

An Overdetermined Problem for an Elliptic Equation

by

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Abstract

We consider the following overdetermined boundary value problem: $\Delta u + \lambda u + \mu = 0$ in Ω , $u = 0$ on $\partial\Omega$ and $\partial u/\partial n = c$ on $\partial\Omega$, where $c \neq 0$, λ and μ are real constants and $\Omega \subset \mathbb{R}^2$ is a smooth bounded convex open set. We first show that it may happen that the problem has no solution. Then we study the existence of solutions for a wide class of domains.

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§1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded simply-connected open set. We consider solutions of the following overdetermined elliptic boundary value problem:

$$(1.1) \quad \Delta u + \lambda u + \mu = 0 \quad \text{in } \Omega,$$

$$(1.2) \quad u = 0 \quad \text{on } \partial\Omega,$$

$$(1.3) \quad \frac{\partial u}{\partial n} = c \quad \text{on } \partial\Omega,$$

where λ , μ and c are real constants and $\partial/\partial n$ is the outward normal derivative.

If $c = 0$ and $\mu \neq 0$ (or equivalently $\mu = 1$) we get as a special case *Schiffer's problem* (Yau [18, p. 688, problem 80]). If $\mu = 0$ and $c \neq 0$ the problem was posed by Berenstein [1].

In 1981 Williams [16] proved that if $\partial\Omega$ is Lipschitz and (1.1)–(1.3) has a solution for $c = 0$ and $\mu = 1$, then $\partial\Omega$ is real analytic. In 2002 Williams [17]

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proved that if $\partial\Omega$ is C^1 and (1.1)–(1.3) has a nonconstant solution $u \in C^2(\overline{\Omega})$, then $\partial\Omega$ is real analytic. In both cases the result holds in any dimension.

The following conjecture is stated in [17] (see also [10]):

Conjecture. *Assume that $\partial\Omega$ is Lipschitz. If (1.1)–(1.3) has a nonconstant solution for some real constants λ , μ and c , then Ω is a disk.*

Assume that Ω is the unit disk. Let J_z denote the z -th Bessel function. For any $\lambda > 0$ such that $\sqrt{\lambda}$ is not a zero of J_1 we define the function

$$u_\lambda(x) = \frac{J_0(\sqrt{\lambda}|x|) - J_0(\sqrt{\lambda})}{\sqrt{\lambda}J_1(\sqrt{\lambda})}, \quad |x| < 1.$$

For $\lambda < 0$ we define the function

$$u_\lambda(x) = -i \frac{J_0(i\sqrt{-\lambda}|x|) - J_0(i\sqrt{-\lambda})}{\sqrt{-\lambda}J_1(i\sqrt{-\lambda})}, \quad |x| < 1.$$

We recall that J_1 has only real zeros [14, pp. 482–483]. We easily verify that u_λ is a solution of (1.1)–(1.3) with $c = -1$ and $\mu = \mu_\lambda$ given by

$$\mu_\lambda = \begin{cases} \frac{\sqrt{\lambda}J_0(\sqrt{\lambda})}{J_1(\sqrt{\lambda})}, & \lambda > 0, \\ \frac{i\sqrt{-\lambda}J_0(i\sqrt{-\lambda})}{J_1(i\sqrt{-\lambda})}, & \lambda < 0. \end{cases}$$

The functions u_λ have a removable singularity at $\lambda = 0$ and the corresponding solution is

$$u_0(x) = \frac{1}{2}(1 - |x|^2), \quad |x| < 1.$$

Therefore, when Ω is the unit disk, there is a continuum of coefficient pairs (λ, μ_λ) and u_λ which solve (1.1)–(1.3) with $c \neq 0$. Notice that when $\lambda > 0$ is such that $J_0(\sqrt{\lambda}) = 0$ we have $\mu_\lambda = 0$. Then the corresponding Dirichlet problem has infinitely many solutions giving rise to the same constant normal derivative on the boundary. Berenstein [1] proved the following converse.

Proposition 1.1. *Let Ω be a simply-connected bounded open subset of \mathbb{R}^2 with $C^{2,\varepsilon}$ boundary ($\varepsilon > 0$). Assume that (1.1)–(1.3) with $\mu = 0$ and $c \neq 0$ has a solution for infinitely many λ . Then Ω is a disk.*

A smooth bounded simply-connected open set $\Omega \subset \mathbb{R}^2$ is said to have the *Schiffer property* if (for any λ) the only solution to (1.1)–(1.3) with $c = 0$ is the trivial solution $u = 0$ (corresponding to $\mu = 0$). It is well known that disks do not have the Schiffer property. Indeed, let $\lambda > 0$ be such that $J_1(\sqrt{\lambda}) = 0$. Then the

function

$$v_\lambda(x) = \frac{1}{\lambda} \left(\frac{J_0(\sqrt{\lambda}|x|)}{J_0(\sqrt{\lambda})} - 1 \right), \quad |x| < 1,$$

satisfies (1.1)–(1.3) with $\mu = 1$ and $c = 0$ when Ω is the unit disk. Berenstein [1] proved the following converse.

Proposition 1.2. *Let Ω be a simply-connected bounded open subset of \mathbb{R}^2 with $C^{2,\varepsilon}$ boundary ($\varepsilon > 0$). Assume that (1.1)–(1.3) with $\mu = 1$ and $c = 0$ has a solution for infinitely many λ . Then Ω is a disk.*

Remark 1. An elementary proof of Propositions 1.1 and 1.2 is given in [5] in the particular case where Ω is a convex set with positive curvature.

