Extending Polynomials
in Maximal and Minimal Ideals

by

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Abstract

Given a homogeneous polynomial on a Banach space E belonging to some maximal or minimal polynomial ideal, we consider its iterated extension to an ultrapower of E and prove that this extension remains in the ideal and has the same ideal norm. As a consequence, we show that the Aron–Berner extension is a well defined isometry for any maximal or minimal ideal of homogeneous polynomials. This allows us to obtain symmetric versions of some basic results of the metric theory of tensor products.

§1. Introduction

Aron and Berner showed in [1] how to extend continuous polynomials (and some holomorphic functions) defined on a Banach space E to the bidual E''. Some time later, Davie and Gamelin [8] proved that this extension preserves the norm. This fact is crucial to showing that some holomorphic functions defined on the unit ball of E can be extended to the ball of E''. A natural question is whether a polynomial ideal is closed under the Aron–Berner extension and, also, if the ideal norm is preserved by this extension. This is easy for nuclear and approximable polynomials, and it is also known to hold, for example, for integral polynomials [7], for extendible polynomials [4], and for the ideal of polynomials that are weakly continuous on bounded sets [21] among others. However, some polynomial ideals

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are not closed under Aron–Berner extension (for example, the ideal of weakly sequentially continuous polynomials).

Floret and Hunfeld showed that there is an extension, the so called uniterated Aron–Berner extension, which is an isometry for maximal polynomial ideals [17]. Although it is easy to prove properties of this uniterated extension, it is hard to compute, since its definition depends on an ultrafilter. On the other hand, the (iterated) Aron–Berner extension is not only easier to compute, but also has a simple characterization that allows one to check if a given extension of a polynomial is actually its Aron–Berner extension [24]. Moreover, this extension is more widely used in the study of polynomials and analytic functions (for example, it allows a description of the spectrum of the algebra of bounded type analytic functions [2]).

Extensions of polynomials to ultrapowers were first studied by Lindström and Ryan in [20] and by Dineen and Timoney in [12], where they show that different extensions preserve the uniform norm. In [17] the authors further developed some of these ideas and proved that maximality and ultrastability are equivalent for a polynomial ideal [17, Theorem 3.2]. Therefore, for maximal ideals, the uniterated extension to the ultrapower turns out to be an isometry.

One of the aims of this note is to prove that the Aron–Berner extension is a well defined isometry on maximal and on minimal ideals. More generally, we study the extension of a polynomial on a Banach space E to some ultrapower $(E)_{\mathfrak{U}}$, and show the following: if $\mathcal{Q}$ is a maximal or a minimal ideal of n-homogeneous polynomials and $P \in \mathcal{Q}(E)$, then the iterated extension $\overline{P}$ of P to the ultrapower $(E)_{\mathfrak{U}}$ is in $\mathcal{Q}((E)_{\mathfrak{U}})$, and $\|P\|_{\mathcal{Q}(E)} = \|\overline{P}\|_{\mathcal{Q}((E)_{\mathfrak{U}})}$. As a contribution to the metric theory of symmetric tensor products, we derive the symmetric versions of the Extension Lemma and the Embedding Lemma [9, 13.2 and 13.3] (which in the non-symmetric setting are much easier to prove). These are Corollaries 3.4 and 3.5 below.

The article is organized as follows. In Section 2 we describe some basic properties of ultrapowers and define the extensions mentioned above. In Section 3 we prove, following [8] and using the representation theorem for maximal polynomial ideals, the results stated in the previous paragraph.

We refer to [13, 14, 15, 16, 17] for the theory of symmetric tensor products and polynomial ideals.

§2. Preliminaries

Throughout the paper, E and F will be Banach spaces, E' the dual space of E, $\kappa_E : E \to E''$ the canonical embedding and B_E the open unit ball of E. We will denote by $\mathrm{FIN}(E)$ the class of all finite-dimensional subspaces of the Banach space E.

