Definition 1.6. The Galois representation $\chi$ corresponds to a Dirichlet character of conductor $\mathfrak{m}$ described in Notation 6.1 by class field theory if an embedding of $K$ into the field of complex numbers is also provided. We let $\chi$ denote this Dirichlet character, too. Moreover the profinite completion of the group $F^*\backslash\mathbb{A}^*/\mathcal{O}_f^*$ and $G_\infty$ are canonically isomorphic by class field theory. In particular there is a bijective correspondence between the finite quotients of these groups. These two sets are going to be identified in all that follows. For every effective divisor $\mathfrak{d}$ on $X$ let $L_{\mathfrak{d}}(E,x)$ be the $L$-function:

$$L_{\mathfrak{d}}(E,t) = L(X(\mathfrak{d}),\rho,t) \in \mathbb{C}[t]$$

where we continue to use the notation introduced in the proof of Proposition 2.4.

**Proposition 7.3.** *We have:*

$$\frac{L_{\mathfrak{m}\infty}(E,t)\mathcal{L}^G_{\mathfrak{m}}(E,\chi,xt)}{L^G_{\mathfrak{m}}(\chi,qxt^2)} = \sum_{\substack{0\neq\mathfrak{q}\triangleleft A\\(\mathfrak{q},\mathfrak{m})=1}} \psi^*_E(\mathfrak{q})\sigma^G_{\mathfrak{m}}(\chi,\mathfrak{q},x)t^{\deg(\mathfrak{q})}.$$

*Proof.* Note that the $l$-adic representation $\rho$ is unramified at every prime ideal $\mathfrak{q}\triangleleft A$ which does not divide $\mathfrak{m}$ therefore the local factor $L_{\mathfrak{q}}(E,t)$ of the Hasse–Weil $L$-function of $E$ at $\mathfrak{q}$ can be written as

$$L_{\mathfrak{q}}(E,t) = \frac{1}{(1-\alpha(\mathfrak{q})t^{\deg(\mathfrak{q})})(1-\beta(\mathfrak{q})t^{\deg(\mathfrak{q})})},$$

where $\alpha(\mathfrak{q})$ and $\beta(\mathfrak{q})$ are complex numbers such that $\alpha(\mathfrak{q})+\beta(\mathfrak{q}) = \psi^*_E(\mathfrak{q})$ and $\alpha(\mathfrak{q})\cdot\beta(\mathfrak{q}) = q^{\deg(\mathfrak{q})}$. On the other hand it is clear from the definition of $\sigma^G_{\mathfrak{m}}(\chi,\mathfrak{q},x)$ that the latter is a $K[G][x]$-valued multiplicative function on the set of non-zero ideals of $A$. Therefore the power series on both sides of the equation in the claim above are Euler products, that is, the left hand side and the right hand side of the equation above are equal to:

$$\prod_{\substack{\mathfrak{q}\in|X|\\ \mathfrak{q}\notin\mathrm{supp}(\mathfrak{m}\infty)}} A_{\mathfrak{q}}(x,y) \quad\text{and}\quad \prod_{\substack{\mathfrak{q}\in|X|\\ \mathfrak{q}\notin supp(\mathfrak{m}\infty)}} B_{\mathfrak{q}}(x,y),$$

respectively, where

$$\begin{aligned} A_{\mathfrak{q}}(x,y) = {} & \frac{1-\chi(\mathfrak{q})(\mathfrak{q}^G)^{-1}(qxt^2)^{\deg(\mathfrak{q})}}{(1-\alpha(\mathfrak{q})t^{\deg(\mathfrak{q})})(1-\beta(\mathfrak{q})t^{\deg(\mathfrak{q})})}\\ & \cdot\frac{1}{(1-\alpha(\mathfrak{q})\chi(\mathfrak{q})(\mathfrak{q}^G)^{-1}(xt)^{\deg(\mathfrak{q})})(1-\beta(\mathfrak{q})\chi(\mathfrak{q})(\mathfrak{q}^G)^{-1}(xt)^{\deg(\mathfrak{q})})}\end{aligned}$$

