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Hecke–Clifford Superalgebras and Crystals of Type $D_l^{(2)}$

Dedicated to Professor Tetsuji Miwa on the occasion of his sixtieth birthday

by

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Abstract

In [BK], Brundan and Kleshchev showed that some parts of the representation theory of the affine Hecke–Clifford superalgebras and its finite-dimensional "cyclotomic" quotients are controlled by the Lie theory of type $A_{2l}^{(2)}$ when the quantum parameter q is a primitive $(2l+1)$-th root of unity. We show that similar theorems hold when q is a primitive $4l$-th root of unity by replacing the Lie theory of type $A_{2l}^{(2)}$ with that of type $D_l^{(2)}$.

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§1. Introduction

It is known that we can sometimes describe the representation theory of "Hecke algebra" by "Lie theory". In this paper, we use the terminology "Lie theory" as a general term for objects related to or arising from Lie algebra, such as highest weight representations, quantum groups, Kashiwara's crystals, etc.

A famous example is Lascoux–Leclerc–Thibon's interpretation [LLT] of Kleshchev's modular branching rule [Kl1]. It asserts that the modular branching graph of the symmetric groups in characteristic p coincides with Kashiwara's crystal associated with the level 1 integrable highest weight representation of the quantum group $U_v(\mathfrak{g}(A_{p-1}^{(1)}))$. Brundan's modular branching rule for the Iwahori–Hecke algebras of type A at the quantum parameter $q = \sqrt[l]{1}$ over $\mathbb{C}$ is a similar result and can be regarded as a q-analogue of the above example [Br1].

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Another beautiful example is Ariki's theorem [Ari] generalizing Lascoux–Leclerc–Thibon's conjecture for the Iwahori–Hecke algebras of type A [LLT]. It relates the decomposition numbers of the Ariki–Koike algebras at $q = \sqrt[l]{1}$ over $\mathbb{C}$ and Kashiwara–Lusztig's canonical basis of an integrable highest weight representation of $U_v(\mathfrak{g}(A^{(1)}_{l-1}))$. Varagnolo–Vasserot's generalization of Ariki's theorem to q-Schur algebras [VV] and Yvonne's conjectural generalization for cyclotomic q-Schur algebras [Yvo] are also examples of connections between Hecke algebras and Lie theory.

However, all the Lie theory involved so far is only that of type $A^{(1)}_n$. Subsequently, based on the work of Grojnowski [Gro] and Grojnowski–Vazirani [GV], Brundan and Kleshchev showed that some parts of the representation theory of the affine Hecke–Clifford superalgebras introduced by Jones and Nazarov [JN] and their finite-dimensional "cyclotomic" quotients[1] introduced by Brundan and Kleshchev [BK, §3,§4-b] are controlled by the Lie theory of type $A^{(2)}_{2l}$ when the quantum parameter q is a primitive $(2l+1)$-th root of unity. Let $\mathcal{H}_n$ be the affine Hecke–Clifford superalgebra (see Definition 3.1) over an algebraically closed field F of characteristic different from 2 and let q be a $(2l+1)$-th primitive root of unity for $l \geq 1$. Their main results are as follows.

(1) The direct sum of the Grothendieck groups $K(\infty) = \bigoplus_{n\geq 0} \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_n)$ of the categories $\mathsf{Rep}\,\mathcal{H}_n$ of integral $\mathcal{H}_n$-supermodules has a natural structure of a commutative graded Hopf $\mathbb{Z}$-algebra under induction and restriction [BK, Theorem 7.1], and the restricted dual $K(\infty)^*$ is isomorphic to the positive part of the Kostant $\mathbb{Z}$-form of the universal enveloping algebra of $\mathfrak{g}(A^{(2)}_{2l})$ [BK, Theorem 7.17].

(2) The disjoint union $B(\infty) = \bigsqcup_{n\geq 0} \mathsf{Irr}(\mathsf{Rep}\,\mathcal{H}_n)$ of the isomorphism classes of irreducible integral $\mathcal{H}_n$-supermodules has a natural crystal structure in the sense of Kashiwara and it is isomorphic to Kashiwara's crystal associated with $U^-_v(\mathfrak{g}(A^{(2)}_{2l}))$ [BK, Theorem 8.10].

(3) For each positive integral weight λ of $A^{(2)}_{2l}$, one can define a finite-dimensional quotient superalgebra $\mathcal{H}^\lambda_n$ of $\mathcal{H}_n$, called the cyclotomic Hecke–Clifford superalgebra [BK, §3, §4-b].

(4) Consider the direct sums $K(\lambda) = \bigoplus_{n\geq 0} \mathsf{K}_0(\mathcal{H}^\lambda_n\text{-}\mathsf{smod})$ of the Grothendieck groups of the categories $\mathcal{H}^\lambda_n$-smod of finite-dimensional $\mathcal{H}^\lambda_n$-supermodules and $K(\lambda)^* = \bigoplus_{n\geq 0} \mathsf{K}_0(\mathsf{Proj}\,\mathcal{H}^\lambda_n)$ of the categories $\mathsf{Proj}\,\mathcal{H}^\lambda_n$ of finite-dimensional projective $\mathcal{H}^\lambda_n$-supermodules. Then $K(\lambda)_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} K(\lambda)$ is naturally identi-

[1] As a special case they include the Hecke–Clifford superalgebras introduced by Olshanski [Ols].

fied[2] with the integrable highest weight $U_{\mathbb{Q}}$-module of highest weight λ where $U_{\mathbb{Q}}$ stands for the $\mathbb{Q}$-form of the universal enveloping algebra of $\mathfrak{g}(A_{2l}^{(2)})$ [BK, Theorem 7.16(i)]. Moreover, the Cartan map $K(\lambda)^* \to K(\lambda)$ is injective [BK, Theorem 7.10] and $K(\lambda)^* \subseteq K(\lambda)$ are dual lattices in $K(\lambda)_{\mathbb{Q}}$ under the Shapovalov form [BK, Theorem 7.16(iii)].

(5) The disjoint union $B(\lambda) = \bigsqcup_{n\geq 0} \mathsf{Irr}(\mathcal{H}_n^\lambda\text{-}\mathsf{smod})$ is isomorphic to Kashiwara's crystal associated with the integrable $U_v(\mathfrak{g}(A_{2l}^{(2)}))$-module of highest weight λ [BK, Theorem 8.11].

Analogous results for the degenerate affine Sergeev superalgebras of Nazarov [Naz] and their cyclotomic quotients [BK, §4-i] over an algebraically closed field F of $\operatorname{char} F = 2l+1$ are also established in [BK] parallel to those for the affine Hecke–Clifford superalgebras and their cyclotomic quotients at $q = \sqrt[2l+1]{1}$ over an algebraically closed field F of $\operatorname{char} F \neq 2$. As a very special corollary of the results for the degenerate superalgebras, they beautifully obtain a modular branching rule of the spin symmetric groups $\widehat{\mathfrak{S}}_n$. This may be the reason why they deal only with the case $q = \sqrt[2l+1]{1}$ for the affine Hecke–Clifford superalgebras in [BK].

Note that exactly the same results hold when q is a primitive $2(2l+1)$-th root of unity for $l \geq 1$. This follows from the fact that $-q$ is a primitive $(2l+1)$-th root of unity and from the superalgebra isomorphism between the affine Hecke–Clifford superalgebras (see Definition 3.1) $\mathcal{H}_n(q)$ and $\mathcal{H}_n(-q)$ given by

$$X_i \mapsto X_i, \quad C_i \mapsto C_i, \quad T_j \mapsto -T_j$$

for $1 \leq i \leq n$ and $1 \leq j < n$. However, the case when the multiplicative order of q is divisible by 4 is yet untouched.

The purpose of this paper is to show that Brundan–Kleshchev's method is still applicable to the case when q is a primitive $4l$-th root of unity for any $l \geq 2$. In this case we have very similar results by replacing $A_{2l}^{(2)}$ with $D_l^{(2)}$ in the above summary. Roughly speaking, we prove the following four statements (for the precise statements, see Corollary 6.11, Corollary 6.12, Theorem 6.13 and Theorem 6.14).

Theorem 1.1. *Let F be an algebraically closed field of characteristic different from 2 and let q be a primitive $4l$-th root of unity for $l \geq 2$. For each positive integral weight λ of $D_l^{(2)}$, we can define a finite-dimensional quotient superalgebra $\mathcal{H}_n^\lambda$ of $\mathcal{H}_n$ (see Definition 4.1) so that the following hold.*

[2] It is not proved so far but expected that the weight space decomposition of $K(\lambda)_{\mathbb{Q}}$ coincides with the block decomposition of $\{\mathcal{H}_n^\lambda\}_{n\geq 0}$ under this identification. In fact, it is settled in the analogous situation when $\mathcal{H}_n^\lambda$ is replaced by the Ariki–Koike algebra [LM], the degenerate Ariki–Koike algebra [Br2] or the odd level cyclotomic quotient of the degenerate affine Sergeev superalgebra [Ruf]. See also [BK′, §2].

(i) *The graded dual of* $K(\infty) = \bigoplus_{n\geq 0} \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_n)$ *is isomorphic to* $U_{\mathbb{Z}}^{+}$ *as a graded* $\mathbb{Z}$*-Hopf algebra* (*see Theorem* 6.14).

(ii) $K(\lambda)_{\mathbb{Q}} = \bigoplus_{n\geq 0} \mathbb{Q} \otimes \mathsf{K}_0(\mathcal{H}_n^{\lambda}\text{-}\mathsf{smod})$ *has a left* $U_{\mathbb{Q}}$*-module structure which is isomorphic to the integrable highest weight* $U_{\mathbb{Q}}$*-module of highest weight* λ (*see Theorem* 6.13 for details).

(iii) $B(\infty) = \bigsqcup_{n\geq 0} \mathsf{Irr}(\mathsf{Rep}\,\mathcal{H}_n)$ *is isomorphic to Kashiwara's crystal associated with* $U_v^{-}(\mathfrak{g}(D_l^{(2)}))$ (*see Corollary* 6.11).

(iv) $B(\lambda) = \bigsqcup_{n\geq 0} \mathsf{Irr}(\mathcal{H}_n^{\lambda}\text{-}\mathsf{smod})$ *is isomorphic to Kashiwara's crystal associated with the integrable* $U_v(\mathfrak{g}(D_l^{(2)}))$*-module of highest weight* λ (*see Corollary* 6.12).

Here $U_{\mathbb{Z}}^{+}$ *is the positive part of the Kostant* $\mathbb{Z}$*-form of the universal enveloping algebra of* $\mathfrak{g}(D_l^{(2)})$ *and* $U_{\mathbb{Q}}$ *is the* $\mathbb{Q}$*-subalgebra of the universal enveloping algebra of* $\mathfrak{g}(D_l^{(2)})$ *generated by the Chevalley generators* (*see* §2.2).

A difference between our paper and [BK] is the consideration of representations of low rank affine Hecke–Clifford superalgebras, treated at length in §5.

Let us explain a reason behind our searching the "missing" connection between Hecke algebra and Lie theory of type $D_{n+1}^{(2)}$. It is well known that the level 1 crystal $\mathbb{B}(\Lambda_0)$ associated with $U_v(A_n^{(1)})$ or $U_v(A_{2n}^{(2)})$ is described by partitions [MM, Kan]. It is interesting that some of the combinatorics appearing in their descriptions had already been studied in the representation theory of the (spin) symmetric groups [Jam, Mor, MY], and such combinatorics controls modular branching of the (spin) symmetric groups [Kl1, Kl2, BK]. Thus, it is natural to ask which level 1 crystal has such a combinatorial realization, i.e., its underlying set is a subset of the set of partitions.

This problem is related to the Kyoto path model [KMN$^2_1$, KMN$^2_2$] or its combinatorial counterpart, Kang's Young wall [Kan]. The key tool underlying their realizations is a notion of perfect crystal [KMN$^2_2$, Definition 1.1.1] which is introduced in [KMN$^2_1$] to compute one-point functions of vertex models in statistical mechanics. As seen in [Kan], in order to realize $\mathbb{B}(\Lambda_0)$ as a subset of the set of partitions, we need a perfect crystal of level 1 which has no branching point.[3] As shown in [KMN$^2_2$], such a perfect crystal of level 1 exists in types $A_n^{(1)}, A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$. Conversely, we can show that a pair of affine type and its perfect crystal of

[3]Let $G = (V, E)$ be a directed graph, meaning that V is the set of vertices and $E \subseteq V \times V$ is the adjacency relation: $(v, w) \in E$ if and only if there exists a directed arrow from v to w. We say that a vertex w is a *branching point* of G if there exist u and v such that $u \neq v$, $u \neq w$, $v \neq w$, $(w, u) \in E$ and $(w, v) \in E$.

level 1 which has no branching point is one of the following:[4]

$$(A_1^{(1)}, B^{1,1}), \quad (A_1^{(1)}, (B^{1,1})^{\otimes 2}), \quad (A_n^{(1)}, B^{1,1})\ (n \geq 2),$$
$$(A_n^{(1)}, B^{n,1})\ (n \geq 2), \quad (A_{2n}^{(2)}, B^{1,1})\ (n \geq 1), \quad (D_{n+1}^{(2)}, B^{1,1})\ (n \geq 2)$$

if we assume the conjecture that any perfect crystal is a finite number of tensor products of Kirillov–Reshetikhin perfect crystals $B^{r,s}$ as stated in the first paragraph of the introduction of [KNO] and also assume the conjectural properties [HKOTY, Conjecture 2.1], [HKOTT, Conjecture 2.1] of Kirillov–Reshetikhin modules $W_s^{(r)}$.

This crystal-theoretic fact distinguishes types $A_n^{(1)}, A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$ from the other affine types and it is a reason behind our searching the "missing" connection between Hecke algebra and Lie theory of type $D_{n+1}^{(2)}$.

Recently, Rouquier [Rou] and Khovanov and Lauda [KL] independently introduced a new family of "quiver Hecke algebras" which categorifies the negative part of the quantized enveloping algebra associated with a symmetrizable Kac–Moody Lie algebra. Subsequently, Brundan and Kleshchev established algebra isomorphisms between blocks of the Ariki–Koike algebras and blocks of cyclotomic quotients of quiver Hecke algebras of cyclic type [BK2]. Thus, it is reasonable to expect that there is a connection such as Morita equivalence between blocks of cyclotomic quotients of the appropriate quiver Hecke algebras and blocks of the cyclotomic quotients of the affine Hecke–Clifford superalgebras.

Organization of the paper. The paper is organized as follows. In §2, we recall our conventions and necessary facts for superalgebras, supermodules and Kashiwara's crystal theory. In §3 (resp. §4), we define the affine Hecke–Clifford superalgebras (resp. the cyclotomic Hecke–Clifford superalgebras) and review fundamental theorems for them from [BK]. In §5, we give some preparatory character calculations concerning the behavior of representations of low rank affine Hecke–Clifford superalgebras $\mathcal{H}_2, \mathcal{H}_3$ and $\mathcal{H}_4$ which are responsible for the appearance of Lie theory of type $D_l^{(2)}$. Finally, in §6 we prove Theorem 1.1.

§2. Preliminaries

§2.1. Superalgebras and supermodules

We briefly recall our conventions and notations for superalgebras and supermodules following [BK, §2-b] (see also the references therein). In the rest of the paper, we always assume that our field F is algebraically closed with $\operatorname{char} F \neq 2$.

[4] $(A_1^{(1)}, (B^{1,1})^{\otimes 2})$ can be interpreted formally as the $n = 1$ case of $(D_{n+1}^{(2)}, B^{1,1})$.

By a *vector superspace*, we mean a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V = V_{\overline{0}} \oplus V_{\overline{1}}$ over F and we denote the parity of a homogeneous vector $v \in V$ by $\overline{v} \in \mathbb{Z}/2\mathbb{Z}$. Given two vector superspaces V and W, an F-linear map $f : V \to W$ is called *homogeneous* if there exists $p \in \mathbb{Z}/2\mathbb{Z}$ such that $f(V_i) \subseteq W_{p+i}$ for $i \in \mathbb{Z}/2\mathbb{Z}$. In this case we call p the parity of f and denote it by $\overline{f}$.

A *superalgebra* A is a vector superspace which is a unital associative F-algebra such that $A_iA_j \subseteq A_{i+j}$ for $i, j \in \mathbb{Z}/2\mathbb{Z}$. By an A-*supermodule*, we mean a vector superspace M which is a left A-module such that $A_iM_j \subseteq M_{i+j}$ for $i, j \in \mathbb{Z}/2\mathbb{Z}$. In the rest of the paper, we only deal with finite-dimensional A-supermodules. Given two A-supermodules V and W, an A-*homomorphism* $f : V \to W$ is an F-linear map such that

$$f(av) = (-1)^{\overline{f}\overline{a}} af(v)$$

for $a \in A$ and $v \in V$. We denote the set of A-homomorphisms from V to W by $\mathrm{Hom}_A(V, W)$. We can thus form a superadditive category $A\text{-}\mathsf{smod}$ whose hom-set is a vector superspace in a way that is compatible with composition. However, we adopt a slightly different definition of isomorphisms from the categorical one.[5] Two A-supermodules V and W are called *evenly isomorphic* (denoted $V \simeq W$) if there exists an even A-homomorphism $f : V \to W$ which is an F-vector space isomorphism. They are called *isomorphic* (denoted $V \cong W$) if $V \simeq W$ or $V \simeq \Pi W$. Here for an A-supermodule M, ΠM is the A-supermodule defined by $(\Pi M)_i = M_{i+\overline{1}}$ for $i \in \mathbb{Z}/2\mathbb{Z}$ and a new action given as follows from the old one:

$$a \cdot_{\mathsf{new}} m = (-1)^{\overline{a}} a \cdot_{\mathsf{old}} m.$$

We denote the isomorphism class of an A-supermodule M by $[M]$ and denote the set of isomorphism classes of irreducible A-supermodules by $\mathsf{Irr}(A\text{-}\mathsf{smod})$.

Given two superalgebras A and B, $A \otimes B$ with multiplication defined by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\overline{b_1}\overline{a_2}}(a_1a_2) \otimes (b_1b_2)$$

for $a_i \in A$, $b_j \in B$ is again a superalgebra. Let V be an A-supermodule and let W be a B-supermodule. Their tensor product $V \otimes W$ is an $A \otimes B$-supermodule by the action given by

$$(a \otimes b)(v \otimes w) = (-1)^{\overline{b}\overline{v}}(av) \otimes (bw)$$

for $a \in A$, $b \in B$, $v \in V$, $w \in W$. Let us assume that V and W are both irreducible. We say that V is *type* Q if $V \simeq \Pi V$; otherwise V is *type* M. If V and W are both

[5]Note that for irreducible A-supermodules V and W, the following statements are equivalent.
(i) There exist $f \in \mathrm{Hom}_A(V, W)$ and $g \in \mathrm{Hom}_A(W, V)$ such that $f \circ g = \mathsf{id}_W$ and $g \circ f = \mathsf{id}_V$.
(ii) There exist $f \in \mathrm{Hom}_A(V, W)$ and $g \in \mathrm{Hom}_A(W, V)$ which are both homogeneous and satisfy $f \circ g = \mathsf{id}_W, g \circ f = \mathsf{id}_V$.

of type Q, then there exists a unique (up to odd isomorphism) irreducible $A \otimes B$-supermodule X of type M such that

$$V \otimes W \simeq X \oplus \Pi X$$

as $A \otimes B$-supermodules. We denote X by $V \circledast W$. Otherwise $V \otimes W$ is irreducible but we also write it as $V \circledast W$. Note that $V \circledast W$ is defined only up to isomorphism in general and $V \circledast W$ is of type M if and only if V and W are of the same type.

We extend the operation $\circledast$ as follows. Let A and B be superalgebras, and let V be an A-supermodule and W a B-supermodule. Consider a pair (V, θ_V) where θ_V is either an odd involution of V or $\theta_V = \mathsf{id}_V$, and also consider a similar pair (W, θ_W). If $\theta_V = \mathsf{id}_V$ or $\theta_W = \mathsf{id}_W$, then we define $(V, \theta_V) \circledast (W, \theta_W) = V \otimes W$. If θ_V and θ_W are both odd involutions, then

$$\theta_V \otimes \theta_W : V \otimes W \to V \otimes W, \qquad v \otimes w \mapsto (-1)^{\overline{v}} \theta_V(v) \otimes \theta_W(w),$$

is an even $A \otimes B$-supermodule homomorphism such that $(\theta_V \otimes \theta_W)^2 = -\,\mathsf{id}_{V \otimes W}$. Thus, $V \otimes W$ decomposes into $\pm\sqrt{-1}$-eigenspaces $X_{\pm\sqrt{-1}}$. Note that $X_{+\sqrt{-1}}$ and $X_{-\sqrt{-1}}$ are oddly isomorphic since

$$(\theta_V \otimes \mathsf{id}_W)(X_{\pm\sqrt{-1}}) = (\mathsf{id}_V \otimes \theta_W)(X_{\pm\sqrt{-1}}) = X_{\mp\sqrt{-1}}.$$

Now we define $(V, \theta_V) \circledast (W, \theta_W) = X_{\sqrt{-1}}$. Of course, we can pick the other summand, but this specification makes arguments simpler when we consider functoriality.

We also introduce a Hom version of the above operation. Assume further that B is a subsuperalgebra of A. If $\theta_V = \mathsf{id}_V$ or $\theta_W = \mathsf{id}_W$, then we define $\overline{\mathrm{Hom}}_B((W, \theta_W), (V, \theta_V)) = \mathrm{Hom}_B(W, V)$, which can be regarded as a supermodule over $C(A, B) := \{a \in A \mid ab = (-1)^{\overline{a}\overline{b}} ba \text{ for all } b \in B\}$ by means of the action $(cf)(v) = c(f(v))$ for $c \in C(A, B)$ and $f \in \mathrm{Hom}_B(W, V)$. If θ_V and θ_W are both odd involutions, then

$$\Theta : \mathrm{Hom}_B(W, V) \longrightarrow \mathrm{Hom}_B(W, V), \qquad f \mapsto (\Theta(f))(v) = (-1)^{\overline{f}} \theta_V(f(\theta_W(v))),$$

is an even $C(A, B)$-supermodule homomorphism such that $\Theta^2 = \mathsf{id}_{\mathrm{Hom}_B(W,V)}$. Thus, $\mathrm{Hom}_B(W, V)$ decomposes into ± 1-eigenspaces $X_{\pm 1}$. Similarly, we see that $X_{\pm 1} \simeq \Pi X_{\mp 1}$, and we define $\overline{\mathrm{Hom}}_B((W, \theta_W), (V, \theta_V)) = X_{+1}$.

For a superalgebra A, we define the Grothendieck group $\mathsf{K}_0(A\text{-}\mathsf{smod})$ to be the quotient of the $\mathbb{Z}$-module freely generated by all finite-dimensional A-supermodules by the $\mathbb{Z}$-submodule generated by

- $V_1 - V_2 + V_3$ for every short exact sequence $0 \to V_1 \to V_2 \to V_3 \to 0$ in $A\text{-}\mathsf{smod}_{\overline{0}}$.
- $M - \Pi M$ for every A-supermodule M.

Here $A\text{-}\mathsf{smod}_{\overline{0}}$ is the abelian subcategory of $A\text{-}\mathsf{smod}$ whose objects are the same but morphisms are even A-homomorphisms. Clearly, $\mathsf{K}_0(A\text{-}\mathsf{smod})$ is a free $\mathbb{Z}$-module with basis $\mathsf{Irr}(A\text{-}\mathsf{smod})$. The importance of the operation $\circledast$ lies in the fact that it gives an isomorphism

$$(1)\quad \mathsf{K}_0(A\text{-}\mathsf{smod}) \otimes_{\mathbb{Z}} \mathsf{K}_0(B\text{-}\mathsf{smod}) \xrightarrow{\sim} \mathsf{K}_0(A \otimes B\text{-}\mathsf{smod}), \qquad [V] \otimes [W] \mapsto [V \circledast W],$$

for two superalgebras A and B.