We now recall some basic properties of ultrapowers. The reader is referred to [18, 19] for further details. Let $\mathfrak{U}$ be an ultrafilter on a set I. Whenever the limit with respect to $\mathfrak{U}$ of a family $\{a_i : i \in I\}$ exists, we denote it by $\lim_{i,\mathfrak{U}} a_i$. For a Banach space E, $(E)_\mathfrak{U}$, the *ultrapower of E with respect to the ultrafilter* $\mathfrak{U}$, consists of the classes of elements of the form $z = (z_i)_\mathfrak{U}$, with $z_i \in E$ for each $i \in I$, where the norm of (z_i) is uniformly bounded, and where we identify (z_i) with (y_i) if $\lim_{i,\mathfrak{U}} \|z_i - y_i\| = 0$. The space $(E)_\mathfrak{U}$ is a Banach space under the norm

$$\|(z_i)_\mathfrak{U}\| = \lim_{i,\mathfrak{U}} \|z_i\|.$$

We may consider E as a subspace of the ultrapower $(E)_\mathfrak{U}$ by means of the canonical embedding $h_E : E \hookrightarrow (E)_\mathfrak{U}$ given by $h_E x = (x_i)_\mathfrak{U}$ where $x_i = x$ for all i.

Let us now define the ultrapower of an operator. If $T : E \to F$ is a bounded linear operator, the *ultrapower operator with respect to the ultrafilter* $\mathfrak{U}$ will be the operator from $(E)_\mathfrak{U}$ to $(F)_\mathfrak{U}$ defined according to the rule $(z_i)_\mathfrak{U} \mapsto (Tz_i)_\mathfrak{U}$. We will denote this operator $(T)_\mathfrak{U}$. It can be seen that $\|(T)_\mathfrak{U}\|$ is $\|T\|$.

We will need a special property of ultrapowers [18, Proposition 6.1], [19, Satz 4.1]:

Proposition 2.1 (Local determination of ultrapowers). *Let E be a Banach space and* $M \in \text{FIN}((E)_\mathfrak{U})$. *For each* $i \in I$ *there exists an operator* $R_i \in \mathcal{L}(M, E)$ *such that*

(1) $z = (R_i z)_\mathfrak{U}$ *for all* $z \in M$.

(2) $\|R_i\| \leq 1$ *for all* $i \in I$ *and there is an* $\mathcal{U} \in \mathfrak{U}$ *with* $\|R_i\| = 1$ *for all* $i \in \mathcal{U}$.

(3) *For all* $\varepsilon > 0$ *there is an* $\mathcal{U}_\varepsilon \in \mathfrak{U}$ *such that the inverse* $R_i^{-1} : R_i(M) \to M$ *exists and* $\|R_i^{-1}\| \leq 1 + \varepsilon$ *for all* $i \in \mathcal{U}_\varepsilon$.

We shall only use (1) and the first part of (2).

There are different ways of extending polynomials from a Banach space to an ultrapower. Two approaches are the iterated and the uniterated extensions. Let $(E)_\mathfrak{U}$ be an ultrapower of a Banach space E. For a continuous n-linear function Φ on E we define an n-linear map $\overline{\Phi}$ on $(E)_\mathfrak{U}$ by

$$\overline{\Phi}(z_1, \ldots, z_n) = \lim_{i_1,\mathfrak{U}} \ldots \lim_{i_n,\mathfrak{U}} \Phi(z_{i_1}^{(1)}, \ldots, z_{i_n}^{(n)})$$

for $z_j = (z_{i_j}^{(j)})_\mathfrak{U} \in (E)_\mathfrak{U}$. If P is an n-homogeneous continuous polynomial and A is its associated symmetric n-linear mapping, the *iterated extension*, $\overline{P}$, of P to $(E)_\mathfrak{U}$ is defined by

$$\overline{P}((z_i)_\mathfrak{U}) := \overline{A}((z_i)_\mathfrak{U}, \ldots, (z_i)_\mathfrak{U}),$$

and this coincides with $\lim_{i_1,\mathfrak{U}} \ldots \lim_{i_n,\mathfrak{U}} A(z_{i_1}, \ldots, z_{i_n})$.