and

$$B_{\mathfrak{q}}(x,y) = \sum_{n=0}^{\infty}\psi^*_E(\mathfrak{q}^n)\sigma^G_{\mathfrak{m}}(\chi,\mathfrak{q}^n,x)t^{\deg(\mathfrak{q})n}$$

for every $\mathfrak{q}\in|X|$ such that $\mathfrak{q}\notin\mathrm{supp}(\mathfrak{m}\infty)$ by the above. Clearly it is sufficient to prove that for every $\mathfrak{q}$ the factors of these Euler products at $\mathfrak{q}$ are equal. But the latter follows at once from Lemma 7.1 and the fact that

$$\sum_{n=0}^{\infty}\sigma^G_{\mathfrak{m}}(\chi,\mathfrak{q}^n,x)t^{\deg(\mathfrak{q})n} = \frac{1}{(1-t^{\deg(\mathfrak{q})})(1-\chi(\mathfrak{q})(\mathfrak{q}^G)^{-1}(xt)^{\deg(\mathfrak{q})})}. \qquad \square$$

**Theorem 7.4.** *We have:*

$$\langle E^G_{\mathfrak{m}}(\chi, g, x, y), \phi_E\rangle = (q-1)\mu(\mathfrak{m})\left(\frac{x}{y}\right)^2 L_{\mathfrak{m}}\left(E, \frac{y}{qx}\right)\mathcal{L}^G_{\mathfrak{m}}(E, \chi, xy).$$

*Proof.* By definition both sides of the equation above are elements of the ring $\mathbb{C}[G][[x, y]][x^{-1}, y^{-1}]$. But in fact the left and the right hand sides are elements of the ring $\mathbb{C}[G][x, y, x^{-1}, y^{-1}]$ by Propositions 3.5 and 2.4, respectively. We also know that after we substitute $q^{-s}$ and $q^{-t}$ into $x$ and $y$, respectively, both sides of the equation above become absolutely convergent when $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(t) > 1$. Therefore it will be sufficient to prove that they are equal after these substitutions by the unique continuation of holomorphic functions. Since $\mathfrak{d}$ is the anticanonical class, its degree is two, so the integral $\langle E^G_{\mathfrak{m}}(\chi, g, q^{-s}, q^{-t}), \phi_E\rangle$ can be rewritten as the infinite sum:

$$\begin{aligned}(q-1)\mu(\mathfrak{m})L^G_{\mathfrak{m}}(\chi, q^{-2t})\frac{|\mathfrak{d}|^{t-s}}{1-q^{s-t-1}}\sum_{0\neq\mathfrak{r}\triangleleft A}|\mathfrak{r}|^{1+t-s}\sigma^G_{\mathfrak{m}}(\chi, \mathfrak{r}, q^{1-2s})\psi^*_E(\mathfrak{r})\\ = (q-1)\mu(\mathfrak{m})q^{2s-2t}L_{\mathfrak{m}}(E, q^{1+s-t})\mathcal{L}^G_{\mathfrak{m}}(E, \chi, q^{-s-t})\end{aligned}$$

by Theorem 6.14 and Proposition 7.3. The claim is now clear. □

**Notation 7.5.** It is clear from Theorem 5.4 that the irreducible components of the curve $Y(\mathfrak{m})_{F_\infty}$ are in a bijective correspondence with the set:

$$GL_2(F)\backslash GL_2(\mathbb{A}_f)/\mathbb{K}_f(\mathfrak{m})$$

of double cosets. In fact for a double coset represented by an element $g \in GL_2(\mathbb{A}_f)$ the corresponding connected component is the image of $\{g\} \times \Omega$ under the uniformization map of Theorem 5.4. Therefore the rule which associates $\chi(\deg(g))^{-1}$ to the irreducible component corresponding to the double coset represented by the element $g \in GL_2(\mathbb{A}_f)$ gives rise to a well-defined $K$-valued function on the irreducible components of the curve $Y(\mathfrak{m})_{F_\infty}$. Actually this function is invariant under the action of the absolute Galois group of the extension $L$ of $F$ we introduced after Proposition 1.9 hence the function above is an algebraic cycle on $Y(\mathfrak{m})_L$ of codimension zero with coefficients in $K$. For every irreducible component $C$ of $Y(\mathfrak{m})_L$ we let $\chi^{-1}(C)$ denote the coefficient of $C$ in this algebraic cycle.