Finally, we make some remarks on projective supermodules. Let A be a superalgebra. A *projective A-supermodule* is, by definition, a projective object in $A\text{-}\mathsf{smod}$; equivalently, it is a projective object in $A\text{-}\mathsf{smod}_{\overline{0}}$ since there are canonical isomorphisms

$$\begin{aligned}&\mathrm{Hom}_{A\text{-}\mathsf{smod}}(V,W)_{\overline{0}} \cong \mathrm{Hom}_{A\text{-}\mathsf{smod}_{\overline{0}}}(V,W),\\ &\mathrm{Hom}_{A\text{-}\mathsf{smod}}(V,W)_{\overline{1}} \cong \mathrm{Hom}_{A\text{-}\mathsf{smod}_{\overline{0}}}(V,\Pi W)\ (\cong \mathrm{Hom}_{A\text{-}\mathsf{smod}_{\overline{0}}}(\Pi V,W)).\end{aligned}$$

We denote by $\mathsf{Proj}\,A$ the full subcategory of $A\text{-}\mathsf{smod}$ consisting of all the projective A-supermodules.

Let us assume further that A is finite-dimensional. Then, as in the usual finite-dimensional algebras, every A-supermodule X has a (unique up to even isomorphism) projective cover P_X in $A\text{-}\mathsf{smod}_{\overline{0}}$. If X is irreducible, then P_X is (evenly) isomorphic to a projective indecomposable A-supermodule. From this, we easily see that $M \cong N$ if and only if $P_M \cong P_N$ for $M, N \in \mathsf{Irr}(A\text{-}\mathsf{smod})$. Thus, $\mathsf{K}_0(\mathsf{Proj}\,A)$ is identified with $\mathsf{K}_0(A\text{-}\mathsf{smod})^* := \mathrm{Hom}_{\mathbb{Z}}(\mathsf{K}_0(A\text{-}\mathsf{smod}), \mathbb{Z})$ through the non-degenerate canonical pairing

$$\begin{aligned}\langle\,,\rangle_A : \mathsf{K}_0(\mathsf{Proj}\,A) \times \mathsf{K}_0(A\text{-}\mathsf{smod}) \to \mathbb{Z},\\ ([P_M],[N]) \mapsto \begin{cases}\dim \mathrm{Hom}_A(P_M,N) & \text{if } \mathsf{type}\,M = \mathsf{M},\\ \frac{1}{2}\dim \mathrm{Hom}_A(P_M,N) & \text{if } \mathsf{type}\,M = \mathsf{Q},\end{cases}\end{aligned}$$

for all $M \in \mathsf{Irr}(A\text{-}\mathsf{smod})$ and $N \in A\text{-}\mathsf{smod}$. Note that the right hand side is nothing but the composition multiplicity $[N : M]$. We also reserve the symbol

$$\omega_A : \mathsf{K}_0(\mathsf{Proj}\,A) \to \mathsf{K}_0(A\text{-}\mathsf{smod})$$

for the natural Cartan map.

§2.2. Lie theory

We review the Lie theory we need. Note that all the Lie-theoretic objects are considered over $\mathbb{C}$ as usual although we are considering representations of "Hecke superalgebra" over F.

Let $A = (a_{ij})_{i,j \in I}$ be a symmetrizable generalized Cartan matrix and let $\mathfrak{g}$ be the corresponding Kac–Moody Lie algebra. We denote the weight lattice by P, the set of simple roots by $\{\alpha_i \mid i \in I\}$ and the set of simple coroots by $\{h_i \mid i \in I\}$, etc. as usual. We denote by $U_{\mathbb{Q}}$ the $\mathbb{Q}$-subalgebra of the universal enveloping algebra of $\mathfrak{g}$ generated by the Chevalley generators $\{e_i, f_i, h_i \mid i \in I\}$. In other words, $U_{\mathbb{Q}}$ is a $\mathbb{Q}$-subalgebra generated by $\{e_i, f_i, h_i \mid i \in I\}$ with the following relations:

$$(2)\qquad \begin{gathered} [h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \\ [e_i, f_j] = \delta_{ij} h_i, \quad (\mathsf{ad}\, e_i)^{1-a_{ik}}(e_k) = (\mathsf{ad}\, f_i)^{1-a_{ik}}(f_k) = 0, \end{gathered}$$

for all $i, j, k \in I$ with $i \neq k$. We also denote by $U_{\mathbb{Z}}^+$ (resp. $U_{\mathbb{Z}}^-$) the positive (resp. negative) part of the Kostant $\mathbb{Z}$-form of $U_{\mathbb{Q}}$, i.e., $U_{\mathbb{Z}}^+$ (resp. $U_{\mathbb{Z}}^-$) is the subalgebra of $U_{\mathbb{Q}}$ generated by the divided powers $\{e_i^{(n)} := e_i^n/n! \mid n \geq 1\}$ (resp. $\{f_i^{(n)} \mid n \geq 1\}$).

Next, we recall the notion of Kashiwara's crystal following [Kas].

Definition 2.1. A $\mathfrak{g}$-*crystal* is a 6-tuple $(B, \mathsf{wt}, \{\varepsilon_i\}_{i\in I}, \{\varphi_i\}_{i\in I}, \{\widetilde{e}_i\}_{i\in I}, \{\widetilde{f}_i\}_{i\in I})$, where

$$\mathsf{wt} : B \to P, \quad \varepsilon_i, \varphi_i : B \to \mathbb{Z} \sqcup \{-\infty\}, \quad \widetilde{e}_i, \widetilde{f}_i : B \sqcup \{0\} \to B \sqcup \{0\},$$

which satisfies the following axioms:

(i) For all $i \in I$, we have $\widetilde{e}_i 0 = \widetilde{f}_i 0 = 0$.

(ii) For all $b \in B$ and $i \in I$, we have $\varphi_i(b) = \varepsilon_i(b) + \mathsf{wt}(b)(h_i)$.

(iii) For all $b \in B$ and $i \in I$, $\widetilde{e}_i b \neq 0$ implies $\varepsilon_i(\widetilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\widetilde{e}_i b) = \varphi_i(b) + 1$ and $\mathsf{wt}(\widetilde{e}_i b) = \mathsf{wt}(b) + \alpha_i$.

(iv) For all $b \in B$ and $i \in I$, $\widetilde{f}_i b \neq 0$ implies $\varepsilon_i(\widetilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\widetilde{f}_i b) = \varphi_i(b) - 1$ and $\mathsf{wt}(\widetilde{f}_i b) = \mathsf{wt}(b) - \alpha_i$.

(v) For all $b, b' \in B$ and $i \in I$, $b' = \widetilde{f}_i b$ is equivalent to $b = \widetilde{e}_i b'$.

(vi) For all $b \in B$ and $i \in I$, $\varphi_i(b) = -\infty$ implies $\widetilde{e}_i b = \widetilde{f}_i b = 0$.

Definition 2.2. Let B be a $\mathfrak{g}$-crystal. The *crystal graph* associated with B (and usually denoted by the same symbol B) is an I-colored directed graph whose vertices are the elements of B and there is an i-colored directed edge from b to b' if and only if $b' = \widetilde{f}_i b$ for $b, b' \in B$ and $i \in I$.

Definition 2.3. Let B and B' be $\mathfrak{g}$-crystals. Their tensor product crystal $B \otimes B'$ is a $\mathfrak{g}$-crystal defined as follows:

$$\begin{aligned} B \otimes B' &= B \times B', \\ \varepsilon_i(b \otimes b') &= \max(\varepsilon_i(b), \varepsilon_i(b') - \mathsf{wt}(b)(h_i)), \\ \varphi_i(b \otimes b') &= \max(\varphi_i(b) + \mathsf{wt}(b')(h_i), \varphi_i(b')), \end{aligned}$$

$$\widetilde{e}_i(b\otimes b') = \begin{cases} \widetilde{e}_i b\otimes b' & \text{if } \varphi_i(b)\geq \varepsilon_i(b'),\\ b\otimes \widetilde{e}_i b' & \text{if } \varphi_i(b)< \varepsilon_i(b'),\end{cases}$$

$$\widetilde{f}_i(b\otimes b') = \begin{cases} \widetilde{f}_i b\otimes b' & \text{if } \varphi_i(b)> \varepsilon_i(b'),\\ b\otimes \widetilde{f}_i b' & \text{if } \varphi_i(b)\leq \varepsilon_i(b'),\end{cases}$$

$$\mathsf{wt}(b\otimes b') = \mathsf{wt}(b)+\mathsf{wt}(b').$$

Here we regard $b\otimes 0$ and $0\otimes b$ as 0.

Definition 2.4. Let B and B' be $\mathfrak{g}$-crystals. A *crystal morphism* $g : B\to B'$ is a map $g : B\sqcup\{0\}\to B'\sqcup\{0\}$ such that

(i) $g(0)=0$.
(ii) If $b\in B$ and $g(b)\in B'$, then we have $\mathsf{wt}(g(b))=\mathsf{wt}(b)$, $\varepsilon_i(g(b))=\varepsilon_i(b)$ and $\varphi_i(g(b))=\varphi_i(b)$ for all $i\in I$.
(iii) For $b\in B$ and $i\in I$, we have $g(\widetilde{e}_i b)=\widetilde{e}_i g(b)$ if $g(b)\in B'$ and $g(\widetilde{e}_i b)\in B'$.
(iv) For $b\in B$ and $i\in I$, we have $g(\widetilde{f}_i b)=\widetilde{f}_i g(b)$ if $g(b)\in B'$ and $g(\widetilde{f}_i b)\in B'$.

If g commutes with all $\widetilde{e}_i$ (resp. $\widetilde{f}_i$), then we call it an *e-strict* (resp. *f-strict*) morphism. We call it a *crystal embedding* if it is injective, e-strict and f-strict.

Example 2.5. For each $\lambda\in P^+$, we denote by $T_\lambda=\{t_\lambda\}$ the $\mathfrak{g}$-crystal defined by

$$\mathsf{wt}(t_\lambda)=\lambda,\qquad \varphi_i(t_\lambda)=\varepsilon_i(t_\lambda)=-\infty,\qquad \widetilde{e}_i t_\lambda=\widetilde{f}_i t_\lambda=0.$$

Example 2.6. For each $i\in I$, we denote by $B_i=\{b_i(n)\mid n\in\mathbb{Z}\}$ the $\mathfrak{g}$-crystal defined by $\mathsf{wt}(b_i(n))=n\alpha_i$ and

$$\varepsilon_j(b_i(n))=\begin{cases}-n & \text{if } j=i,\\ -\infty & \text{if } j\neq i,\end{cases}\qquad \varphi_j(b_i(n))=\begin{cases}n & \text{if } j=i,\\ -\infty & \text{if } j\neq i,\end{cases}$$

$$\widetilde{e}_j(b_i(n))=\begin{cases}b_i(n+1) & \text{if } j=i,\\ 0 & \text{if } j\neq i,\end{cases}\qquad \widetilde{f}_j(b_i(n))=\begin{cases}b_i(n-1) & \text{if } j=i,\\ 0 & \text{if } j\neq i.\end{cases}$$

These pathological $\mathfrak{g}$-crystals are utilized in the following characterizations [KS, Proposition 3.2.3], [Sai, Proposition 2.3.1].

Proposition 2.7. *Denote by $\mathbb{B}(\infty)$ the associated $\mathfrak{g}$-crystal with the crystal base of $U_v^-(\mathfrak{g})$. Let B be a $\mathfrak{g}$-crystal and b_0 an element of B with $\mathsf{wt}(b_0)=0$. If the following conditions hold, then B is isomorphic to $\mathbb{B}(\infty)$:*

(i) $\mathsf{wt}(B)\subseteq\sum_{i\in I}\mathbb{Z}_{\leq 0}\alpha_i$.
(ii) b_0 *is a unique element of B such that* $\mathsf{wt}(b_0)=0$.

(iii) $\varepsilon_i(b_0) = 0$ *for every* $i \in I$.

(iv) $\varphi_i(b) \in \mathbb{Z}$ *for any* $b \in B$ *and* $i \in I$.

(v) *For every* $i \in I$, *there exists a crystal embedding* $\Psi_i : B \to B \otimes B_i$ *such that* $\Psi_i(B) \subseteq B \times \{\widetilde{f}_i^n b_i(0) \mid n \geq 0\}$.

(vi) *For any* $b \in B$ *such that* $b \neq b_0$, *there exists* $i \in I$ *such that* $\Psi_i(b) = b' \otimes \widetilde{f}_i^n b_i(0)$ *with* $n > 0$.

Proposition 2.8. *Denote by* $\mathbb{B}(\lambda)$ *the associated* $\mathfrak{g}$*-crystal with the crystal base of the integrable highest* $U_v(\mathfrak{g})$*-module of highest weight* $\lambda \in P^+$. *Let* B *be a* $\mathfrak{g}$*-crystal and* b_λ *an element of* B *with* $\mathsf{wt}(b_\lambda) = \lambda$. *If the following conditions hold, then* B *is isomorphic to* $\mathbb{B}(\lambda)$:

(i) b_λ *is a unique element of* B *such that* $\mathsf{wt}(b_\lambda) = \lambda$.

(ii) *There is an* f*-strict crystal morphism* $\Phi : B(\infty) \otimes T_\lambda \to B$ *such that* $\Phi(b_0 \otimes t_\lambda) = b_\lambda$ *and* $\mathsf{Im}\,\Phi = B \sqcup \{0\}$. *Here* b_0 *is the unique element of* $B(\infty)$ *with* $\mathsf{wt}(b_0) = 0$.

(iii) *The sets* $\{b \in B(\infty) \otimes T_\lambda \mid \Phi(b) \neq 0\}$ *and* B *are isomorphic through* Φ.

(iv) *For any* $b \in B$ *and* $i \in I$, $\varepsilon_i(b) = \max\{k \geq 0 \mid \widetilde{e}_i^k(b) \neq 0\}$ *and* $\varphi_i(b) = \max\{k \geq 0 \mid \widetilde{f}_i^k(b) \neq 0\}$.

§3. Affine Hecke–Clifford superalgebras of Jones and Nazarov

§3.1. Definition and vector superspace structure

From now on, we fix a non-zero quantum parameter $q \in F^\times$ and set $\xi = q - q^{-1}$ for convenience. Let us define our main ingredient $\mathcal{H}_n$, the affine Hecke–Clifford superalgebra [JN, §3]. Although Jones and Nazarov introduced it under the name of affine Sergeev algebra, we call it the affine Hecke–Clifford superalgebra following [BK, §2-d].

Definition 3.1. Let $n \geq 0$ be an integer. The *affine Hecke–Clifford superalgebra* $\mathcal{H}_n$ is defined by even generators $X_1^{\pm 1}, \ldots, X_n^{\pm 1}, T_1, \ldots, T_{n-1}$ and odd generators $C_1, \ldots, C_n$ with the following relations:

(i) $X_i X_i^{-1} = X_i^{-1} X_i = 1$, $X_i X_j = X_j X_i$ for all $1 \leq i, j \leq n$.

(ii) $C_i^2 = 1$, $C_i C_j + C_j C_i = 0$ for all $1 \leq i \neq j \leq n$.

(iii) $T_i^2 = \xi T_i + 1$, $T_i T_j = T_j T_i$, $T_k T_{k+1} T_k = T_{k+1} T_k T_{k+1}$ for all $1 \leq k \leq n-2$ and $1 \leq i, j \leq n-1$ with $|i-j| \geq 2$.

(iv) $C_i X_i^{\pm 1} = X_i^{\mp 1} C_i$, $C_i X_j^{\pm 1} = X_j^{\pm 1} C_i$ for all $1 \leq i \neq j \leq n$.

(v) $T_iC_i = C_{i+1}T_i$, $(T_i + \xi C_iC_{i+1})X_iT_i = X_{i+1}$ for all $1 \le i \le n-1$.

(vi) $T_iC_j = C_jT_i$, $T_iX_j^{\pm 1} = X_j^{\pm 1}T_i$ for all $1 \le i \le n-1$ and $1 \le j \le n$ with $j \ne i, i+1$.

Note that the relations in Definition 3.1 imply the following for $1 \le i \le n-1$:

$$T_iC_{i+1} = C_iT_i - \xi(C_i - C_{i+1}), \tag{3}$$

$$T_iX_i = X_{i+1}T_i - \xi(X_{i+1} + C_iC_{i+1}X_i), \tag{4}$$

$$T_iX_i^{-1} = X_{i+1}^{-1}T_i + \xi(X_i^{-1} + X_{i+1}^{-1}C_iC_{i+1}). \tag{5}$$

We define the Clifford superalgebra $\mathcal{C}_n$ by odd generators $C_1, \dots, C_n$ with relation (ii) and also define the *Iwahori–Hecke (super)algebra* $\mathcal{H}_n^{\mathsf{IW}}$ *of type* A by (even) generators $T_1, \dots, T_{n-1}$ with relations (iii). By [BK, Theorem 2.2], the natural superalgebra homomorphisms

$$\alpha_A : F[X_1^{\pm 1}, \dots, X_n^{\pm 1}] \to \mathcal{H}_n, \quad \alpha_B : \mathcal{C}_n \to \mathcal{H}_n, \quad \alpha_C : \mathcal{H}_n^{\mathsf{IW}} \to \mathcal{H}_n$$

are all injective and we have the following isomorphism of vector superspaces:

$$F[X_1^{\pm 1}, \dots, X_n^{\pm 1}] \otimes \mathcal{C}_n \otimes \mathcal{H}_n^{\mathsf{IW}} \xrightarrow{\sim} \mathcal{H}_n, \quad x \otimes c \otimes t \mapsto \alpha_A(x)\alpha_B(c)\alpha_C(t). \tag{6}$$

In what follows, we identify $f \in F[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ with $\alpha_A(f) \in \mathcal{H}_n$ and omit α_A, etc. By (6), we easily see that the natural superalgebra homomorphisms

$$\mathcal{H}_0 \to \mathcal{H}_1 \to \mathcal{H}_2 \to \cdots$$

are all injective and they form a tower of superalgebras. We also see that for each composition $\mu = (\mu_1, \dots, \mu_\alpha)$ of n, the parabolic subsuperalgebra $\mathcal{H}_\mu$ generated by

$$\{X_i^{\pm 1}, C_i \mid 1 \le i \le n\} \cup \bigcup_{k=1}^{\alpha-1} \{T_j \mid \mu_1 + \cdots + \mu_k \le j < \mu_1 + \cdots + \mu_{k+1}\}$$

in $\mathcal{H}_n$ is isomorphic to $\mathcal{H}_{\mu_1} \otimes \cdots \otimes \mathcal{H}_{\mu_\alpha}$ as a superalgebra.

§3.2. Automorphism and antiautomorphism

It is easily checked that there exist an automorphism σ of $\mathcal{H}_n$ and an antiautomorphism τ of $\mathcal{H}_n$ defined by

$$\sigma : T_i \mapsto -T_{n-i} + \xi, \quad C_j \mapsto C_{n+1-j}, \quad X_j \mapsto X_{n+1-j},$$
$$\tau : T_i \mapsto T_i + \xi C_iC_{i+1}, \quad C_j \mapsto C_j, \quad X_j \mapsto X_j$$

for $1 \le i \le n-1$ and $1 \le j \le n$ [BK, §2-i].

Let M be an $\mathcal{H}_n$-supermodule. The dual space M^* has again an $\mathcal{H}_n$-supermodule structure by $(hf)(m) = f(\tau(h)m)$ for $f \in M^*$, $m \in M$ and $h \in \mathcal{H}_n$. We denote this $\mathcal{H}_n$-supermodule by M^τ. We also denote by M^σ the $\mathcal{H}_n$-supermodule obtained by twisting the action of $\mathcal{H}_n$ through σ. Then we have the following [BK, Lemma 2.9, Theorem 2.14].

Lemma 3.2. *Let M be an $\mathcal{H}_m$-supermodule and let N be an $\mathcal{H}_n$-supermodule. Then*

(i) $(\mathsf{Ind}_{\mathcal{H}_{m,n}}^{\mathcal{H}_{m+n}} M \otimes N)^\sigma \cong \mathsf{Ind}_{\mathcal{H}_{n,m}}^{\mathcal{H}_{m+n}} N^\sigma \otimes M^\sigma$.

(ii) $(\mathsf{Ind}_{\mathcal{H}_{m,n}}^{\mathcal{H}_{m+n}} M \otimes N)^\tau \cong \mathsf{Ind}_{\mathcal{H}_{n,m}}^{\mathcal{H}_{m+n}} N^\tau \otimes M^\tau$.

Moreover, if M and N are both irreducible, the same holds for $\circledast$ in place of $\otimes$.

§3.3. Cartan subsuperalgebra $\mathcal{A}_n$

The subsuperalgebra

$$\mathcal{A}_n := \langle X_i^{\pm}, C_i \rangle_{1 \le i \le n} \ (\subseteq \mathcal{H}_n)$$

plays the role of a "Cartan subalgebra" in the rest of the paper.

Definition 3.3. For each integer $i \in \mathbb{Z}$, we define

$$q(i) = 2 \cdot \frac{q^{2i+1} + q^{-(2i+1)}}{q + q^{-1}}, \qquad b_{\pm}(i) = \frac{q(i)}{2} \pm \sqrt{\frac{q(i)^2}{4} - 1}$$

and choose a subset $I_q \subseteq \mathbb{Z}$ such that the map $I_q \to \{q(i) \mid i \in \mathbb{Z}\}$, $i \mapsto q(i)$, gives a bijection. An $\mathcal{A}_n$-supermodule M is called *integral* if the set of eigenvalues of $X_j + X_j^{-1}$ is a subset of $\{q(i) \mid i \in I_q\}$ for all $1 \le j \le n$ (equivalently, the set of eigenvalues of $X_1 + X_1^{-1}$ is a subset of $\{q(i) \mid i \in I_q\}$ by [BK, Lemma 4.4]). Let μ be a composition of n. An $\mathcal{H}_\mu$-supermodule M is called *integral* if $\mathsf{Res}_{\mathcal{A}_n}^{\mathcal{H}_\mu} M$ is integral.

We denote the full subcategory of $\mathcal{A}_n$-smod (resp. $\mathcal{H}_\mu$-smod) consisting of integral representations by $\mathsf{Rep}\,\mathcal{A}_n$ (resp. $\mathsf{Rep}\,\mathcal{H}_\mu$). We also denote by ch_μ the $\mathbb{Z}$-linear homomorphism induced by the restriction functor $\mathsf{Res}_{\mathcal{A}_n}^{\mathcal{H}_\mu}$

$$\mathsf{ch}_\mu : \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_\mu) \to \mathsf{K}_0(\mathsf{Rep}\,\mathcal{A}_n)$$

between the Grothendieck groups. We always write ch instead of ch_n and call $\mathsf{ch}\,M$ the *formal character* of the $\mathcal{H}_n$-supermodule M.

We recall a special case of covering modules [BK, §4-h].

Definition 3.4. Let $m \ge 1$ and let $i \in I_q$. We define a $2m$-dimensional $\mathcal{H}_1$-supermodule $L_m^{\pm}(i)$ with an even basis $\{w_1, \ldots, w_m\}$ and an odd basis $\{w_1', \ldots, w_m'\}$

and the following matrix representations of actions of generators with respect to this basis:

$$X_1 : \begin{pmatrix} J(b_\pm(i); m) & O \\ O & J(b_\pm(i); m)^{-1} \end{pmatrix}, \qquad C_1 : \begin{pmatrix} O & E_m \\ E_m & O \end{pmatrix}.$$

Here $J(\alpha; m) := (\delta_{i,j}\alpha + \delta_{i,j+1})_{1 \le i,j \le m}$ stands for the Jordan matrix of size m.

We also define, for $m \ge 1$, $\mathcal{H}_1$-homomorphisms $g_m^\pm : L_{m+1}^\pm(i) \twoheadrightarrow L_m^\pm(i)$ by

$$w_k \mapsto \begin{cases} w_k & \text{if } 1 \le k \le m, \\ 0 & \text{if } k = m+1, \end{cases} \qquad w'_k \mapsto \begin{cases} w'_k & \text{if } 1 \le k \le m, \\ 0 & \text{if } k = m+1. \end{cases}$$

Here w_k and w'_k on the left hand side are those of $L_{m+1}^\pm(i)$ whereas w_k and w'_k on the right hand side are those of $L_m^\pm(i)$. Note that there is an odd isomorphism $g_m^\circ : L_m^+(i) \xrightarrow{\sim} L_m^-(i)$ since $J(b_+(i); m)$ and $J(b_-(i); m)^{-1}$ are similar. For convenience, we abbreviate $L_m^+(i)$ (resp. $L_1^+(i)$) to $L_m(i)$ (resp. $L(i)$) and g_m^+ to g_m.