On the other hand the *uniterated extension*, $\widetilde{P}$, is defined by

$$\widetilde{P}((z_i)_{\mathfrak{U}}) := \lim_{i,\mathfrak{U}} P(z_i).$$

Similarly, there are two analogous ways of extending a polynomial from a Banach space into its bidual. The classical Aron–Berner extension [1] which is, by construction, iterated (the associated multilinear form is extended from the last variable to the first one by weak-star continuity) and the uniterated Aron–Berner extension (a term coined in [13]). We need to construct a local ultrapower of E in order to define this uniterated extension, so we give some details.

First, we recall the *Principle of Local Reflexivity*: for each $M \in \text{FIN}(E'')$, $N \in \text{FIN}(E')$ and $\varepsilon > 0$, there exists an operator $T \in \mathcal{L}(M, E)$ such that

1. T is an ε-isometry, that is, $(1 - \varepsilon)\|x''\| \leq \|T(x'')\| \leq (1 + \varepsilon)\|x''\|$.
2. $T(x'') = x''$ for every $x'' \in M \cap E$.
3. $x'(T(x'')) = x''(x')$ for $x'' \in M$ and $x' \in N$.

Let I be the set of all triples (M, N, ε), where M and N are finite-dimensional subspaces of E'' and E' respectively and $\varepsilon > 0$. For each $i \in I$, we denote by M_i, N_i and ε_i the three elements of the triple. We define an ordering on I by setting $i < j$ if $M_i \subset M_j$, $N_i \subset N_j$ and $\varepsilon_i > \varepsilon_j$. The sets of the form $B_i = \{j \in I : i \leq j\}$ form a filterbase. Let $\mathfrak{U}$ be an ultrafilter on I which contains this filterbase. The filter $\mathfrak{U}$ thus constructed is called a *local ultrafilter for E*, and $(E)_{\mathfrak{U}}$ is called a *local ultrapower of E*.

Finally, let us fix, for each $i \in I$, an operator $T_i : M_i \to E$ in accordance with the Principle of Local Reflexivity. The canonical embedding of E into the ultrapower $(E)_{\mathfrak{U}}$ extends to a canonical embedding $J_E : E'' \to (E)_{\mathfrak{U}}$ defined by $J_E(x'') = (x_i)$, where x_i is equal to $T_i(x'')$ if $x'' \in M_i$ and 0 otherwise. In this way, $J_E(E'')$ is the range of a norm one projection on $(E)_{\mathfrak{U}}$. This projection $\text{Pr} : (E)_{\mathfrak{U}} \to J_E(E'')$ is given by $\text{Pr}((x_i)_{\mathfrak{U}}) = J_E(w^*\text{-}\lim_{i,\mathfrak{U}} x_i)$ (the weak-star limit in E'' of the collection (x_i)).

For a polynomial $P \in \mathcal{P}^n(E)$ its *uniterated Aron–Berner extension* to E'' is defined by $\widetilde{P} \circ J \in \mathcal{P}^n(E'')$.

Let us recall some definitions from the theory of Banach polynomial ideals [16]. A *Banach ideal of continuous scalar valued n-homogeneous polynomials* is a pair $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ such that:

(i) $\mathcal{Q}(E) = \mathcal{Q} \cap \mathcal{P}^n(E)$ is a linear subspace of $\mathcal{P}^n(E)$ and $\|\cdot\|_{\mathcal{Q}}$ is a norm which makes the pair $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ a Banach space.

(ii) If $T \in \mathcal{L}(E_1, E)$, $P \in \mathcal{Q}(E)$ then $P \circ T \in \mathcal{Q}(E_1)$ and

$$\|P \circ T\|_{\mathcal{Q}(E_1)} \leq \|P\|_{\mathcal{Q}(E)} \|T\|^n.$$

(iii) $z \mapsto z^n$ belongs to $\mathcal{Q}(\mathbb{K})$ and has norm 1.

Let $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ be the Banach ideal of continuous scalar valued n-homogeneous polynomials and, for $P \in \mathcal{P}^n(E)$, define $\|P\|_{\mathcal{Q}^{\max}(E)} := \sup\{\|P|_M\|_{\mathcal{Q}(M)} : M \in \mathrm{FIN}(E)\} \in [0, \infty]$. The *maximal kernel* of $\mathcal{Q}$ is the ideal given by $\mathcal{Q}^{\max} := \{P \in \mathcal{P}^n : \|P\|_{\mathcal{Q}^{\max}} < \infty\}$. An ideal $\mathcal{Q}$ is said to be *maximal* if $\mathcal{Q} \stackrel{1}{=} \mathcal{Q}^{\max}$.