**Definition 7.6.** For every $C, D \in \mathbb{Z}[\mathcal{V}_{\mathfrak{m}}]_0$ let $\kappa_{\mathfrak{m}}(C, D)$ denote the element:

$$\epsilon_{\mathfrak{m}}(C) \otimes \epsilon_{\mathfrak{m}}(D) \in H^2_{\mathcal{M}}(Y(\mathfrak{m}), \mathbb{Z}(2))$$

where we use the notation of Lemma 5.2. By Proposition 5.5 the pull-back of $\kappa_{\mathfrak{m}}(C, D)$ with respect to the uniformization map of Theorem 5.4 is the element of $K_2(GL_2(\mathbb{A}_f) \times \Omega)$ introduced in Definition 4.9 which is denoted by the same symbol hence our new notation will not cause any confusion. Clearly $\kappa_{\mathfrak{m}}(C,D)$ is linear in the variables $C$ and $D$. Let the same symbol denote by abuse of notation the unique $\Delta$-bilinear extension:

$$\kappa_{\mathfrak{m}}(\cdot,\cdot) : \Delta[\mathcal{V}_{\mathfrak{m}}]_0 \times \Delta[\mathcal{V}_{\mathfrak{m}}]_0 \to H^2_{\mathcal{M}}(Y(\mathfrak{m}),\Delta(2))$$

of this pairing. Let $\kappa_{\mathfrak{m}}(\chi)$ denote the unique element of $H^2_{\mathcal{M}}(Y(\mathfrak{m})_L,\Delta(2))$ whose restriction to every irreducible component $C$ of $Y(\mathfrak{m})_L$ is $\chi^{-1}(C)\kappa_{\mathfrak{m}}(\chi_1,\chi_2)|_C$.

**Theorem 7.7.** *We have*

$$\langle\{\kappa_{\mathfrak{m}}(\chi)\},\phi_E\rangle = b(E,\mathfrak{m})L(E,q^{-1})\mathcal{L}_{\mathfrak{m}}(E,\chi)'$$

*in* $F^*_\infty \otimes K$ *where* $b(E,\mathfrak{m}) \in K^*$.

*Proof.* The Hecke eigenform $\phi_E$ is locally constant and has compact support as a function on $GL_2(F)\backslash GL_2(\mathbb{A})$ hence it takes only finitely many values. In particular there is a positive $n \in \mathbb{N}$ such that $n\phi_E$ takes integer values. Let $C, D \in \Delta[\mathcal{V}_{\mathfrak{m}}]_0$ be two functions such that the function $\chi(\det(g_f))^{-1}E^G_{\mathfrak{m}}(C,D,\cdot,x,y)$ is right $Z(\mathbb{A})\mathbb{K}(\mathfrak{m}\infty)$-invariant. Then the integral:

$$P^G_E(C,D,x,y) = n\langle\chi(\det(g_f))^{-1}E^G_{\mathfrak{m}}(C,D,\cdot,x,y),\phi_E\rangle \in \Delta[[x,y]](x^{-1},y^{-1})$$

is well-defined and it is in fact an element of $\Delta[x,y,x^{-1},y^{-1}]$ according to Proposition 3.5. Therefore we may evaluate $P^G_E(C,D,x,y)$ at $x=y=1$. As we already noted at the end of the proof of the limit formula 4.10 we have:

$$E^G_{\mathfrak{m}}(C,D,g) = (-1)^G E^G_{\mathfrak{m}}(D,C,g\Pi)$$

for every $g \in GL_2(\mathbb{A})$ (using the notation of that proof). Therefore we get the equality $P^G_E(C,D,1,1) = -(-1)^G P^G_E(D,C,1,1)$ because $\phi_E$ is harmonic and the Petersson product is translation-invariant. The elements $P^G_E(C,D,1,1)$ satisfy the obvious compatibility: let $P_E(C,D)$ denote their limit. Then $P_E(C,D) \in \Delta[[G_\infty]]$ lies in $I$ by Proposition 3.8. Moreover we have:

$$2P_E(\chi_1,\chi_2)' = P_E(\chi_1,\chi_2)'/P_E(\chi_2,\chi_1)' = \langle\{\kappa_{\mathfrak{m}}(\chi)\},\phi_E\rangle^n \in G_\infty \otimes K$$