The *Schiffer conjecture* asserts that disks are the only smooth bounded simply-connected open sets for which (1.1)–(1.3) with $\mu = 1$ and $c = 0$ has a solution for even a single value of λ . Williams [15] established that for smooth bounded simply-connected open sets the Schiffer property is equivalent to the Pompeiu property. We shall not define the latter, instead we refer the reader to the bibliographic survey of the Pompeiu problem ([19]). Wide classes of smooth bounded simply-connected open sets in \mathbb{R}^2 having the Schiffer property were studied in [9] and the references therein. In [3] and [4] we gave some elementary results allowing us to exhibit very simple examples of planar domains having the Schiffer property. However, when $c \neq 0$, we do not know of any example supporting the above conjecture, even in the particular case studied by Berenstein. We first examine this problem.

Proposition 1.3. *There exist smooth bounded simply-connected open sets $\Omega \subset \mathbb{R}^2$ such that, for any fixed constant $c \neq 0$, (1.1)–(1.3) has no solution.*

The *width* of a convex planar domain in a given direction is the distance between two parallel supporting lines perpendicular to that direction. A set of *constant width* has the same width in all directions. Clearly disks have constant width. However there are plenty of smooth domains which have constant width but which are not disks: see [2], [5], [11] and [13].

Now we can state our main result.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex open set satisfying the following conditions:*

- (i) $\partial\Omega$ is a C^∞ curve with positive curvature;
- (ii) Ω has the Schiffer property;
- (iii) Ω is not of constant width.

Let c be a nonzero constant. Then there exist at most finitely many different pairs of coefficients $(\lambda_m, \mu_m) \in \mathbb{R}^2$ such that the Cauchy problem

$$\Delta u + \lambda_m u + \mu_m = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{and} \quad \frac{\partial u}{\partial n} = c \quad \text{on } \partial\Omega,$$

has a solution.

Remark 2. A true ellipse satisfies the conditions of Theorem 1.1. Indeed, it is well known that a true ellipse has the Schiffer property: a very simple proof is given in [6].

§2. Preliminaries

We assume first that $\Omega \subset \mathbb{R}^2$ is a bounded simply-connected open set with C^∞ boundary $\partial\Omega$. Let $x = x(s) = (x_1(s), x_2(s))$, $s \in [0, L]$, be a parametrization of $\partial\Omega$ by arc length. We denote by $\tau(s) = (\tau_1(s), \tau_2(s))$ the tangent to $\partial\Omega$ at $x(s)$ and by $\nu(s) = (\nu_1(s), \nu_2(s))$ the exterior normal to $\partial\Omega$ at $x(s)$. We have

$$\tau_1(s) = x'_1(s), \quad \tau_2(s) = x'_2(s), \quad s \in [0, L],$$

and

$$\nu_1(s) = x'_2(s), \quad \nu_2(s) = -x'_1(s), \quad s \in [0, L].$$

The Frenet formulas are

$$x''(s) = -\kappa(s)\nu(s), \quad \nu'(s) = \kappa(s)x'(s), \quad s \in [0, L],$$

where $\kappa = \kappa(s)$ is the curvature.

Now suppose that there exists $u \in C^\infty(\overline{\Omega})$ satisfying (1.1)–(1.3). We shall use some formulas established in a more general situation [7, Lemma 2.5 p. 101 and Lemma 2.6 p. 104] (see also [5]).

Lemma 2.1.

(1) *We have*

$$(\lambda c^2 + \mu^2) \int_0^L (x_1 \nu_2 - x_2 \nu_1)(\nu_1 + i\nu_2)^2 ds + A\mu + B = 0,$$

where A and B are independent of λ and μ . Moreover

$$A = 2ic \int_0^L (2\kappa(x \cdot \nu + i(x_1 \nu_2 - x_2 \nu_1)) - 1)(\nu_1 + i\nu_2)^2 ds,$$

$$B = -2ic^2 \int_0^L \kappa(\kappa(x \cdot \nu + i(x_1 \nu_2 - x_2 \nu_1)) - 1)(\nu_1 + i\nu_2)^2 ds.$$

(2) Let $k = 2p$ with $p \geq 2$. We have

$$0 = (-1)^p \lambda^{p-1} (\lambda c^2 + \mu^2) \int_0^L (x_1 \nu_2 - x_2 \nu_1) (\nu_1 + i\nu_2)^k ds + \mu^2 \sum_{j=2}^p A_{j,k} \lambda^{p-j} + \mu \sum_{j=1}^p B_{j,k} \lambda^{p-j} + \sum_{j=1}^p C_{j,k} \lambda^{p-j},$$

where $A_{j,k}$, $B_{j,k}$ and $C_{j,k}$ are independent of λ and μ . Moreover

$$B_{1,k} = (-1)^p i k c \int_0^L (\kappa(2x \cdot \nu + ik(x_1 \nu_2 - x_2 \nu_1)) - 1) (\nu_1 + i\nu_2)^k ds.$$

Lemma 2.2.

