

# Non-negativity of the Fourier Coefficients of Eta Products Associated to Regular Systems of Weights

by

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## Abstract

We prove Saito's conjecture [9, Conjecture 13.5] about the non-negativity of the Fourier coefficients of the eta products associated to regular systems of weights.

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## §1. Introduction

Let  $W$  be a regular system of weights (see Section 2 for the definitions). Let  $\eta(\tau)$  be the Dedekind eta function. In [9], Kyoji Saito introduced an integer  $\nu_W$  and a holomorphic function  $\eta_W(\tau)$  on the upper half plane  $\mathfrak{H}$ . The function  $\eta_W(\tau)$  is a finite product of functions of the form  $\eta(a\tau)^b$  for some integers  $a, b$  with  $a \geq 1$ .

In [9], Saito made the following conjecture:

**Conjecture 1.1** ([9, Conjecture 13.5]). *Let  $W = (a, b, c, h)$  be a regular system of weights. Then for any integer  $\nu$ , all the coefficients in the  $q$ -expansion of  $\eta(h\tau)^\nu \eta_W(\tau)$  are non-negative if and only if  $\nu \leq \nu_W$ .*

The aim of this paper is to prove the following theorem.

**Theorem 1.2.** *Conjecture 1.1 is true.*

**Notation.** In this paper we use the following notation. Let  $\mathbb{Z}$ ,  $\mathbb{Q}$  and  $\mathbb{C}$  denote the ring of rational integers, the field of rational numbers, and the field of complex numbers, respectively. For an integer  $a \in \mathbb{Z}$ , we let  $\mathbb{Z}_{<a} \subset \mathbb{Z}$  denote the

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subset of the integers smaller than  $a$ . For integers  $a_1, \dots, a_r \in \mathbb{Z}$ , we denote by  $\gcd(a_1, \dots, a_r)$  (resp. by  $\text{lcm}(a_1, \dots, a_r)$ ) the greatest common divisor (resp. the least common multiple) of  $a_1, \dots, a_r$ .

**§2. Eta products associated to regular systems of weights**

We recall the notion of regular systems of weights, which was introduced by Kyoji Saito [8], and the definition of  $\eta_W(\tau)$  and  $\nu_W$  in the statement of Conjecture 1.1.

A *regular system of weights* is a quadruple  $W = (a, b, c, h)$  of integers  $a, b, c, h \in \mathbb{Z}$  satisfying the following two conditions:

- (1)  $1 \leq a, b, c < h$ .
- (2) The rational function

$$\chi_W(T) = T^{-h} \cdot \frac{(T^h - T^a)(T^h - T^b)(T^h - T^c)}{(T^a - 1)(T^b - 1)(T^c - 1)}$$

belongs to  $\mathbb{Z}[T, T^{-1}]$ .

According to [9, Section 1], we say that a regular system  $W = (a, b, c, h)$  of weights is *primitive* if it satisfies the following two conditions:

- (3)  $\gcd(a, b, c, h) = 1$ .
- (4) If  $h \in \{a + b, b + c, a + c\}$  then  $a + b + c = h + 1$ .

For a regular system  $W = (a, b, c, h)$  of weights, we use the notations  $\mu_W, \varphi_W(\lambda)$  of [9]. Let us briefly recall them. We put

$$\mu_W = \frac{(h - a)(h - b)(h - c)}{abc}.$$

It was shown in [8, (1.3) Theorem 1] that  $\mu_W$  is a positive integer and that there exist integers  $m_1, \dots, m_{\mu_W}$  such that  $\chi_W(T) = \sum_{j=1}^{\mu_W} T^{m_j}$ . We let  $\varphi_W(\lambda)$  denote the polynomial

$$\varphi_W(\lambda) = \prod_{j=1}^{\mu_W} (\lambda - e^{2\pi m_j \sqrt{-1}/h}).$$

It was shown in [9, Section 2] that all the coefficients of  $\varphi_W(\lambda)$  are integers. Hence for each  $d | h$  with  $d \geq 1$ , there exists a unique integer  $e_W(d)$  such that

$$\varphi_W(\lambda) = \prod_{d \geq 1, d|h} (\lambda^d - 1)^{e_W(d)}.$$

Let  $\mathfrak{H} = \{\tau \in \mathbb{C} \mid \text{Im } \tau > 0\}$  be the complex upper half plane. For  $a \in \mathbb{Q}$ , let  $q^a$  denote the holomorphic function  $\tau \mapsto e^{2\pi a \sqrt{-1}\tau}$  on  $\mathfrak{H}$ . Let  $\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n)$

be the Dedekind eta function. We define the holomorphic function  $\eta_W(\tau)$  to be the product

$$\eta_W(\tau) = \prod_{d \geq 1, d|h} \eta(d\tau)^{e_W(d)}.$$

We define the integer  $\nu_W$  as the sum

$$\nu_W = - \sum_{d \geq 1, d|h} de_W(h/d).$$

We easily see that, for each regular system  $W$  of weights, there exists a primitive regular system  $W'$  of weights and a positive integer  $d$  such that  $\eta_W(\tau) = \eta_{W'}(d\tau)$ . Therefore, in the proof of Theorem 1.2, we may assume that  $W$  is primitive.

### §3. A classification of primitive regular systems of weights

Let  $W = (a, b, c, h)$  be a primitive regular system of weights. As is remarked in [8, (1.6)], for any  $x \in \{a, b, c\}$ , there exists  $y \in \{a, b, c\}$  such that  $x$  divides  $h - y$ . If we choose such a  $y$  for each  $x \in \{a, b, c\}$ , then we have a map  $\phi$  from  $\{a, b, c\}$  to itself which sends  $x$  to  $y$ . We say that we are in *case  $i$*  if we can choose the map  $\phi$  so that the image of  $\phi$  has cardinality  $i$ . By permuting  $a, b, c$  if necessary, we may assume without loss of generality that we are in one of the following seven cases:

**Case 1:**  $\text{lcm}(a, b, c) \mid (h - a)$  and  $\text{gcd}(b, c) = 1$ .

**Case 2a:**  $\text{lcm}(b, c) \mid (h - a)$ ,  $a \mid (h - b)$  and  $\text{gcd}(b, c) = 1$ .

**Case 2b:**  $\text{lcm}(a, c) \mid (h - a)$  and  $b \mid (h - b)$ .

**Case 2c:**  $\text{lcm}(a, b) \mid (h - a)$  and  $c \mid (h - b)$ .

**Case 3a:**  $\text{lcm}(a, b, c) \mid h$ .

**Case 3b:**  $b \mid (h - a)$ ,  $a \mid (h - b)$  and  $c \mid (h - c)$ .

**Case 3c:**  $c \mid (h - a)$ ,  $a \mid (h - b)$  and  $b \mid (h - c)$ .

Here we explain the reason why the equality  $\text{gcd}(b, c) = 1$  holds in case 1. Suppose that we are in case 1 and that  $\text{gcd}(b, c) = d > 1$ . Let  $\zeta \in \mathbb{C}$  be a primitive  $d$ -th root of unity. The denominator  $(T^a - 1)(T^b - 1)(T^c - 1)$  in the definition of  $\chi_W(T)$  has vanishing order at least 2 at  $T = \zeta$ , while  $T^h - T^a$  has vanishing order 1. Hence  $T^h - T^b$  or  $T^h - T^c$  must have a zero at  $T = \zeta$ , which implies that  $d$  divides  $h - b$  or  $h - c$ . Since  $d$  divides  $b, c$  and  $h - a$ , it follows that  $d$  divides  $a, b, c$  and  $h$ . This gives a contradiction. We can check the equality  $\text{gcd}(b, c) = 1$  in case 2a by a similar argument.