The *minimal kernel* of $\mathcal{Q}$ is defined as the composition ideal $\mathcal{Q}^{\min} := \mathcal{Q} \circ \overline{\mathfrak{F}}$, where $\overline{\mathfrak{F}}$ stands for the ideal of approximable operators. In other words, a polynomial P belongs to $\mathcal{Q}^{\min}(E)$ if it admits a factorization

(2.1)

where F is a Banach space, $T : E \to F$ is an approximable operator and Q is in $\mathcal{Q}(F)$. The minimal norm is given by $\|P\|_{\mathcal{Q}^{\min}} := \inf\{\|Q\|_{\mathcal{Q}(F)} \|T\|^n\}$, where the infimum runs over all possible factorizations as in (2.1). An ideal $\mathcal{Q}$ is said to be *minimal* if $\mathcal{Q} \stackrel{1}{=} \mathcal{Q}^{\min}$.

For properties of maximal and minimal ideals of homogeneous polynomials and examples see [14, 17] and the references therein.

Floret and Hunfeld proved the following result [17, Theorems 3.2 and 3.3].

Theorem 2.2. *Let $\mathcal{Q}$ be a maximal ideal of n-homogeneous polynomials, $Q \in \mathcal{P}^n(E)$ and $\mathfrak{U}$ an ultrafilter of E. Then P belongs to $\mathcal{Q}(E)$ if and only if the uniterated extension $\widetilde{P}$ is in $\mathcal{Q}((E)_\mathfrak{U})$. In this case $\|\widetilde{P}\|_{\mathcal{Q}((E)_\mathfrak{U})} = \|P\|_{\mathcal{Q}(E)}$.*

If $\mathfrak{U}$ is a local ultrafilter for E then we also have that P belongs to $\mathcal{Q}(E)$ if and only if the uniterated Aron–Berner extension $\widetilde{P} \circ J$ belongs to $\mathcal{Q}(E'')$, and moreover $\|\widetilde{P} \circ J\|_{\mathcal{Q}(E'')} = \|P\|_{\mathcal{Q}(E)}$.

We will present a similar theorem for the iterated extension to the ultrapower and for the Aron–Berner extension. We will also conclude that the same holds in the case that $\mathcal{Q}$ is a minimal ideal of homogeneous polynomials.

The following proposition is due to Lindström and Ryan [20, Proposition 2.1]. It states that the Aron–Berner extension can be recovered from the iterated extension to a local ultrapower of E:

Proposition 2.3. *If* $(E)_\mathfrak{U}$ *is a local ultrapower of* E, *then the restriction of* $\overline{P}$ *to the canonical image of* E'' *in* $(E)_\mathfrak{U}$ *coincides with the Aron–Berner extension of* P *to* E''.

§3. The results

Maximal and minimal ideals of homogeneous polynomials are easily seen to be closed under the Aron–Berner extension: just use a multilinear version of the Extension Lemma [9, 13.2] (whose proof is identical) and the main result of [15]. In this section we will show that this extension is actually an isometry.

First, let A be the symmetric multilinear form associated to a polynomial P. For each fixed j, $1 \leq j \leq n$, $x_1, \ldots, x_{j-1} \in E$, and $z_j, z_{j+1}, \ldots z_n \in (E)_\mathfrak{U}$, we have

$$\overline{A}(h_E x_1, \ldots, h_E x_{j-1}, z_j, z_{j+1}, \ldots, z_n)$$
$$= \lim_{i_j, \mathfrak{U}} \overline{A}(h_E x_1, \ldots, h_E x_{j-1}, h_E z_{i_j}^{(j)}, z_{j+1}, \ldots, z_n),$$

where $\overline{A}$ is the iterated extension of A to a local ultrapower.