(1) We have

$$(\lambda c^2 + \mu^2) \int_0^L (\nu_1 + i\nu_2)^3 ds + a\mu + b = 0,$$

where a and b are independent of λ and μ . Moreover

$$a = -8c \int_0^L \kappa (\nu_1 + i\nu_2)^3 ds \quad \text{and} \quad b = 4c^2 \int_0^L \kappa^2 (\nu_1 + i\nu_2)^3 ds.$$

(2) Let $k = 2p + 1$ with $p \geq 2$. We have

$$0 = (-1)^p \lambda^{p-1} (\lambda c^2 + \mu^2) \int_0^L (\nu_1 + i\nu_2)^k ds + \mu^2 \sum_{j=2}^p \alpha_{j,k} \lambda^{p-j} + \mu \sum_{j=1}^p \beta_{j,k} \lambda^{p-j} + \sum_{j=1}^p \gamma_{j,k} \lambda^{p-j},$$

where $\alpha_{j,k}$, $\beta_{j,k}$ and $\gamma_{j,k}$ are independent of λ and μ . Moreover

$$\begin{aligned} \alpha_{2,k} &= (-1)^p \frac{2}{3} (p-1)p(p+1)(p+2) \int_0^L \kappa^2 (\nu_1 + i\nu_2)^k ds, \\ \beta_{1,k} &= (-1)^{p-1} (k^2 - 1)c \int_0^L \kappa (\nu_1 + i\nu_2)^k ds, \\ \gamma_{1,k} &= (-1)^p \frac{2}{3} p(p+1)(p^2 + p + 1)c^2 \int_0^L \kappa^2 (\nu_1 + i\nu_2)^k ds, \\ \beta_{2,5} &= -16c \int_0^L \kappa^3 (\nu_1 + i\nu_2)^5 ds, \\ \gamma_{2,5} &= -16c^2 \left(\frac{8}{3} \int_0^L \kappa^4 (\nu_1 + i\nu_2)^5 ds + \int_0^L \kappa'^2 (\nu_1 + i\nu_2)^5 ds \right). \end{aligned}$$

Remark 3. In [7], B , $\beta_{2,5}$ and $\gamma_{2,5}$ are not given explicitly, but they can be easily obtained from the proof.

Lemma 2.3. *Assume that Ω has the Schiffer property. Given any $c \in \mathbb{R} \setminus \{0\}$ and $\lambda \in \mathbb{R}$, there exists at most one $\mu \in \mathbb{R}$ such that the Cauchy problem*

$$\Delta v + \lambda v + \mu = 0 \quad \text{in } \Omega, \quad v = 0 \quad \text{and} \quad \frac{\partial v}{\partial n} = c \quad \text{on } \partial\Omega,$$

has a solution.

Proof. Suppose there exist $c \in \mathbb{R} \setminus \{0\}$ and $\lambda \in \mathbb{R}$ for which the above problem has a solution for two different values μ_1 and μ_2 . We denote by v_1 and v_2 two solutions corresponding to μ_1 and μ_2 respectively. The function $u = (v_1 - v_2)/(\mu_1 - \mu_2)$ satisfies (1.1)–(1.3) with $\mu = 1$ and $c = 0$ and we reach a contradiction.

Assume moreover that $\partial\Omega$ has positive curvature and that $0 \in \Omega$. Since the curve $\partial\Omega$ turns continuously, to each point $x = x(s) \in \partial\Omega$ we can associate a unique θ (modulo 2π) and θ describes a complete circuit $0 \leq \theta \leq 2\pi$ as $0 \leq s \leq L$. For each angle θ , $0 \leq \theta < 2\pi$, let $h(\theta)$ denote the distance from the origin to the supporting line of Ω with outward normal $\nu = (\cos \theta, \sin \theta)$. We have

$$h(\theta) = x \cdot \nu,$$

and h has period 2π . From the Serret–Frenet formulas we can derive the following second order ordinary differential equation involving the support function h and the radius of curvature ρ :

$$h(\theta) + h''(\theta) = \rho(\theta).$$

When $0 \notin \Omega$, the support function is defined in the following way. By translation there exists $a = (a_1, a_2) \in \mathbb{R}^2$ such that $0 \in \tilde{\Omega} = a + \Omega$. If \tilde{h} denotes the support function of $\tilde{\Omega}$ we have

$$h(\theta) = -a_1 \cos \theta - a_2 \sin \theta + \tilde{h}(\theta).$$

We refer the reader to Flanders [8] and the references therein for a detailed discussion.

For any $f : [0, 2\pi] \rightarrow \mathbb{C}$ such that $f \in L^2[0, 2\pi]$ we denote by

$$c_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-in\theta} d\theta, \quad n \in \mathbb{Z},$$

the Fourier coefficients.

§3. Proof of Proposition 1.3

Let $r > 32$. Define

$$h(\theta) = r + \cos 3\theta + \cos 5\theta, \quad 0 \leq \theta \leq 2\pi.$$

Then h is of class C^∞ and has period 2π . Since

$$\rho(\theta) = r - 8 \cos 3\theta - 24 \cos 5\theta > 0, \quad 0 \leq \theta \leq 2\pi,$$

h must be the support function of a convex set Ω .