We also mention without proof that the converse is true in the following sense, although we will not use it: if four integers  $a, b, c, h$  are in one of the seven cases

above, and if they satisfy conditions (1), (3) and (4) in the definition of a primitive regular system of weights, then  $W = (a, b, c, h)$  is a primitive regular system of weights.

**§4. A formula for  $\eta(h\tau)^{\nu w} \eta_W(\tau)$**

Let  $H_+$  denote the set of holomorphic functions  $f$  on  $\mathfrak{H}$  satisfying the following property:  $f$  is not identically zero and there exists an integer  $m \geq 1$  such that  $f$  is written as  $f(\tau) = \sum_{n \gg -\infty} a_n q^{n/m}$  with  $a_n \in \mathbb{Z}$ ,  $a_n \geq 0$  for all  $n \in \mathbb{Z}$ . Let  $H_+^0 \subset H_+$  denote the subset of the functions  $f = \sum_{n \gg -\infty} a_n q^{n/m} \in H_+$  satisfying  $\lim_{n \rightarrow \infty} a_n/n^k = 0$  for some  $k \geq 1$ . The set  $H_+$  forms a multiplicative monoid and  $H_+^0$  is a submonoid of  $H_+$ . The complement  $H_+ \setminus H_+^0$  is an ideal of  $H_+$  in the following sense: if  $f \in H_+ \setminus H_+^0$  and  $g \in H_+$  then  $fg \in H_+ \setminus H_+^0$ . In fact, let  $f, g$  in  $H_+$  and write  $f = \sum_{n \gg -\infty} a_n q^{n/m}$  and  $g = \sum_{n \gg -\infty} b_n q^{n/m'}$ . Take an integer  $n_0$  such that  $b_{n_0} > 0$ . Then the coefficient of  $q^{n/m+n_0/m'}$  in the  $q$ -expansion of  $fg$  is larger than or equal to  $a_n b_{n_0}$  for each  $n$ . Hence  $f \notin H_+^0$  implies  $fg \notin H_+^0$ .

For an integer  $\alpha \geq 1$ , we put

$$\eta_\alpha(\tau) = \frac{\eta(\alpha\tau)^\alpha}{\eta(\tau)}.$$

From the well-known relation between  $\eta_\alpha(\tau)$  and the generating function of the  $\alpha$ -cores ([3, (2.1)], originally due to [7]) or Klyachko’s identity ([6], see [3, Section 2] and [5] for different proofs), we see that  $\eta_\alpha(\tau) \in H_+^0$ . For integers  $\alpha_1, \dots, \alpha_m \geq 1$  such that  $\text{lcm}(\alpha_1, \dots, \alpha_m)$  divides  $h$ , we put

$$\eta_{[\alpha_1, \dots, \alpha_m; h]}(\tau) = \frac{\eta(h\tau)^{h - \sum_j \alpha_j} \prod_j \eta(h\tau/\alpha_j)}{\eta(\tau)}.$$

For a regular system of weights  $W = (a, b, c, h)$ , we put

$$\begin{aligned} f_{a,b,h} &= \eta_{\text{gcd}(a,b,h)}(h\tau/\text{gcd}(a, b, h)), \\ f_{b,c,h} &= \eta_{\text{gcd}(b,c,h)}(h\tau/\text{gcd}(b, c, h)), \\ f_{a,c,h} &= \eta_{\text{gcd}(a,c,h)}(h\tau/\text{gcd}(a, c, h)). \end{aligned}$$

**Lemma 4.1.** *Let  $W = (a, b, c, h)$  be a primitive regular system of weights.*

- (1) *In case 1, we have  $\eta(h\tau)^{\nu w} \eta_W(\tau) = f_{a,b,h}^{\frac{h-a}{\text{lcm}(a,b)}} f_{a,c,h}^{\frac{h-a}{\text{lcm}(a,c)}} \eta_{[a;h]}(\tau)$ .*
- (2) *In case 2a, we have  $\eta(h\tau)^{\nu w} \eta_W(\tau) = f_{a,b,h}^{\frac{h-a-b}{\text{lcm}(a,b)}} \eta_h(\tau)$ .*
- (3) *In case 2b, we have  $\eta(h\tau)^{\nu w} \eta_W(\tau) = f_{a,b,h}^{\frac{h}{\text{lcm}(a,b)}} f_{a,c,h}^{\frac{h-a}{\text{lcm}(a,c)}} \eta_{[a,b;h]}(\tau)$ .*
- (4) *In case 2c, we have  $\eta(h\tau)^{\nu w} \eta_W(\tau) = f_{a,b,h}^{\frac{h-a}{\text{lcm}(a,b)}} \eta_{[a;h]}(\tau)$ .*

- (5) In case 3a, we have  $\eta(h\tau)^{\nu w} \eta_W(\tau) = f_{a,b,h}^{\frac{h}{\text{lcm}(a,b)}} f_{b,c,h}^{\frac{h}{\text{lcm}(b,c)}} f_{a,c,h}^{\frac{h}{\text{lcm}(a,c)}} \eta_{[a,b,c;h]}(\tau)$ .
- (6) In case 3b, we have  $\eta(h\tau)^{\nu w} \eta_W(\tau) = f_{a,b,h}^{\frac{h-a-b}{\text{lcm}(a,b)}} \eta_{[c;h]}(\tau)$ .
- (7) In case 3c, we have  $\eta(h\tau)^{\nu w} \eta_W(\tau) = \eta_h(\tau)$ .

Before giving a proof of Lemma 4.1, we need some preparation. For  $f \in \mathbb{Q}[T, T^{-1}]$  let  $S_{(f)} \subset \mathbb{Q}[T, T^{-1}]$  denote the multiplicative set of the elements  $g \in \mathbb{Q}[T, T^{-1}]$  such that  $f$  and  $g$  have no common zero in  $\mathbb{C}$  except for  $T = 0$ . Let  $\mathbb{Q}[T, T^{-1}]_{(f)}$  denote the localization of  $\mathbb{Q}[T, T^{-1}]$  with respect to  $S_{(f)}$ . Then the canonical homomorphism  $\mathbb{Q}[T, T^{-1}] \rightarrow \mathbb{Q}[T, T^{-1}]_{(f)}$  induces an isomorphism  $\mathbb{Q}[T, T^{-1}]/(f) \xrightarrow{\cong} \mathbb{Q}[T, T^{-1}]_{(f)}/(f)$  of residue rings.