Definition 3.5. For $i \in I_q$ we define an $\mathcal{H}_1$-supermodule $R_m(i) = \mathcal{H}_1/N(i)$ where $N(i)$ is the two-sided ideal generated by

$$f(i) = \begin{cases} (X_1 + X_1^{-1} - q(i))^m & \text{if } q(i) \ne \pm 2, \\ (X_1 - b_+(i))^m \ (= (X_1 - b_-(i))^m) & \text{if } q(i) = \pm 2. \end{cases}$$

As in [BK, §4-h] (or by elementary linear algebra), we have the following.

Lemma 3.6. *Let $i \in I_q$.*

(i) *If $q(i) \ne \pm 2$, then there exists an even isomorphism $R_m(i) \simeq L_m^+(i) \oplus L_m^-(i)$ for $m \ge 1$ which commutes with the obvious surjection $R_m(i) \twoheadleftarrow R_{m+1}(i)$.*

(7)
$$\begin{array}{ccccccc} R_1(i) & \twoheadleftarrow & R_2(i) & \twoheadleftarrow & R_3(i) & \twoheadleftarrow & \cdots \\ \wr\Big| & & \wr\Big| & & \wr\Big| & & \\ L_1(i) \oplus \Pi L_1(i) & \overset{g_1 + \Pi g_1}{\twoheadleftarrow} & L_2(i) \oplus \Pi L_2(i) & \overset{g_2 + \Pi g_2}{\twoheadleftarrow} & L_3(i) \oplus \Pi L_3(i) & \twoheadleftarrow & \cdots \end{array}$$

(ii) *If $q(i) = \pm 2$, then $R_m(i) \simeq L_m^+(i) = L_m^-(i)$ and there exist odd involutions g_k° for $k \ge 1$ that make the following diagram commute:*

(8)
$$\begin{array}{ccccccc} R_1(i) & \twoheadleftarrow & R_2(i) & \twoheadleftarrow & R_3(i) & \twoheadleftarrow & \cdots \\ \wr\Big| & & \wr\Big| & & \wr\Big| & & \\ L_1(i) & \twoheadleftarrow & L_2(i) & \twoheadleftarrow & L_3(i) & \twoheadleftarrow & \cdots \\ \circlearrowleft g_1^\circ & & \circlearrowleft g_2^\circ & & \circlearrowleft g_3^\circ & & \end{array}$$

In virtue of $\mathcal{A}_n \cong \mathcal{A}_1^{\otimes n}$ and (1), we have the following (see [BK, Lemma 4.8]).

Lemma 3.7. *We have* $\mathsf{Irr}(\mathsf{Rep}\,\mathcal{A}_n) = \{L(i_1) \circledast \cdots \circledast L(i_n) \mid (i_1, \ldots, i_n) \in I_q^n\}$*. For* $(i_1, \ldots, i_n) \in I_q^n$*,* $L(i_1) \circledast \cdots \circledast L(i_n)$ *is of type* Q *if and only if* $\#\{1 \le k \le n \mid q(i_k) = \pm 2\}$ *is odd.*

§3.4. Block decomposition

The (super)center $Z(\mathcal{H}_n)$ of $\mathcal{H}_n$ is naturally identified with the algebra of symmetric polynomials of $X_1 + X_1^{-1}, \ldots, X_n + X_n^{-1}$ [JN, Proposition 3.2(b)], [BK, Theorem 2.3] via

$$F[X_1 + X_1^{-1}, \ldots, X_n + X_n^{-1}]^{\mathfrak{S}_n} \xrightarrow{\sim} Z(\mathcal{H}_n), \qquad f \mapsto f.$$

Thus, $\mathcal{H}_n$ is a finite $Z(\mathcal{H}_n)$-module and this implies that all irreducible $\mathcal{H}_n$-supermodules are finite-dimensional. For any $M \in \mathsf{Rep}\,\mathcal{H}_n$, we have a decomposition $M = \bigoplus_{\gamma \in I_q^n/\mathfrak{S}_n} M[\gamma]$ with

$$M[\gamma] = \{m \in M \mid \forall f \in Z(\mathcal{H}_n), \exists N \in \mathbb{Z}_{>0}, (f - \chi_\gamma(f))^N m = 0\}$$

in $\mathsf{Rep}\,\mathcal{H}_n$. Here χ_γ is a central character attached to $\gamma = [(\gamma_1, \ldots, \gamma_n)]$ by

$$\chi_\gamma : Z(\mathcal{H}_n) \to F, \qquad f(X_1 + X_1^{-1}, \ldots, X_n + X_n^{-1}) \mapsto f(q(\gamma_1), \ldots, q(\gamma_n)).$$

Note that if $\gamma_1 \ne \gamma_2$ in $I_q^n/\mathfrak{S}_n$, then $\chi_{\gamma_1} \ne \chi_{\gamma_2}$.

Definition 3.8. Let $M \in \mathsf{Irr}(\mathsf{Rep}\,\mathcal{H}_n)$. Then there exists a unique $\gamma \in I_q^n/\mathfrak{S}_n$ such that $M = M[\gamma]$. In this case, we say that M *belongs to the block* γ.

We remark that this terminology coincides with the usual notion of block. This follows from a general result of Müller [BG, III.9.2] for an algebra which is finite over its center and the fact that the set $\{\chi_\gamma \mid \gamma \in I_q^n/\mathfrak{S}_n\}$ exhausts the possible central characters arising from $\mathsf{Rep}\,\mathcal{H}_n$. In fact, for any $\gamma = [(\gamma_1, \ldots, \gamma_n)] \in I_q^n/\mathfrak{S}_n$, all the composition factors of $\mathsf{Ind}_{\mathcal{A}_n}^{\mathcal{H}_n} L(\gamma_1) \circledast \cdots \circledast L(\gamma_n)$ belong to γ since

$$\mathsf{ch}\,\mathsf{Ind}_{\mathcal{A}_n}^{\mathcal{H}_n} L(i_1) \circledast \cdots \circledast L(i_n) = \sum_{w \in \mathfrak{S}_n} [L(i_{w(1)}) \circledast \cdots \circledast L(i_{w(n)})].$$

This identity [BK, Lemma 4.10] follows from the Mackey theorem [BK, Theorem 2.8].

§3.5. Kashiwara operators

Recall the Kato supermodules $L(i^n) := \mathsf{Ind}_{\mathcal{A}_n}^{\mathcal{H}_n} L(i)^{\circledast n}$ [BK, §4-g]. Using them, we can introduce Kashiwara operators $\widetilde{e}_i$ and $\widetilde{f}_i$ that send an irreducible supermodule to another one (if defined). We first recall a fundamental property of Kato's modules [BK, Theorem 4.16(i)].

Theorem 3.9. *For $i \in I_q$ and $n \geq 1$, $L(i^n)$ is irreducible of the same type as $L(i)^{\circledast n}$ and it is the only irreducible supermodule in its block of* $\mathsf{Rep}\,\mathcal{H}_n$.

Definition 3.10. For $i \in I_q$, $0 \leq m \leq n$ and $M \in \mathsf{Rep}\,\mathcal{H}_n$, we denote by $\Delta_{i^m} M$ the simultaneous generalized $q(i)$-eigenspace of the commuting operators $X_k + X_k^{-1}$ for all $n-m < k \leq n$. Note that $\Delta_{i^m} M$ is an $\mathcal{H}_{n-m,m}$-supermodule. We also define $\varepsilon_i(M) = \max\{m \geq 0 \mid \Delta_{i^m} M \neq 0\}$.

By [BK, §5-a], we have the following [BK, Lemma 5.5, Theorem 5.6, Corollary 5.8].

Theorem 3.11. *Let $i \in I_q$, $m \geq 0$ and $M \in \mathsf{Irr}(\mathsf{Rep}\,\mathcal{H}_n)$.*

(i) *$N := \mathsf{Cosoc}\,\mathsf{Ind}_{\mathcal{H}_{n,m}}^{\mathcal{H}_{n+m}} M \circledast L(i^m)$ is irreducible with $\varepsilon_i(N) = \varepsilon_i(M) + m$, and any other irreducible composition factor L of $\mathsf{Ind}_{\mathcal{H}_{n,m}}^{\mathcal{H}_{n+m}} M \circledast L(i^m)$ satisfies $\varepsilon_i(L) < \varepsilon_i(M) + n$.*

(ii) *Assume that $0 \leq m \leq \varepsilon_i(M)$. There exists (up to isomorphism) an irreducible $\mathcal{H}_{n-m}$-supermodule L such that $\mathsf{type}\,L = \mathsf{type}\,M$, $\varepsilon_i(L) = \varepsilon_i(M) - m$ and $\mathsf{Soc}\,\Delta_{i^m} M \cong L \circledast L(i^m)$.*

(iii) *Assume that $\varepsilon_i(M) > 0$. Then*

$$\mathsf{Soc}\,\mathsf{Res}_{\mathcal{H}_{n-1}}^{\mathcal{H}_{n-1,1}} \Delta_i(M) \simeq \begin{cases} L \oplus \Pi L & \text{if } \mathsf{type}\,M = \mathsf{Q} \text{ or } q(i) \neq \pm 2, \\ L & \text{if } \mathsf{type}\,M = \mathsf{M} \text{ and } q(i) = \pm 2, \end{cases}$$

for some irreducible $\mathcal{H}_{n-1}$-module L of the same type as M if $q(i) \neq \pm 2$ and of the opposite type to M if $q(i) = \pm 2$.

Definition 3.12. Let us write $B(\infty) := \bigsqcup_{n \geq 0} \mathsf{Irr}(\mathsf{Rep}\,\mathcal{H}_n)$. For $i \in I_q$, we define maps $\widetilde{e}_i, \widetilde{f}_i : B(\infty) \sqcup \{0\} \to B(\infty) \sqcup \{0\}$ as follows:

- $\widetilde{e}_i 0 = \widetilde{f}_i 0 = 0$.
- For $M \in \mathsf{Irr}(\mathsf{Rep}\,\mathcal{H}_n)$, we set $\widetilde{f}_i M = \mathsf{Cosoc}\,\mathsf{Ind}_{\mathcal{H}_{n,1}}^{\mathcal{H}_{n+1}} M \circledast L(i)$.
- For $M \in \mathsf{Irr}(\mathsf{Rep}\,\mathcal{H}_n)$, we set $\widetilde{e}_i M = 0$ if $\varepsilon_i(M) = 0$, otherwise $\widetilde{e}_i M = L$ for a unique $L \in \mathsf{Irr}(\mathsf{Rep}\,\mathcal{H}_{n-1})$ with $\mathsf{Soc}\,\Delta_i M \cong L \circledast L(i)$.

Note that $\varepsilon_i(M) = \max\{m \geq 0 \mid (\widetilde{e}_i)^m M \neq 0\}$ for $M \in \mathsf{Irr}(\mathsf{Rep}\,\mathcal{H}_n)$ and $i \in I_q$ by Theorem 3.11(ii). By [BK, Lemma 5.10], $\widetilde{e}_i$ and $\widetilde{f}_i$ satisfy one of the axioms of Kashiwara's crystal (see Definition 2.1(v)):

Lemma 3.13. *For $M, N \in B(\infty)$ and $i \in I_q$, $\widetilde{f}_i M = N$ is equivalent to $\widetilde{e}_i N = M$.*

Definition 3.14. For $\boldsymbol{i} = (i_1, \ldots, i_n) \in I_q^n$, we define $L(\boldsymbol{i}) = \widetilde{f}_{i_n}\widetilde{f}_{i_{n-1}} \cdots \widetilde{f}_{i_2}\widetilde{f}_{i_1}\mathbf{1}$. Here $\mathbf{1}$ is the trivial representation of $\mathcal{H}_0 = F$.

Note that $L(\boldsymbol{i})$ for $\boldsymbol{i} = (i, \ldots, i)$ coincides with the Kato supermodule $L(i^n)$ by Theorem 3.9. By an inductive use of Lemma 3.13, we have the following [BK, §5-d, Lemma 5.15].

Corollary 3.15. *For any $L \in \mathsf{Irr}(\mathsf{Rep}\,\mathcal{H}_n)$ there exists $\boldsymbol{i} \in I_q^n$ such that $L \cong L(\boldsymbol{i})$. $\mathsf{Res}^{\mathcal{H}_n}_{\mathcal{A}_n} L(\boldsymbol{i})$ has a submodule isomorphic to $L(i_1) \circledast \cdots \circledast L(i_n)$.*

Also a repeated use of Theorem 3.11(ii) implies the following [BK, Lemma 5.14].

Corollary 3.16. *Let $M \in \mathsf{Irr}(\mathsf{Rep}\,\mathcal{H}_n)$ and let μ be a composition of n. For any irreducible composition factor N of $\mathsf{Res}^{\mathcal{H}_n}_{\mathcal{H}_\mu} M$, we have $\mathsf{type}\, M = \mathsf{type}\, N$.*

§3.6. Root operators

We shall define root operators e_i as direct summands of $\mathsf{Res}^{\mathcal{H}_{n-1,1}}_{\mathcal{H}_{n-1}} \Delta_i$. Note that for any $M \in \mathsf{Rep}\,\mathcal{H}_n$ and $i \in I_q$, we have a natural identification

$$\mathsf{Res}^{\mathcal{H}_{n-1,1}}_{\mathcal{H}_{n-1}} \Delta_i M \simeq \varinjlim_m \mathrm{Hom}_{\mathcal{H}'_1}(R_m(i), M). \tag{9}$$

Here $\mathcal{H}'_1$ stands for a subsuperalgebra in $\mathcal{H}_n$ generated by $\{X_n^{\pm 1}, C_n\}$ isomorphic to $\mathcal{H}_1$. Considering (7) or (8), we can chose a summand of $\mathsf{Res}^{\mathcal{H}_{n-1,1}}_{\mathcal{H}_{n-1}} \Delta_i M$ appropriately as follows.

Definition 3.17. For $M \in \mathsf{Irr}(\mathsf{Rep}\,\mathcal{H}_n)$ and $i \in I_q$, we define

$$e_i M = \varinjlim_m \overline{\mathrm{Hom}}_{\mathcal{H}'_1}((L_m(i), \theta_m^\circ), (M, \theta_M)) \ (\in \mathsf{Rep}\,\mathcal{H}_{n-1}).$$

Here the θ's are defined as follows.

- $\theta_m^\circ = \mathsf{id}_{L_m(i)}$ if $q(i) \neq \pm 2$, and $\theta_m^\circ = g_m^\circ$ otherwise.
- $\theta_M = \mathsf{id}_M$ if $\mathsf{type}\, M = \mathsf{M}$, and θ_M is an odd involution of M otherwise.

Thus, by Theorem 3.11(iii), we have

$$\mathsf{Res}^{\mathcal{H}_{n-1,1}}_{\mathcal{H}_{n-1}} \Delta_i(M) \simeq \begin{cases} e_i M & \text{if } \mathsf{type}\, M = \mathsf{M} \text{ and } q(i) = \pm 2, \\ e_i M \oplus \Pi e_i M & \text{if } \mathsf{type}\, M = \mathsf{Q} \text{ or } q(i) \neq \pm 2. \end{cases}$$

By the commutativity of $\mathsf{Res}^{\mathcal{H}_n}_{\mathcal{H}_{n-1}}$ and τ-duality, we obtain the following [BK, Lemma 6.6(i)].

Corollary 3.18. *Let $M \in \mathsf{Irr}(\mathsf{Rep}\,\mathcal{H}_n)$ and $i \in I_q$. Then e_iM is non-zero if and only if $\widetilde{e}_iM$ is non-zero, in which case e_iM is a self-dual indecomposable module with irreducible socle and cosocle isomorphic to $\widetilde{e}_iM$.*

Also, as seen in [BK, §6-d], we have the following [BK, Theorem 6.11].

Theorem 3.19. *Let $M \in \mathsf{Irr}(\mathsf{Rep}\,\mathcal{H}_n)$ and $i \in I_q$.*

(i) *In $\mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_n)$, we have $[e_iM] = \varepsilon_i(M)[\widetilde{e}_iM] + \sum c_a[N_a]$ where N_a are irreducibles with $\varepsilon_i(N_a) < \varepsilon_i(M) - 1$.*
(ii) *If $q(i) \neq \pm 2$, then $\varepsilon_i(M)$ is the maximal size of a Jordan block of $X_n + X_n^{-1}$ on M with eigenvalue $q(i)$.*
(iii) *If $q(i) = \pm 2$, then $\varepsilon_i(M)$ is the maximal size of a Jordan block of X_n on M with eigenvalue $b_+(i) = b_-(i)$.*
(iv) $\mathrm{End}_{\mathcal{H}_{n-1}}(e_iM) \simeq \mathrm{End}_{\mathcal{H}_{n-1}}(\widetilde{e}_iM)^{\oplus \varepsilon_i(M)}$ *as vector superspaces.*

§3.7. Kashiwara's crystal structure

In this subsection, let $A = (a_{ij})_{i,j \in I_q}$ be an arbitrary symmetrizable generalized Cartan matrix indexed by I_q. We identify $I_q^n/\mathfrak{S}_n$ and $\Gamma_n := \{\sum_{i \in I_q} k_i\alpha_i \in \sum_{i \in I_q} \mathbb{Z}_{\geq 0}\alpha_i \mid \sum_{i \in I_q} k_i = n\}$ by

$$b_A : I_q^n/\mathfrak{S}_n \xrightarrow{\sim} \Gamma_n, \qquad [(\gamma_1, \ldots, \gamma_n)] \mapsto \sum_{k=1}^{n} \alpha_{\gamma_k}.$$

For $M \in \mathsf{Irr}(\mathsf{Rep}\,\mathcal{H}_n)$ belonging to a block $\gamma \in I_q^n/\mathfrak{S}_n$ and $i \in I_q$, we define

$$\mathsf{wt}(M) = -b_A(\gamma), \qquad \varphi_i(M) = \varepsilon_i(M) + \langle h_i, \mathsf{wt}(M)\rangle.$$

By Theorem 3.11 and Lemma 3.13, we can check the following [BK, Lemma 8.5].

Lemma 3.20. *The 6-tuple $(B(\infty), \mathsf{wt}, \{\varepsilon_i\}_{i \in I_q}, \{\varphi_i\}_{i \in I_q}, \{\widetilde{e}_i\}_{i \in I_q}, \{\widetilde{f}_i\}_{i \in I_q})$ is a $\mathfrak{g}(A)$-crystal.*

Finally, we introduce the σ-version of the above operations for $M \in B(\infty)$ and $i \in I_q$:

$$\widetilde{e}_i^*M = (\widetilde{e}_i(M^\sigma))^\sigma, \qquad \widetilde{f}_i^*M = (\widetilde{f}_i(M^\sigma))^\sigma, \qquad \varepsilon_i^*(M) = \varepsilon_i(M^\sigma).$$

Of course, we have $\varepsilon_i^*(M) = \max\{k \geq 0 \mid (\widetilde{e}_i^*)^k M \neq 0\}$. However, $\varepsilon_i^*(M)$ has another description as follows by Theorem 3.19(ii)&(iii).

Lemma 3.21. *Let $i \in I_q$ and $M \in \mathsf{Irr}(\mathsf{Rep}\,\mathcal{H}_n)$.*

- *If $q(i) \neq \pm 2$, then $\varepsilon_i^*(M)$ is the maximal size of a Jordan block of $X_1 + X_1^{-1}$ on M with eigenvalue $q(i)$.*
- *If $q(i) = \pm 2$, then $\varepsilon_i^*(M)$ is the maximal size of a Jordan block of X_1 on M with eigenvalue $b_+(i) = b_-(i)$.*

We also quote results [BK, Lemmas 8.1, 8.2, 8.4] concerning the commutativity of $\widetilde{e}_i$ and $\widetilde{f}_j^*$.

Lemma 3.22. *Let $M \in \mathsf{Irr}(\mathsf{Rep}\,\mathcal{H}_n)$ and $i, j \in I_q$.*

(i) $\varepsilon_i(\widetilde{f}_i^* M) = \varepsilon_i(M)$ *or* $\varepsilon_i(\widetilde{f}_i^* M) = \varepsilon_i(M) + 1$.

(ii) *If $i \neq j$, then $\varepsilon_i(\widetilde{f}_j^* M) = \varepsilon_i(M)$.*

(iii) *If $\varepsilon_i(\widetilde{f}_j^* M) = \varepsilon_i(M)$ (denoted by ε), then $\widetilde{e}_i^{\varepsilon} \widetilde{f}_j^* M \cong \widetilde{f}_j^* \widetilde{e}_i^{\varepsilon} M$.*

(iv) *If $\varepsilon_i(\widetilde{f}_i^* M) = \varepsilon_i(M) + 1$, then $\widetilde{e}_i \widetilde{f}_i^* M \cong M$.*

§3.8. Hopf algebra structure

Consider the graded $\mathbb{Z}$-free module

$$K(\infty) = \bigoplus_{n \geq 0} \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_n)$$

with natural basis $B(\infty)$ and define $\mathbb{Z}$-linear maps

$$\diamond_{m,n} : \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_m) \otimes \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_n) \xrightarrow{\sim} \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_{m,n}) \xrightarrow{\mathsf{Ind}_{\mathcal{H}_{m,n}}^{\mathcal{H}_{m+n}}} \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_{m+n}),$$

$$\Delta_{m,n} : \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_{m+n}) \xrightarrow{\mathsf{Res}_{\mathcal{H}_{m,n}}^{\mathcal{H}_{m+n}}} \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_{m,n}) \xrightarrow{\sim} \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_m) \otimes \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_n),$$

$$\diamond = \sum_{m,n \geq 0} \diamond_{m,n} : K(\infty) \otimes K(\infty) \to K(\infty), \qquad \iota : \mathbb{Z} \xrightarrow{\sim} \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_0) \xhookrightarrow{\text{inj}} K(\infty),$$

$$\Delta = \sum_{m,n \geq 0} \Delta_{m,n} : K(\infty) \to K(\infty) \otimes K(\infty), \qquad \varepsilon : K(\infty) \xrightarrow{\text{proj}} \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_0) \xrightarrow{\sim} \mathbb{Z}.$$

Note that $\diamond_{m,n}$ is well-defined since for any $M \in \mathsf{Rep}\,\mathcal{H}_{m,n}$ we have $\mathsf{Ind}_{\mathcal{H}_{m,n}}^{\mathcal{H}_{m+n}} M \in \mathsf{Rep}\,\mathcal{H}_{m+n}$ by [BK, Lemma 4.6].

Transitivity of induction and restriction makes $(K(\infty), \diamond, \iota)$ a graded $\mathbb{Z}$-algebra and $(K(\infty), \Delta, \varepsilon)$ a graded $\mathbb{Z}$-coalgebra. Injectivity of the formal character map $\mathsf{ch} : \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_n) \hookrightarrow \mathsf{K}_0(\mathsf{Rep}\,\mathcal{A}_n)$ [BK, Theorem 5.12] implies $L \cong L^{\tau}$ for all $L \in B(\infty)$ [BK, Corollary 5.13]. Combining this with Lemma 3.2(ii), we see that the multiplication of $(K(\infty), \diamond, \iota)$ is commutative. By the Mackey theorem [BK, Theorem 2.8], we see that $(K(\infty), \diamond, \Delta, \iota, \varepsilon)$ is a graded $\mathbb{Z}$-bialgebra.[6] Since a connected (non-negatively) graded bialgebra is a Hopf algebra [Swe, p. 238], we get the following [BK, Theorem 7.1].

Theorem 3.23. *$(K(\infty), \diamond, \Delta, \iota, \varepsilon)$ is a commutative graded Hopf algebra over $\mathbb{Z}$. Thus, $K(\infty)^*$ is a cocommutative graded Hopf algebra over $\mathbb{Z}$.*

Here $K(\infty)^*$ is a graded dual of $K(\infty)$, i.e.,

$$K(\infty)^* = \bigoplus_{n \geq 0} \mathrm{Hom}_{\mathbb{Z}}(\mathsf{K}_0(\mathsf{Rep}\, \mathcal{H}_n), \mathbb{Z}).$$

$K(\infty)^*$ has a natural $\mathbb{Z}$-free basis $\{\delta_M \mid M \in B(\infty)\}$ defined by $\delta_M([M]) = 1$ and $\delta_M([N]) = 0$ for all $[N] \in B(\infty)$ with $N \not\cong M$.