Lemma 3.1. *Let Ω be as above. Assume that there exist $c \neq 0$, $\lambda, \mu \in \mathbb{R}$ and $u \in C^\infty(\bar{\Omega})$ satisfying (1.1)–(1.3). With the notations of Lemmas 2.1 and 2.2, b/c^2 and B/c^2 depend only on r and we have*

$$\int_0^L (x_1 \nu_2 - x_2 \nu_1)(\nu_1 + i\nu_2)^2 ds = 16i\pi, \quad \int_0^L (\nu_1 + i\nu_2)^3 ds = -8\pi, \quad A = a = 0,$$

and

$$\frac{B}{c^2} = -\frac{2^6 \cdot 7i\pi}{r^2} + o\left(\frac{1}{r^2}\right), \quad \frac{b}{c^2} = \frac{2^5\pi}{r^2} + o\left(\frac{1}{r^3}\right), \quad \text{as } r \rightarrow +\infty.$$

Proof. We have

$$\begin{aligned} \int_0^L (\nu_1 + i\nu_2)^3 ds &= \int_0^{2\pi} \rho(\theta) e^{3i\theta} d\theta = -8\pi, \\ \int_0^L (x_1 \nu_2 - x_2 \nu_1)(\nu_1 + i\nu_2)^2 ds &= -\int_0^{2\pi} h'(\theta) \rho(\theta) e^{2i\theta} d\theta, \end{aligned}$$

and

$$\begin{aligned} -\int_0^{2\pi} h'(\theta) \rho(\theta) e^{2i\theta} d\theta &= \int_0^{2\pi} (3 \sin 3\theta + 5 \sin 5\theta)(r - 8(\cos 3\theta + 3 \cos 5\theta)) e^{2i\theta} d\theta \\ &= -2^3 \cdot 3^2 \int_0^{2\pi} e^{2i\theta} \sin 3\theta \cos 5\theta d\theta \\ &\quad - 2^3 \cdot 5 \int_0^{2\pi} e^{2i\theta} \sin 5\theta \cos 3\theta d\theta = 16i\pi, \\ a &= -8c \int_0^{2\pi} e^{3i\theta} d\theta = 0, \\ A &= 2ic \int_0^{2\pi} (2(h(\theta) - ih'(\theta)) - \rho(\theta)) e^{2i\theta} d\theta = 0. \end{aligned}$$

Next as $r \rightarrow +\infty$ we have

$$\frac{B}{c^2} = -2i \int_0^{2\pi} \frac{h(\theta) - ih'(\theta)}{\rho(\theta)} e^{2i\theta} d\theta$$

$$\begin{aligned}
&= -\frac{2i}{r} \sum_{n \geq 0} \frac{8^n}{r^n} \int_0^{2\pi} (r + \cos 3\theta + \cos 5\theta + 3i \sin 3\theta + 5i \sin 5\theta) \\
&\quad \times (\cos 3\theta + 3 \cos 5\theta)^n e^{2i\theta} d\theta \\
&= -\frac{2^4 i}{r^2} \int_0^{2\pi} (2^2 \cdot 13 \cos 3\theta \cos 5\theta + 3^2 i \sin 3\theta \cos 5\theta \\
&\quad + 5i \cos 3\theta \sin 5\theta) e^{2i\theta} d\theta + o\left(\frac{1}{r^2}\right) \\
&= -\frac{2^6 \cdot 7i\pi}{r^2} + o\left(\frac{1}{r^2}\right),
\end{aligned}$$

and

$$\begin{aligned}
\frac{b}{c^2} &= 4 \int_0^{2\pi} \frac{e^{3i\theta}}{\rho(\theta)} d\theta = \frac{4}{r} \sum_{n \geq 0} \frac{8^n}{r^n} \int_0^{2\pi} (\cos 3\theta + 3 \cos 5\theta)^n e^{3i\theta} d\theta \\
&= \frac{2^5}{r^2} \int_0^{2\pi} e^{3i\theta} \cos 3\theta d\theta + o\left(\frac{1}{r^3}\right) = \frac{2^5 \pi}{r^2} + o\left(\frac{1}{r^3}\right).
\end{aligned}$$

Now we can complete the proof of Proposition 1.3. Choose Ω as in Lemma 3.1 with r sufficiently large. Using Lemmas 2.1(1), 2.2(1) and 3.1 we get

$$\lambda + \frac{\mu^2}{c^2} - \frac{28}{r^2} + o\left(\frac{1}{r^2}\right) = 0 \quad \text{and} \quad \lambda + \frac{\mu^2}{c^2} - \frac{4}{r^2} + o\left(\frac{1}{r^3}\right) = 0,$$

and we obtain a contradiction.

Remark 4. For any $r > 32$, Ω has the Schiffer property and Ω is of constant width. Indeed, if (1.1)–(1.3) has a nontrivial solution when $c = 0$, then Lemma 2.2(1) implies that

$$\int_0^L (\nu_1 + i\nu_2)^3 ds = 0,$$

and we have a contradiction. On the other hand $h(\theta) + h(\theta + \pi) = 2r$ for $\theta \in [0, 2\pi]$, hence Ω is of constant width.

Remark 5. Notice that in the case considered by Berenstein ($\mu = 0$) we can easily give examples of sets Ω satisfying the conclusion of Proposition 1.3 and such that:

- (i) Ω have the Schiffer property;
- (ii) Ω are not of constant width.