**Sublemma 4.2.** *Let  $i \geq 1$  be a positive integer and let  $n$  be a non-zero integer. Then  $\frac{T^i-1}{T^n-1}$  is in  $\mathbb{Q}[T, T^{-1}]_{(T^i-1)}$ , and*

$$\frac{T^i - 1}{T^n - 1} \equiv \frac{\text{gcd}(i, n)}{n} \cdot \frac{T^i - 1}{T^{\text{gcd}(i,n)} - 1} \pmod{(T^i - 1)}.$$

*Proof.* Since  $\frac{T^i-1}{T^n-1} = -T^n \cdot \frac{T^i-1}{T^n-1}$ , we may assume that  $n \geq 1$ . Put  $d = \text{gcd}(i, n)$ . We have  $\frac{T^i-1}{T^n-1} = \frac{T^i-1}{T^d-1} \cdot \frac{T^d-1}{T^n-1}$ . Since  $\frac{T^n-1}{T^d-1} = T^{n-d} + \dots + T^d + 1$ , we have

$$\frac{T^n - 1}{T^d - 1} \equiv \frac{n}{d} \pmod{(T^d - 1)}.$$

Hence  $\frac{T^n-1}{T^d-1} \in S_{(T^d-1)}$  and  $\frac{T^d-1}{T^n-1} \equiv \frac{d}{n} \pmod{(T^d-1)}$ . The  $\mathbb{Q}$ -linear map  $\mathbb{Q}[T, T^{-1}]_{(T^d-1)} \rightarrow \mathbb{Q}[T, T^{-1}]_{(T^i-1)}$  which sends  $f \in \mathbb{Q}[T, T^{-1}]_{(T^d-1)}$  to  $\frac{T^i-1}{T^d-1} \cdot f$  induces a  $\mathbb{Q}$ -linear map  $\mathbb{Q}[T, T^{-1}]_{(T^d-1)}/(T^d-1) \rightarrow \mathbb{Q}[T, T^{-1}]_{(T^i-1)}/(T^i-1)$ . Hence the claim follows.  $\square$

**Sublemma 4.3.** *Let  $i \geq 1$  be a positive integer and let  $d_1, d_2$  be positive divisors of  $i$ . Then*

$$\frac{T^i - 1}{T^{d_1} - 1} \cdot \frac{T^i - 1}{T^{d_2} - 1} \equiv \frac{i}{\text{lcm}(d_1, d_2)} \cdot \frac{T^i - 1}{T^{\text{gcd}(d_1, d_2)} - 1} \pmod{(T^i - 1)}.$$

*Proof.* We have  $\frac{T^i-1}{T^{d_2-1}} = \frac{T^i-1}{T^{d_1-1}} \cdot \frac{T^{d_1-1}}{T^{d_2-1}}$ . We put  $d = \text{gcd}(i, d)$ . By Sublemma 4.2, the element  $\frac{T^{d_1-1}}{T^{d_2-1}}$  is in  $\mathbb{Q}[T, T^{-1}]_{(T^{d_1-1})}$  and

$$\frac{T^{d_1} - 1}{T^{d_2} - 1} \equiv \frac{d}{d_2} \cdot \frac{T^{d_1} - 1}{T^d - 1} \pmod{(T^{d_1} - 1)}.$$

Hence

$$\frac{T^i - 1}{T^{d_2} - 1} \equiv \frac{i}{\text{lcm}(d_1, d_2)} \cdot \frac{T^{d_1} - 1}{T^d - 1} \pmod{(T^{d_1} - 1)}.$$

The  $\mathbb{Q}$ -linear map  $\mathbb{Q}[T, T^{-1}]_{(T^{d_1-1})} \rightarrow \mathbb{Q}[T, T^{-1}]_{(T^i-1)}$  which sends  $f \in \mathbb{Q}[T, T^{-1}]_{(T^{d_1-1})}$  to  $\frac{T^i-1}{T^{d_1-1}} \cdot f$  induces a  $\mathbb{Q}$ -linear map  $\mathbb{Q}[T, T^{-1}]_{(T^{d_1-1})}/(T^{d_1}-1) \rightarrow \mathbb{Q}[T, T^{-1}]_{(T^i-1)}/(T^i-1)$ . Hence the claim follows.  $\square$

*Proof of Lemma 4.1.* Let  $W = (a, b, c, h)$  be a primitive regular system of weights. By Sublemma 4.2 we have

$$\frac{T^h - T^x}{T^x - 1} \equiv \frac{\gcd(x, h)}{x} \cdot \frac{T^h - 1}{T^{\gcd(x, h)} - 1} - 1 \pmod{(T^h - 1)}$$

for  $x \in \{a, b, c\}$ . Hence  $\chi_W(T)$  is congruent modulo  $(T^h - 1)$  to

$$\begin{aligned} & \left( \frac{\gcd(a, h)}{a} \frac{T^h - 1}{T^{\gcd(a, h)} - 1} - 1 \right) \left( \frac{\gcd(b, h)}{b} \frac{T^h - 1}{T^{\gcd(b, h)} - 1} - 1 \right) \\ & \quad \times \left( \frac{\gcd(c, h)}{c} \frac{T^h - 1}{T^{\gcd(c, h)} - 1} - 1 \right). \end{aligned}$$

By Sublemma 4.3, we have

$$\frac{T^h - 1}{T^{\gcd(x, h)} - 1} \cdot \frac{T^h - 1}{T^{\gcd(y, h)} - 1} \equiv \frac{h \cdot \gcd(x, y, h)}{\gcd(x, h) \gcd(y, h)} \cdot \frac{T^h - 1}{T^{\gcd(x, y, h)} - 1} \pmod{(T^h - 1)}$$

for  $x, y \in \{a, b, c\}$  and

$$\begin{aligned} & \frac{T^h - 1}{T^{\gcd(a, h)} - 1} \cdot \frac{T^h - 1}{T^{\gcd(b, h)} - 1} \cdot \frac{T^h - 1}{T^{\gcd(a, h)} - 1} \\ & \quad \equiv \frac{h^2}{\gcd(a, h) \gcd(b, h) \gcd(c, h)} \cdot \frac{T^h - 1}{T - 1} \pmod{(T^h - 1)}. \end{aligned}$$

Hence

$$\begin{aligned} (4.1) \quad \chi_W(T) & \equiv \frac{h^2}{abc} \frac{T^h - 1}{T - 1} - \frac{h \gcd(a, b, h)}{ab} \frac{T^h - 1}{T^{\gcd(a, b, h)} - 1} \\ & \quad - \frac{h \gcd(b, c, h)}{bc} \frac{T^h - 1}{T^{\gcd(b, c, h)} - 1} - \frac{h \gcd(a, c, h)}{ac} \frac{T^h - 1}{T^{\gcd(a, c, h)} - 1} \\ & \quad + \frac{\gcd(a, h)}{a} \frac{T^h - 1}{T^{\gcd(a, h)} - 1} + \frac{\gcd(b, h)}{b} \frac{T^h - 1}{T^{\gcd(b, h)} - 1} \\ & \quad + \frac{\gcd(c, h)}{c} \frac{T^h - 1}{T^{\gcd(c, h)} - 1} - 1 \pmod{(T^h - 1)}. \end{aligned}$$

This implies

$$\begin{aligned} (4.2) \quad \nu_W & = -\frac{h^2}{abc} + \frac{h \gcd(a, b, h)^2}{ab} + \frac{h \gcd(b, c, h)^2}{bc} + \frac{h \gcd(a, c, h)^2}{ac} \\ & \quad - \frac{\gcd(a, h)^2}{a} - \frac{\gcd(b, h)^2}{b} - \frac{\gcd(c, h)^2}{c} + h. \end{aligned}$$

We note that each coefficient on the right hand side of (4.1) is a rational number and may not be an integer. In each of the seven cases listed in Section 3, we can rewrite the right hand side of (4.1) in such a way that each term has coefficients in  $\mathbb{Z}$ , and then the claim of Lemma 4.1 follows from this new description of  $\chi_W(T) \pmod{(T^h - 1)}$ . In the next two paragraphs we explain the details in case 3a and case 3b, which gives proofs of the claims (5) and (6). The other five claims can be proved in a similar manner and the details are left to the reader.