by the Kronecker limit formula 4.10 using the notation we introduced in Notation 6.1. Therefore we get that

$$\langle\{\kappa_{\mathfrak{m}}(\chi)\},\phi_E\rangle = \frac{q-1}{2}\mu(\mathfrak{m})L_{\mathfrak{m}}(E,q^{-1})\mathcal{L}_{\mathfrak{m}}(E,\chi)'$$

using Theorem 7.4. Since $L_{\mathfrak{m}}(E,q^{-1}) = a(E,\mathfrak{m})L(E,q^{-1})$ for some $a(E,\mathfrak{m}) \in \mathbb{Q}^*$ the claim follows. $\square$

The function field analogue of the Shimura–Taniyama–Weil conjecture claims the following:

**Theorem 7.8.** *There is a non-trivial map $\pi : X_0(\mathfrak{n}) \to E$ defined over $F$.*

*Proof.* Although this theorem is certainly very well known and has been stated in the literature several times already, in some cases with an indication of proof, its complete proof have not been written yet; we will present it now for the sake of record. Let $l$ be a prime different from $p$ and let $V_{\overline{F}}$ denote the base change of any algebraic variety $V$ over $F$ to the separable closure $\overline{F}$ of the field $F$. The $\mathrm{Gal}(\overline{F}|F)$-module $H^1(E_{\overline{F}}, \mathbb{Q}_l)$ is absolutely irreducible, because the curve $E$ is not isotrivial. By the global Langlands correspondence for function fields (see [17], proved in this case in [3] already) there is a corresponding cuspidal automorphic representation $\pi$ of $GL_2(\mathbb{A})$. Let $\omega$ denote the grössencharacter of $F$ which assigns to each idele its normalized absolute value. By the compatibility of the local and global Langlands correspondences the $\infty$-adic component of $\pi \otimes \omega^{-1}$ is isomorphic to the Steinberg representation. Also the conductor of $\pi$ is $\mathfrak{n}\infty$, so there is a non-zero automorphic form $\phi$ of level $\mathfrak{n}\infty$ and trivial central character which is an element of $\pi \otimes \omega^{-1}$. By the above $\phi$ is also harmonic, so by the main theorem of [4] there is an absolutely irreducible $\mathrm{Gal}(\overline{F}|F)$ submodule of $H^1(X_0(\mathfrak{n})_{\overline{F}}, \mathbb{Q}_l)$ corresponding to the representation $\pi$. This representation must be isomorphic to $H^1(E_{\overline{F}}, \mathbb{Q}_l)$, because the Langlands correspondence is a bijection. By Zarhin's theorem (see [26] and [27]) there is a homomorphism from the Jacobian of $X_0(\mathfrak{n})$ onto $E$ which induces this isomorphism. We get the map of the claim by composing the map above with a finite-to-one map from $X_0(\mathfrak{n})$ into its Jacobian. $\square$

Our next goal is to give an explicit description of the relation between the modular parameterization of the elliptic curve $E$ in the theorem above and the normalized Hecke eigenform attached to $E$, due to Gekeler and Reversat [9].

**Definition 7.9.** Let $\deg(u) : GL_2(F_\infty) \to \mathbb{Z}$ denote the unique function for every holomorphic function $u : \Omega \to \mathbb{C}_\infty^*$ such that the regulator $\{c \otimes f\}$ introduced in Definition 4.5 is equal to $c^{\deg(u)}$ for every $c \in \mathbb{C}_\infty^*$. Then $\deg(u)$ is just the van der Put logarithmic derivative of $u$ introduced in [5]. Similarly to the notation we introduced in Definition 4.7 let $\deg(u) : GL_2(\mathbb{A}) \to \mathbb{Z}$ be the function given by the formula $\deg(g_f, g_\infty) = \deg(u(g_f, \cdot))(g_\infty)$ for each $g \in GL_2(\mathbb{A}_f)$ if $u : GL_2(\mathbb{A}_f) \times \Omega \to \mathbb{C}_\infty^*$ is holomorphic in the second variable. Recall that $\theta : \mathbb{C}_\infty^* \to E(\mathbb{C}_\infty)$ denotes the Tate uniformization of $E$. A t*heta function* attached to $E$ (and the modular parameterization $\pi$) is a function $u_E : GL_2(\mathbb{A}_f) \times \Omega \to \mathbb{C}_\infty^*$ holomorphic in the second variable for each $g \in GL_2(\mathbb{A}_f)$ if it satisfies the following properties:

(a) we have $u_E(gk, z) = u_E(g, z)$ for each $g \in GL_2(\mathbb{A}_f)$, $z \in \Omega$ and $k \in \mathbb{K}_0(\mathfrak{n}) \cap GL_2(\mathbb{A}_f)$,
(b) the harmonic cochain $\deg(u_E)$ is $c_E\phi_E$, where $c_E$ is a positive integer.

(c) the diagram:

$$\begin{tikzcd} GL_2(\mathbb{A}_f) \times \Omega \arrow[r, "u_E"'] \arrow[d] & \mathbb{C}_\infty^* \arrow[d, "\theta"] \\ Y_0(\mathfrak{n}) \arrow[r, "\pi"] & E(\mathbb{C}_\infty) \end{tikzcd}$$

is commutative where the vertical map on the left is the uniformization map mentioned in Notation 5.10.

**Theorem 7.10** (Gekeler–Reversat). *There is a theta function attached to $E$.*

*Proof.* See [9], Section 9.5, pages 86–88. □

*Proof of Theorem 1.10.* Let $\kappa_{\mathfrak{m},\mathfrak{n}}(\chi) \in H^2_{\mathcal{M}}(Y_0(\mathfrak{n})_L, K(2))$ denote the push-forward of the element $\kappa_{\mathfrak{m}}(\chi)$ with respect to the map $Y(\mathfrak{m}) \to Y_0(\mathfrak{m})$ induced by the forgetful map between the functors represented by these moduli curves. Then we have $\langle\{\kappa_{\mathfrak{m}}(\chi)\}, \psi\rangle = \langle\{\kappa_{\mathfrak{m},\mathfrak{n}}(\chi)\}, \psi\rangle$ for every $\psi \in \mathcal{H}_0(\mathfrak{n}, \mathbb{Q})$ by the invariance theorem (Theorem 3.11 of [22]). Moreover there is a $\kappa_{\mathfrak{m},\mathfrak{n}}(\chi)' \in H^2_{\mathcal{M}}(X_0(\mathfrak{n})_L, K(2))$ such that $\langle\{\kappa_{\mathfrak{m},\mathfrak{n}}(\chi)\}, \psi\rangle = \langle\{\kappa_{\mathfrak{m},\mathfrak{n}}(\chi)'\}, \psi\rangle$ for every $\psi$ as above by Lemma 5.15. Let $C \subset X_0(\mathfrak{n}) \times X_0(\mathfrak{n})$ denote the correspondence which is the composition of the uniformization map $\pi : X_0(\mathfrak{n}) \to E$ of Theorem 7.8 and its graph $\Gamma(\pi) \subset E \times X_0(\mathfrak{n})$ considered as a correspondence from $E$ to $X_0(\mathfrak{n})$. Then the endomorphism $J(C)$ : $J_0(\mathfrak{n}) \to J_0(\mathfrak{n})$ induced by $C$ is equal to $d(E)T_E$ where $d(E)$ is a non-zero rational number and $T_E \in \mathbb{T}(\mathfrak{n}) \otimes \mathbb{Q}$ is a projection operator. Moreover we have $T_E(\phi_E) = \phi_E$ therefore

$$\begin{aligned} b(E,\mathfrak{m})L(E,q^{-1})\mathcal{L}_{\mathfrak{m}}(E,\chi)' &= \langle\{\kappa_{\mathfrak{m}}(\chi)\}, \phi_E\rangle = \langle\{\kappa_{\mathfrak{m},\mathfrak{n}}(\chi)'\}, \phi_E\rangle \\ &= \langle\{\kappa_{\mathfrak{m},\mathfrak{n}}(\chi)'\}, T_E(\phi_E)\rangle = \langle T_E\{\kappa_{\mathfrak{m},\mathfrak{n}}(\chi)'\}, \phi_E\rangle \\ &= d(E)^{-1}\langle\{C_*(\kappa_{\mathfrak{m},\mathfrak{n}}(\chi)')\}, \phi_E\rangle \end{aligned}$$