Now, we will imitate the procedure used by Davie and Gamelin in [8]. Denote by A the symmetric n-linear form associated to P. We have the following lemma:

Lemma 3.1. *Let* $M \in \mathrm{FIN}((E)_\mathfrak{U})$ *and* $z_1, \ldots, z_r \in M$. *For a given natural number* m *and* $\varepsilon > 0$ *there exist operators* $R_1, \ldots, R_m \in \mathcal{L}(M, E)$ *with norm less than or equal to* 1 *such that*

$$(3.1) \qquad |A(R_{i_1} z_k, \ldots, R_{i_n} z_k) - \overline{A}(z_k, \ldots, z_k)| < \varepsilon$$

for any distinct indices $i_1, \ldots, i_n$ *between* 1 *and* m *and every* $k = 1, \ldots, r$.

Proof. Since A is symmetric, in order to prove the lemma it suffices to obtain (3.1) for $i_1 < \cdots < i_n$. We will select the operators $R_1, \ldots, R_m$ inductively by the following procedure. By Proposition 2.1, for each $\mathfrak{i} \in I$ there exists an operator $R_\mathfrak{i} \in \mathcal{L}(M, E)$ with norm less than or equal to 1 such $z_k = (R_\mathfrak{i} z_k)_\mathfrak{U}$.

Since $z_k = (R_\mathfrak{i} z_k)_\mathfrak{U}$ for each k, the set $\{\mathfrak{i} \in I : \overline{A}(h_E R_\mathfrak{i} z_k, z_k, \ldots, z_k) - \overline{A}(z_k, z_k, \ldots, z_k)| < \varepsilon/n\}$ belongs to the filter $\mathfrak{U}$. Therefore, we can pick $R_1 \in \mathcal{L}(M, E)$ such that

$$|\overline{A}(h_E R_1 z_k, z_k, \ldots, z_k) - \overline{A}(z_k, z_k, \ldots, z_k)| < \varepsilon/n$$

for every $k = 1, \ldots, r$.

In a similar way we can choose R_2 such that

$$|\overline{A}(h_E R_2 z_k, z_k, \ldots, z_k) - \overline{A}(z_k, z_k, \ldots, z_k)| < \varepsilon/n,$$

and moreover,

$$|\overline{A}(h_E R_1 z_k, h_E R_2 z_k, z_k, \ldots, z_k) - \overline{A}(h_E R_1 z_k, z_k, \ldots, z_k)| < \varepsilon/n$$

for every k. Proceeding in this way, we get R_l's so that

$$|\overline{A}(h_E R_{i_1} z_k, \ldots, h_E R_{i_{r-1}} z_k, h_E R_{i_r} z_k, z_k, \ldots, z_k)$$
$$- \overline{A}(h_E R_{i_1} z_k, \ldots, h_E R_{i_{r-1}} z_k, z_k, \ldots, z_k)| < \varepsilon/n$$

whenever $i_1 < \cdots < i_r$ and $k = 1, \ldots, r$. Then

$$|\overline{A}(h_E R_{i_1} z_k, \ldots, h_E R_{i_n} z_k) - \overline{A}(z_k \ldots, z_k)|$$

is estimated by the sum of n terms

$$|\overline{A}(h_E R_{i_1} z_k, \ldots, h_E R_{i_n} z_k) - \overline{A}(h_E R_{i_1} z_k, \ldots, h_E R_{i_{n-1}} z_k, z_k)| + \cdots$$
$$+ |\overline{A}(h_E R_{i_1} z_k, z_k \ldots, z_k) - \overline{A}(z_k, \ldots, z_k)|,$$

each smaller than ε/n, for all $k = 1, \ldots, r$. $\qquad \square$

Proposition 3.2. *Let $M \in \mathrm{FIN}((E)_{\mathfrak{U}})$ and $z_1, \ldots, z_r \in M$, $P : E \to \mathbb{K}$ a continuous polynomial and $\varepsilon > 0$. There exist a finite subset $\mathfrak{F}$ of $\mathbb{N}$ and operators $(R_i)_{i \in \mathfrak{F}}$ in $\mathcal{L}(M, E)$ with norm less than or equal to 1 such that*

$$\left| \sum_{k=1}^r \overline{P}(z_k) - \sum_{k=1}^r P\left(\frac{1}{|\mathfrak{F}|} \sum_{i \in \mathfrak{F}} R_i z_k \right) \right| < \varepsilon.$$