Suppose that we are in case 3a. We have  $\gcd(a, h) = a$ ,  $\gcd(b, h) = b$ , and  $\gcd(c, h) = c$ . Hence  $\gcd(a, b, h) = \gcd(a, b)$ ,  $\gcd(b, c, h) = \gcd(b, c)$ , and  $\gcd(a, c, h) = \gcd(a, c)$ . By (4.1) and (4.2) we have

$$(4.3) \quad \chi_W(T) \equiv \frac{h^2}{abc} \frac{T^h - 1}{T - 1} - \frac{h}{\text{lcm}(a, b)} \frac{T^h - 1}{T^{\gcd(a, b)} - 1} - \frac{h}{\text{lcm}(b, c)} \frac{T^h - 1}{T^{\gcd(b, c)} - 1} - \frac{h}{\text{lcm}(a, c)} \frac{T^h - 1}{T^{\gcd(a, c)} - 1} + \frac{T^h - 1}{T^a - 1} + \frac{T^h - 1}{T^b - 1} + \frac{T^h - 1}{T^c - 1} - 1 \pmod{(T^h - 1)}$$

and

$$\nu_W = -\frac{h^2}{abc} - a - b - c + \frac{h \gcd(a, b)}{\text{lcm}(a, b)} + \frac{h \gcd(b, c)}{\text{lcm}(b, c)} + \frac{h \gcd(a, c)}{\text{lcm}(a, c)} + h.$$

We note that every term on the right hand side of (4.3) has coefficients in  $\mathbb{Z}$ . Hence

$$\varphi_W(\lambda) = \frac{(\lambda^h - 1)^{h^2/abc} (\lambda^{h/a} - 1) (\lambda^{h/b} - 1) (\lambda^{h/c} - 1)}{(\lambda^{\frac{h}{\gcd(a, b)}} - 1)^{\frac{h}{\text{lcm}(a, b)}} (\lambda^{\frac{h}{\gcd(b, c)}} - 1)^{\frac{h}{\text{lcm}(b, c)}} (\lambda^{\frac{h}{\gcd(a, c)}} - 1)^{\frac{h}{\text{lcm}(a, c)}} (\lambda - 1)}$$

and

$$\eta_W(\tau) = \frac{\eta(h\tau)^{h^2/abc} \eta(h\tau/a) \eta(h\tau/b) \eta(h\tau/c)}{\eta\left(\frac{h\tau}{\gcd(a, b)}\right)^{\frac{h}{\text{lcm}(a, b)}} \eta\left(\frac{h\tau}{\gcd(b, c)}\right)^{\frac{h}{\text{lcm}(b, c)}} \eta\left(\frac{h\tau}{\gcd(a, c)}\right)^{\frac{h}{\text{lcm}(a, c)}} \eta(\tau)}.$$

Therefore

$$\eta(h\tau)^{\nu_W} \eta_W(\tau) = f_{a, b, h}^{\frac{h}{\text{lcm}(a, b)}} f_{b, c, h}^{\frac{h}{\text{lcm}(b, c)}} f_{a, c, h}^{\frac{h}{\text{lcm}(a, c)}} \eta_{[a, b, c; h]}(\tau).$$

Suppose that we are in case 3b. We have  $\gcd(a, h) = \gcd(b, h) = \gcd(a, b)$  and  $\gcd(c, h) = c$ . Hence  $\gcd(a, b, h) = \gcd(a, b)$  and  $\gcd(b, c, h) = \gcd(a, c, h) = 1$ . By (4.1) we have

$$(4.4) \quad \chi_W(T) \equiv \frac{h^2}{abc} \frac{T^h - 1}{T - 1} - \frac{h}{\text{lcm}(a, b)} \frac{T^h - 1}{T^{\gcd(a, b)} - 1} - \frac{h}{bc} \frac{T^h - 1}{T - 1} - \frac{h}{ac} \frac{T^h - 1}{T - 1} + \frac{\gcd(a, b)}{a} \frac{T^h - 1}{T^{\gcd(a, b)} - 1} + \frac{\gcd(a, b)}{b} \frac{T^h - 1}{T^{\gcd(a, b)} - 1} + \frac{T^h - 1}{T^c - 1} - 1 \equiv \frac{h(h - a - b)}{abc} \frac{T^h - 1}{T - 1} - \frac{h - a - b}{\text{lcm}(a, b)} \frac{T^h - 1}{T^{\gcd(a, b)} - 1} + \frac{T^h - 1}{T^c - 1} - 1 \pmod{(T^h - 1)}$$

and

$$\nu_W = -\frac{h(h-a-b)}{abc} + \frac{(h-a-b)\gcd(a,b)}{\text{lcm}(a,b)} - c + h.$$

We note that every term in the last line of (4.4) has coefficients in  $\mathbb{Z}$ . Hence

$$\varphi_W(\lambda) = \frac{(\lambda^h - 1)^{h(h-a-b)/abc}(\lambda^{h/c} - 1)}{(\lambda^{h/\gcd(a,b)} - 1)^{(h-a-b)/\text{lcm}(a,b)}(\lambda - 1)}.$$

and

$$\eta_W(\tau) = \frac{\eta(h\tau)^{h(h-a-b)/abc}\eta(h\tau/c)}{\eta(h\tau/\gcd(a,b))^{(h-a-b)/\text{lcm}(a,b)}\eta(\tau)}.$$

Therefore

$$\eta(h\tau)^{\nu_W}\eta_W(\tau) = f_{a,b,h}^{\frac{h-a-b}{\text{lcm}(a,b)}}\eta_{[c;h]}(\tau). \quad \square$$

**Proposition 4.4.** *Let  $r \geq 2$  and let  $a_1, \dots, a_r \geq 1$  be positive integers satisfying  $\gcd(a_i, a_j) = 1$  for  $1 \leq i < j \leq \max(r-1, 2)$ . Put  $h = \prod_{j=1}^r a_j$ . Then the eta product*

$$\eta_{(a_1, \dots, a_r)}(\tau) = \frac{\eta(h\tau)^{\prod_{j=1}^r (a_j-1)}}{\prod_{J \subset \{1, \dots, r\}} \eta((\prod_{j \in J} a_j)\tau)^{(-1)^{\#J}}}$$

belongs to  $H_+^0$ . Here we understand  $\prod_{j \in J} a_j = 1$  for  $J = \emptyset$ .