§3.9. Left $K(\infty)^*$-module structure on $K(\infty)$

By [Swe, Proposition 2.1.1], for a coalgebra C and a right C-comodule $\omega : M \to M \otimes C$, M is turned into a left C^*-module by

$$C^* \otimes M \xrightarrow{\mathsf{id}_{C^*} \otimes \omega} C^* \otimes M \otimes C \xrightarrow{\mathsf{swap} \otimes \mathsf{id}_C} M \otimes C^* \otimes C \xrightarrow{\mathsf{id}_M \otimes \langle , \rangle} M \otimes \mathbb{Z} \xrightarrow{\sim} M.$$

This implies that each coalgebra C is naturally regarded as a left C^*-module. It is easily seen that if C is a connected (non-negatively) graded coalgebra then the left action of C^* is faithful. Thus, $K(\infty)$ has a natural faithful left $K(\infty)^*$-module structure and it coincides with the root operators e_i in the following sense [BK, Lemmas 7.2 and 7.4].

Lemma 3.24. *For $i \in I_q, r \geq 1$ and $M \in K(\infty)$, we have $\delta_{L(i^r)} \cdot M = e_i^{(r)} M$.*

Note that $e_i^{(r)}$ is a priori an operator on $K(\infty)_{\mathbb{Q}} := \mathbb{Q} \otimes K(\infty)$, however as seen in Lemma 3.24 it is a well-defined operator on $K(\infty)$. We can prove this directly by defining a divided power root operator $e_i^{(r)}$ in a module-theoretic way [BK, §6-c].

§4. Cyclotomic Hecke–Clifford superalgebra

§4.1. Definition and vector superspace structure

Definition 4.1. Let $n \geq 1$ and assume that $R = a_d X_1^d + \cdots + a_0 \in F[X_1]$ ($\subseteq \mathcal{H}_n$) satisfies $C_1 R = a_0 X_1^{-d} R C_1$ (equivalently, the coefficients $\{a_i\}_{i=0}^d$ of R

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[6] In checking the details, we need the commutativity of the following diagrams for $m \geq k$ and $n \geq l$, which follows from Corollary 3.16:

$$\begin{array}{ccc} \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_{m,n}) & \xrightarrow{\sim} & \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_m) \otimes \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_n) \\ \downarrow \mathsf{Res}^{\mathcal{H}_{m,n}}_{\mathcal{H}_{k,l}} & & \downarrow \mathsf{Res}^{\mathcal{H}_m}_{\mathcal{H}_k} \otimes \mathsf{Res}^{\mathcal{H}_n}_{\mathcal{H}_l} \\ \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_{k,l}) & \xrightarrow{\sim} & \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_k) \otimes \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_l) \end{array} \qquad \begin{array}{ccc} \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_{m,n}) & \xrightarrow{\sim} & \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_m) \otimes \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_n) \\ \uparrow \mathsf{Ind}^{\mathcal{H}_{m,n}}_{\mathcal{H}_{k,l}} & & \uparrow \mathsf{Ind}^{\mathcal{H}_m}_{\mathcal{H}_k} \otimes \mathsf{Ind}^{\mathcal{H}_n}_{\mathcal{H}_l} \\ \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_{k,l}) & \xrightarrow{\sim} & \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_k) \otimes \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_l) \end{array}$$

satisfy $a_d = 1$ and $a_i = a_0 a_{d-i}$ for all $0 \leq i \leq d$). We define the *cyclotomic Hecke–Clifford superalgebra* $\mathcal{H}_n^R = \mathcal{H}_n/\langle R \rangle$ for $n \geq 1$ and set $\mathcal{H}_0^R = F$.

Note that the antiautomorphism τ of $\mathcal{H}_n$ induces an antiautomorphism of $\mathcal{H}_n^R$ also denoted by τ. As in the affine case, for an $\mathcal{H}_n^R$-supermodule M we write M^τ for the dual space M^* with $\mathcal{H}_n^R$-supermodule structure induced by τ.

By [BK, Theorem 3.6], $\mathcal{H}_n^R$ is a finite-dimensional superalgebra whose basis is the canonical image of the elements

$$\{X_1^{\alpha_1} \cdots X_n^{\alpha_n} C_1^{\beta_1} \cdots C_n^{\beta_n} T_w \mid 0 \leq \alpha_k < d,\, \beta_k \in \mathbb{Z}/2\mathbb{Z},\, w \in \mathfrak{S}_n\}.$$

Thus, we have the following commutativity between towers of superalgebras:

$$\begin{array}{ccccccc}
\mathcal{H}_0 & \hookrightarrow & \mathcal{H}_1 & \hookrightarrow & \mathcal{H}_2 & \hookrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{H}_0^R & \hookrightarrow & \mathcal{H}_1^R & \hookrightarrow & \mathcal{H}_2^R & \hookrightarrow & \cdots
\end{array}$$

It makes it possible to define inductions and restrictions for $\{\mathcal{H}_n^R\}_{n\geq 0}$ as well as M^τ and we have the following [BK, Theorem 3.9, Corollary 3.15].

Theorem 4.2. *Let M be an $\mathcal{H}_n^R$-supermodule.*

(i) *There is a natural isomorphism of $\mathcal{H}_n^R$-modules*

$$\mathsf{Res}_{\mathcal{H}_n^R}^{\mathcal{H}_{n+1}^R} \mathsf{Ind}_{\mathcal{H}_n^R}^{\mathcal{H}_{n+1}^R} M \simeq (M \oplus \Pi M)^d \oplus \mathsf{Ind}_{\mathcal{H}_{n-1}^R}^{\mathcal{H}_n^R} \mathsf{Res}_{\mathcal{H}_{n-1}^R}^{\mathcal{H}_n^R} M.$$

(ii) *The functors $\mathsf{Res}_{\mathcal{H}_n^R}^{\mathcal{H}_{n+1}^R}$ and $\mathsf{Ind}_{\mathcal{H}_n^R}^{\mathcal{H}_{n+1}^R}$ are left and right adjoint to each other.*

(iii) *There is a natural isomorphism $\mathsf{Ind}_{\mathcal{H}_n^R}^{\mathcal{H}_{n+1}^R}(M^\tau) \simeq (\mathsf{Ind}_{\mathcal{H}_n^R}^{\mathcal{H}_{n+1}^R} M)^\tau$ of $\mathcal{H}_{n+1}^R$-modules.*

We also define two natural functors (note that pr^R is a left adjoint to infl^R)

$$\mathsf{pr}^R : \mathcal{H}_n\text{-}\mathsf{smod} \to \mathcal{H}_n^R\text{-}\mathsf{smod}, \qquad M \mapsto M/\langle R \rangle M,$$
$$\mathsf{infl}^R : \mathcal{H}_n^R\text{-}\mathsf{smod} \to \mathcal{H}_n\text{-}\mathsf{smod}, \qquad M \mapsto \mathsf{Res}_{\mathcal{H}_n}^{\mathcal{H}_n^R} M.$$

In the following, we assume that the functor infl^R factors through the forgetful functor $\mathsf{Rep}\,\mathcal{H}_n \to \mathcal{H}_n\text{-}\mathsf{smod}$. By [BK, Lemma 4.4], this is equivalent to assuming that the set of roots of R is a subset of $\{b_\pm(i) \mid i \in I_q\}$. Thus, in the following, every $\mathcal{H}_n^R$-module M is automatically integral and has a decomposition $M = \bigoplus_{\gamma \in I_q^n/\mathfrak{S}_n} \mathsf{pr}^R((\mathsf{infl}^R M)[\gamma])$ as an $\mathcal{H}_n^R$-module.

§4.2. Kashiwara operators

Kashiwara operators for cyclotomic superalgebras are defined using those defined for affine superalgebras as follows. By Lemma 3.13, $\widetilde{e}_i^R$ and $\widetilde{f}_i^R$ clearly satisfy Definition 2.1(v).

Definition 4.3. Let us write $B(R) := \bigsqcup_{n\geq 0} \mathsf{Irr}(\mathcal{H}_n^R\text{-}\mathsf{smod})$. For $i \in I_q$, we define maps $\widetilde{e}_i^R, \widetilde{f}_i^R : B(R) \sqcup \{0\} \to B(R) \sqcup \{0\}$ as follows:

- $\widetilde{e}_i^R 0 = \widetilde{f}_i^R 0 = 0$.
- For $M \in \mathsf{Irr}(\mathcal{H}_n^R\text{-}\mathsf{smod})$, we set $\widetilde{e}_i^R M = (\mathsf{pr}^R \circ \widetilde{e}_i \circ \mathsf{infl}^R)M$ and $\widetilde{f}_i^R M = (\mathsf{pr}^R \circ \widetilde{f}_i \circ \mathsf{infl}^R)M$.

We also define, for $M \in B(R)$ and $i \in I_q$,

$$\varepsilon_i^R(M) = \max\{k \geq 0 \mid (\widetilde{e}_i^R)^k(M) \neq 0\} \ (= \varepsilon_i(\mathsf{infl}^R M)),$$
$$\varphi_i^R(M) = \max(\{k \geq 0 \mid (\widetilde{f}_i^R)^k(M) \neq 0\} \sqcup \{+\infty\}).$$

Note that although $\varphi_i^R(M)$ might take the value $+\infty$, it always takes a finite value as seen in Lemma 4.9(ii) below.

§4.3. Root operators

Definition 4.4. For $M \in \mathcal{H}_n^R\text{-}\mathsf{smod}$ such that $\mathsf{infl}^R M$ belongs to a block $\gamma \in I_q^n/\mathfrak{S}_n$ with $b_A(\gamma) = \sum_{i\in I_q} k_i\alpha_i$, we define

$$\mathsf{Res}_i^R M = \begin{cases} \mathsf{pr}^R((\mathsf{infl}^R \mathsf{Res}_{\mathcal{H}_{n-1}^R}^{\mathcal{H}_n^R} M)[b_A^{-1}(\gamma - \alpha_i)]) & \text{if } k_i > 0, \\ 0 & \text{if } k_i = 0, \end{cases}$$
$$\mathsf{Ind}_i^R M = \mathsf{pr}^R((\mathsf{infl}^R \mathsf{Ind}_{\mathcal{H}_n^R}^{\mathcal{H}_{n+1}^R} M)[b_A^{-1}(\gamma + \alpha_i)]).$$

In general, for $M \in \mathcal{H}_n^R\text{-}\mathsf{smod}$ we define $\mathsf{Res}_i^R M$ (resp. $\mathsf{Ind}_i^R M$) by applying Res_i^R (resp. Ind_i^R) for each summand of $M = \bigoplus_{\gamma\in I_q^n/\mathfrak{S}_n} \mathsf{pr}^R((\mathsf{infl}^R M)[\gamma])$.

By Theorem 4.2 and central character considerations, we get the following [BK, Lemma 6.1].

Corollary 4.5. *Let $i \in I_q$.*

(i) Res_i^R *and* Ind_i^R *are left and right adjoint to each other.*

(ii) *For each $M \in \mathcal{H}_n^R\text{-}\mathsf{smod}$ there are natural isomorphisms*

$$\mathsf{Ind}_i^R(M^\tau) \simeq (\mathsf{Ind}_i^R M)^\tau, \qquad \mathsf{Res}_i^R(M^\tau) \simeq (\mathsf{Res}_i^R M)^\tau.$$

Note that Res_i^R is nothing but $\mathsf{pr}^R \circ \mathsf{Res}_{\mathcal{H}_{n-1}}^{\mathcal{H}_{n-1,1}} \circ \Delta_i \circ \mathsf{infl}^R$ and it can be described as follows (see also (9)). Replacing each operator with its left adjoint and checking the well-definedness, we have the following [BK, Lemma 6.2].

Lemma 4.6. *Let $M \in \mathcal{H}_n^R$-smod and $i \in I_q$. There are natural isomorphisms*

$$\mathsf{Res}_i^R M \simeq \varinjlim_m \mathsf{pr}^R \operatorname{Hom}_{\mathcal{H}_1'}(R_m(i), \mathsf{infl}^R M),$$
$$\mathsf{Ind}_i^R M \simeq \varprojlim_m \mathsf{pr}^R \mathsf{Ind}_{\mathcal{H}_n \otimes \mathcal{H}_1}^{\mathcal{H}_{n+1}}((\mathsf{infl}^R M) \otimes R_m(i)).$$

Here both sequences stabilize after finitely many terms.

As in the affine case, we can choose a suitable summand of $\mathsf{Res}_i^R M$ and $\mathsf{Ind}_i^R M$ using (7) or (8).

Definition 4.7. Let $M \in \mathsf{Irr}(\mathcal{H}_n^R\text{-smod})$. We define

$$e_i^R X = \varinjlim_m \mathsf{pr}^R \overline{\operatorname{Hom}}_{\mathcal{H}_1'}((L_m(i), \theta_m^\circ), (\mathsf{infl}^R X, \mathsf{infl}^R \theta_X)),$$
$$f_i^R X = \varprojlim_m \mathsf{pr}^R \mathsf{Ind}_{\mathcal{H}_n \otimes \mathcal{H}_1}^{\mathcal{H}_{n+1}}(\mathsf{infl}^R X, \mathsf{infl}^R \theta_X) \circledast (L_m(i), \theta_m^\circ)$$

for each $X = M$ or $X = P := P_M$ and $i \in I_q$. Here θ's are defined as follows:

- $\theta_m^\circ = \mathsf{id}_{L_m(i)}$ if $q(i) \neq \pm 2$, and $\theta_m^\circ = g_m^\circ$ otherwise.
- $\theta_M = \mathsf{id}_M$ if $\mathsf{type}\, M = \mathsf{M}$, and θ_M is an odd involution of M otherwise.
- $\theta_P = \mathsf{id}_P$ if $\mathsf{type}\, M = \mathsf{M}$, and θ_P is an odd involution of P whose existence is guaranteed by [Kl2, Lemma 12.2.16][7] otherwise.

Note that for a projective indecomposable P and $i \in I_q$, $e_i^R P$ and $f_i^R P$ are again projectives since they are summands of Res_i^R and Ind_i^R respectively (see also Corollary 4.5). Thus, we define operators e_i^R and f_i^R on $K(R) := \bigoplus_{n\geq 0} \mathsf{K}_0(\mathcal{H}_n^R\text{-smod})$ and $K(R)^* \cong \bigoplus_{n \geq 0} \mathsf{K}_0(\mathsf{Proj}\, \mathcal{H}_n^R)$.

Lemma 4.8. *For any projective indecomposable $\mathcal{H}_n^R$-supermodule P and $i \in I_q$, we have in $\mathsf{K}_0(\mathcal{H}_{n-1}^R\text{-smod})$ and $\mathsf{K}_0(\mathcal{H}_{n+1}^R\text{-smod})$ respectively*

$$e_i^R(\omega_{\mathcal{H}_n^R}[P]) = \omega_{\mathcal{H}_{n-1}^R}([e_i^R P]), \qquad f_i^R(\omega_{\mathcal{H}_n^R}[P]) = \omega_{\mathcal{H}_{n+1}^R}([f_i^R P]).$$

[7] In [BK, §6-c], the authors claim that for $\mathsf{type}\, M = \mathsf{Q}$ a lift θ_P which is also an odd involution of the odd involution θ_M is unique. However, this is not true in general. Note that any odd involution of P works in the rest of this paper since our aim is to halve $\mathsf{Res}_i^R P$ or $\mathsf{Ind}_i^R P$ in the same way as $\mathsf{Res}_i^R M$ or $\mathsf{Ind}_i^R M$ to obtain Lemma 4.8.

Proof. Let A and B be superalgebras and consider an (even) exact functor X : A-smod $\to$ B-smod which sends every projective to a projective. Then for any projective indecomposable projective A-supermodule P, we easily see $X(\omega_A[P]) = \omega_B([XP])$ in $\mathsf{K}_0(B\text{-}\mathsf{smod})$. By Corollary 4.5(i), this implies that

$$\mathsf{Res}_i^R(\omega_{\mathcal{H}_n^R}[P]) = \omega_{\mathcal{H}_{n-1}^R}([\mathsf{Res}_i^R P]), \qquad \mathsf{Ind}_i^R(\omega_{\mathcal{H}_n^R}[P]) = \omega_{\mathcal{H}_{n+1}^R}([\mathsf{Ind}_i^R P]).$$

We shall only show $e_i^R(\omega_{\mathcal{H}^R}[P]) = \omega_{\mathcal{H}_{n-1}^R}([e^R P])$ in $\mathsf{K}_0(\mathcal{H}_{n-1}^R\text{-}\mathsf{smod})$ because the other is similar. By (7), (8), Lemma 4.6 and Definition 4.7, we have

$$[e_i^R P] = \begin{cases} [\mathsf{Res}_i^R P] & \text{if } q(i) = \pm 2 \text{ and } \mathsf{type}\,\mathsf{Cosoc}\,P = \mathsf{M}, \\ \frac{1}{2}[\mathsf{Res}_i^R P] & \text{otherwise} \end{cases}$$

in $\mathsf{K}_0(\mathsf{Proj}\,\mathcal{H}_{n-1}^R)$. Similarly, for $M \in \mathsf{Irr}(\mathcal{H}_{n-1}^R\text{-}\mathsf{smod})$ we have

$$[e_i^R M] = \begin{cases} [\mathsf{Res}_i^R M] & \text{if } q(i) = \pm 2 \text{ and } \mathsf{type}\,M = \mathsf{M}, \\ \frac{1}{2}[\mathsf{Res}_i^R M] & \text{otherwise} \end{cases}$$

in $\mathsf{K}_0(\mathcal{H}_{n-1}^R\text{-}\mathsf{smod})$. Thus, it is enough to show that for each irreducible factor N of P we have $\mathsf{type}\,N = \mathsf{type}\,\mathsf{Cosoc}\,P$. Take a unique $\gamma \in I_q^n/\mathfrak{S}_n$ such that $P = P[\gamma]$. It is clear that N also belongs to the block γ. By Corollary 3.16, $\mathsf{type}\,N$ is determined by its central character. □

Since $e_i^R = \mathsf{pr}^R \circ e_i \circ \mathsf{infl}^R$ and $\widetilde{e}_i^R = \mathsf{pr}^R \circ \widetilde{e}_i \circ \mathsf{infl}^R$, Corollary 3.18 and Theorem 3.19 hold for $M \in \mathsf{Rep}\,\mathcal{H}_n^R$ and $i \in I_q$ by replacing e_i, $\widetilde{e}_i$ and ε_i appearing there with e_i^R, $\widetilde{e}_i^R$ and ε_i^R respectively. We quote the corresponding properties of f_i^R, $\widetilde{f}_i^R$ and φ_i^R [BK, Theorem 6.6(ii), Lemma 6.18, Corollary 6.24].

Lemma 4.9. *Let $M \in \mathsf{Irr}(\mathcal{H}_n^R\text{-}\mathsf{smod})$ and $i \in I_q$.*

- (i) *$f_i^R M$ is non-zero if and only if $\widetilde{f}_i^R M$ is non-zero, in which case it is a self-dual indecomposable module with irreducible socle and cosocle isomorphic to $\widetilde{f}_i M$.*
- (ii) *$\varphi_i^R(M)$ is the smallest $m \geq 1$ (thus, takes a finite value by Lemma 4.6) such that $f_i^R M = \mathsf{pr}^R\,\mathsf{Ind}_{\mathcal{H}_n \otimes \mathcal{H}_1}^{\mathcal{H}_{n+1}}(\mathsf{infl}^R M, \mathsf{infl}^R \theta_M) \circledast (L_m(i), \theta_m^\circ)$ if $f_i^R M \neq 0$. If $f_i^R M = 0$ then $\varphi_i^R(M) = 0$.*
- (iii) *In $\mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_n)$, we have $[f_i^R M] = \varphi_i^R(M)[\widetilde{f}_i M] + \sum c_a[N_a]$ where N_a are irreducibles with $\varepsilon_i^R(N_a) < \varepsilon_i^R(M) + 1$.*
- (iv) *$\mathrm{End}_{\mathcal{H}_{n-1}^R}(f_i^R M) \simeq \mathrm{End}_{\mathcal{H}_{n-1}^R}(\widetilde{f}_i^R M)^{\oplus \varphi_i^\lambda(M)}$ as vector superspaces.*

Corollary 4.10. *For any $M \in \mathsf{Irr}(\mathcal{H}_n^R\text{-}\mathsf{smod})$ and $i \in I_q$, we have $(e_i^R)^{\varepsilon_i^R(M)+1}[M] = (f_i^R)^{\varphi_i^R(M)+1}[M] = 0$ in $K(R)$.*

Proof. The equality $(e_i^R)^{\varepsilon_i^R(M)+1}[M] = 0$ follows from Theorem 3.19(i). To prove $(f_i^R)^{\varphi_i^R(M)+1}[M]=0$, it is enough to show that $(f_i^R)^m[M]\neq 0$ implies $(\widetilde{f}_i^R)^m M\neq 0$ for any $m \geq 0$. By the definition, $(f_i^R)^m[M] \neq 0$ is equivalent to $[(\mathsf{Ind}_i^R)^m M] \neq 0$. By Corollary 4.5(i), we have

$$\begin{aligned}(10)\qquad \mathrm{Hom}_{\mathcal{H}_{n+m}^R}((\mathsf{Ind}_i^R)^m M, N) &\cong \mathrm{Hom}_{\mathcal{H}_n^R}(M, (\mathsf{Res}_i^R)^m N)\\ &= \mathrm{Hom}_{\mathcal{H}_n}(\mathsf{infl}^R M, \mathsf{Res}^{\mathcal{H}_{n,m}}_{\mathcal{H}_n} \Delta_{i^m} \mathsf{infl}^R N)\end{aligned}$$

for any $N\in\mathcal{H}_{n+m}^R$-smod. Since $(\mathsf{Ind}_i^R)^m M\neq 0$, there exists an $N\in\mathsf{Irr}(\mathcal{H}_{n+m}^R\text{-}\mathsf{smod})$ such that (10) is non-zero. Take any irreducible sub-$\mathcal{H}_n$-supermodule $X \cong \mathsf{infl}^R M$ of $\mathsf{Res}^{\mathcal{H}_{n,m}}_{\mathcal{H}_n} \Delta_{i^m} \mathsf{infl}^R N$ and consider the $\mathcal{H}_{n,m}$-supermodule $X' := \mathcal{H}'_m X$ where $\mathcal{H}'_m$ stands for the subsuperalgebra in $\mathcal{H}_{n+m}$ generated by $\{X_k^{\pm 1}, C_k, T_l \mid n < k \leq n{+}m,\ n < l < n{+}m\}$ isomorphic to $\mathcal{H}_m$. Then $\mathsf{ch}_{(n,m)} X' = c\cdot[X\circledast L(i^m)]$ for some $c \in \mathbb{Z}_{\geq 1}$ by Theorem 3.9. Comparing with $\mathsf{Soc}\,\Delta_{i^m} \mathsf{infl}^R N \cong (\widetilde{e}_i^m \mathsf{infl}^R N) \circledast L(i^m)$ by Theorem 3.11(ii) (see also [BK, Lemma 5.9(i)]), we see that $(\mathsf{infl}^R M \cong)\ X \cong \widetilde{e}_i^m \mathsf{infl}^R N$, which implies $(\widetilde{f}_i^R)^m M \cong N \neq 0$. □

As proved in [BK, Lemma 7.14], $[\mathsf{Res}_i^R \mathsf{Ind}_j^R M] - [\mathsf{Ind}_j^R \mathsf{Res}_i^R M]$ is a multiple of $[M]$ for any $M \in \mathsf{Irr}(\mathcal{H}_n^R\text{-}\mathsf{smod})$. By Theorem 3.19(i) and Lemma 4.9(iii), this implies the following.

Corollary 4.11. *For any $M \in \mathsf{Irr}(\mathcal{H}_n^R\text{-}\mathsf{smod})$ and $i, j \in I_q$, we have $e_i^R(f_j^R[M]) - f_j^R(e_i^R[M]) = \delta_{i,j}(\varphi_i^R(M) - \varepsilon_i^R(M)) \cdot [M]$ in $K(R)$.*

By Schur's lemma, Theorem 4.2(i), Theorem 3.19(iv), Lemma 4.9(ii) and Lemma 4.9(iv), we have the following. See also [BK, Lemma 6.20].