using the self-adjointness of the operator $T_E$ with respect to the Petersson product in the fourth equation and Lemma 5.12 in the last equation. By the invariance theorem (Theorem 3.11 of [22]) the harmonic form $\{C_*(\kappa_{\mathfrak{m},\mathfrak{n}}(\chi)')\}$ is equal to $\{\pi_*(\kappa_{\mathfrak{m},\mathfrak{n}}(\chi)')\}\deg(u_E)$ where $\pi_*(\kappa_{\mathfrak{m},\mathfrak{n}}(\chi)') \in H^2_{\mathcal{M}}(E_L, K(2))$ is the push-forward of $\kappa_{\mathfrak{m},\mathfrak{n}}(\chi)'$ with respect to the uniformization $\pi : X_0(\mathfrak{n}) \to E$. Hence we have

$$L(E,q^{-1})\mathcal{L}_{\mathfrak{m}}(E,\chi)' = \frac{c(E)}{b(E,\mathfrak{m})d(E)}\{\pi_*(\kappa_{\mathfrak{m},\mathfrak{n}}(\chi)')\}\langle\phi_E,\phi_E\rangle$$

by the definition of theta functions. As the Petersson product is positive definite restricted to $\mathcal{H}_0(\mathfrak{n}, \mathbb{Q})$ the claim is now obvious. □

## 8. The action of certain correspondences on $K_2$

In this chapter the notation used will be somewhat independent of the one used in the rest of the paper.

**Notation 8.1.** Let $l$ be a prime number and for every scheme $S$ on which $l$ is invertible let

$$c_{2,2} : H^2_{\mathcal{M}}(S, \mathbb{Q}(2)) \to H^2_{et}(S, \mathbb{Q}_l(2))$$

denote the étale Chern class map. Let $L$ be a field complete with respect to a discrete valuation and let $\mathcal{O}$ denote its valuation ring. Assume that the residue field of $\mathcal{O}$ is a finite field of characteristic $p \neq l$. Let $\mathfrak{X} \to \mathrm{Spec}(\mathcal{O})$ be a flat, regular, proper and semi-stable scheme over $\mathrm{Spec}(\mathcal{O})$ such that its generic fiber $X$ is a smooth, geometrically irreducible curve over $\mathrm{Spec}(L)$. Let $Y$ denote the special fiber of $\mathfrak{X}$ and let

$$\partial : H^2_{\mathcal{M}}(X, \mathbb{Q}(2)) \to H^1_{\mathcal{M}}(Y, \mathbb{Q}(1))$$

denote the boundary map furnished by the localization sequence for the pair $(\mathfrak{X}, Y)$.

**Lemma 8.2.** *For every element $k \in H^2_{\mathcal{M}}(X, \mathbb{Q}(2))$ such that $c_{2,2}(x) = 0$ we have $\partial(k) = 0$.*

*Proof.* Let $R$ denote the ring of global sections of the sheaf of total quotient rings of $\mathcal{O}_Y$. Since the residue field of any closed point $y$ of $Y$ is a finite field, the homomorphism

$$j^* : H^1_{\mathcal{M}}(Y, \mathbb{Q}(1)) \to H^1_{\mathcal{M}}(\mathrm{Spec}(R), \mathbb{Q}(1)) = R^* \otimes \mathbb{Q}$$

induced by the natural map $j : \mathrm{Spec}(R) \to Y$ is injective. Let

$$c_{1,1} : R^* \otimes \mathbb{Q} \to H^1_{et}(\mathrm{Spec}(R), \mathbb{Q}(1))$$