*Proof.* We proceed in a manner similar to [1, Section 3]. If  $a_j = 1$  for some  $j$ , we easily see that  $\eta_{(a_1, \dots, a_r)} = 1$ . We assume  $a_j \geq 2$  for all  $j$ . If  $r = 2$ , we may assume without loss of generality that  $a_1$  is odd. We put  $h' = \prod_{j=1}^{r-1} a_j$ . Let

$$M = \{j \in \mathbb{Z} \mid 1 \leq j < h'/2, a_i \nmid j \text{ for } i = 1, \dots, r-1\}.$$

Then

$$\begin{aligned} &\eta_{(a_1, \dots, a_r)}(\tau) \\ &= q^{b/24} \prod_{j \in M} \left( \eta_{a_r}(h'\tau)\eta(h'\tau)\eta(h\tau)^{a_r-2} \prod_{n \geq 1} \frac{(1 - q^{a_r j} q^{(n-1)h})(1 - q^{-a_r j} q^{nh})}{(1 - q^j q^{(n-1)h'})(1 - q^{-j} q^{nh'})} \right), \end{aligned}$$

where  $b = (h - (-1)^r) \prod_{j=1}^r (a_j - 1) - h(2a_r - 2)$ . Hence the claim follows from [1, Theorem 1.2]. □

### §5. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 assuming the following theorem, whose proof will be given in Section 6.3.

**Theorem 5.1.** *Let  $h \geq 1$  be an integer.*

- (1) *For any  $a \geq 1$  dividing  $h$ , we have  $\eta_{[a;h]}(\tau) \in H_+^0$ .*



- (2) For any  $a, b \geq 1$  such that  $h > \max(a, b)$  and  $\text{lcm}(a, b)$  divides  $h$ , we have  $\eta_{[a,b;h]}(\tau) \in H_+^0$ .
- (3) Let  $a, b, c \geq 1$  be three integers such that  $\text{lcm}(a, b, c)$  divides  $h$ . Suppose that there exist  $\alpha, \beta, \gamma \in \mathbb{Z}$  satisfying  $\alpha \not\equiv \beta \pmod{\frac{h}{\text{lcm}(a,b)}}$ ,  $\beta \not\equiv \gamma \pmod{\frac{h}{\text{lcm}(b,c)}}$ , and  $\alpha \not\equiv \gamma \pmod{\frac{h}{\text{lcm}(a,c)}}$ . Then  $\eta_{[a,b,c;h]}(\tau) \in H_+^0$ .

**Corollary 5.2.** Let  $W = (a, b, c, h)$  be a primitive regular system of weights. Then  $\eta(h\tau)^{\nu_W} \eta_W(\tau) \in H_+^0$ .

*Proof of Corollary 5.2.* If  $\text{lcm}(a, b, c) \nmid h$ , then the claim follows immediately from Lemma 4.1 and Theorem 5.1. Thus we may assume that  $\text{lcm}(a, b, c) \mid h$ .

First suppose that all of  $\frac{h}{\text{lcm}(a,b)}, \frac{h}{\text{lcm}(b,c)}, \frac{h}{\text{lcm}(a,c)}$  are greater than one and  $\max(\frac{h}{\text{lcm}(a,b)}, \frac{h}{\text{lcm}(b,c)}, \frac{h}{\text{lcm}(a,c)}) \geq 3$ . By permuting  $a, b, c$  if necessary, we may assume that  $\frac{h}{\text{lcm}(a,c)} \geq 3$ . Then the claim follows from Lemma 4.1(5) and Theorem 5.1(3) for  $(\alpha, \beta, \gamma) = (-1, 0, 1)$ .

Next suppose that  $\min(\frac{h}{\text{lcm}(a,b)}, \frac{h}{\text{lcm}(b,c)}, \frac{h}{\text{lcm}(a,c)}) = 1$ . In this case, we have  $h = \text{lcm}(a, b, c)$ . We may assume that  $h/\text{lcm}(a, b) = 1$ . We put  $d = \text{gcd}(a, b)$ ,  $a' = a/d, b' = b/d, a'_1 = \text{gcd}(a, c)$ , and  $b'_1 = \text{gcd}(b, c)$ . We have  $h = a'b'd$ . Since  $\text{gcd}(a, b, c) = \text{gcd}(a, b, c, h) = 1$ , any two of  $d, a'_1$ , and  $b'_1$  are relatively prime. Hence the quotients  $a'_2 = a'/a'_1 = a/(da'_1)$  and  $b'_2 = b'/b'_1 = b/(db'_1)$  are integers and we have  $a'_1 b'_1 = \text{gcd}(ab, c)$ . Since  $c$  divides  $h$  and  $h$  divides  $ab$ , we have  $a'_1 b'_1 = c$ . Since we are in case 3a and since  $\text{lcm}(a, c) = ac/a'_1 = a'b'_1 d$  and  $\text{lcm}(b, c) = bc/b'_1 = a'_1 b'd$ , it follows from Lemma 4.1(5) that

$$\eta(h\tau)^{\nu_W} \eta_W(\tau) = f_{a,b,h} f_{b,c,h}^{a'_2} f_{a,c,h}^{b'_2} \eta_{[a,b,c;h]}(\tau).$$

Since

$$\begin{aligned} & f_{a,b,h} f_{b,c,h} f_{a,c,h} \eta_{[a,b,c;h]}(\tau) \\ &= \eta_d(a'b'\tau) \eta_{b'_1}(a'b'_2 d\tau) \eta_{a'_1}(a'_2 b'd\tau) \cdot \frac{\eta(h\tau)^{h-a-b-a'_1 b'_1} \eta(a'\tau) \eta(b'\tau) \eta(a'_2 b'_2 d\tau)}{\eta(\tau)} \\ &= \frac{\eta(h\tau)^{(a'-1)(b'-1)d+a'_1+b'_1-a'_1 b'_1} \eta(a'\tau) \eta(b'\tau) \eta(a'_2 b'_2 d\tau)}{\eta(\tau) \eta(a'b'\tau) \eta(a'b'_2 d\tau) \eta(a'_2 b'd\tau)} \\ &= \eta_{\langle a', b', d \rangle}(\tau) \cdot \frac{\eta(h\tau)^{(a'-1)(b'-1)-(a'_1-1)(b'_1-1)} \eta(a\tau) \eta(b\tau) \eta(a'_2 b'_2 d\tau)}{\eta(d\tau) \eta(a'b'_2 d\tau) \eta(a'_2 b'd\tau)} \\ &= \eta_{\langle a', b', d \rangle}(\tau) \eta_{\langle a'_1, b'_1, a'_2 b'_2 \rangle}(d\tau) \cdot \frac{\eta(h\tau)^{a'b'_2+a'_2 b'-a'-b'-a'_2 b'_2} \eta(a\tau) \eta(b\tau) \eta(a'_1 b'_1 d\tau)}{\eta(a'_1 d\tau) \eta(b'_1 d\tau)} \\ &= \eta_{\langle a', b', d \rangle}(\tau) \eta_{\langle a'_1, b'_1, a'_2 b'_2 \rangle}(d\tau) \eta_{[a'_2 b'_2, b'; a'_2 b']}(a'_1 d\tau) \eta_{[a'; a' b'_2]}(b'_1 d\tau), \end{aligned}$$

we have

$$(5.1) \quad \eta(h\tau)^{\nu w} \eta_W(\tau) = f_{b,c,h}^{a'_2-1} f_{a,c,h}^{b'_2-1} \eta_{\langle a',b',d \rangle}(\tau) \eta_{\langle a'_1,b'_1,a'_2b'_2 \rangle}(d\tau) \\ \times \eta_{[a'_2b'_2,b';a'_2b']}(a'_1 d\tau) \eta_{[a';a'b'_2]}(b'_1 d\tau).$$