Corollary 4.12. *For any $M \in \mathsf{Irr}(\mathcal{H}_n^R\text{-}\mathsf{smod})$, we have*

$$\sum_{i\in I_q}(2 - \delta_{b_+(i),b_-(i)})(\varphi_i^R(M) - \varepsilon_i^R(M)) = d.$$

§4.4. Left $K(\infty)^*$-module structure on $K(R)$

Clearly, infl^R induces an injection $K(R) \hookrightarrow K(\infty)$ and a map $\Delta^R : K(R) \to K(R) \otimes K(\infty)$ with the following commutative diagram:

$$\begin{array}{ccc} K(\infty) & \xrightarrow{\ \Delta\ } & K(\infty) \otimes K(\infty) \\ {\scriptstyle \mathsf{infl}^R}\uparrow & & \uparrow{\scriptstyle \mathsf{infl}^R \otimes \mathsf{id}_{K(\infty)}} \\ K(R) & \xrightarrow{\ \Delta^R\ } & K(R) \otimes K(\infty) \end{array}$$

Thus, $K(R)$ is a subcomodule of the right regular $K(\infty)$-comodule. This implies that $K(R)$ is a $K(\infty)^*$-submodule of a left $K(\infty)^*$-module $K(\infty)$ in §3.9 where an operator $(e_i^R)^{(r)}$ acts as $\delta_{L(i^r)}$ by Lemma 3.24 for $i \in I_q$ and $r \geq 1$.

§4.5. Injectivity of the Cartan map

The purpose of this subsection is to show the injectivity of the Cartan map $\omega_{\mathcal{H}_n^R}$ of $\mathcal{H}_n^R$ [BK, Theorem 7.10]. It is essentially the same as [BK, §7-c] but arguments are slightly different because we do not define divided power operators $e_i^{(r)}, (e_i^R)^{(r)}$ and $(f_i^R)^{(r)}$ in a module-theoretic way as [BK, §6-c].

We first recall the following formula [BK, Lemma 7.6] which follows from the definitions that e_i^R and f_i^R are suitable summands of Res_i^R and Ind_i^R respectively.

Lemma 4.13. *For any $x \in \mathsf{K}_0(\mathsf{Proj}\,\mathcal{H}_n^R)$ and $y_\pm \in \mathsf{K}_0(\mathcal{H}_{n\pm1}^R\text{-}\mathsf{smod})$, we have*

$$\langle e_i^R x, y_-\rangle_{\mathcal{H}_{n-1}^R} = \langle x, f_i^R y_-\rangle_{\mathcal{H}_n^R}, \quad \langle f_i^R x, y_+\rangle_{\mathcal{H}_{n+1}^R} = \langle x, e_i^R y_+\rangle_{\mathcal{H}_n^R}.$$

Since $(e_i^R)^{(r)}$ is a well-defined operator on $K(R)$, we have the following. See also [BK, Corollary 7.7].

Corollary 4.14. *$(f_i^R)^{(r)}$ is a well-defined operator on $K(R)^*$ for any $i \in I_q$ and $r \geq 1$. More precisely, if*

$$(e_i^R)^{(r)}[M] = \sum_{N \in \mathsf{Irr}(\mathcal{H}_{n-r}^R\text{-}\mathsf{smod})} a_{M,N}[N], \quad (f_i^R)^{(r)}[M] = \sum_{N \in \mathsf{Irr}(\mathcal{H}_{n+r}^R\text{-}\mathsf{smod})} b_{M,N}[N]$$

in $\mathsf{K}_0(\mathcal{H}_{n-r}^R\text{-}\mathsf{smod})$ and $\mathbb{Q} \otimes \mathsf{K}_0(\mathcal{H}_{n+r}^R\text{-}\mathsf{smod})$ respectively, then

$$(f_i^R)^{(r)}[P_N] = \sum_{M \in \mathsf{Irr}(\mathcal{H}_{n+r}^R\text{-}\mathsf{smod})} a_{M,N}[P_M],$$

$$(e_i^R)^{(r)}[P_N] = \sum_{M \in \mathsf{Irr}(\mathcal{H}_{n-r}^R\text{-}\mathsf{smod})} b_{M,N}[P_M]$$

in $\mathsf{K}_0(\mathsf{Proj}\,\mathcal{H}_{n+r}^R)$ and $\mathbb{Q} \otimes \mathsf{K}_0(\mathsf{Proj}\,\mathcal{H}_{n-r}^R)$ respectively.

Lemma 4.15. *Let $M \in \mathsf{Irr}(\mathcal{H}_n^R\text{-}\mathsf{smod})$ and $i \in I_q$. For $m \leq \varepsilon := \varepsilon_i^R(M)$, we have*

$$(e_i^R)^m[P_M] = \sum_{\substack{L \in \mathsf{Irr}(\mathcal{H}_{n-m}^R\text{-}\mathsf{smod}) \\ \varepsilon_i^R(L) \geq \varepsilon - m}} b_L[P_L] \tag{11}$$

in $\mathsf{K}_0(\mathsf{Proj}\,\mathcal{H}_{n-m}^R)$. Moreover, in case $m = \varepsilon$, we have

$$(e_i^R)^\varepsilon[P_M] = \varepsilon! \binom{\varepsilon + \varphi_i^R(M)}{\varepsilon}[P_{(\tilde{e}_i^R)^\varepsilon M}] + \sum_{\substack{L \in \mathsf{Irr}(\mathcal{H}_{n-\varepsilon}^R\text{-}\mathsf{smod}) \\ \varepsilon_i^R(L) > 0}} b_L[P_L].$$

Proof. By Corollary 4.14, b_L is the coefficient of $[M]$ in $(f_i^R)^m[L]$ in $\mathsf{K}_0(\mathcal{H}_n^R\text{-smod})$. By Lemma 4.9(iii), we have

$$(f_i^R)^m[L] \in \sum_{\substack{N\in \mathsf{Irr}(\mathcal{H}_n^R\text{-smod}) \\ \varepsilon_i^R(N)\le m+\varepsilon_i^R(L)}} \mathbb{Z}_{\ge 0}[N].$$

This implies $\varepsilon \le m + \varepsilon_i^R(L)$ if $b_L \neq 0$ and completes the proof of (11).

Suppose $b_L \neq 0$ and $\varepsilon_i^R(L) = 0$. Again, by Lemma 4.9(iii), we have $(\widetilde{f}_i^R)^\varepsilon L \cong M$ and $b_L = \varepsilon!\binom{\varphi_i^R(L)}{\varepsilon}$. Thus, $L \cong (\widetilde{e}_i^R)^\varepsilon M$ and $b_L = \varepsilon!\binom{\varepsilon+\varphi_i^R(M)}{\varepsilon}$. ☐

Theorem 4.16. $\omega_{\mathcal{H}_n^R} : \mathsf{K}_0(\mathsf{Proj}\,\mathcal{H}_n^R) \to \mathsf{K}_0(\mathcal{H}_n^R\text{-smod})$ *is injective for all* $n \ge 0$.

Proof. We argue by induction on n. The case $n = 0$ is clear.

Suppose $n > 0$ and $\omega_{\mathcal{H}_{n'}^R}$ is injective for all smaller $n' < n$. We show that if

$$\omega_{\mathcal{H}_n^R}\Big(\sum_{M\in \mathsf{Irr}(\mathcal{H}_n^R\text{-smod})} a_M[P_M]\Big) = 0 \tag{12}$$

for $a_M \in \mathbb{Z}$, then $a_M = 0$ for all $M \in \mathsf{Irr}(\mathcal{H}_n^R\text{-smod})$. To prove this, it is enough to show that for each $i \in I_q$ we have $a_M = 0$ for all $M \in \mathsf{Irr}(\mathcal{H}_n^R\text{-smod})$ with $\varepsilon_i^R(M) > 0$. This is because there exists some $i \in I_q$ such that $\varepsilon_i^R(M) > 0$ for any $M \in \mathsf{Irr}(\mathcal{H}_n^R\text{-smod})$ if $n > 0$.

For each $i \in I_q$, we use induction on $\varepsilon_i^R(M) > 0$. Suppose that for a given M with $\varepsilon := \varepsilon_i^R(M) > 0$ we have $a_N = 0$ for all N with $0 < \varepsilon_i^R(N) < \varepsilon$. Applying $(e_i^R)^\varepsilon$ to (12), we have

$$0 = \sum_{\substack{L\in \mathsf{Irr}(\mathcal{H}_n^R\text{-smod}) \\ \varepsilon_i^R(L)=\varepsilon}} \varepsilon!\binom{\varepsilon+\varphi_i^R(L)}{\varepsilon} a_L \omega_{\mathcal{H}_{n-\varepsilon}^R}([P_{(\widetilde{e}_i^R)^\varepsilon L}]) + \omega_{\mathcal{H}_{n-\varepsilon}^R}(X)$$

where $X \in \sum_{L' \in \mathsf{Irr}(\mathcal{H}_{n-\varepsilon}^R\text{-smod}) \text{ with } \varepsilon_i^R(L')>0} \mathbb{Z}[P_{L'}]$ by Lemmas 4.8 and 4.15. By the induction hypothesis, we have $a_M = 0$. ☐

§4.6. Symmetric non-degenerate bilinear form on $K(R)_\mathbb{Q}$

By Theorem 4.16, $\bigoplus_{n\ge 0}\mathsf{K}_0(\mathsf{Proj}\,\mathcal{H}_n^R) \cong K(R)^* \subseteq K(R)$ are two integral lattices of $K(R)_\mathbb{Q} := \mathbb{Q}\otimes K(R)$. Thus, by tensoring with $\mathbb{Q}$, $\bigoplus_{n\ge 0}\langle\,,\rangle_{\mathcal{H}_n^R} : K(R)^*\times K(R) \to \mathbb{Z}$ induces a non-degenerate bilinear form on $K(R)_\mathbb{Q}$ which we denote by $\langle\,,\rangle_R$.

Lemma 4.17. *Let* $M \in \mathsf{Irr}(\mathcal{H}_n^R\text{-smod})$ *and* $i \in I_q$. *Then*

$$[P_M] = (f_i^R)^{(\varepsilon)}[P_{(\widetilde{e}_i^R)^\varepsilon M}] - \sum_{\substack{L\in \mathsf{Irr}(\mathcal{H}_n^R\text{-smod}) \\ \varepsilon_i^R(L)>\varepsilon}} a_L[P_L]$$

for $\varepsilon = \varepsilon_i^R(M)$ *in* $\mathsf{K}_0(\mathsf{Proj}\,\mathcal{H}_n^R)$.

Proof. Write $(f_i^R)^{(\varepsilon)}[P_{(\widetilde{e}_i^R)^\varepsilon M}] = \sum_{L \in \mathsf{Irr}(\mathcal{H}_n^R\text{-}\mathsf{smod})} b_L[P_L]$ in $\mathsf{K}_0(\mathsf{Proj}\,\mathcal{H}_n^R)$. By Corollary 4.14, b_L is the coefficient of $[(\widetilde{e}_i^R)^\varepsilon M]$ of $(e_i^R)^{(\varepsilon)}[L]$ in $\mathsf{K}_0(\mathcal{H}_{n-\varepsilon}^R\text{-}\mathsf{smod})$. Thus, $b_L \neq 0$ implies $\varepsilon_i^R(L) \geq \varepsilon$. Finally, suppose $b_L \neq 0$ and $\varepsilon_i^R(L) = \varepsilon$. By Theorem 3.19(i), we have $b_L = 1$ and $(\widetilde{e}_i^R)^\varepsilon L \cong (\widetilde{e}_i^R)^\varepsilon M$, i.e., $L \cong M$. □

A repeated use of Lemma 4.17 implies the following [BK, Theorem 7.9].

Theorem 4.18. *We have* $\bigoplus_{n\geq 0} \mathsf{K}_0(\mathsf{Proj}\,\mathcal{H}_n^R) = U_{\mathbb{Z}}^-[\mathbf{1}_R]$ *where* $\mathbf{1}_R$ *is the trivial supermodule of* $\mathcal{H}_0^R = F$*.*

Proof. We prove $[P_M] \in U_{\mathbb{Z}}^-[\mathbf{1}_R]$ for all $M \in B(R)$. Suppose for contradiction that there exists an $M \in \mathsf{Irr}(\mathcal{H}_n^R\text{-}\mathsf{smod})$ such that $[P_M] \notin U_{\mathbb{Z}}^-[\mathbf{1}_R]$. We take such an M with minimum n. Since $n > 0$, there exists an $i \in I_q$ with $\varepsilon_i^R(M) > 0$. We take N with maximum $\varepsilon_i^R(N)$ $(\geq \varepsilon_i^R(M) > 0)$ in $\{N \in \mathsf{Irr}(\mathcal{H}_n^R\text{-}\mathsf{smod}) \mid [P_N] \notin U_{\mathbb{Z}}^-[\mathbf{1}_R]\}$ $(\neq \emptyset)$. However, $[P_N] \in U_{\mathbb{Z}}^-[\mathbf{1}_R]$ by the choice of N and Lemma 4.17, a contradiction. □

Using Lemma 4.13 inductively together with the equality $\mathsf{K}_0(\mathcal{H}_{n+1}^R\text{-}\mathsf{smod})_{\mathbb{Q}} = \sum_{i\in I_q} f_i^R\,\mathsf{K}_0(\mathcal{H}_n^R\text{-}\mathsf{smod})_{\mathbb{Q}}$ by Theorem 4.18, we get the following result [BK, Theorem 7.11].

Corollary 4.19. *The non-degenerate bilinear form* $\langle\, ,\rangle_R$ *on* $K(R)_{\mathbb{Q}}$ *is symmetric.*

§5. Character calculations

The purpose of this section is to give preparatory character calculations concerning the behavior of representations of low rank affine Hecke–Clifford superalgebras $\mathcal{H}_2, \mathcal{H}_3$ and $\mathcal{H}_4$ for §6.2. Since they are responsible for the appearance of Lie theory of type $D_l^{(2)}$ and omitted[8] in [BK], we give detailed and self-contained calculations.

§5.1. Preparations

We note that if a given $M \in \mathsf{Irr}(\mathsf{Rep}\,\mathcal{H}_n)$ has a formal character of the form $\mathsf{ch}\,M = c \cdot [L(i_i) \circledast \cdots \circledast L(i_n)]$ for some $c \in \mathbb{Z}_{\geq 1}$ then $M \cong L(i_1, \ldots, i_n)$ by Corollary 3.15. We also recall the shuffle lemma [BK, Lemma 4.11] to compute the formal characters.

Lemma 5.1. *For* $M \in \mathsf{Irr}(\mathsf{Rep}\,\mathcal{H}_m)$ *and* $N \in \mathsf{Irr}(\mathsf{Rep}\,\mathcal{H}_n)$ *with*

$$\mathsf{ch}\,M = \sum_{\boldsymbol{i}\in I_q^m} a_{\boldsymbol{i}}[L(i_1) \circledast \cdots \circledast L(i_m)] \quad \textit{and} \quad \mathsf{ch}\,N = \sum_{\boldsymbol{j}\in I_q^n} b_{\boldsymbol{j}}[L(j_1) \circledast \cdots \circledast L(j_n)],$$

[8] For degenerate affine Sergeev superalgebras, detailed character calculations can be found in [Kl2, Chapter 18].

we have

$$\operatorname{ch}\operatorname{Ind}_{\mathcal{H}_{m,n}}^{\mathcal{H}_{m+n}} M \circledast N = \sum_{\substack{\boldsymbol{i}\in I_q^m \\ \boldsymbol{j}\in I_q^n}} a_{\boldsymbol{i}} b_{\boldsymbol{j}} \Big(\sum_{\boldsymbol{k}\in I_q^{m+n}} [L(k_1) \circledast \cdots \circledast L(k_{m+n})] \Big).$$

Here we sum over $\boldsymbol{k} \in I_q^{m+n}$ *satisfying the following condition: there exist* $1 \le u_1 < \cdots < u_m \le m+n$ *and* $1 \le v_1 < \cdots < v_n \le m+n$ *such that* $(k_{u_1}, \dots, k_{u_m}) = (i_1, \dots, i_m)$, $(k_{v_1}, \dots, k_{v_n}) = (j_1, \dots, j_n)$ *and* $\{u_1, \dots, u_m\} \sqcup \{v_1, \dots, v_n\} = \{1, \dots, m+n\}$.

We also need the following [BK, Lemma 4.3], which is proved by direct calculation.

Lemma 5.2. *Let* $a, b \in F^\times$ *with* $a + a^{-1} = q(i)$ *and* $b + b^{-1} = q(j)$ *for some* $i, j \in I_q$. *If* $|i - j| \le 1$, *then*

$$a^{-2}(ab-1)^2(ab^{-1}-1)^2 \cdot (a^{-2}(ab-1)^2(ab^{-1}-1)^2 - \xi^2 a^{-1}b^{-1}(ab-1)^2 - \xi^2 a^{-1} b(ab^{-1}-1)^2) = 0.$$

Corollary 5.3. *For any* $i, j \in \mathbb{Z}$ *with* $|i - j| = 1$ *and* $q(j) \ne q(i)$, *we have*

$$\frac{\xi^2}{(q(j) - q(i))^2}(q(i)q(j) - 4) = 1.$$

Proof. We take a and b satisfying $a + a^{-1} = q(i)$ and $b + b^{-1} = q(j)$. We have

$$a^{-2}(ab-1)^2(ab^{-1}-1)^2 - \xi^2(a^{-1}b^{-1}(ab-1)^2 + a^{-1}b(ab^{-1}-1)^2) = 0$$

by Lemma 5.2 and $q(i) \ne q(j)$. A direct calculation shows that the left hand side is equal to $(q(i) - q(j))^2 - \xi^2(q(i)q(j) - 4)$. □

In the rest of this section, for each $i \in I_q$ we write the basis elements w_1 and w_1' of $L(i)$ $(= L_1^+(i))$ in Definition 3.4 as $v_{\bar{0}}^i$ and $v_{\bar{1}}^i$ respectively. Recall that the irreducible $\mathcal{H}_1$-supermodule $L(i) = Fv_{\bar{0}}^i \oplus Fv_{\bar{1}}^i$ is given by the grading $L(i)_j = Fv_j^i$ for $j \in \mathbb{Z}/2\mathbb{Z}$ and the following action:

$$X_1^{\pm} v_{\bar{0}}^i = b_{\pm}(i) v_{\bar{0}}^i, \quad X_1^{\pm} v_{\bar{1}}^i = b_{\mp}(i) v_{\bar{1}}^i, \quad C_1 v_{\bar{0}}^i = v_{\bar{1}}^i, \quad C_1 v_{\bar{1}}^i = v_{\bar{0}}^i.$$

§5.2. On the block $[(i,j)]$ with $|i-j| = 1$

Lemma 5.4. *For any* $i, j \in \mathbb{Z}$ *such that*

$$|i - j| = 1, \quad q(j) \ne q(i), \quad (\operatorname{type} L(i), \operatorname{type} L(j)) \ne (\mathsf{Q}, \mathsf{Q}),$$

define an $\mathcal{H}_2$*-supermodule* M *and an* $\mathcal{A}_2$*-supermodule* N *as follows:*

$$M := \operatorname{Ind}_{\mathcal{H}_{1,1}}^{\mathcal{H}_2} L(j) \otimes L(i), \quad N := (X_2 + X_2^{-1} - q(i))M \subseteq \operatorname{Res}_{\mathcal{H}_{1,1}}^{\mathcal{H}_2} M.$$

Then:

(i) *N is T_1-invariant, i.e., N is an $\mathcal{H}_2$-supermodule.*

(ii) $\mathsf{ch}\, N = [L(i) \otimes L(j)]$.

Proof. Note that $\mathsf{ch}_{1,1} N = [L(i) \otimes L(j)]$ because $0 \subsetneq N \subsetneq M$ and $\mathsf{ch}\, M = [L(i) \otimes L(j)] + [L(j) \otimes L(i)]$ by Lemma 5.1 and $\mathsf{ch}\,\mathsf{Cosoc}(M) = \mathsf{ch}\, L(ji)$ contains a term $[L(j) \otimes L(i)]$ by Corollary 3.15. Thus, it is enough to show that $T_1 N \subseteq N$.

By (3) and (4), we have

$$(X_2 + X_2^{-1} - q(i))T_1 = T_1(X_1 + X_1^{-1} - q(i)) \\ + \xi(X_2 + C_1C_2X_1 - X_1^{-1} - X_2^{-1}C_1C_2).$$

From this, we see that X and Y defined below form a basis of $N_{\overline{0}}$:

$$\begin{aligned} X &:= (X_2 + X_2^{-1} - q(i))T_1 \otimes v_{\overline{0}}^{j} \otimes v_{\overline{0}}^{i} \\ &= (q(j) - q(i))T_1 \otimes v_{\overline{0}}^{j} \otimes v_{\overline{0}}^{i} \\ &\quad + \xi((b_+(i) - b_-(j))1 \otimes v_{\overline{0}}^{j} \otimes v_{\overline{0}}^{i} - (b_+(i) - b_+(j))1 \otimes v_{\overline{1}}^{j} \otimes v_{\overline{1}}^{i}), \\ Y &:= (X_2 + X_2^{-1} - q(i))T_1 \otimes v_{\overline{1}}^{j} \otimes v_{\overline{1}}^{i} \\ &= (q(j) - q(i))T_1 \otimes v_{\overline{1}}^{j} \otimes v_{\overline{1}}^{i} \\ &\quad + \xi((b_-(i) - b_-(j))1 \otimes v_{\overline{0}}^{j} \otimes v_{\overline{0}}^{i} + (b_-(i) - b_+(j))1 \otimes v_{\overline{1}}^{j} \otimes v_{\overline{1}}^{i}). \end{aligned}$$

To show $T_1 N \subseteq N$, it is enough to show $T_1 N_{\overline{0}} \subseteq N_{\overline{0}}$. For this purpose, it is enough to show the following equalities which follow from Corollary 5.3:

$$\begin{aligned} T_1 X &= \xi\left(1 + \frac{b_+(i) - b_-(j)}{q(j) - q(i)}\right)X - \xi\frac{b_+(i) - b_+(j)}{q(j) - q(i)}Y, \\ T_1 Y &= \xi\frac{b_-(i) - b_-(j)}{q(j) - q(i)}X + \xi\left(1 + \frac{b_-(i) - b_+(j)}{q(j) - q(i)}\right)Y. \end{aligned}$$ □

Corollary 5.5. *For any $i, j \in \mathbb{Z}$ such that*

$$|i - j| = 1, \qquad q(j) \neq q(i), \qquad (\mathsf{type}\, L(i), \mathsf{type}\, L(j)) \neq (\mathsf{Q}, \mathsf{Q}),$$

we have:

(i) $\mathsf{ch}\, L(ij) = [L(i) \otimes L(j)]$.

(ii) *There exists a basis $\{X, Y\}$ of $L(ij)_{\overline{0}}$ such that the matrix representations of $L(ij)$ with respect to the basis $\{X, Y, C_1X, C_1Y\}$ are as follows:*

$$X_1^{\pm 1} : \begin{pmatrix} b_\pm(i) & 0 & 0 & 0 \\ 0 & b_\mp(i) & 0 & 0 \\ 0 & 0 & b_\pm(i) & 0 \\ 0 & 0 & 0 & b_\mp(i) \end{pmatrix}, \qquad X_2^{\pm 1} : \begin{pmatrix} b_\pm(j) & 0 & 0 & 0 \\ 0 & b_\mp(j) & 0 & 0 \\ 0 & 0 & b_\pm(j) & 0 \\ 0 & 0 & 0 & b_\mp(j) \end{pmatrix},$$

$$C_1 : \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad C_2 : \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$T_1 : \frac{\xi}{q(j)-q(i)} \begin{pmatrix} b_+(j)-b_-(i) & b_-(i)-b_-(j) & 0 & 0 \\ b_+(j)-b_+(i) & b_-(j)-b_+(i) & 0 & 0 \\ 0 & 0 & b_+(j)-b_+(i) & b_-(j)-b_+(i) \\ 0 & 0 & b_-(i)-b_+(j) & b_-(j)-b_-(i) \end{pmatrix}.$$

§5.3. On the block $[(i,i,j)]$ with $|i-j|=1$

Lemma 5.6. *For any $i, j \in \mathbb{Z}$ such that*

$$|i-j| = 1, \quad q(j) \neq q(i), \quad (\mathsf{type}\, L(i), \mathsf{type}\, L(j)) = (\mathsf{M}, \mathsf{M}),$$

define an $\mathcal{H}_3$-supermodule M and an $\mathcal{H}_{2,1}$-supermodule N as follows:

$$M := \mathsf{Ind}_{\mathcal{H}_{2,1}}^{\mathcal{H}_3} L(ij) \otimes L(i), \quad N := (X_3 + X_3^{-1} - q(i))M \subseteq \mathsf{Res}_{\mathcal{H}_{2,1}}^{\mathcal{H}_3} M.$$

If $q(i)q(j) + q(j)^2 - 8 \neq 0$, then $T_2 N \not\subseteq N$ and M is irreducible.