Indeed, let $r > 11$ and define

$$h(\theta) = r + \cos 2\theta + \cos 3\theta, \quad \theta \in [0, 2\pi].$$

Then h is of class C^∞ and has period 2π . Since

$$\rho(\theta) = r - 3 \cos 2\theta - 8 \cos 3\theta > 0, \quad 0 \leq \theta \leq 2\pi,$$

h must be the support function of a convex set Ω . As in Lemma 3.1 we have

$$\int_0^L (x_1\nu_2 - x_2\nu_1)(\nu_1 + i\nu_2)^2 ds = 2ir\pi, \quad \int_0^L (\nu_1 + i\nu_2)^3 ds = -8\pi,$$

and

$$\frac{B}{c^2} = -\frac{4i\pi}{r} + o\left(\frac{1}{r^2}\right), \quad \frac{b}{c^2} = \frac{2^5\pi}{r^2} + o\left(\frac{1}{r^2}\right), \quad \text{as } r \rightarrow +\infty.$$

Then using Lemmas 2.1(1) and 2.2(1) we obtain

$$\lambda - \frac{2}{r^2} + o\left(\frac{1}{r^3}\right) = 0 \quad \text{and} \quad \lambda - \frac{4}{r^2} + o\left(\frac{1}{r^3}\right) = 0,$$

and we have a contradiction. On the other hand, since

$$\int_0^L (\nu_1 + \nu_2)^3 ds \neq 0 \quad \text{and} \quad c_2(h) \neq 0,$$

(i) and (ii) are satisfied.

In fact we could also treat the case $\mu \neq 0$. From Lemmas 2.1(1) and 2.2(1) we get two equations for λ , μ/c and μ^2/c^2 . Then Lemma 2.2(2) with $p = 2$ gives a third equation:

$$\alpha_{2,5} \frac{\mu^2}{c^2} + \frac{\beta_{2,5} \mu}{c} + \frac{\gamma_{1,5}}{c^2} \lambda + \frac{\gamma_{2,5}}{c^2} = 0.$$

However, to get a contradiction requires tedious calculations; we leave the details to the reader.

§4. Proof of Theorem 1.1

Let $c \neq 0$ be a given constant. Assume that (1.1)–(1.3) has a solution corresponding to a sequence of infinitely many different pairs $(\lambda_m, \mu_m) \in \mathbb{R}^2$, $m \in \mathbb{N}$. Lemma 2.3 implies that infinitely many of the constants λ_m are different. Then we may assume that all the constants λ_m are different. Now we have two cases to consider:

Case 1: There exists a subsequence of (λ_m) that we still denote under (λ_m) such that $\lambda_m \rightarrow \pm\infty$ as $m \rightarrow \infty$.

Since the problem is invariant under translation we can assume that $c_1(h) = c_{-1}(h) = 0$. We have

$$(4.1) \quad c_{2p+1}(\rho) = 0 \quad \forall p \in \mathbb{Z}.$$

Indeed, suppose the contrary. Then there exists $p \geq 1$ such that $c_{2p+1}(\rho) \neq 0$. Suppose first that $c_3(\rho) \neq 0$. By Lemma 2.2(1) we have

$$(4.2) \quad \lambda_m c^2 + \mu_m^2 + \frac{b}{2\pi c_3(\rho)} = 0 \quad \forall m \in \mathbb{N}.$$

(4.2) implies that $|\mu_m| \rightarrow +\infty$ and $\lambda_m \rightarrow -\infty$ as $m \rightarrow \infty$. Then using (4.2) and Lemma 2.1 we get

$$(4.3) \quad A = B_{1,2p} = 0 \quad \forall p \geq 2.$$

Since

$$(4.4) \quad A = -2\pi c \overline{c_2(h')} = 4i\pi c \overline{c_2(h)},$$

and

$$(4.5) \quad B_{1,2p} = (-1)^{p-1} 2\pi c \overline{c_{2p}(h')} = (-1)^p 4ip\pi c \overline{c_{2p}(h)},$$

we deduce that

$$(4.6) \quad c_{2n}(h) = 0 \quad \forall n \in \mathbb{Z} \setminus \{0\}.$$

But (4.6) implies that Ω is of constant width and we reach a contradiction. Now if $c_3(\rho) = 0$, there exists $p \geq 2$ such that

$$c_{2j+1}(\rho) = 0, \quad j = 1, \dots, p-1, \quad \text{and} \quad c_{2p+1}(\rho) \neq 0.$$

By Lemma 2.2(2), μ_m satisfies a polynomial equation of degree 2 with coefficients depending on λ_m . It is quite easy to see that $|\mu_m| \rightarrow +\infty$ as $m \rightarrow \infty$. Then we deduce that

$$\frac{\lambda_m}{\mu_m^2} c^2 + 1 \rightarrow 0 \quad \text{as } m \rightarrow \infty,$$

from which we derive

$$(4.7) \quad \lambda_m c^2 + \mu_m^2 + \frac{(-1)^p}{2\pi c_{2p+1}(\rho)} (\gamma_{1,2p+1} - c^2 \alpha_{2,2p+1}) \rightarrow 0$$

as $m \rightarrow \infty$ because $\beta_{1,2q+1} = 0$ for $q \geq 2$. Using (4.7) and Lemma 2.1 we deduce that (4.3) holds and we conclude as before. Thus (4.1) is proved.