Hence if  $a'_2 \geq 2$  and  $b'_1 \geq 2$ , then  $\eta(h\tau)^{\nu w} \eta_W(\tau) \in H_+^0$  by Proposition 4.4 and Theorem 5.1. Interchanging the roles of  $a$  and  $b$ , we see that  $\eta(h\tau)^{\nu w} \eta_W(\tau) \in H_+^0$  if  $a'_1 \geq 2$  and  $b'_2 \geq 2$ . If  $a'_1 = a'_2 = 1$  (resp.  $b'_1 = b'_2 = 1$ ), then  $h = b$  (resp.  $h = a$ ), which is a contradiction. If  $a'_2 = b'_2 = 1$ , then  $\eta_{\langle a'_1,b'_1,a'_2b'_2 \rangle} = \eta_{[a'_2b'_2,b';a'_2b]} = \eta_{[a';a'b'_2]} = 1$ . It follows from (5.1) that  $\eta(h\tau)^{\nu w} \eta_W(\tau) = \eta_{\langle a',b',d \rangle}(\tau)$ . Hence  $\eta(h\tau)^{\nu w} \eta_W(\tau) \in H_+^0$  by Proposition 4.4. If  $a'_1 = b'_1 = 1$  and  $a'_2, b'_2 \geq 2$ , then  $a'_2 = a', b'_2 = b'$  and  $f_{b,c,h} = f_{a,c,h} = \eta_{\langle a'_1,b'_1,a'_2b'_2 \rangle} = 1$ . It follows from (5.1) that

$$\eta(h\tau)^{\nu w} \eta_W(\tau) = \eta_{\langle a',b',d \rangle}(\tau) \eta_{[a'b',b';a'b']}(d\tau) \eta_{[a';a'b']}(d\tau) \\ = \frac{\eta(h\tau)^{(a'-1)(b'-1)(d-1)}}{\eta(\tau) \eta(a'\tau)^{-1} \eta(b'\tau)^{-1} \eta(d\tau)^{-1} \eta(a\tau) \eta(b\tau) \eta(a'b'\tau) \eta(h\tau)^{-1}} \\ \times \frac{\eta(a\tau)}{\eta(h\tau)^{b'}} \cdot \frac{\eta(h\tau)^{a'b'-a'} \eta(b\tau)}{\eta(d\tau)} \\ = \frac{\eta(h\tau)^{(a'-1)(b'-1)d} \eta(a'\tau) \eta(b'\tau)}{\eta(\tau) \eta(a'b'\tau)} \\ = \eta_{[a,b;h]}(\tau) \eta_d(a'b'\tau).$$

Hence  $\eta(h\tau)^{\nu w} \eta_W(\tau) \in H_+^0$  by Theorem 5.1(2).

Next suppose that  $\frac{h}{\text{lcm}(a,b)} = \frac{h}{\text{lcm}(b,c)} = \frac{h}{\text{lcm}(a,c)} = 2$ . Then  $h = 2 \cdot \text{lcm}(a, b, c)$ . We put  $a'_3 = \text{lcm}(a, b, c)/a$ ,  $b'_3 = \text{lcm}(a, b, c)/b$ , and  $c'_3 = \text{lcm}(a, b, c)/c$ . Then  $\text{gcd}(a'_3, b'_3) = \text{gcd}(b'_3, c'_3) = \text{gcd}(a'_3, c'_3) = 1$  and  $\text{lcm}(a, b, c) = a'_3 b'_3 c'_3$ . It follows from Lemma 4.1(5) that

$$\eta(h\tau)^{\nu w} \eta_W(\tau) = f_{a,b,h}^2 f_{b,c,h}^2 f_{a,c,h}^2 \eta_{[a,b,c;h]}(\tau).$$

Since

$$f_{a,b,h} f_{b,c,h} f_{a,c,h} \eta_{[a,b,c;h]}(\tau) \\ = \eta_{c'_3}(2a'_3 b'_3 \tau) \eta_{b'_3}(2a'_3 c'_3 \tau) \eta_{a'_3}(2b'_3 c'_3 \tau) \cdot \frac{\eta(h\tau)^{h-a-b-c} \eta(2a'_3 \tau) \eta(2b'_3 \tau) \eta(2c'_3 \tau)}{\eta(\tau)} \\ = \frac{\eta(h\tau)^{h-a-b-c+a'_3+b'_3+c'_3} \eta(2a'_3 \tau) \eta(2b'_3 \tau) \eta(2c'_3 \tau)}{\eta(\tau) \eta(2a'_3 b'_3 \tau) \eta(2b'_3 c'_3 \tau) \eta(2a'_3 c'_3 \tau)} \\ = \eta_{\langle a'_3, b'_3, c'_3 \rangle}(2\tau) \cdot \frac{\eta(h\tau)^{h/2} \eta(2\tau)}{\eta(\tau)} = \eta_{\langle a'_3, b'_3, c'_3 \rangle}(2\tau) \eta_{[h/2;h]}(\tau),$$

we have

$$\eta(h\tau)^{\nu w} \eta_W(\tau) = f_{a,b,h} f_{b,c,h} f_{a,c,h} \eta_{\langle a'_3, b'_3, c'_3 \rangle}(2\tau) \eta_{[h/2;h]}(\tau).$$

Hence  $\eta(h\tau)^{\nu w} \eta_W(\tau) \in H_+^0$  by Proposition 4.4 and Theorem 5.1(1). This completes the proof.  $\square$

*Proof of Theorem 1.2.* The “if” part follows from Corollary 5.2. Let  $W$  be a regular system of weights. To prove the “only if” part for  $W$ , it suffices to prove that  $\eta(h\tau)^{\nu w+1} \eta_W(\tau) \notin H_+$ . Assume that  $\eta(h\tau)^{\nu w+1} \eta_W(\tau) \in H_+$ . The asymptotics  $p(n) \sim e^{\pi\sqrt{2n/3}}/(4n\sqrt{3})$  of the partition function  $p(n)$  (see [4], [10]) shows that  $\eta(h\tau)^{-1} \in H_+ \setminus H_+^0$ . Hence  $\eta(h\tau)^{\nu w} \eta_W(\tau) = \eta(h\tau)^{-1} \cdot \eta(h\tau)^{\nu w+1} \eta_W(\tau) \in H_+ \setminus H_+^0$ . This gives a contradiction. Thus  $\eta(h\tau)^{\nu w+1} \eta_W(\tau) \notin H_+$ .  $\square$

### §6. Proof of Theorem 5.1

#### §6.1. AP-coverings

An *AP-subset* is a subset  $L \subset \mathbb{Z}$  of the form  $L = a + b\mathbb{Z}$  for some  $a, b \in \mathbb{Z}$  with  $b > a \geq 0$ . The integers  $a, b$  are uniquely determined by  $L$  and are denoted by  $a(L), i(L)$ , respectively. For an AP-subset  $L$ , let  $\psi_L : \mathbb{Z} \rightarrow \mathbb{Z}$  denote the unique order-preserving injection whose image is equal to  $L$  and which sends  $0 \in \mathbb{Z}$  to  $a(L)$ . An *AP-covering* of  $\mathbb{Z}$  is a family  $\mathfrak{L} = \{L_j\}_{j \in J}$  of AP-subsets satisfying  $\mathbb{Z} = \coprod_{j \in J} L_j$ .