Proof. Since $\mathsf{ch}\,\mathsf{Cosoc}\, M = L(iji)$ contains a term $[L(i) \otimes L(j) \otimes L(i)]$ by Corollary 3.15 and $\mathsf{ch}\, M = [L(i) \otimes L(j) \otimes L(i)] + 2[L(i)^{\otimes 2} \otimes L(j)]$ by Lemma 5.1, if M is reducible then M has a unique irreducible submodule M' with $\mathsf{Res}_{\mathcal{H}_{2,1}}^{\mathcal{H}_3} M' \cong L(i^2) \otimes L(j)$ by Theorem 3.9. Thus, if M is reducible then $\mathsf{Res}_{\mathcal{H}_{2,1}}^{\mathcal{H}_3} M' = N$. This implies that if $T_2 N \not\subseteq N$ then M is irreducible.

In the rest of the proof, we show that $T_2 N \not\subseteq N$ if $q(i)q(j) + q(j)^2 - 8 \neq 0$. We take a basis $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) := (X, Y, C_1 X, C_1 Y)$ of $L(ij)$ as in Corollary 5.5. Then a basis of M is given by

$$\{X_{\beta,k,l} := \beta \otimes \alpha_k \otimes v_l^i \mid \beta \in \{1, T_2, T_1 T_2\},\ k \in \{1,2,3,4\},\ l \in \mathbb{Z}/2\mathbb{Z}\}$$

and a basis of $N_{\overline{0}}$ is given by $\{Y_k, Z_k \mid 1 \le k \le 4\}$ where

$$Y_k := (X_3 + X_3^{-1} - q(i)) X_{T_2,k,f(k)}, \quad Z_k := (X_3 + X_3^{-1} - q(i)) X_{T_1 T_2,k,f(k)} (= T_1 Y_k)$$

for $k = 1,2,3,4$ and $f(1) = f(2) = \overline{0}$ and $f(3) = f(4) = \overline{1}$. More explicitly,

$$\begin{aligned} Y_1 &= (q(j) - q(i)) T_2 \otimes \alpha_1 \otimes v_{\overline{0}}^i \\ &\quad + \xi((b_+(i) - b_-(j)) 1 \otimes \alpha_1 \otimes v_{\overline{0}}^i + (b_+(i) - b_+(j)) 1 \otimes \alpha_4 \otimes v_{\overline{1}}^i), \\ Y_2 &= (q(j) - q(i)) T_2 \otimes \alpha_2 \otimes v_{\overline{0}}^i \\ &\quad + \xi((b_+(i) - b_+(j)) 1 \otimes \alpha_2 \otimes v_{\overline{0}}^i + (b_-(j) - b_+(i)) 1 \otimes \alpha_3 \otimes v_{\overline{1}}^i), \end{aligned}$$

$$\begin{aligned}
Y_3 &= (q(j)-q(i))T_2 \otimes \alpha_3 \otimes v^i_{\overline{1}} \\
&\quad + \xi((b_-(i)-b_-(j))1 \otimes \alpha_3 \otimes v^i_{\overline{1}} + (b_-(i)-b_+(j))1 \otimes \alpha_2 \otimes v^i_{\overline{0}}), \\
Y_4 &= (q(j)-q(i))T_2 \otimes \alpha_4 \otimes v^i_{\overline{1}} \\
&\quad + \xi((b_-(i)-b_+(j))1 \otimes \alpha_4 \otimes v^i_{\overline{1}} + (b_-(j)-b_-(i))1 \otimes \alpha_1 \otimes v^i_{\overline{0}}), \\
Z_1 &= (q(j)-q(i))T_1T_2 \otimes \alpha_1 \otimes v^i_{\overline{0}} \\
&\quad + \frac{\xi^2}{q(j)-q(i)}((b_+(i)-b_-(j))(b_+(j)-b_-(i))1 \otimes \alpha_1 \otimes v^i_{\overline{0}} \\
&\quad + (b_+(i)-b_-(j))(b_+(j)-b_+(i))1 \otimes \alpha_2 \otimes v^i_{\overline{0}} \\
&\quad + (b_+(i)-b_+(j))(b_-(j)-b_+(i))1 \otimes \alpha_3 \otimes v^i_{\overline{1}} \\
&\quad + (b_+(i)-b_+(j))(b_-(j)-b_-(i))1 \otimes \alpha_4 \otimes v^i_{\overline{1}}), \\
Z_2 &= (q(j)-q(i))T_1T_2 \otimes \alpha_2 \otimes v^i_{\overline{0}} \\
&\quad + \frac{\xi^2}{q(j)-q(i)}((b_+(i)-b_+(j))(b_-(i)-b_-(j))1 \otimes \alpha_1 \otimes v^i_{\overline{0}} \\
&\quad + (b_+(i)-b_+(j))(b_-(j)-b_+(i))1 \otimes \alpha_2 \otimes v^i_{\overline{0}} \\
&\quad + (b_-(j)-b_+(i))(b_+(j)-b_+(i))1 \otimes \alpha_3 \otimes v^i_{\overline{1}} \\
&\quad + (b_-(j)-b_+(i))(b_-(i)-b_+(j))1 \otimes \alpha_4 \otimes v^i_{\overline{1}}).
\end{aligned}$$

To prove $T_2 N_{\overline{0}} \not\subseteq N_{\overline{0}}$ it is enough to show $T_2 Z_1 \notin N_{\overline{0}}$. Note that

$$T_2 Z_1 = \xi((b_+(j)-b_-(i))T_1T_2 \otimes \alpha_1 \otimes v^i_{\overline{0}} + (b_+(j)-b_+(i))T_1T_2 \otimes \alpha_2 \otimes v^i_{\overline{0}}) + \Delta$$

for a suitable $\Delta \in \mathsf{span}\{X_{T_2,k,l} \mid 1 \le k \le 4,\ l \in \mathbb{Z}/2\mathbb{Z}\}$. Thus, if $T_2 Z_1 \in N_{\overline{0}}$, then we must have

$$\begin{aligned}
T_2 Z_1 &= \xi \frac{b_+(j)-b_-(i)}{q(j)-q(i)} Z_1 + \xi \frac{b_+(j)-b_+(i)}{q(j)-q(i)} Z_2 \\
&\quad + \frac{\xi^2}{(q(j)-q(i))^2}\big((b_+(i)-b_-(j))(b_+(j)-b_-(i))Y_1 \\
&\qquad\qquad + (b_+(i)-b_-(j))(b_+(j)-b_+(i))Y_2 \\
&\quad + (b_+(i)-b_+(j))(b_-(j)-b_+(i))Y_3 + (b_+(i)-b_+(j))(b_-(j)-b_-(i))Y_4\big).
\end{aligned}$$

In particular, the coefficient of $1 \otimes \alpha_1 \otimes v^i_{\overline{0}}$ on the right hand side must be 0. This gives us

$$\frac{\xi^3}{(q(j)-q(i))^2}(b_+(i)-b_-(i))(q(i)q(j)+q(j)^2-8) = 0.$$

Thus, we have $T_2 Z_1 \notin N_{\overline{0}}$ if $q(i)q(j)+q(j)^2-8 \neq 0$. □

Corollary 5.7. *Assume q is a primitive $4l$-th root of unity for $l \geq 3$ and assume $i, j \in \mathbb{Z}$ satisfy*

$$|i-j| = 1, \quad q(j) \neq q(i), \quad (\mathsf{type}\, L(i), \mathsf{type}\, L(j)) = (\mathsf{M}, \mathsf{M}).$$

Then:

(i) $L(iji) \cong L(iij) \cong \mathsf{Ind}_{2,1}^{3} L(ij) \otimes L(i)$.
(ii) $\mathsf{ch}\, L(iji) = \mathsf{ch}\, L(iij) = 2[L(i)^{\otimes 2} \otimes L(j)] + [L(i) \otimes L(j) \otimes L(i)]$.
(iii) $\mathsf{ch}\, L(jii) = 2[L(j) \otimes L(i)^{\otimes 2}] + [L(i) \otimes L(j) \otimes L(i)]$.

Proof. $q(i)q(j) + q(j)^2 - 8 = 0$ is equivalent to $q^{4i+3\pm 3} = 1$ or $q^{4i+1\pm 3} = 1$ since

$$\begin{aligned}(q^{2j+1} &+ q^{-2(j+1)})^2 + (q^{2i+1} + q^{-2(i+1)})(q^{2j+1} + q^{-2(j+1)}) - 2(q+q^{-1})^2 \\ &= (q^{2(i\pm 1)+1} + q^{-2((i\pm 1)+1)})^2 + (q^{2i+1} + q^{-2(i+1)})(q^{2(i\pm 1)+1} + q^{-2((i\pm 1)+1)}) \\ &\quad - 2(q+q^{-1})^2 \\ &= (q+q^{-1})(q^{2i+1.5\pm 1.5} - q^{-(2i+1.5\pm 1.5)})(q^{2i+0.5\pm 1.5} - q^{-(2i+0.5\pm 1.5)}).\end{aligned}$$

Since $\mathsf{type}\, L(i) = \mathsf{M}$, we have $l \geq 3$ and $1 \leq i \leq l-2$. Thus $2 \leq 4i - 2 < 4i + 6 \leq 4l - 2$ and we see that $q^{4i+3\pm 3} \neq 1$ and $q^{4i+1\pm 3} \neq 1$.

By Lemma 5.6, $L(iji) \cong M := \mathsf{Ind}_{2,1}^{3} L(ij) \otimes L(i)$. Thus, $\mathsf{ch}\, L(iji) = 2[L(i)^{\otimes 2} \otimes L(j)] + [L(i) \otimes L(j) \otimes L(i)]$ by Lemma 5.1. This implies $\Delta_j M \neq 0$ and $\widetilde{e}_j M \cong L(i^2)$ by Theorem 3.9. Thus, we have $M \cong L(iij)$.

Finally, consider the irreducible supermodule $L(iij)^\sigma$. It belongs to the same block as $L(iij) \cong L(iji)$, but it is not isomorphic to $L(iij) \cong L(iji)$ in virtue of their formal characters. Thus, we have $L(iij)^\sigma \cong L(jii)$. □

Lemma 5.8. *For any $i, j \in \mathbb{Z}$ such that*

$$|i-j| = 1, \quad q(j) \neq q(i), \quad (\mathsf{type}\, L(i), \mathsf{type}\, L(j)) = (\mathsf{Q}, \mathsf{M}),$$

define an $\mathcal{H}_3$-supermodule M and an $\mathcal{H}_{2,1}$-supermodule N as follows:

$$M := \mathsf{Ind}_{\mathcal{H}_{2,1}}^{\mathcal{H}_3} L(ij) \circledast L(i), \qquad N := (X_3 + X_3^{-1} - q(i))M \subseteq \mathsf{Res}_{\mathcal{H}_{2,1}}^{\mathcal{H}_3} M.$$

Then:

(i) *N is T_2-invariant, i.e., N is an $\mathcal{H}_3$-supermodule.*
(ii) $\mathsf{ch}\, N = 2[L(i)^{\circledast 2} \circledast L(j)]$ *and* $\mathsf{ch}\, M/N = [L(i) \circledast L(j) \circledast L(i)]$.

Proof. As in the first paragraph of the proof of Lemma 5.6, if N is T_2-invariant then $\mathsf{ch}\, N = 2[L(i)^{\circledast 2} \circledast L(j)]$ and $\mathsf{ch}\, M/N = [L(i) \circledast L(j) \circledast L(i)]$. Thus, it is enough to show that N is T_2-invariant.

In the rest of the proof, we write a instead of $b_+(i) = b_-(i)$ and take a basis $\{X, Y, C_1X, C_1Y\}$ of $L(ij)$ as in Corollary 5.5.

We can take a realization of $L(ij) \circledast L(i)$ as an $\mathcal{H}_{2,1}$-submodule W of $L(ij) \otimes L(i)$ given as follows because a direct calculation shows that W is $\mathcal{H}_{2,1}$-invariant:

$$W := W_{\overline{0}} \oplus W_{\overline{1}}, \quad W_{\overline{0}} := FX' \oplus FY', \quad W_{\overline{1}} := F(C_1X') \oplus F(C_1Y'),$$
$$X' := X \otimes v^i_{\overline{0}} + \sqrt{-1}(C_1X) \otimes v^i_{\overline{1}}, \quad Y' := Y \otimes v^i_{\overline{0}} - \sqrt{-1}(C_1Y) \otimes v^i_{\overline{1}}.$$

More precisely, we can check that the matrix representations with respect to the basis $\{X', Y', C_1X', C_1Y'\}$ are given as follows.

$$X_1^{\pm 1} : \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad X_2^{\pm 1} : \begin{pmatrix} b_\pm(j) & 0 & 0 & 0 \\ 0 & b_\mp(j) & 0 & 0 \\ 0 & 0 & b_\pm(j) & 0 \\ 0 & 0 & 0 & b_\mp(j) \end{pmatrix}, \quad X_3^{\pm 1} : \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix},$$
$$C_1 : \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad C_2 : \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad C_3 : \begin{pmatrix} 0 & 0 & \sqrt{-1} & 0 \\ 0 & 0 & 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 & 0 & 0 \\ 0 & \sqrt{-1} & 0 & 0 \end{pmatrix},$$
$$T_1 : \frac{\xi}{q(j)-q(i)} \begin{pmatrix} b_+(j)-a & a-b_-(j) & 0 & 0 \\ b_+(j)-a & b_-(j)-a & 0 & 0 \\ 0 & 0 & b_+(j)-a & b_-(j)-a \\ 0 & 0 & a-b_+(j) & b_-(j)-a \end{pmatrix}.$$

Hereafter, we put $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) := (X', Y', C_1X', C_1Y')$. Then a basis of M is given by $\{X_{\beta,k} := \beta \otimes \alpha_k \mid \beta \in \{1, T_2, T_1T_2\}, k \in \{1,2,3,4\}\}$. It is enough to show that $T_2 N_{\overline{0}} \subseteq N_{\overline{0}}$. We can choose

$$\{Y_k := (X_3 + X_3^{-1} - q(i))X_{T_2,k}, Y_{k+2} := (X_3 + X_3^{-1} - q(i))X_{T_1T_2,k} \mid 1 \le k \le 2\}$$

as a basis of $N_{\overline{0}}$. More explicitly, we have

$$\begin{aligned}
Y_1 &= (q(j)-q(i))T_2 \otimes \alpha_1 + \xi((a-b_-(j))1 \otimes \alpha_1 + \sqrt{-1}(a-b_+(j))1 \otimes \alpha_2), \\
Y_2 &= (q(j)-q(i))T_2 \otimes \alpha_2 + \xi(\sqrt{-1}(a-b_-(j))1 \otimes \alpha_1 + (a-b_+(j))1 \otimes \alpha_2), \\
Y_3 &= (q(j)-q(i))T_1T_2 \otimes \alpha_1 \\
&\quad + \frac{\xi^2}{q(j)-q(i)}(b_+(j)-a)(a-b_-(j))((1-\sqrt{-1})1 \otimes \alpha_1 + (1+\sqrt{-1})1 \otimes \alpha_2), \\
Y_4 &= (q(j)-q(i))T_1T_2 \otimes \alpha_2 \\
&\quad + \frac{\xi^2}{q(j)-q(i)}(b_+(j)-a)(a-b_-(j))((-1+\sqrt{-1})1 \otimes \alpha_1 + (1+\sqrt{-1})1 \otimes \alpha_2).
\end{aligned}$$

Now we can check the following relations using Corollary 5.3:

$$\begin{aligned}
T_2Y_1 &= \xi\frac{b_+(j)-a}{q(j)-q(i)}Y_1 + \xi\frac{(a-b_+(j))\sqrt{-1}}{q(j)-q(i)}Y_2,\\
T_2Y_2 &= \xi\frac{(a-b_-(j))\sqrt{-1}}{q(j)-q(i)}Y_1 + \xi\frac{b_-(j)-a}{q(j)-q(i)}Y_2,\\
T_2Y_3 &= \frac{\xi(b_+(j)-a)}{q(j)-q(i)}(Y_3+Y_4)\\
&\quad + \frac{\xi^2(b_+(j)-a)(a-b_-(j))}{(q(j)-q(i))^2}((1-\sqrt{-1})Y_1+(1+\sqrt{-1})Y_2),\\
T_2Y_4 &= \frac{\xi(a-b_-(j))}{q(j)-q(i)}(Y_3-Y_4)\\
&\quad + \frac{\xi^2(b_+(j)-a)(a-b_-(j))}{(q(j)-q(i))^2}((-1+\sqrt{-1})Y_1+(1+\sqrt{-1})Y_2).
\end{aligned}$$
□

Corollary 5.9. *For any $i, j \in \mathbb{Z}$ such that*

$$|i-j| = 1, \qquad q(j) \neq q(i), \qquad (\mathsf{type}\, L(i), \mathsf{type}\, L(j)) = (\mathsf{Q}, \mathsf{M}),$$

setting $a = b_+(i) = b_-(i)$, we have:

(i) $\mathrm{ch}\, L(iij) = 2[L(i)^{\circledast 2} \circledast L(j)]$ *and* $\mathrm{ch}\, L(iji) = [L(i) \circledast L(j) \circledast L(i)]$.

(ii) *There exists a basis $\{Y_1, Y_2, Y_3, Y_4\}$ of $L(iij)_{\bar{0}}$ such that*

$$\begin{aligned}
Y_3 &= T_1Y_1, \qquad Y_4 = T_1Y_2,\\
X_3^{\pm 1}Y_1 &= b_\pm(j)Y_1, \qquad X_3^{\pm 1}Y_2 = b_\mp(j)Y_2,\\
X_3^{\pm 1}Y_3 &= b_\pm(j)Y_3, \qquad X_3^{\pm 1}Y_4 = b_\mp(j)Y_4,\\
T_2Y_1 &= \frac{\xi(b_+(j)-a)}{q(j)-q(i)}(Y_1-\sqrt{-1}Y_2), \qquad T_2Y_2 = \frac{\xi(a-b_-(j))}{q(j)-q(i)}(\sqrt{-1}Y_1-Y_2),\\
T_2Y_3 &= \frac{\xi(b_+(j)-a)}{q(j)-q(i)}(Y_3+Y_4)\\
&\quad + \frac{\xi^2(b_+(j)-a)(a-b_-(j))}{(q(j)-q(i))^2}((1-\sqrt{-1})Y_1+(1+\sqrt{-1})Y_2),\\
T_2Y_4 &= \frac{\xi(a-b_-(j))}{q(j)-q(i)}(Y_3-Y_4)\\
&\quad + \frac{\xi^2(b_+(j)-a)(a-b_-(j))}{(q(j)-q(i))^2}((-1+\sqrt{-1})Y_1+(1+\sqrt{-1})Y_2),\\
C_3Y_1 &= -C_1Y_2, \qquad C_3Y_2 = C_1Y_1,\\
C_3Y_3 &= \sqrt{-1}(C_1Y_4) - \xi(1+\sqrt{-1})(C_1Y_2),\\
C_3Y_4 &= \sqrt{-1}(C_1Y_3) + \xi(1-\sqrt{-1})(C_1Y_1).
\end{aligned}$$

Proof. It is enough to show the last four relations. Direct calculations using (3) give us

$$\begin{aligned}
C_1Y_1 &= (q(j)-q(i))T_2\otimes\alpha_3 + \xi((a-b_-(j))1\otimes\alpha_3 + \sqrt{-1}(a-b_+(j))1\otimes\alpha_4),\\
C_1Y_2 &= (q(j)-q(i))T_2\otimes\alpha_4 + \xi(\sqrt{-1}(a-b_-(j))1\otimes\alpha_3 + (a-b_+(j))1\otimes\alpha_4),\\
C_1Y_3 &= -\sqrt{-1}(q(j)-q(i))T_1T_2\otimes\alpha_3 + (q(j)-q(i))\xi(1+\sqrt{-1})T_2\otimes\alpha_3 + \Delta_1,\\
C_1Y_4 &= \sqrt{-1}(q(j)-q(i))T_1T_2\otimes\alpha_4 + (q(j)-q(i))\xi(1-\sqrt{-1})T_2\otimes\alpha_4 + \Delta_2,\\
C_3Y_1 &= (q(j)-q(i))T_2\otimes(-\alpha_4) + \Delta_3 = -C_1Y_2,\\
C_3Y_2 &= (q(j)-q(i))T_2\otimes\alpha_3 + \Delta_4 = C_1Y_1,\\
C_3Y_3 &= -(q(j)-q(i))T_1T_2\otimes\alpha_4 + \Delta_5 = \sqrt{-1}(C_1Y_4) - \sqrt{-1}\xi(1-\sqrt{-1})(C_1Y_2),\\
C_3Y_4 &= (q(j)-q(i))T_1T_2\otimes\alpha_3 + \Delta_6 = \sqrt{-1}(C_1Y_3) - \sqrt{-1}\xi(1+\sqrt{-1})(C_1Y_1).
\end{aligned}$$

Here $\Delta_1,\dots,\Delta_6$ are suitable elements in $\mathsf{span}\{1\otimes\alpha_k \mid 1\le k\le 4\}$ $(\subseteq M)$. $\square$

§5.4. On the block $[(i,i,i,j)]$ with $|i-j|=1$ and $(\mathsf{type}\, L(i), \mathsf{type}\, L(j)) = (\mathsf{Q},\mathsf{M})$

Lemma 5.10. *For any $i,j\in\mathbb{Z}$ such that*

$$|i-j|=1,\qquad q(j)\neq q(i),\qquad (\mathsf{type}\, L(i), \mathsf{type}\, L(j)) = (\mathsf{Q},\mathsf{M}),$$

define an $\mathcal{H}_4$-supermodule M and an $\mathcal{H}_{3,1}$-supermodule N as follows:

$$M := \mathsf{Ind}_{\mathcal{H}_{3,1}}^{\mathcal{H}_4} L(iij)\otimes L(i),\qquad N := (X_4 + X_4^{-1} - q(i))M \subseteq \mathsf{Res}_{\mathcal{H}_{3,1}}^{\mathcal{H}_4} M.$$

If $q(j)+2q(i)\neq 0$, then $T_3N\not\subseteq N$ and M is irreducible.

Proof. By the same reasoning as in Lemma 5.6, if $T_3N\not\subseteq N$ then M is irreducible. In the rest of the proof, we show that if $q(j)+2q(i)\neq 0$ then $T_3N\not\subseteq N$.