Now we claim that there exists $p \in \mathbb{N} \setminus \{0\}$ such that

$$\int_0^L (x_1 \nu_2 - x_2 \nu_1) (\nu_1 + i\nu_2)^{2p} ds \neq 0.$$

Indeed, suppose the contrary. Then

$$\int_0^L (x_1 \nu_2 - x_2 \nu_1) (\nu_1 + i\nu_2)^{2p} ds = -2\pi \overline{c_{2p}(h'\rho)} = 0 \quad \forall p \in \mathbb{Z}.$$

Using (4.1) and the fact that $c_n(\rho) = (1 - n^2)c_n(h)$ for $n \in \mathbb{Z}$, we can write

$$\begin{aligned} c_{2p+1}(h'\rho) &= \sum_{j=-\infty}^{+\infty} c_{2p+1-j}(h')c_j(\rho) = \sum_{j=-\infty}^{+\infty} c_{2(p-j)+1}(h')c_{2j}(\rho) \\ &= i \sum_{j=-\infty}^{+\infty} (2(p-j) + 1)c_{2(p-j)+1}(h)c_{2j}(\rho) = 0 \end{aligned}$$

for all $p \in \mathbb{Z}$ (recall that $c_1(h) = c_{-1}(h) = 0$). Therefore we have

$$c_n(h'\rho) = 0 \quad \forall n \in \mathbb{Z},$$

hence $h'\rho = 0$. Since ρ is positive we deduce that h is constant, contrary to (iii), and our claim is proved.

Now assume that $c_2(h'\rho) \neq 0$. Then Lemma 2.1(1) implies that

$$(4.8) \quad \lambda_m c^2 + \mu_m^2 + \frac{\overline{c_2(h')}}{c_2(h'\rho)} c \mu_m - \frac{B}{2\pi c_2(h'\rho)} = 0 \quad \forall m \in \mathbb{N}.$$

Therefore $|\mu_m| \rightarrow +\infty$ as $m \rightarrow \infty$. Using (4.8), Lemma 2.1(2) and (4.5) we deduce that

$$(4.9) \quad c_{2p}(h') = \frac{c_2(h')}{c_2(h'\rho)} c_{2p}(h'\rho) \quad \forall p \geq 2.$$

Clearly (4.9) also holds for $p = 0, 1$. Since $c_{2p+1}(h') = c_{2p+1}(h'\rho) = 0$ for all $p \in \mathbb{Z}$, we conclude that

$$(4.10) \quad h' = \frac{c_2(h')}{c_2(h'\rho)} h'\rho.$$

It is quite elementary to see that (4.10) implies that ρ is constant, contradicting (iii).

Now, as $c_2(h'\rho) = 0$, there exists $p \geq 2$ such that

$$(4.11) \quad c_{2j}(h'\rho) = 0, \quad j = 1, \dots, p-1, \quad \text{and} \quad c_{2p}(h'\rho) \neq 0.$$

By Lemma 2.1(2), μ_m satisfies a polynomial equation of degree 2 with coefficients depending on λ_m . Clearly $|\mu_m| \rightarrow +\infty$ as $m \rightarrow \infty$. Then we deduce that

$$\frac{\lambda_m}{\mu_m^2} c^2 + 1 \rightarrow 0 \quad \text{as } m \rightarrow \infty,$$

from which we derive

$$(4.12) \quad \lambda_m c^2 + \mu_m^2 + (-1)^{p-1} \frac{B_{1,2p}}{2\pi c_{2p}(h'\rho)} \mu_m + \frac{(-1)^{p-1}}{2\pi c_{2p}(h'\rho)} (C_{1,2p} - c^2 A_{2,2p}) \rightarrow 0$$

as $m \rightarrow \infty$. Since $|\mu_m| \rightarrow +\infty$, (4.11) and Lemma 2.1 imply that

$$A = 0 \quad \text{and} \quad B_{1,4} = \cdots = B_{1,2p-2} = 0 \quad \text{if } p \geq 3.$$

Then using (4.12), Lemma 2.1(2) and (4.5) we deduce that

$$(4.13) \quad c_{2q}(h') = \frac{c_{2p}(h')}{c_{2p}(h'\rho)} c_{2q}(h'\rho) \quad \forall q \geq p+1.$$

Since (4.13) also holds for $q = 0, \dots, p$, we conclude as before.

Case 2: The sequence (λ_m) is bounded.

Let (ε_j) and (φ_j) , $j \in \mathbb{N}$, denote the eigenvalues and a complete orthonormal system of real eigenvectors for $-\Delta$ with Dirichlet boundary conditions. For any $z \in \mathbb{C} \setminus \{\varepsilon_j; j \in \mathbb{N}\}$ we define v_z as the solution of

$$\Delta v_z + z v_z + 1 = 0 \quad \text{in } \Omega, \quad v_z = 0 \quad \text{on } \partial\Omega.$$

Since

$$\int_{\partial\Omega} \frac{\partial v_z}{\partial n} d\sigma(x) = \sum_{j=0}^{+\infty} \frac{\varepsilon_j}{z - \varepsilon_j} \left(\int_{\Omega} \varphi_j dx \right)^2,$$

the function

$$z \mapsto \int_{\partial\Omega} \frac{\partial v_z}{\partial n} d\sigma(x)$$

is meromorphic on \mathbb{C} , nontrivial, and has a pole at ε_j if and only if the corresponding eigenspace is not orthogonal to the constants. Now we define the function

$$(4.14) \quad \mu(z) = \frac{c|\partial\Omega|}{\int_{\partial\Omega} \frac{\partial v_z}{\partial n} d\sigma(x)}.$$