We put  $E(q) = q^{-1/24} \eta(\tau) = \prod_{n \geq 1} (1 - q^n)$ .

**Proposition 6.1.** *Let  $\mathfrak{L} = \{L_j\}_{j \in J}$  be an AP-covering of  $\mathbb{Z}$ . Then the function*

$$E_{\mathfrak{L}}(\tau) = \frac{\prod_{j \in J} E(q^{i(L_j)})}{E(q)}$$

*belongs to  $H_+$ . Moreover if  $J$  is a finite set, then  $E_{\mathfrak{L}}(\tau)$  belongs to  $H_+^0$ .*

We make the following conjecture which generalizes Proposition 6.1.

**Conjecture 6.2.** *Let  $J$  be a finite or countable set. Suppose that for each  $j \in J$  a positive integer  $m_j \geq 1$  is given such that  $\sum_{j \in J} 1/m_j \leq 1$ . Then  $\frac{\prod_{j \in J} E(q^{m_j})}{E(q)} \in H_+$ . Moreover, if  $J$  is a finite set and  $\sum_{j \in J} 1/m_j = 1$ , then  $\frac{\prod_{j \in J} E(q^{m_j})}{E(q)} \in H_+^0$ .*

We note that Corollary 5.2 is an immediate consequence of Lemma 4.1 if we assume Conjecture 6.2.

#### §6.2. Maya diagrams

We prove Proposition 6.1 by using the notion of Maya diagrams. Our argument can be regarded as a generalization of the argument in [3, Section 2]. A *Maya*

*diagram* (cf. [2, §4.1]) is a subset  $S \subset \mathbb{Z}$  such that  $x \in S$  for  $x \ll 0$  and  $x \notin S$  for  $x \gg 0$ . A Maya diagram  $S$  is said to be of *minimum energy* if there exists  $a \in \mathbb{Z}$  such that  $S = \mathbb{Z}_{<a}$ . We say that two Maya diagrams  $S, S'$  are *equivalent* if  $S' = S + x$  for some  $x \in \mathbb{Z}$ . Let  $S$  be a Maya diagram and take a subset  $S'$  of  $S \cap \mathbb{Z}_{<0}$  whose complement  $(S \cap \mathbb{Z}_{<0}) \setminus S'$  is a finite set. The integer

$$c(S) = \#(S \setminus S') - \#(S \setminus \mathbb{Z}_{<0})$$

depends only on  $S$  and is independent of the choice of  $S'$ . We call the integer  $e(S)$  the *charge* of  $S$ . A Maya diagram  $S$  is said to be of *charge zero* if  $c(S) = 0$ . For a Maya diagram  $S$ , we let  $S^\dagger$  denote the unique Maya diagram of minimal energy with  $c(S) = c(S^\dagger)$ . Explicitly  $S^\dagger = \mathbb{Z}_{<c(S)}$ . For any Maya diagram  $S$ , there exists a unique Maya diagram, which we denote by  $S^b$ , of charge zero which is equivalent to  $S$ , since  $c(S + x) = c(S) + x$  for  $x \in \mathbb{Z}$ .

Let  $S$  be a Maya diagram and take a subset  $S'$  of  $S \cap S^\dagger$  whose complement  $(S \cap S^\dagger) \setminus S'$  is a finite set. The integer

$$e(S) = \sum_{y \in S \setminus S'} y - \sum_{y \in S^\dagger \setminus S'} y$$

depends only on  $S$  and is independent of the choice of  $S'$ . We call it the *energy* of  $S$ . It follows immediately from the definition that  $e(S) \geq 0$ , and equality holds if and only if  $S = S^\dagger$ . For each integer  $n \geq 0$ , there exists a canonical one-to-one correspondence between the partitions of  $n$  and Maya diagrams  $S$  of charge zero with  $e(S) = n$ . For a given Maya diagram  $M = \{m_1, m_2, \dots\}$  with  $m_1 > m_2 > \dots$  of charge zero with  $e(S) = n$ , the corresponding partition of  $n$  is given as follows. For  $i = 1, 2, \dots$  we put  $n_i = m_i + i$ . We have  $n_1 \geq n_2 \geq \dots$  and  $n_i = 0$  for  $i \gg 0$ . Let  $r \geq 0$  denote the smallest non-negative integer such that  $n_{i+1} = 0$ . Then  $n = n_1 + \dots + n_r$ , which gives the partition corresponding to  $M$ . Hence

$$(6.1) \quad E(q)^{-1} = \sum_{S \in \mathbf{Maya}^0} q^{e(S)},$$

where  $\mathbf{Maya}^0$  denotes the set of Maya diagrams of charge zero.

*Proof of Proposition 6.1.* Let  $\mathfrak{L} = (L_j)_{j \in J}$  be an AP-covering of  $\mathbb{Z}$ . Let  $S$  be a Maya diagram. For each  $j \in J$ , the inverse image  $\psi_{L_j}^{-1}(S)$  is also a Maya diagram. We say that  $S$  is  $\mathfrak{L}$ -*reduced* if  $\psi_{L_j}^{-1}(S)$  is of minimum energy for every  $j \in J$ . For a Maya diagram  $S$ , we let  $S_{\mathfrak{L}}$  denote the unique Maya diagram which is  $\mathfrak{L}$ -reduced and  $c(\psi_{L_j}^{-1}(S)) = c(\psi_{L_j}^{-1}(S_{\mathfrak{L}}))$  for every  $j \in J$ . Explicitly,

$$S_{\mathfrak{L}} = \prod_{j \in J} \psi_{L_j}(\psi_{L_j}^{-1}(S)^\dagger).$$

Since  $\psi_{L_j}^{-1}(\mathbb{Z}_{<0}) = \mathbb{Z}_{<0}$ , it follows from the definition of  $c(S)$  that  $c(S) = \sum_{j \in J} c(\psi_{L_j}^{-1}(S))$ . Hence  $c(S) = c(S_{\mathfrak{L}})$ .