We write a for $b_+(i)=b_-(i)$ as in the proof of Lemma 5.8 and we take a basis $\{Y_1,Y_2,Y_3,Y_4\}$ of $L(iij)_{\overline{0}}$ as in Corollary 5.9. Thus, we can choose

$$\{Z_{\beta,k} := \beta\otimes Y_k\otimes v_{\overline{0}}^i, W_{\beta,k} := \beta\otimes C_1Y_k\otimes v_{\overline{1}}^i \mid \beta\in\{1,T_3,T_2T_3,T_1T_2T_3\},\ k\in\{1,2,3,4\}\}$$

as a basis of $M_{\overline{0}}$ and

$$\left\{\begin{matrix} Z'_{\beta,k} := (X_4+X_4^{-1}-q(i))Z_{\beta,k}, \\ W'_{\beta,k} := (X_4+X_4^{-1}-q(i))W_{\beta,k}\end{matrix}\ \middle|\ \beta\in\{T_3,T_2T_3,T_1T_2T_3\},\ k\in\{1,2,3,4\}\right\}$$

as a basis of $N_{\overline{0}}$. More explicitly, we have

$$\begin{aligned}
Z'_{T_3,1} &= (q(j)-q(i))T_3\otimes Y_1\otimes v_{\overline{0}}^i\\
&\quad + \xi((a-b_-(j))1\otimes Y_1\otimes v_{\overline{0}}^i + (a-b_+(j))1\otimes C_1Y_2\otimes v_{\overline{1}}^i),
\end{aligned}$$

$$
\begin{aligned}
Z'_{T_3,2} &= (q(j)-q(i))T_3 \otimes Y_2 \otimes v^i_{\bar{0}} \\
&\quad + \xi((a-b_+(j))1 \otimes Y_2 \otimes v^i_{\bar{0}} + (b_-(j)-a)1 \otimes C_1Y_1 \otimes v^i_{\bar{1}}), \\
Z'_{T_3,3} &= (q(j)-q(i))T_3 \otimes Y_3 \otimes v^i_{\bar{0}} + \xi((a-b_-(j))1 \otimes Y_3 \otimes v^i_{\bar{0}} \\
&\quad + \sqrt{-1}(b_+(j)-a)1 \otimes C_1Y_4 \otimes v^i_{\bar{1}} - \xi(1+\sqrt{-1})(b_+(j)-a)1 \otimes C_1Y_2 \otimes v^i_{\bar{1}}), \\
Z'_{T_3,4} &= (q(j)-q(i))T_3 \otimes Y_4 \otimes v^i_{\bar{0}} + \xi((a-b_+(j))1 \otimes Y_4 \otimes v^i_{\bar{0}} \\
&\quad + \sqrt{-1}(b_-(j)-a)1 \otimes C_1Y_3 \otimes v^i_{\bar{1}} + \xi(1-\sqrt{-1})(b_-(j)-a)1 \otimes C_1Y_1 \otimes v^i_{\bar{1}}), \\
W'_{T_3,1} &= (q(j)-q(i))T_3 \otimes C_1Y_1 \otimes v^i_{\bar{1}} \\
&\quad + \xi((a-b_-(j))1 \otimes C_1Y_1 \otimes v^i_{\bar{1}} + (a-b_+(j))1 \otimes Y_2 \otimes v^i_{\bar{0}}), \\
W'_{T_3,2} &= (q(j)-q(i))T_3 \otimes C_1Y_2 \otimes v^i_{\bar{1}} \\
&\quad + \xi((a-b_+(j))1 \otimes C_1Y_2 \otimes v^i_{\bar{1}} + (b_-(j)-a)1 \otimes Y_1 \otimes v^i_{\bar{0}}), \\
W'_{T_3,3} &= (q(j)-q(i))T_3 \otimes C_1Y_3 \otimes v^i_{\bar{1}} + \xi((a-b_-(j))1 \otimes C_1Y_3 \otimes v^i_{\bar{1}} \\
&\quad + \sqrt{-1}(b_+(j)-a)1 \otimes Y_4 \otimes v_{i,0} - \xi(1+\sqrt{-1})(b_+(j)-a)1 \otimes Y_2 \otimes v_{i,0}), \\
W'_{T_3,4} &= (q(j)-q(i))T_3 \otimes C_1Y_4 \otimes v^i_{\bar{1}} + \xi((a-b_+(j))1 \otimes C_1Y_4 \otimes v^i_{\bar{1}} \\
&\quad + \sqrt{-1}(b_-(j)-a)1 \otimes Y_3 \otimes v^i_{\bar{0}} + \xi(1-\sqrt{-1})(b_-(j)-a)1 \otimes Y_1 \otimes v^i_{\bar{0}}), \\
Z'_{T_2T_3,k} &= T_2Z'_{T_3,k} = (q(j)-q(i))T_2T_3 \otimes Y_k \otimes v^i_{\bar{0}} + \Delta_k \quad (1 \le k \le 4), \\
W'_{T_2T_3,k} &= T_2W'_{T_3,k} = (q(j)-q(i))T_2T_3 \otimes C_1Y_k \otimes v^i_{\bar{1}} + \Delta_{k+4} \quad (1 \le k \le 4).
\end{aligned}
$$

Here each Δ_m for $1 \le m \le 8$ is a suitable element in $\mathsf{span}\{1 \otimes C_1^dY_k \otimes v^i_e \mid k \in \{1,2,3,4\},\ d \in \{0,1\},\ e \in \mathbb{Z}/2\mathbb{Z}\}$ ($\subseteq M$). We write $\Delta_3 = \sum_{k=1}^4 P_k 1 \otimes Y_k \otimes v^i_{\bar{0}} + \sum_{k=1}^4 Q_k 1 \otimes C_1Y_k \otimes v^i_{\bar{1}}$ with suitable coefficients. We define Ω_m, $\Omega_{Z,k}$ and $\Omega_{W,k}$ to be the coefficients of $1 \otimes Y_1 \otimes v^i_{\bar{0}}$ in Δ_m, $Z'_{T_3,k}$ and $W'_{T_3,k}$ respectively. Now $T_3Z'_{T_2T_3,3}$ is expanded as follows:

$$
\begin{aligned}
&\xi(b_+(j)-a)(T_2T_3 \otimes Y_3 \otimes v^i_{\bar{0}} + T_2T_3 \otimes Y_4 \otimes v^i_{\bar{0}}) \\
&+ \frac{\xi^2(b_+(j)-a)(a-b_-(j))}{q(j)-q(i)}((1-\sqrt{-1})T_2T_3 \otimes Y_1 \otimes v^i_{\bar{0}} + (1+\sqrt{-1})T_2T_3 \otimes Y_2 \otimes v^i_{\bar{0}}) \\
&+ \sum_{k=1}^4 P_kT_3 \otimes Y_k \otimes v^i_{\bar{0}} + \sum_{k=1}^4 Q_kT_3 \otimes C_1Y_k \otimes v^i_{\bar{1}}.
\end{aligned}
$$

Thus, if $T_3Z'_{T_2T_3,3} \in N_{\bar{0}}$, then we must have

$$
\begin{aligned}
T_3Z'_{T_2T_3,3} &= \frac{\xi(b_+(j)-a)}{q(j)-q(i)}(Z'_{T_2T_3,3} + Z'_{T_2T_3,4}) + \sum_{k=1}^4 \frac{P_kZ'_{T_3,k} + Q_kW'_{T_3,k}}{q(j)-q(i)} \\
&\quad + \frac{\xi^2(b_+(j)-a)(a-b_-(j))}{(q(j)-q(i))^2}((1-\sqrt{-1})Z'_{T_2T_3,1} + (1+\sqrt{-1})Z'_{T_2T_3,2}).
\end{aligned}
$$

In particular, the coefficient of $1 \otimes \alpha_1 \otimes v_{i,0}$ on the right hand side must be 0, in other words

$$\begin{aligned} S &:= \frac{\xi(b_+(j)-a)}{q(j)-q(i)}(\Omega_3+\Omega_4) + \sum_{k=1}^{4} \frac{P_k\Omega_{Z,k}+Q_k\Omega_{W,k}}{q(j)-q(i)} \\ &\quad + \frac{\xi^2(b_+(j)-a)(a-b_-(j))}{(q(j)-q(i))^2}((1-\sqrt{-1})\Omega_1+(1+\sqrt{-1})\Omega_2) = 0. \end{aligned}$$

Note that $\Omega_{Z,2} = \Omega_{Z,3} = \Omega_{Z,4} = \Omega_{W,1} = \Omega_{W,3} = 0$ and the necessary data are calculated as follows:

$$\begin{aligned} \Omega_1 &= \frac{\xi^2}{q(j)-q(i)}(a-b_-(j))(b_+(j)-a), \\ \Omega_2 &= \frac{\xi^2\sqrt{-1}}{q(j)-q(i)}(a-b_+(j))(a-b_-(j)), \\ \Omega_3 &= \frac{\xi^3(1-\sqrt{-1})}{(q(j)-q(i))^2}(a-b_-(j))^2(b_+(j)-a), \\ \Omega_4 &= \frac{\xi^3(1-\sqrt{-1})}{(q(j)-q(i))^2}(b_+(j)-a)^2(a-b_-(j)), \end{aligned}$$

$$\Omega_{Z,1} = \xi(a-b_-(j)), \quad \Omega_{W,2} = \xi(b_-(j)-a), \quad \Omega_{W,4} = \xi^2(1-\sqrt{-1})(b_-(j)-a),$$

$$\begin{aligned} P_1 = \Omega_3 &= \frac{\xi^3(1-\sqrt{-1})}{(q(j)-q(i))^2}(a-b_-(j))^2(b_+(j)-a), \\ Q_4 &= \frac{-\sqrt{-1}\xi^2}{q(j)-q(i)}(b_+(j)-a)(a-b_-(j)), \\ Q_2 &= \frac{\xi^3(1+\sqrt{-1})\sqrt{-1}}{(q(j)-q(i))^2}(b_+(j)-a)^2(a-b_-(j)) \\ &\quad + \frac{\xi^3(1+\sqrt{-1})}{q(j)-q(i)}(b_+(j)-a)(a-b_-(j)). \end{aligned}$$

Using them, we have

$$\begin{aligned} S &= \frac{\xi^4(1-\sqrt{-1})}{(q(j)-q(i))^3}(a-b_-(j))(b_+(j)-a) \\ &\quad \cdot (4(a-b_-(j))(b_+(j)-a)+(b_+(j)-a)^2+(a-b_-(j))^2). \end{aligned}$$

Note that $(a-b_-(j))(b_+(j)-a) = aq(j)-2 \neq 0$ since $q(j) \neq \pm 2$. Thus, we have

$$4(a-b_-(j))(b_+(j)-a)+(b_+(j)-a)^2+(a-b_-(j))^2 = (q(j)+4a)(q(j)-2a) = 0.$$

Again, by $q(j) \neq \pm 2$, we have $q(j)+4a = q(j)+2q(i) = 0$ if $T_3Z'_{T_2T_3,3} \in N_{\overline{0}}$. □

Corollary 5.11. *Assume q is a primitive $4l$-th root of unity for $l \geq 3$ and $i, j \in \mathbb{Z}$ satisfy*

$$|i-j| = 1, \quad q(j) \neq q(i), \quad (\mathsf{type}\, L(i), \mathsf{type}\, L(j)) = (\mathsf{Q}, \mathsf{M}).$$

Then:

(i) $L(iiji) \cong L(iiij) \cong \mathsf{Ind}_{\mathcal{H}_{3,1}}^{\mathcal{H}_4} L(iij) \otimes L(i)$.

(ii) $\mathsf{ch}\, L(iiji) = \mathsf{ch}\, L(iiij) = 6[L(i)^{\circledast 3} \circledast L(j)] + 2[L(i)^{\circledast 2} \circledast L(j) \circledast L(i)]$.

(iii) $\mathsf{ch}\, L(jiii) = 6[L(j) \circledast L(i)^{\circledast 3}] + 2[L(i) \circledast L(j) \circledast L(i)^{\circledast 2}]$.

(iv) $\mathsf{ch}\, L(ijii) = 2[L(i) \circledast L(j) \circledast L(i)^{\circledast 2}] + 2[L(i)^{\circledast 2} \circledast L(j) \circledast L(i)]$.

Proof. We only need to consider the cases $(i, j) = (0, 1), (l-1, l-2)$. In each case, we see that $q(j)+2q(i) = 0$ implies $q^6 = 1$. Thus, we have $L(iiji) \cong \mathsf{Ind}_{\mathcal{H}_{3,1}}^{\mathcal{H}_4} L(iij) \otimes L(i)$ by Lemma 5.10. By the same reasoning as in Corollary 5.7, we have $L(iiij) \cong L(iiji)$. Note that $L(jiii) \ncong L(ijii)$ since $L(ji) \ncong L(ij)$ by Corollary 5.5. Since $\varepsilon_i(L(iiij)^\sigma) = 3$, we see that $L(jiii) \cong L(iiij)^\sigma$. Now it is easily seen that $L(iiji) \cong \mathsf{Ind}_{\mathcal{H}_{3,1}}^{\mathcal{H}_4} L(iij) \circledast L(i)$. □

§5.5. The case when q is a primitive 8-th root of unity

Lemma 5.12. *Let q be a primitive 8-th root of unity. There is a basis $B = \{w_1, w_2\}$ of $L(01)$ such that w_1 is even and w_2 is odd and the matrix representations with respect to B are as follows:*

$$X_1^{\pm 1} : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X_2^{\pm 1} : \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, C_1 : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, C_2 : \begin{pmatrix} 0 & -q^2 \\ q^2 & 0 \end{pmatrix}, T_1 : \begin{pmatrix} q & 0 \\ 0 & q^3 \end{pmatrix}.$$

Proof. We can check by direct calculation that these matrices satisfy the defining relations of $\mathcal{H}_2$. It is clearly irreducible and note that the whole space is a simultaneous $(2, -2) = (q(0), q(1))$-eigenspace of $(X_1 + X_1^{-1}, X_2 + X_2^{-1})$. □

Corollary 5.13. *We have* $\mathsf{ch}\, L(01) = [L(0) \circledast L(1)]$ *and* $\mathsf{ch}\, L(10) = [L(1) \circledast L(0)]$.

Lemma 5.14. *Let q be a primitive 8-th root of unity. We can take a basis $B = \{w_i \mid 1 \leq i \leq 8\}$ of $L(001)$ such that w_i is even and w_{i+4} is odd for $1 \leq i \leq 4$ and the matrix representations with respect to B are as follows:*

$$X_i : \begin{pmatrix} M_{X_i} & O \\ O & M_{X_i} \end{pmatrix}, \quad X_3^{\pm 1} : -E_8, \quad X_1^{-1} : 2E_8 - X_1, \quad X_2^{-1} : 2E_8 - X_2,$$

$$C_j : \begin{pmatrix} O & M_{C_j} \\ -M_{C_j} & O \end{pmatrix}, \quad T_1 : \frac{1}{1+q^2} \begin{pmatrix} M_{T_1} & O \\ O & M_{T_1} \end{pmatrix},$$

$$T_2 : \begin{pmatrix} M_{T_2} & O & O & O \\ O & M_{T_2} & O & O \\ O & O & M_{T_2} & O \\ O & O & O & M_{T_2} \end{pmatrix},$$

for $1 \leq i \leq 2$ *and* $1 \leq j \leq 3$ *where*

$$M_{X_1} = \begin{pmatrix} 1 & 0 & -2 & 2q \\ 0 & 1 & 2q & -2q^2 \\ 2 & 2q^{-1} & 1 & 0 \\ 2q^{-1} & -2q^2 & 0 & 1 \end{pmatrix}, \quad M_{X_2} = \begin{pmatrix} -1 & -2q^{-1} & 0 & 0 \\ 2q & 3 & 0 & 0 \\ 0 & 0 & -1 & 2q \\ 0 & 0 & -2q^{-1} & 3 \end{pmatrix},$$

$$M_{C_1} = \begin{pmatrix} q^2 & 0 & 2q^2 & 2q^{-1} \\ 0 & q^2 & 2q^{-1} & -2 \\ 2q^2 & 2q & -q^2 & 0 \\ 2q & 2 & 0 & -q^2 \end{pmatrix}, \quad M_{C_2} = \begin{pmatrix} 0 & 0 & q^2 & 0 \\ 0 & 0 & 2q^{-1} & -1 \\ q^2 & 0 & 0 & 0 \\ 2q & 1 & 0 & 0 \end{pmatrix},$$

$$M_{C_3} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & q^2 \\ 1 & 0 & 0 & 0 \\ 0 & q^2 & 0 & 0 \end{pmatrix},$$

$$M_{T_1} = \begin{pmatrix} q^3 & q^2 & -q^3 & -1 \\ 0 & q^3 & 0 & q \\ q^3 & q^2 & q^3 & 1 \\ 0 & q & 0 & q^3 \end{pmatrix}, \quad M_{T_2} = \begin{pmatrix} q^3 + q & 1 \\ 1 & 0 \end{pmatrix}.$$

Proof. We can check by direct calculation that these matrices satisfy the defining relations of $\mathcal{H}_3$ and the whole space is a simultaneous $(2, 2, -2) = (q(0), q(0), q(1))$-eigenspace of $(X_1 + X_1^{-1}, X_2 + X_2^{-1}, X_3 + X_3^{-1})$. Since $\dim L(0^2) \circledast L(1) = 8$, by Theorem 3.9 this supermodule is irreducible. □

Corollary 5.15. *Let* q *be a primitive* 8*-th root of unity. Then*

$$\begin{aligned} \mathsf{ch}\, L(001) &= 2[L(0)^{\circledast 2} \circledast L(1)], \\ \mathsf{ch}\, L(010) &= [L(0) \circledast L(1) \circledast L(0)], \\ \mathsf{ch}\, L(100) &= 2[L(1) \circledast L(0)^{\circledast 2}]. \end{aligned}$$

Proof. Since $\mathsf{ch}\, L(001) = 2[L(0)^{\circledast 2} \circledast L(1)]$, we have $L(100) \cong L(001)^{\sigma}$. Consider $M = \mathsf{Ind}_{\mathcal{H}_{2,1}}^{\mathcal{H}_3} L(01) \circledast L(0)$. By Corollary 5.13 and Lemma 5.1, we have $\mathsf{ch}\, M = [L(0) \circledast L(1) \circledast L(0)] + 2[L(0)^{\circledast 2} \circledast L(1)]$. Applying Theorem 3.11(i), we see that $L(010) \cong \mathsf{Cosoc}\, M$ with $\mathsf{ch}\, L(010) = [L(0) \circledast L(1) \circledast L(0)]$. □

Corollary 5.16. *Let* q *be a primitive* 8*-th root of unity. Then* $M := \mathsf{Ind}_{\mathcal{H}_{3,1}}^{\mathcal{H}_4} L(001) \circledast L(0)$ *is an irreducible* $\mathcal{H}_4$*-supermodule.*

Proof. Take a basis $\{w_i \mid 1 \le i \le 8\}$ as in Lemma 5.14. Consider the following linear transformations with respect to this basis:

$$X_4^{\pm 1} : E_8, \qquad C_4 : \begin{pmatrix} O & -E_4 \\ -E_4 & O \end{pmatrix}.$$

We can check that the matrix representations of $\{X_i^{\pm 1}, C_i, T_j \mid 1 \le i \le 4,\ 1 \le j \le 3\}$ satisfy the defining relations of $\mathcal{H}_{3,1}$. Thus, they are also matrix representations of $L(001) \circledast L(0)$.

To prove that M is irreducible, it is enough to show that the $\mathcal{H}_{3,1}$-supermodule $N := (X_4 + X_4^{-1} - q(0))M$ is not T_3-invariant as in the proof of Lemma 5.10. Thus, it is enough to show that $T_3 Z \neq (Z - W)/2$ where

$$\begin{aligned} Z &:= (X_4 + X_4^{-1} - 2)T_3 \otimes w_1 = -4T_3 \otimes w_1 + 2\xi(w_1 + w_3), \\ W &:= (X_4 + X_4^{-1} - 2)T_3 \otimes w_3 = -4T_3 \otimes w_3 + 2\xi(w_3 - w_1), \\ T_3 Z &= -2\xi(T_3 \otimes w_1 - T_3 \otimes w_3) - 4 \cdot 1 \otimes w_1. \end{aligned}$$

This follows from $2\xi \neq -4$. □

Corollary 5.17. *Let q be a primitive 8-th root of unity. Then:*

(i) $L(0010) \cong L(0001) \cong \mathsf{Ind}_{\mathcal{H}_{3,1}}^{\mathcal{H}_4} L(001) \circledast L(0)$.
(ii) $\mathsf{ch}\, L(0010) = \mathsf{ch}\, L(0001) = 6[L(0)^{\circledast 3} \circledast L(1)] + 2[L(0)^{\circledast 2} \circledast L(1) \circledast L(0)]$.
(iii) $\mathsf{ch}\, L(1000) = 6[L(1) \circledast L(0)^{\circledast 3}] + 2[L(0) \circledast L(1) \circledast L(0)^{\circledast 2}]$.
(iv) $\mathsf{ch}\, L(0100) = 2[L(0) \circledast L(1) \circledast L(0)^{\circledast 2}] + 2[L(0)^{\circledast 2} \circledast L(1) \circledast L(0)]$.

Proof. Same as the proof of Corollary 5.11. □

§6. Hecke–Clifford superalgebras and crystals of type $D_l^{(2)}$

Recall that F is an algebraically closed field of characteristic different from 2. From now on, we assume that q is a primitive $4l$-th root of unity for $l \ge 2$ and choose $\{0, 1, \dots, l-1\}$ as I_q. Note that $q(0) = 2$ and $q(l-1) = -2$.

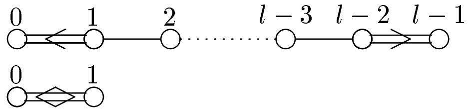
§6.1. Lie theory of type $D_l^{(2)}$

Consider the Dynkin diagram and the affine Cartan matrix indexed by I_q of type $D_l^{(2)}$ as follows.[9]

[9]According to Kac's notation [Kac, Table Aff 1-3], $D_2^{(2)}$ should be regarded as $A_1^{(1)}$.

$l \geq 3$

$l = 2$



$$\begin{pmatrix} 2 & -2 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -2 & 2 \end{pmatrix}$$

In the rest of this section, let $\mathfrak{g}$ be the corresponding Kac–Moody Lie algebra and apply definitions in §3.7 for $A = D_l^{(2)}$.

§6.2. Representations of low rank affine Hecke–Clifford superalgebras

The purpose of this subsection is to show that [BK, Lemmas 5.19 and 5.20] still hold in our setting, i.e., when q is a primitive $4l$-th root of unity for $l \geq 2$. This fact is responsible for the appearance of the Lie theory of type $D_l^{(2)}$.

Lemma 6.1. *Let $i, j \in I_q$ with $|i-j| = 1$. Then, for all $a, b \geq 0$ with $a + b < -\langle h_i, \alpha_j \rangle$, there is a non-split short exact sequence*

$$0 \to L(i^{a+1}ji^b) \to \mathsf{Ind}_{\mathcal{H}_{a+b+1,1}}^{\mathcal{H}_{a+b+2}} L(i^a ji^b) \circledast L(i) \to L(i^a ji^{b+1}) \to 0. \tag{13}$$

Moreover, for every $a, b \geq 0$ with $a + b \leq -\langle h_i, \alpha_j \rangle$, we have

$$\mathsf{ch}\, L(i^a ji^b) = a!b![L(i)^{\circledast a} \circledast L(j) \circledast L(i)^{\circledast b}]. \tag{14}$$

Proof. (14) is established in Corollaries 5.5, 5.9, 5.13 and 5.15. The existence of a non-split short exact sequence (13) follows from Lemma 5.1, Theorem 3.11(i), Definition 3.14 and the injectivity of the formal character map $\mathsf{ch} : \mathsf{K}_0(\mathsf{Rep}\, \mathcal{H}_n) \hookrightarrow \mathsf{K}_0(\mathsf{Rep}\, \mathcal{A}_n)$ [BK, Theorem 5.12]. □

Lemma 6.2. *Let $i, j \in I_q$ with $|i-j| = 1$ and set $n = 1 - \langle h_i, \alpha_j \rangle$. Then $L(i^n j) \cong L(i^{n-1} ji)$. Moreover, for every $a, b \geq 0$ with $a + b = -\langle h_i, \alpha_j \rangle$, we have*

$$L(i^a ji^{b+1}) \cong \mathsf{Ind}_{\mathcal{H}_{n,1}}^{\mathcal{H}_{n+1}} L(i^a ji^b) \circledast L(i) \cong \mathsf{Ind}_{\mathcal{H}_{1,n}}^{\mathcal{H}_{n+1}} L(i) \circledast L(i^a ji^b)$$

with character

$$a!(b+1)![L(i)^{\circledast a} \circledast L(j) \circledast L(i)^{\circledast (b+1)}] + (a+1)!b![L(i)^{\circledast (a+1)} \circledast L(j) \circledast L(i)^{\circledast b}].$$

Proof. Character formulas are established in Corollaries 5.7, 5.11 and 5.16. The rest of the reasoning is the same as the proof of Lemma 6.1. □

Corollary 6.3. *The operators $\{e_i : K(\infty) \to K(\infty) \mid i \in I_q\}$ satisfy the Serre relations, i.e.,*

$$\begin{cases} e_i e_j = e_j e_i & \text{if } |i-j| > 1, \\ e_i^2 e_j + e_j e_i^2 = 2 e_i e_j e_i & \text{if } |i-j| = 1 \text{ and } i \neq 0 \text{ and } i \neq l-1, \\ e_i^3 e_j + 3 e_i e_j e_i^2 = 3 e_i^2 e_j e_i + e_j e_i^3 & \text{otherwise.} \end{cases} \tag{15}$$

Proof. By Lemma 3.24 and coassociativity of Δ, it is enough to check the same relations on $\mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_2), \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_3)$ and $\mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_4)$ respectively. This is achieved using the character formulas in Lemmas 6.1 and 6.2. □

The same argument using Lemmas 6.1 and 6.2 establishes the following [BK′, Lemma 5.23].