Lemma 4.1. *In the above setting μ is a meromorphic function on \mathbb{C} and $\mu(\lambda_m) = \mu_m$ for every $m \in \mathbb{N}$ such that $\lambda_m \in \mathbb{R} \setminus \{\varepsilon_j; j \in \mathbb{N}\}$*

Proof. μ is clearly a meromorphic function on \mathbb{C} . If u_m is a solution of (1.1)–(1.3) corresponding to (λ_m, μ_m) , where $\lambda_m \in \mathbb{R} \setminus \{\varepsilon_j; j \in \mathbb{N}\}$, then $u_m = \mu_m v_{\lambda_m}$. Therefore

$$c = \mu_m \frac{\partial v_{\lambda_m}}{\partial n},$$

and we get $\mu(\lambda_m) = \mu_m$.

Now, by taking a subsequence if necessary, we may assume that $\lambda_m \in \mathbb{R} \setminus \{\varepsilon_j; j \in \mathbb{N}\}$ for all $m \in \mathbb{N}$ and that there exists $\lambda_\star \in \mathbb{R}$ such that

$$\lambda_m \rightarrow \lambda_\star \quad \text{as } m \rightarrow \infty.$$

With the notations of Lemmas 2.1 and 2.2, for each $p \geq 1$ we define meromorphic functions on \mathbb{C} by setting

$$\begin{aligned}
 F_1(z) &= (c^2z + \mu(z)^2) \int_0^L (x_1\nu_2 - x_2\nu_1)(\nu_1 + i\nu_2)^2 ds + A\mu(z) + B, \\
 G_1(z) &= (c^2z + \mu(z)^2) \int_0^L (\nu_1 + i\nu_2)^3 ds + a\mu(z) + b, \\
 F_p(z) &= (-1)^p z^{p-1} (c^2z + \mu(z)^2) \int_0^L (x_1\nu_2 - x_2\nu_1)(\nu_1 + i\nu_2)^{2p} ds \\
 &\quad + \mu(z)^2 \sum_{j=2}^p A_{j,2p} z^{p-j} + \mu(z) \sum_{j=1}^p B_{j,2p} z^{p-j} + \sum_{j=1}^p C_{j,2p} z^{p-j}, \\
 G_p(z) &= (-1)^p z^{p-1} (c^2z + \mu(z)^2) \int_0^L (\nu_1 + i\nu_2)^{2p+1} ds \\
 &\quad + \mu(z)^2 \sum_{j=2}^p \alpha_{j,2p+1} z^{p-j} + \mu(z) \sum_{j=1}^p \beta_{j,2p+1} z^{p-j} + \sum_{j=1}^p \gamma_{j,2p+1} z^{p-j},
 \end{aligned}$$

if $p \geq 2$, where μ is defined in (4.14). By Lemmas 2.1, 2.2 and 4.1, for every $p \geq 1$, we have

$$F_p(\lambda_m) = G_p(\lambda_m) = 0 \quad \forall m \in \mathbb{N}.$$

Notice that, for each $p \geq 1$, the poles of F_p and G_p are included in $\{\varepsilon_j; j \in \mathbb{N}\}$. Since a nontrivial meromorphic function defined on all of \mathbb{C} except at its poles cannot have a sequence of zeros with a finite limit point, we deduce that $F_p \equiv 0$ and $G_p \equiv 0$ for all $p \geq 1$. Then there exists a sequence $(\tilde{\lambda}_m)$ increasing to $+\infty$ in $\mathbb{R} \setminus \{\varepsilon_j; j \in \mathbb{N}\}$ such that $F_p(\tilde{\lambda}_m) = G_p(\tilde{\lambda}_m) = 0$ for $p \geq 1$ and $m \in \mathbb{N}$. Now we can argue as in Case 1.

§5. Concluding remarks

The following theorem was proved in [13, Theorem 3.2 p. 1198].

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex open set of class $C^{3,\alpha}$ ($\alpha \in (0, 1]$). Assume that Ω is not of constant width and that Ω has the Schiffer property. Let $\psi \in C^1(\partial\Omega)$ be such that:*

- (i) ψ is not identically constant;
- (ii) ψ has at most countably many zeros.

Then there exist at most finitely many different pairs of coefficients $(\lambda_m, \mu_m) \in \mathbb{R}^2$ such that the Cauchy problem

$$\Delta u + \lambda_m u + \mu_m = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{and} \quad \frac{\partial u}{\partial n} = \psi \quad \text{on } \partial\Omega,$$

has a solution.

Using a completely different approach we proved the same theorem in [5, Theorem 1.3 p. 774] under slightly different assumptions. However the conditions there imply that Ω is not of constant width and that Ω has the Schiffer property. On the other hand we only assume in [5] that $\psi \in C^\infty(\partial\Omega)$ is not identically constant. Theorem 1.1 shows that this assumption is not necessary.

The proof of Theorem 1.1 in Case 1 shows that $\lambda_m \rightarrow +\infty$ cannot occur. Then the arguments in Case 2 easily lead to a contradiction.