We let  $\mathbf{Maya}_{\mathfrak{L}}^0$  denote the set of  $\mathfrak{L}$ -reduced Maya diagrams of charge zero. Let

$$F_{\mathfrak{L}} : \mathbf{Maya}^0 \rightarrow \mathbf{Maya}_{\mathfrak{L}}^0 \times \prod_{j \in J} \mathbf{Maya}^0$$

denote the map which sends  $S \in \mathbf{Maya}^0$  to  $(S_{\mathfrak{L}}, ((\psi_{L_j}^{-1}(S))^b)_{j \in J})$ . We claim that  $F_{\mathfrak{L}}$  is bijective. For  $\mathcal{S} = (S', (S_j)_{j \in J}) \in \mathbf{Maya}_{\mathfrak{L}}^0 \times \prod_{j \in J} \mathbf{Maya}^0$ , we put

$$G_{\mathfrak{L}}(\mathcal{S}) = \prod_{j \in J} \psi_{L_j}(S_j + c(\psi_{L_j}^{-1}(S'))),$$

which is a Maya diagram. We have

$$c(G_{\mathfrak{L}}(\mathcal{S})) = \sum_{j \in J} c(S_j + c(\psi_{L_j}^{-1}(S'))) = \sum_{j \in J} c(\psi_{L_j}^{-1}(S')) = c(S') = 0.$$

Hence  $G_{\mathfrak{L}}(\mathcal{S}) \in \mathbf{Maya}^0$ . It is immediate from the definition of  $G_{\mathfrak{L}}$  that  $F_{\mathfrak{L}}(G_{\mathfrak{L}}(\mathcal{S})) = \mathcal{S}$ . For  $S \in \mathbf{Maya}^0$ , we have  $G_{\mathfrak{L}}(F_{\mathfrak{L}}(S)) = S$  since  $(\psi_{L_j}^{-1}(S))^{\dagger} + c(\psi_{L_j}^{-1}(S_{\mathfrak{L}})) = \psi_{L_j}^{-1}(S)$  or each  $j \in J$ . Since the map  $G_{\mathfrak{L}}$  is the inverse of  $F_{\mathfrak{L}}$ , the map  $F_{\mathfrak{L}}$  is bijective.

Let  $S \in \mathbf{Maya}^0$  and take a subset  $S'$  of  $S \cap S_{\mathfrak{L}} \cap \mathbb{Z}_{<0}$  whose complement is a finite set. Then

$$\begin{aligned} e(S) - e(S_{\mathfrak{L}}) &= \sum_{j \in J} \left( \sum_{y \in (S \setminus S') \cap L_j} y - \sum_{y \in (S_{\mathfrak{L}} \setminus S') \cap L_j} y \right) \\ &= \sum_{j \in J} i(L_j) \cdot \left( \sum_{y \in \psi_{L_j}^{-1}(S) \setminus \psi_{L_j}^{-1}(S')} y - \sum_{y \in (\psi_{L_j}^{-1}(S_{\mathfrak{L}}) \setminus \psi_{L_j}^{-1}(S'))} y \right) \\ &= \sum_{j \in J} i(L_j) e(\psi_{L_j}^{-1}(S)). \end{aligned}$$

Hence if we put  $F_{\mathfrak{L}}(S) = (S_{\mathfrak{L}}, (S_j)_{j \in J})$ , then

$$(6.2) \quad e(S) = e(S_{\mathfrak{L}}) + \sum_{j \in J} i(L_j) e(S_j).$$

Since the map  $F_{\mathfrak{L}}$  is bijective, it follows from the equalities (6.1) and (6.2) that

$$E(q)^{-1} = \left( \sum_{S' \in \mathbf{Maya}_{\mathfrak{L}}^0} q^{e(S')} \right) \cdot \prod_{j \in J} E(q^{i(L_j)})^{-1}.$$

Hence

$$E_{\mathfrak{L}}(\tau) = \sum_{S' \in \mathbf{Maya}_{\mathfrak{L}}^0} q^{e(S')} \in H_+.$$

Let  $S' \in \mathbf{Maya}_{\mathfrak{L}}^0$ . For each  $j \in J$  we put  $m_j = c(\psi_{L_i}^{-1}(S'))$ . Then  $\sum_{j \in J} m_j = 0$  and  $e(S') = \sum_j (a(L_j)m_j + i(L_j)m_j(m_j - 1)/2)$ . Therefore,

$$E_{\mathfrak{L}}(\tau) = \sum_{(m_j)_{j \in J}} q^{\sum_j (a(L_j)m_j + i(L_j)m_j(m_j - 1)/2)},$$

where the summation is over the systems of integers  $(m_j)_{j \in J}$  such that  $m_j = 0$  for all but finitely many  $j$  and  $\sum_j m_j = 0$ . Hence  $E_{\mathfrak{L}}(\tau) \in H_+^0$  if  $J$  is a finite set. This completes the proof.  $\square$

**§6.3. Proof of Theorem 5.1**

The claim (1) follows from Proposition 6.1 for the AP-covering

$$\mathbb{Z} = \frac{h}{a}\mathbb{Z} \amalg \prod_{\substack{1 \leq j \leq h-1 \\ \frac{h}{a} \nmid j}} (j + h\mathbb{Z}).$$

Let  $a, b, h$  be as in claim (2). First suppose that  $h > \text{lcm}(a, b)$ . Then claim (2) follows from Proposition 6.1 for the AP-covering

$$\mathbb{Z} = \frac{h}{a}\mathbb{Z} \amalg \left(1 + \frac{h}{b}\mathbb{Z}\right) \amalg \prod_{\substack{1 \leq j \leq h-1 \\ \frac{h}{a} \nmid j, \frac{h}{b} \nmid (j-1)}} (j + h\mathbb{Z}).$$

Next suppose that  $h = \text{lcm}(a, b)$ . Then  $\text{gcd}(a, b) \neq \min(a, b)$ . We put  $a' = a/\text{gcd}(a, b)$ ,  $b' = b/\text{gcd}(a, b)$ . Then  $a', b' \geq 2$  and we have

$$\eta_{[a,b;h]}(\tau) = \eta_{[a',b';a'b']}(\tau) f_{a,b,h}^{a'b' - a' - b'}.$$

Hence to prove (2), we may assume that  $\text{gcd}(a, b) = 1$  and  $h = ab$ . Since

$$\eta_{[a,b;ab]}(\tau) = \frac{\eta(ab\tau)^{(a-1)(b-1)}\eta(a\tau)\eta(b\tau)}{\eta(ab\tau)\eta(\tau)},$$

we have  $\eta_{[a,b;ab]}(\tau) \in H_+^0$  by Proposition 4.4. This proves claim (2).

Let  $\alpha, \beta, \gamma$  be as in claim (3). Then it follows from Proposition 6.1 for the AP-covering

$$\mathbb{Z} = \left(\alpha + \frac{h}{a}\mathbb{Z}\right) \amalg \left(\beta + \frac{h}{b}\mathbb{Z}\right) \amalg \left(\gamma + \frac{h}{c}\mathbb{Z}\right) \amalg \prod_{\substack{0 \leq j \leq h-1 \\ \frac{h}{a} \nmid (j-\alpha), \frac{h}{b} \nmid (j-\beta), \frac{h}{c} \nmid (j-\gamma)}} (j + h\mathbb{Z})$$

that  $\eta_{[a,b,c;h]}(\tau) \in H_+^0$ , which proves claim (3).  $\square$

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