Proof. For $\varepsilon > 0$, fix m large enough and choose $R_1, \ldots, R_m$ as in the previous lemma, such that

$$|\overline{A}(z_k, \ldots, z_k) - A(R_{i_1} z_k, \ldots, R_{i_n} z_k)| < \varepsilon/2r$$

for any distinct indices $i_1, \ldots, i_n$ between 1 and m and every $k = 1, \ldots, r$. Now, we set $\mathfrak{F} = \{1, \ldots, m\}$ and define $R \in \mathcal{L}(M, E)$ by $R := \frac{1}{|\mathfrak{F}|} \sum_{i \in \mathfrak{F}} R_i$. For $k \in \{1, \ldots, r\}$, we have

$$|\overline{P}(z_k) - P(R z_k)| = \left| \frac{1}{m^n} \sum_{i_1, \ldots, i_n = 1}^m [\overline{A}(z_k, \ldots, z_k) - A(R_{i_1} z_k, \ldots, R_{i_n} z_k)] \right|$$
$$\leq |\Sigma_1^k| + |\Sigma_2^k|,$$

where Σ_1^k is the sum over the n-tuples of non-repeated indices (which is less than $\varepsilon/2r$) and Σ_2^k is the sum over the remaining indices. It is easy to show that there

are exactly $m^n - \prod_{j=0}^{n-1}(m-j)$ summands in Σ_2^k, each bounded by a constant $C > 0$ (obviously we can assume that C is independent of k), thus

$$|\Sigma_2^k| \leq \frac{1}{m^n}\left(m^n - \prod_{j=0}^{n-1}(m-j)\right)C = \left[1 - \left(1 - \frac{1}{m}\right)\cdots\left(1 - \frac{n-1}{m}\right)\right]C.$$

For m sufficiently large this is less than $\varepsilon/2r$. $\square$

Recall that an s-tensor norm α is called *finitely generated* if for every Banach space E and $z \in \otimes^{n,s}E$, we have $\alpha(z, \otimes^{n,s}E) = \inf\{\alpha(z, \otimes^{n,s}M) : M \in \mathrm{FIN}(E), z \in \otimes^{n,s}M\}$. Now we can state the main theorem:

Theorem 3.3. *Let α be a finitely generated s-tensor norm and $P \in (\widetilde{\otimes}_\alpha^{n,s}E)'$ a polynomial. The iterated extension $\overline{P}$ of P to the ultrapower $(E)_\mathfrak{U}$ belongs to $(\widetilde{\otimes}_\alpha^{n,s}(E)_\mathfrak{U})'$ and*

$$\|P\|_{(\widetilde{\otimes}_\alpha^{n,s}E)'} = \|\overline{P}\|_{(\widetilde{\otimes}_\alpha^{n,s}(E)_\mathfrak{U})'}.$$

Equivalently, if $\mathcal{Q}$ is a maximal ideal of n-homogeneous polynomials and $P \in \mathcal{Q}(E)$, the iterated extension $\overline{P}$ of P to the ultrapower $(E)_\mathfrak{U}$ belongs to $\mathcal{Q}((E)_\mathfrak{U})$ and

$$\|P\|_{\mathcal{Q}(E)} = \|\overline{P}\|_{\mathcal{Q}((E)_\mathfrak{U})}.$$

Proof. Thanks to the representation theorem for maximal polynomial ideals [17, Section 3.2] (see also [14, Section 4]), it is enough to show the first statement.