be the connecting homomorphism of the long cohomological exact sequence attached to Kummer's short exact sequence. Let $\pi : \widetilde{Y} \to Y$ be the normalization of $Y$, and let $\mathrm{Div}(\widetilde{Y})$ denote the group of divisors on $\widetilde{Y}$. Since the homomorphism $R^* \to \mathrm{Div}(\widetilde{Y})$ which assigns to every $r \in R^*$ the divisor of $\pi^*(r)$ has a finite kernel and the group $\mathrm{Div}(\widetilde{Y})$ is a free abelian group, the intersection $\bigcap_{n \in \mathbb{N}} (R^*)^{l^n}$ is finite. Hence the homomorphism $c_{1,1}$ is injective. Therefore for every $k \in H^2_{\mathcal{M}}(X, \mathbb{Q}(2))$ we have $\partial(k) = 0$ if the equation $c_{1,1} \circ j_* \circ \partial(k) = 0$ holds. Let $K$ be the function field of the curve $X$ and let $i : \mathrm{Spec}(K) \to X$ be the generic point. The claim now follows from the fact that the diagrams:

$$\begin{tikzcd}
H^2_{\mathcal{M}}(X,\mathbb{Q}(2)) \arrow[r, "i^*"] \arrow[d, "c_{2,2}"'] & H^2_{\mathcal{M}}(\mathrm{Spec}(K),\mathbb{Q}(2)) \arrow[r, "\partial"] \arrow[d, "c_{2,2}"'] & R^*\otimes\mathbb{Q} \arrow[d, "c_{2,1}"'] \\
H^2_{et}(X,\mathbb{Q}_l(2)) \arrow[r, "i^*"] & H^2_{et}(\mathrm{Spec}(K),\mathbb{Q}_l(2)) \arrow[r, "\partial"] & H^1_{et}(\mathrm{Spec}(R),\mathbb{Q}_l(1))
\end{tikzcd}$$

and

$$\begin{tikzcd}
H^2_{\mathcal{M}}(X,\mathbb{Q}(2)) \arrow[r, "i^*"] \arrow[d, "\partial"'] & H^2_{\mathcal{M}}(\mathrm{Spec}(K),\mathbb{Q}(2)) \arrow[d, "\partial"'] \\
H^1_{\mathcal{M}}(Y,\mathbb{Q}(1)) \arrow[r, "j^*"] & R^*\otimes\mathbb{Q}
\end{tikzcd}$$

are commutative, where the symbols $\partial$ denote the respective localisation maps everywhere in the diagrams. □

**Notation 8.3.** For every smooth, projective, geometrically irreducible curve $Z$ defined over a field $K$ let $\mathrm{Jac}(Z)$ denote the Jacobian of $Z$, as usual. Moreover for every correspondence $C \subset Z \times Z$ let

$$C_* : H^2_{\mathcal{M}}(Z,\mathbb{Q}(2)) \to H^2_{\mathcal{M}}(Z,\mathbb{Q}(2))$$

and

$$J(C) : \mathrm{Jac}(Z) \to \mathrm{Jac}(Z)$$

denote the endomorphisms induced by $C$ on $H^2_{\mathcal{M}}(Z,\mathbb{Q}(2))$ and $\mathrm{Jac}(Z)$, respectively. Let $L$ and $X$ be as in Notation 8.1 and let $C \subset X \times X$ be a correspondence.

**Lemma 8.4.** *We have $c_{2,2}(C_*(k)) = 0$ for every $k \in H^2_{\mathcal{M}}(X,\mathbb{Q}(2))$ if the endomorphism $J(C)$ is zero.*

*Proof.* Let $\overline{X}$ denote the base change of $X$ to the separable closure $\overline{L}$ of $L$. Note that there is a Hochschild–Serre spectral sequence:

$$H^i(\mathrm{Gal}(\overline{L}|L), H^j_{et}(\overline{X},\mathbb{Q}_l(2))) \Rightarrow H^{i+j}_{et}(X,\mathbb{Q}_l(2)).$$

Because $H^0_{et}(\overline{X},\mathbb{Q}_l(2)) = \mathbb{Q}_l(2)$ and $H^2_{et}(\overline{X},\mathbb{Q}_l(2)) = \mathbb{Q}_l(1)$ we have

$$H^2(\mathrm{Gal}(\overline{L}|L), H^0_{et}(\overline{X},\mathbb{Q}_l(2))) = 0 = H^0(\mathrm{Gal}(\overline{L}|L), H^2_{et}(\overline{X},\mathbb{Q}_l(2)))$$