We have seen in the proof of Theorem 1.1 that if (1.1)–(1.3) has a solution corresponding to a sequence of infinitely many different pairs $(\lambda_m, \mu_m) \in \mathbb{R}^2$, $m \in \mathbb{N}$, and if there exists $p \geq 1$ such that $c_{2p+1}(\rho) \neq 0$, then Ω is of constant width. A class of smooth bounded convex sets of constant width for which there exist at most finitely many different pairs of coefficients $(\lambda_m, \mu_m) \in \mathbb{R}^2$ such that (1.1)–(1.3) has a solution is given in [7, Proposition 6.2 p. 118]. We have the following stronger result, where we use the notations introduced in Section 2.

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex open set satisfying the following conditions:*

- (i) $\partial\Omega$ is a C^∞ curve with positive curvature;
- (ii) Ω has the Schiffer property;
- (iii) Ω has constant width;
- (iv) $\{p \in \mathbb{N} \setminus \{0\}; c_{2p+1}(\rho) \neq 0\}$ is finite.

Let c be a nonzero constant. Then there exist at most finitely many different pairs of coefficients $(\lambda_m, \mu_m) \in \mathbb{R}^2$ such that the Cauchy problem

$$\Delta u + \lambda_m u + \mu_m = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{and} \quad \frac{\partial u}{\partial n} = c \quad \text{on } \partial\Omega,$$

has a solution.

Proof. We shall need a simple lemma proved in [7, Lemma 6.1 p. 118].

Lemma 5.1. *In the setting of Theorem 5.2, for all $p \in \mathbb{N}$, there exists $q \geq p$ such that $c_{2q+1}(1/\rho) \neq 0$.*

Now as in the proof of Theorem 1.1 we are led to Case 1. Assume first that $c_3(\rho) \neq 0$. Then (4.2) holds, hence $|\mu_m| \rightarrow +\infty$ and $\lambda_m \rightarrow -\infty$ as $m \rightarrow \infty$. Using Lemma 2.2(2) we deduce that

$$(5.1) \quad c_{2n+1}\left(\frac{1}{\rho}\right) = \frac{2}{n(n+1)} \frac{c_3(1/\rho)}{c_3(\rho)} c_{2n+1}(\rho) \quad \forall n \geq 2.$$

(Notice that (5.1) also holds for $n = 1$.) Then Lemma 5.1 and (iv) give a contradiction. Now if $c_3(\rho) = 0$, as in the proof of Theorem 1.1 we arrive at (4.7). Since

for all $n \geq 2$,

$$\gamma_{1,2n+1} - c^2 \alpha_{2,2n+1} = (-1)^n 4\pi n(n+1) c^2 \overline{c_{2n+1}(1/\rho)},$$

using Lemma 2.2(2) we deduce that if $c_{2j+1}(\rho) = 0$, $j = 1, \dots, p-1$, and $c_{2p+1}(\rho) \neq 0$, then

$$(5.2) \quad c_{2n+1} \left(\frac{1}{\rho} \right) = \frac{p(p+1)}{n(n+1)} \frac{c_{2p+1}(1/\rho)}{c_{2p+1}(\rho)} c_{2n+1}(\rho) \quad \forall n \geq p+1$$

(clearly (5.2) also holds for $n = p$ and in fact also for $n = 1, \dots, p-1$), and we conclude as before. The arguments in Case 2 are the same as in the proof of Theorem 1.1.

Notice that the class \mathcal{B} in [7, Definition 1.1 p. 94 and Proposition 6.2 p. 118] is strictly contained in the class defined in Theorem 5.2. Indeed, if $c_3(\rho) \neq 0$, then Ω has the Schiffer property: see Remark 4 or [7, Proposition 6.1 p. 117]. However there are plenty of smooth domains of constant width having the Schiffer property and such that $c_3(\rho) = 0$. Let us give an example. Let $r > 72$ and define

$$h(\theta) = r + \cos 5\theta + \cos 7\theta, \quad \theta \in [0, 2\pi].$$

Then h is of class C^∞ and has period 2π . Since

$$\rho(\theta) = r - 24 \cos 5\theta - 48 \cos 7\theta > 0, \quad 0 \leq \theta \leq 2\pi,$$

h must be the support function of a convex set Ω . Clearly $c_3(\rho) = 0$. Since $h(\theta) + h(\theta + \pi) = 2r$, Ω has constant width. Assume that (1.1)–(1.3) has a solution when $c = 0$ and $\mu = 1$. In Lemma 2.1(1) we have $A = B = 0$ and

$$\int_0^L (x_1 \nu_2 - x_2 \nu_1)(\nu_1 + i\nu_2)^2 ds = - \int_0^{2\pi} h'(\theta) \rho(\theta) e^{2i\theta} d\theta = 36i\pi \neq 0,$$

and we reach a contradiction.

Finally let u be a solution of (1.1)–(1.3). With the notations of Section 4, the eigenvalue ε_0 is simple and the eigenfunction φ_0 is of constant sign. Integrating by parts we obtain

$$\varepsilon_0 \int_{\Omega} u \varphi_0 dx = - \int_{\Omega} u \Delta \varphi_0 dx = - \int_{\Omega} \varphi_0 \Delta u dx = \lambda \int_{\Omega} u \varphi_0 dx + \mu \int_{\Omega} \varphi_0 dx.$$

If $\lambda = \varepsilon_0$, we deduce that $\mu = 0$. Then by the classical result of Serrin [12], Ω is a disk.

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