Let $w \in \otimes^{n,s}M$, where $M \in \mathrm{FIN}((E)_\mathfrak{U})$. Since α is finitely generated, we only have to show that

$$|\langle \overline{P}, w\rangle| \leq \|P\|_{(\widetilde{\otimes}_\alpha^{n,s}E)'}\alpha(w, \otimes^{n,s}M).$$

Now, $w = \sum_{k=1}^r \otimes^n z_k$ with $z_k \in M$. Given $\varepsilon > 0$, by Proposition 3.2 we can take a finite set $\mathfrak{F}$ and operators $(R_i)_{i\in\mathfrak{F}}$ with $\|R_i\|_{\mathcal{L}(M,E)} \leq 1$ such that

$$\left|\sum_{k=1}^r \overline{P}(z_k) - \sum_{k=1}^r P\left(\frac{1}{|\mathfrak{F}|}\sum_{i\in\mathfrak{F}} R_i z_k\right)\right| < \varepsilon.$$

Therefore,

$$\begin{aligned}
|\langle \overline{P}, w\rangle| &= \left|\sum_{k=1}^r \overline{P}(z_k)\right| \\
&\leq \left|\sum_{k=1}^r \overline{P}(z_k) - \sum_{k=1}^r P\left(\frac{1}{|\mathfrak{F}|}\sum_{i\in\mathfrak{F}} R_i z_k\right)\right| + \left|\sum_{k=1}^r P\left(\frac{1}{|\mathfrak{F}|}\sum_{i\in\mathfrak{F}} R_i z_k\right)\right| \\
&\leq \varepsilon + \left|\left\langle P, \sum_{k=1}^r \otimes^n \frac{1}{|\mathfrak{F}|}\sum_{i\in\mathfrak{F}} R_i z_k\right\rangle\right|
\end{aligned}$$

$$\leq \varepsilon + \|P\|_{(\widetilde{\otimes}_\alpha^{n,s} E)'} \alpha\Big(\sum_{k=1}^r \otimes^n \frac{1}{|\mathfrak{F}|}\sum_{i\in\mathfrak{F}} R_i z_k, \otimes^{n,s} E\Big)$$

$$\leq \varepsilon + \|P\|_{(\widetilde{\otimes}_\alpha^{n,s} E)'} \alpha\Big(\otimes^{n,s} R\Big(\sum_{k=1}^r z_k\Big), \otimes^{n,s} E\Big),$$

where $R = \frac{1}{|\mathfrak{F}|}\sum_{i\in\mathfrak{F}} R_i$ (note that $\|R\|_{\mathcal{L}(M,E)} \leq 1$ since each $\|R_i\|_{\mathcal{L}(M,E)} \leq 1$). By the metric mapping property of α and the previous inequality we get

$$|\langle \overline{P}, w\rangle| \leq \varepsilon + \|P\|_{(\widetilde{\otimes}_\alpha^{n,s} E)'} \alpha\Big(\sum_{k=1}^r \otimes^n z_k, \otimes^{n,s} M\Big),$$

which ends the proof. $\square$

The following result can be seen as a symmetric version of the Extension Lemma [9, 13.2].

Corollary 3.4. *Let α be a finitely generated s-tensor norm of order n and $P \in (\widetilde{\otimes}_\alpha^{n,s} E)'$ be a polynomial. Then the Aron–Berner extension $\mathrm{AB}(P)$ of P is in $(\widetilde{\otimes}_\alpha^{n,s} E'')'$ and*

$$\|P\|_{(\widetilde{\otimes}_\alpha^{n,s} E)'} = \|\mathrm{AB}(P)\|_{(\widetilde{\otimes}_\alpha^{n,s} E'')'}.$$

Therefore, if $\mathcal{Q}$ is a maximal ideal of n-homogeneous polynomials and we take $P \in \mathcal{Q}(E)$, then its Aron–Berner extension is in $\mathcal{Q}(E'')$ and

$$\|P\|_{\mathcal{Q}(E)} = \|\mathrm{AB}(P)\|_{\mathcal{Q}(E'')}.$$

Proof. Let $(E)_{\mathfrak{U}}$ be a local ultrapower of E and $J_E : E'' \to (E)_{\mathfrak{U}}$ the canonical embedding. By Proposition 2.3 the iterated extension to the local ultrapower of E restricted to E'' coincides with the Aron–Berner extension of P. In other words, $\mathrm{AB}(P) = \overline{P} \circ J_E$. Therefore,

$$\|\mathrm{AB}(P)\|_{\mathcal{Q}(E'')} = \|\overline{P}\circ J_E\|_{\mathcal{Q}(E'')} \leq \|\overline{P}\|_{\mathcal{Q}((E)_{\mathfrak{U}})}\|J_E\|^n = \|P\|_{\mathcal{Q}(E)}.$$