by local class field theory. In particular $E^{2,0}_\infty = E^{0,2}_\infty = 0$ for the spectral sequence mentioned above. Moreover $H^3(\mathrm{Gal}(\overline{L}|L), M) = 0$ for every $\mathrm{Gal}(\overline{L}|L)$-module $M$ hence $E^{3,0}_2 = 0$. Therefore we have $E^{1,1}_\infty = E^{1,1}_2 = H^1(\mathrm{Gal}(\overline{L}|L), H^1(\overline{X},\mathbb{Q}_l(2)))$ and so there is an isomorphism:

$$\iota_X : H^2_{et}(X,\mathbb{Q}_l(2)) \to H^1(\mathrm{Gal}(\overline{L}|L), H^1(\overline{X},\mathbb{Q}_l(2))).$$

Let

$$T_l(C) : H^1(\overline{X}, \mathbb{Q}_l(2))) \to H^1(\overline{X}, \mathbb{Q}_l(2)))$$

be the endomorphism of $H^1(\overline{X}, \mathbb{Q}_l(2)))$ induced by $C$. The map $T_l(C)$ induces a homomorphism on cohomology:

$$T_l(C)_* : H^1(\mathrm{Gal}(\overline{L}|L), H^1(\overline{X}, \mathbb{Q}_l(2))) \to H^1(\mathrm{Gal}(\overline{L}|L), H^1(\overline{X}, \mathbb{Q}_l(2)))$$

by functoriality. Then we have the following commutative diagram:

$$\begin{tikzcd} H^2_{\mathcal{M}}(X, \mathbb{Q}(2)) \arrow[r, "c_{2,2}"] \arrow[d, "C_*"'] & H^2_{et}(X, \mathbb{Q}_l(2)) \arrow[r, "\iota_X"] \arrow[d, "C_*"'] & H^1(\mathrm{Gal}(\overline{L}|L), H^1(\overline{X}, \mathbb{Q}_l(2))) \arrow[d, "T_l(C)_*"'] \\ H^2_{\mathcal{M}}(X, \mathbb{Q}(2)) \arrow[r, "c_{2,2}"] & H^2_{et}(X, \mathbb{Q}_l(2)) \arrow[r, "\iota_X"] & H^1(\mathrm{Gal}(\overline{L}|L), H^1(\overline{X}, \mathbb{Q}_l(2))) \end{tikzcd}$$

Since we have $H^1(\overline{X}, \mathbb{Q}_l(2)) = H^1(\mathrm{Jac}(\overline{X}), \mathbb{Q}_l(2))$ the map $T_l(C)$ is zero when the endomorphism $J(C)$ is. In this case $T_l(C)_*$ is also zero, so the claim is now clear. $\square$

**Notation 8.5.** As in the introduction, let $F$ denote the function field of $X$, where the latter is a geometrically connected smooth projective curve defined over the finite field $\mathbb{F}_q$ of characteristic $p$. Let $Z$ be a smooth, projective, geometrically irreducible curve defined over $F$ and let $C \subset Z \times Z$ be a correspondence. Assume that $Z$ has a flat, regular, proper and semi-stable model $\mathfrak{Z} \to X$ over $X$.

**Proposition 8.6.** *Assume that the endomorphism $J(C)$ is zero. Then for every $k \in H^2_{\mathcal{M}}(Z, \mathbb{Q}(2))$ the element $C_*(k) \in H^2_{\mathcal{M}}(Z, \mathbb{Q}(2))$ lies in the image of the natural map $H^2_{\mathcal{M}}(\mathfrak{Z}, \mathbb{Q}(2)) \to H^2_{\mathcal{M}}(Z, \mathbb{Q}(2))$.*

*Proof.* For every closed point $x$ of $X$ let $\mathfrak{Z}_x$ denote the fiber of $\mathfrak{Z}$ at $x$. By the exactness of the localisation sequence it will be sufficient to show that the image of $C_*(k)$ under the boundary map:

$$\partial : H^2_{\mathcal{M}}(Z, \mathbb{Q}(2)) \to H^1_{\mathcal{M}}(\mathfrak{Z}_x, \mathbb{Q}(1))$$

is zero for every $x$ as above. But this follows at once from Lemmas 8.2 and 8.4 applied to the base change of $Z$ to the completion of $F$ with respect to $x$. $\square$

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