Lemma 6.4. *Let $M \in \mathsf{Irr}(\mathsf{Rep}\,\mathcal{H}_n)$ and $i, j \in I_q$ with $i \neq j$. Then the following hold where $k = -\langle h_i, \alpha_j\rangle$ and $\varepsilon = \varepsilon_i(M)$:*

(i) *There exists a unique pair (a, b) of non-negative integers with $a + b = k$ such that for every $m \geq 0$ we have $\varepsilon_i(\tilde{f}_i^m \tilde{f}_j M) = m + \varepsilon - a$.*
(ii) $[\mathsf{Cosoc}\,\mathsf{Ind}\,\tilde{f}_i^{m-k} M \circledast L(i^a j i^b) : \tilde{f}_i^m \tilde{f}_j M] > 0$ *for* $m \geq k$.
(iii) $[\mathsf{Cosoc}\,\mathsf{Ind}\,\tilde{e}_i^{k-m} M \circledast L(i^a j i^b) : \tilde{f}_i^m \tilde{f}_j M] > 0$ *for* $0 \leq m < k \leq m + \varepsilon$.

Note that Lemma 6.4(ii)&(iii) is equivalent to saying that

$$[\mathsf{Cosoc}\,\mathsf{Ind}(\tilde{f}_i^{\varepsilon+m-k}\tilde{e}_i^{\varepsilon} M) \circledast L(i^a j i^b) : \tilde{f}_i^m \tilde{f}_j M] > 0$$

for every $m \geq 0$ with $k \leq m + \varepsilon$.

Keep the setting of Lemma 6.4. Since there are surjections

$$\mathsf{Ind}\,\tilde{e}_i^{\varepsilon} M \circledast L(i^{\varepsilon+m-k}) \twoheadrightarrow \tilde{f}_i^{\varepsilon+m-k}\tilde{e}_i^{\varepsilon} M, \qquad \mathsf{Ind}\,L(i^a) \circledast L(j i^b) \twoheadrightarrow L(i^a j i^b)$$

by Theorem 3.11(i) and Lemma 6.1 respectively, we have

$$[\mathsf{Cosoc}\,\mathsf{Ind}(\tilde{e}_i^{\varepsilon} M \circledast L(i^{\varepsilon+m-b}) \circledast L(j i^b)) : \tilde{f}_i^m \tilde{f}_j M] > 0.$$

By Frobenius reciprocity there is a non-zero injective homomorphism

$$\tilde{e}_i^{\varepsilon} M \circledast L(i^{\varepsilon+m-b}) \circledast L(j i^b) \hookrightarrow \mathsf{Res}_{\mathcal{H}_{n-\varepsilon,\varepsilon+m-b,b+1}}\, \tilde{f}_i^m \tilde{f}_j M.$$

Thus, we also have a non-zero injective homomorphism

$$\tilde{e}_i^{\varepsilon} M \circledast L(i^{\varepsilon+m-b}) \hookrightarrow \mathsf{Res}_{\mathcal{H}_{n-\varepsilon,\varepsilon+m-b}}\, \tilde{f}_i^m \tilde{f}_j M.$$

Again by Frobenius reciprocity, for every $m \geq 0$ with $k \leq m + \varepsilon$ we have

$$[\mathsf{Res}_{\mathcal{H}_{n+m-b}}\, \tilde{f}_i^m \tilde{f}_j M : \tilde{f}_i^{m-b} M] > 0. \tag{16}$$

§6.3. Cyclotomic Hecke–Clifford superalgebra

Definition 6.5. For each positive integral weight $\lambda \in P^+$, we define a polynomial

$$f^{\lambda} = (X_1 - 1)^{\lambda(h_0)}(X_1 + 1)^{\lambda(h_{l-1})} \prod_{i=1}^{l-2} (X_1^2 - q(i) X_1 + 1)^{\lambda(h_i)}.$$

Note that since the canonical central element is $c = h_0 + h_{l-1} + \sum_{i=1}^{l-2} 2h_i$, the degree of f^λ is $\lambda(c)$. It is clear that the set of roots of f^λ is a subset of $\{b_\pm(i) \mid i \in I_q\}$ and we can easily check that f^λ satisfies the assumption in Definition 4.1. From now on, we apply all the constructions in §4 for $R = f^\lambda$ and abbreviate $K(R)$, e_i^R, etc. to $K(\lambda)$, e_i^λ, etc. respectively.

As a corollary of Lemma 3.21, we have the following characterization of $\mathsf{Im}(\mathsf{infl}^\lambda : B(\lambda) \hookrightarrow B(\infty))$ [BK, Corollary 6.13].

Corollary 6.6. *Let $\lambda \in P^+$ and $M \in B(\infty)$. We have $\mathsf{pr}^\lambda M = M$ if and only if $\varepsilon_i^*(M) \leq \lambda(h_i)$ for all $i \in I_q$.*

Lemma 6.7. *Let $i, j \in I_q$ with $i \neq j$ and $M \in \mathsf{Irr}(\mathcal{H}_n^\lambda\text{-}\mathsf{smod})$ such that $\varphi_j^\lambda(M) > 0$. Then $\varphi_i^\lambda(\widetilde{f}_j^\lambda M) - \varepsilon_i^\lambda(\widetilde{f}_j^\lambda M) \leq \varphi_i^\lambda(M) - \varepsilon_i^\lambda(M) - a_{ij}$.*

Proof. Put $\varepsilon = \varepsilon_i^\lambda(M) = \varepsilon_i(\mathsf{infl}^\lambda M)$. Apply Lemma 6.4 to $\mathsf{infl}^\lambda M$ and take a pair (a, b) as in Lemma 6.4(i). Since $\varepsilon_i^\lambda(\widetilde{f}_j^\lambda M) = \varepsilon_i(\widetilde{f}_j \,\mathsf{infl}^\lambda M) = \varepsilon - a$, it is enough to show that $\varphi_i^\lambda(\widetilde{f}_j^\lambda M) \leq \varphi_i^\lambda(M) + b$. Note that $m > \varphi_i^\lambda(M) + b$ implies that $-a_{ij} \leq m + \varepsilon$ since $m + \varepsilon + a_{ij} > \varphi_i^\lambda(M) + (\varepsilon - a)$. Thus, we have

$$\varepsilon_i^*(\widetilde{f}_i^m \widetilde{f}_j \,\mathsf{infl}^\lambda M) \geq \varepsilon_i^*(\widetilde{f}_i^{m-b} \,\mathsf{infl}^\lambda M) > \lambda(h_i).$$

Here the first inequality follows from (16) and the second inequality follows from Corollary 6.6 and the σ-version of Lemma 3.22(ii). Again by Corollary 6.6, we have $\mathsf{pr}^\lambda \widetilde{f}_i^m \widetilde{f}_j \,\mathsf{infl}^\lambda M = 0$ for each $m > \varphi_i^\lambda(M) + b$, i.e., $\varphi_i^\lambda(\widetilde{f}_j^\lambda M) \leq \varphi_i^\lambda(M) + b$. □

Theorem 6.8. *For any $M \in \mathsf{Irr}(\mathcal{H}_n^\lambda\text{-}\mathsf{smod})$ and $i \in I_q$, we have $\varphi_i^\lambda(M) - \varepsilon_i^\lambda(M) = \langle h_i, \lambda + \mathsf{wt}(\mathsf{infl}^\lambda M)\rangle$.*

Proof. By Corollary 6.6, we have $\varphi_i^\lambda(\mathbf{1}_\lambda) = \lambda(h_i)$. Combining this with the obvious $\varepsilon_i^\lambda(\mathbf{1}_\lambda) = 0$ and Lemma 6.7, we inductively have $\varphi_i^\lambda(M) - \varepsilon_i^\lambda(M) \leq \langle h_i, \lambda + \mathsf{wt}(\mathsf{infl}^\lambda M)\rangle$. Thus, it is enough to show that

$$(\varphi_0^\lambda(M) - \varepsilon_0^\lambda(M)) + (\varphi_{l-1}^\lambda(M) - \varepsilon_{l-1}^\lambda(M)) + \sum_{i=1}^{l-2} 2(\varphi_i^\lambda(M) - \varepsilon_i^\lambda(M)) = \lambda(h_i),$$

which is the same thing as Corollary 4.12. □

Corollary 6.9. *The 6-tuple $(B(\lambda), \mathsf{wt}^\lambda, \{\varepsilon_i^\lambda\}_{i \in I_q}, \{\varphi_i^\lambda\}_{i \in I_q}, \{\widetilde{e}_i^\lambda\}_{i \in I_q}, \{\widetilde{f}_i^\lambda\}_{i \in I_q})$ is a $\mathfrak{g}$-crystal by defining $\mathsf{wt}^\lambda(M) = \lambda + \mathsf{wt}(\mathsf{infl}^\lambda M)$ for $M \in B(\lambda)$.*

§6.4. Lie-theoretic descriptions of $B(\infty)$ and $B(\lambda)$

Theorem 6.10. *For each $i \in I_q$, the map*

$$\Psi_i : B(\infty) \to B(\infty) \otimes B_i, \qquad [M] \mapsto [(\widetilde{e}_i^*)^{\varepsilon_i^*(M)} M] \otimes b_i(-\varepsilon_i^*(M)),$$

is a crystal embedding.

Proof. We prove $\Psi_i([\widetilde{f}_j M]) = \widetilde{f}_j \Psi_i([M])$ for any $i, j \in I_q$ and $[M] \in B(\infty)$. In case $i \neq j$, this follows from the σ-versions of Lemma 3.22(ii)&(iii).

Let us assume $i = j$ and put $a = \varepsilon_i^*(M)$. By Definition 2.3,

$$\widetilde{f}_i \Psi_i([M]) = \begin{cases} [\widetilde{f}_i (\widetilde{e}_i^*)^a M] \otimes b_i(-a) & \text{if } \varepsilon_i((\widetilde{e}_i^*)^a M) + a + \langle h_i, \mathsf{wt}(M) \rangle > 0, \\ [(\widetilde{e}_i^*)^a M] \otimes b_i(-a-1) & \text{if } \varepsilon_i((\widetilde{e}_i^*)^a M) + a + \langle h_i, \mathsf{wt}(M) \rangle \leq 0. \end{cases}$$

Comparing with the σ-versions of Lemma 3.22(i)&(iii)&(iv), it is enough to show

$$\varepsilon_i^*(\widetilde{f}_i M) = \begin{cases} a & \text{if } \varepsilon_i((\widetilde{e}_i^*)^a M) + a + \langle h_i, \mathsf{wt}(M) \rangle > 0, \\ a+1 & \text{if } \varepsilon_i((\widetilde{e}_i^*)^a M) + a + \langle h_i, \mathsf{wt}(M) \rangle \leq 0. \end{cases}$$

Consider the case $\varepsilon_i((\widetilde{e}_i^*)^a M) + a + \langle h_i, \mathsf{wt}(M) \rangle > 0$ and take $\lambda_1 \in P^+$ such that $\lambda_1(h_j)$ is large enough for any $j \neq i$ and $\lambda_1(h_i) = a$. Note that M can be regarded as an element of $B(\lambda_1)$ by Corollary 6.6. By Theorem 6.8, we have

$$\begin{aligned} \varphi_i^{\lambda_1}(\mathsf{pr}^{\lambda_1} M) &= \varepsilon_i^{\lambda_1}(\mathsf{pr}^{\lambda_1} M) + \langle h_i, \lambda_1 + \mathsf{wt}(M) \rangle = \varepsilon_i(M) + a + \langle h_i, \mathsf{wt}(M) \rangle \\ &\geq \varepsilon_i((\widetilde{e}_i^*)^a M) + a + \langle h_i, \mathsf{wt}(M) \rangle \geq 1. \end{aligned}$$

Thus, $\varepsilon_i^*(\widetilde{f}_i M) \leq \lambda_1(h_i) = a$ by Corollary 6.6. This implies $\varepsilon_i^*(\widetilde{f}_i M) = a$ by the σ-version of Lemma 3.22(i).

Finally, consider the case $\varepsilon_i((\widetilde{e}_i^*)^a M) + a + \langle h_i, \mathsf{wt}(M) \rangle \leq 0$, i.e.,

$$\varepsilon_i^*((\widetilde{e}_i)^a M^\sigma) + a + \langle h_i, \mathsf{wt}(M^\sigma) \rangle = \varepsilon_i^*((\widetilde{e}_i)^a M^\sigma) - a + \langle h_i, \mathsf{wt}((\widetilde{e}_i)^a M^\sigma) \rangle \leq 0.$$

Take $\lambda_2 \in P^+$ such that $\lambda_2(h_j)$ is large enough for any $j \neq i$ and $\lambda_2(h_i) = r + \varepsilon_i^*((\widetilde{e}_i)^a M^\sigma)$ for $r = a - \varepsilon_i^*((\widetilde{e}_i)^a M^\sigma) - \langle h_i, \mathsf{wt}((\widetilde{e}_i)^a M^\sigma) \rangle$ (≥ 0). Again $(\widetilde{e}_i)^a M^\sigma$ can be regarded as an element of $B(\lambda_2)$ and we have

$$\begin{aligned} \varphi_i^{\lambda_2}(\mathsf{pr}^{\lambda_2}(\widetilde{e}_i)^a M^\sigma) &= \varepsilon_i^{\lambda_2}(\mathsf{pr}^{\lambda_2}(\widetilde{e}_i)^a M^\sigma) + \langle h_i, \lambda_2 + \mathsf{wt}((\widetilde{e}_i)^a M^\sigma) \rangle \\ &= \langle h_i, \lambda_2 + \mathsf{wt}((\widetilde{e}_i)^a M^\sigma) \rangle = a \end{aligned}$$

by Theorem 6.8. Combined with Corollary 6.6, this implies

$$\begin{cases} \varepsilon_i(M) = \varepsilon_i^*(M^\sigma) = \varepsilon_i^*(\widetilde{f}_i^a (\widetilde{e}_i)^a M^\sigma) \leq \lambda_2(h_i), \\ \varepsilon_i(\widetilde{f}_i^* M) = \varepsilon_i^*(\widetilde{f}_i M^\sigma) = \varepsilon_i^*(\widetilde{f}_i^{a+1} (\widetilde{e}_i)^a M^\sigma) \geq \lambda_2(h_i) + 1. \end{cases}$$

Thus, by Lemma 3.22(i), we have

$$\varepsilon_i(M) = \lambda_2(h_i) = a - \langle h_i, \mathsf{wt}((\widetilde{e}_i)^a M^\sigma) \rangle = -a - \langle h_i, \mathsf{wt}(M) \rangle.$$

Take $\lambda_3 \in P^+$ such that $\lambda_3(h_j)$ is large enough for any $j \neq i$ and $\lambda_3(h_i) = a$. Again M can be regarded as an element of $B(\lambda_3)$ and we have

$$\varphi_i^{\lambda_3}(\mathsf{pr}^{\lambda_3} M) = \varepsilon_i^{\lambda_3}(\mathsf{pr}^{\lambda_3} M) + \langle h_i, \lambda_3 + \mathsf{wt}(M) \rangle = \varepsilon_i(M) + a + \langle h_i, \mathsf{wt}(M) \rangle = 0$$

by Theorem 6.8. Thus, $\varepsilon_i^*(\widetilde{f}_i M) > \lambda_3(h_i) = a$ by Corollary 6.6. This implies $\varepsilon_i^*(\widetilde{f}_i M) = a+1$ by the σ-version of Lemma 3.22(i). □

Corollary 6.11. *The* $\mathfrak{g}$*-crystal* $B(\infty)$ *is isomorphic to* $\mathbb{B}(\infty)$*.*

Proof. Apply Proposition 2.7 to $B = B(\infty)$ and $b_0 = [\mathbf{1}]$. □

Corollary 6.12. *For each* $\lambda \in P^+$*, the* $\mathfrak{g}$*-crystal* $B(\lambda)$ *is isomorphic to* $\mathbb{B}(\lambda)$*.*

Proof. Apply Proposition 2.8 to $B = B(\lambda)$, $b_\lambda = [\mathbf{1}_\lambda]$ and the map

$$\Phi : B(\infty) \otimes T_\lambda \to B(\lambda), \qquad [M] \otimes t_\lambda \mapsto [\mathsf{pr}^\lambda M].$$

The latter is an f-strict crystal morphism since $\widetilde{f}_i^\lambda = \mathsf{pr}^\lambda \circ \widetilde{f}_i \circ \mathsf{infl}^\lambda$ by Definition 4.3 and $\widetilde{f}_i M \neq 0$ for any $M \in B(\infty)$ by Definition 3.12. □

§6.5. Lie-theoretic descriptions of $K(\infty)_{\mathbb{Q}}$ and $K(\lambda)_{\mathbb{Q}}$

Theorem 6.13. *For each* $\lambda \in P^+$*, we have the following.*

(i) $K(\lambda)_{\mathbb{Q}}$ *has a left* $U_{\mathbb{Q}}(= \langle e_i, f_i, h_i \mid (2)\rangle_{i \in I_q})$*-module structure by*

$$e_i[M] = [e_i^\lambda M], \qquad f_i[M] = [f_i^\lambda M], \qquad h_i[M] = \langle h_i, \mathsf{wt}^\lambda(M)\rangle [M],$$

and it is isomorphic to the integrable highest weight $U_{\mathbb{Q}}$*-module of highest weight* λ *with highest weight vector* $[\mathbf{1}_\lambda]$*.*

(ii) *The symmetric non-degenerate bilinear form* $\langle\, , \rangle_\lambda$ *on* $K(\lambda)_{\mathbb{Q}}$ *from* §4.6 *coincides with the usual Shapovalov form satisfying* $\langle [\mathbf{1}_\lambda], [\mathbf{1}_\lambda]\rangle_\lambda = 1$ *under the above identification.*

(iii) $\bigoplus_{n \geq 0} \mathsf{K}_0(\mathsf{Proj}\, \mathcal{H}_n^\lambda) \cong K(\lambda)^* \subseteq K(\lambda)$ *are two integral lattices of* $K(\lambda)_{\mathbb{Q}}$ *containing* $[\mathbf{1}_\lambda]$ *with* $K(\lambda)^* = U_{\mathbb{Z}}^-[\mathbf{1}_\lambda]$ *and* $K(\lambda)$ *being its dual under the Shapovalov form.*

Proof. By §4.4 and Corollary 6.3, the operators $\{e_i^\lambda : K(\lambda) \to K(\lambda) \mid i \in I_q\}$ satisfy the Serre relations (15). This implies that the operators $\{f_i^\lambda : K(\lambda)^* \to K(\lambda)^* \mid i \in I_q\}$ satisfy the Serre relations by Lemma 4.13. Thus, both operators satisfy the Serre relations on $K(\lambda)_{\mathbb{Q}}$ by Theorem 4.16. By Corollary 4.11 and Theorem 6.8, we have $[e_i^\lambda, f_j^\lambda] = \delta_{i,j} h_i$ as operators on $K(\lambda)_{\mathbb{Q}}$. Since other relations of (2) are immediately deduced from the definition of the action of h_i, $K(\lambda)_{\mathbb{Q}}$ has a left $U_{\mathbb{Q}}$-module structure by the above actions. By Corollary 4.10, e_i^λ and f_i^λ are both nilpotent operators on $K(\lambda)_{\mathbb{Q}}$. Since the action of $\{h_i \mid i \in I_q\}$ is diagonalized with finite-dimensional weight spaces by the definition, $K(\lambda)_{\mathbb{Q}}$ is an integrable $U_{\mathbb{Q}}$-module. By Theorem 4.18, $K(\lambda)_{\mathbb{Q}} = U_{\mathbb{Q}}^-[\mathbf{1}_\lambda]$ is a highest weight $U_{\mathbb{Q}}$-module of highest weight λ with highest weight vector $[\mathbf{1}_\lambda]$. Now (ii) is a direct consequence of Lemma 4.13 and Corollary 4.19, and (iii) is a restatement of Theorem 4.16 and Corollary 4.18. □

Theorem 6.14. *There exists a graded $\mathbb{Z}$-Hopf algebra isomorphism $U_{\mathbb{Z}}^{+} \xrightarrow{\sim} K(\infty)^{*}$ which takes $e_i^{(r)}$ to $\delta_{L(i^r)}$ for each $i \in I_q$ and $r \geq 0$.*

Proof. By §3.9 and Corollary 6.3, there exists a graded $\mathbb{Z}$-algebra map $\pi : U_{\mathbb{Z}}^{+} \to K(\infty)^{*}$ which takes $e_i^{(r)}$ to $\delta_{L(i^r)}$ for each $i \in I_q$ and $r \geq 0$. It is easily checked that it is a graded $\mathbb{Z}$-coalgebra map since $\delta_{L(i)}$ is mapped to $\delta_{L(i)} \otimes 1 + 1 \otimes \delta_{L(i)}$ via the comultiplication of $K(\infty)^{*}$. Thus, π is a graded $\mathbb{Z}$-Hopf algebra map by [Swe, Lemma 4.0.4].

It is enough to show that π is an isomorphism of graded $\mathbb{Z}$-modules. By Corollary 6.6, we have a natural isomorphism $\varinjlim_{\lambda \in P^{+}} \mathsf{K}_0(\mathcal{H}_n^{\lambda}\text{-}\mathsf{smod}) \xrightarrow{\sim} \mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_n)$. Combined with Theorem 4.18, this gives us

$$\begin{aligned}\operatorname{Hom}_{\mathbb{Z}}(\mathsf{K}_0(\mathsf{Rep}\,\mathcal{H}_n), \mathbb{Z}) &\cong \varprojlim_{\lambda \in P^{+}} \operatorname{Hom}_{\mathbb{Z}}(\mathsf{K}_0(\mathcal{H}_n^{\lambda}\text{-}\mathsf{smod}), \mathbb{Z}) \\ &\cong \varprojlim_{\lambda \in P^{+}} \mathsf{K}_0(\mathsf{Proj}\,\mathcal{H}_n^{\lambda}) = \varprojlim_{\lambda \in P^{+}} (U_{\mathbb{Z}}^{-})_n[\mathbf{1}_{\lambda}] \xleftarrow{\sim} (U_{\mathbb{Z}}^{-})_n,\end{aligned}$$

where $(U_{\mathbb{Z}}^{-})_n$ is the set of homogeneous elements of $U_{\mathbb{Z}}^{-}$ of degree n via the principal grading, i.e., $\deg f_i^{(r)} = r$ for all $i \in I_q$ and $r \geq 0$. The last isomorphism follows easily from the fact $(U_{\mathbb{Z}}^{-})_n[\mathbf{1}_{\lambda}] \subseteq K(\lambda)_{\mathbb{Q}} \cong U_{\mathbb{Q}}^{-} / \sum_{i \in I} U_{\mathbb{Q}}^{-} f_i^{\lambda(h_i)+1}$ as shown in Theorem 6.13. By tracing this isomorphism, we see that the graded $\mathbb{Z}$-module isomorphism $K(\infty)^{*} \cong U_{\mathbb{Z}}^{-}$ is given by the composite

$$U_{\mathbb{Z}}^{-} \xrightarrow{\sim} U_{\mathbb{Z}}^{+} \xrightarrow{\pi} K(\infty)^{*}$$

where $U_{\mathbb{Z}}^{-} \xrightarrow{\sim} U_{\mathbb{Z}}^{+}$ is the algebra anti-isomorphism given by $f_i \mapsto e_i$ for all $i \in I_q$. See also the proof of [BK, Theorem 7.17] in [BK′, §3]. □

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