The other inequality is immediate. $\square$

As a direct consequence we also obtain a symmetric version of the Embedding Lemma [9, 13.3]:

Corollary 3.5. *The natural maps*

$$\otimes^{n,s} J_E : \otimes_\alpha^{n,s} E \to \otimes_\alpha^{n,s} (E)_{\mathfrak{U}},$$

$$\otimes^{n,s} \kappa_E : \otimes_\alpha^{n,s} E \to \otimes_\alpha^{n,s} E''$$

are isometries for every finitely generated tensor norm α.

Proof. Clearly $\|\otimes^{n,s} J_E\| \le 1$, and $\langle P, w \rangle = \langle \overline{P}, \otimes^{n,s} J_E w \rangle$ gives the remaining inequality. Similarly we infer that $\otimes^{n,s} \kappa_E$ is an isometry. $\square$

Now we turn our attention to minimal ideals of polynomials. In order to show that the Aron–Berner extension is also an isometry for minimal ideals, we first need the following simple result:

Lemma 3.6. *Let $T : E \to F$ be an approximable operator. Then $(T)_{\mathfrak{U}} : (E)_{\mathfrak{U}} \to (F)_{\mathfrak{U}}$ is also approximable.*

Proof. It is sufficient to show that if $T : E \to F$ is a rank-one operator, then so too is $(T)_{\mathfrak{U}}$. Let $x' \in E'$ be such that $T(x) = x'(x)y$. If we denote $\lambda := \lim_{i,\mathfrak{U}} x'(x_i)$ we see easily that $(T)_{\mathfrak{U}}(x_i)_{\mathfrak{U}} = (x'(x_i)y)_{\mathfrak{U}} = \lambda h_F y$. $\square$

Theorem 3.7. *Let $\mathcal{Q}$ be a minimal ideal. If $P \in \mathcal{Q}(E)$, then the iterated extension $\overline{P}$ belongs to $\mathcal{Q}((E)_{\mathfrak{U}})$ and*

$$\|P\|_{\mathcal{Q}(E)} = \|\overline{P}\|_{\mathcal{Q}((E)_{\mathfrak{U}})}.$$

Proof. Since $P \in \mathcal{Q}(E) \overset{1}{=} ((\mathcal{Q}^{\max})^{\min})(E)$ (see [14, 3.4]), given $\varepsilon > 0$ there exist a Banach space F, an approximable operator $T : E \to F$ and a polynomial $Q \in \mathcal{Q}^{\max}(F)$ such that $P = Q \circ T$ and $\|Q\|_{\mathcal{Q}^{\max}(F)} \|T\|^n \le \|P\|_{\mathcal{Q}(E)} + \varepsilon$ (as in (2.1)). Notice that $\overline{P} = \overline{Q} \circ (T)_{\mathfrak{U}}$. By Theorem 3.3 we have $\|Q\|_{\mathcal{Q}^{\max}(F)} = \|\overline{Q}\|_{\mathcal{Q}^{\max}((F)_{\mathfrak{U}})}$. Since by Lemma 3.6, $(T)_{\mathfrak{U}}$ is also approximable, we conclude that

$$\|\overline{P}\|_{\mathcal{Q}((E)_{\mathfrak{U}})} \le \|\overline{Q}\|_{\mathcal{Q}^{\max}((F)_{\mathfrak{U}})} \|(T)_{\mathfrak{U}}\|^n = \|Q\|_{\mathcal{Q}^{\max}(F)} \|T\|^n \le \|P\|_{\mathcal{Q}(E)} + \varepsilon.$$

The reverse inequality is immediate. $\square$

Similarly to Corollary 3.4 we have:

Corollary 3.8. *Let $\mathcal{Q}$ be a minimal ideal. For $P \in \mathcal{Q}(E)$, its Aron–Berner extension $\mathrm{AB}(P)$ belongs to $\mathcal{Q}(E'')$ and*

$$\|P\|_{\mathcal{Q}(E)} = \|\mathrm{AB}(P)\|_{\mathcal{Q}(E'')}.$$

We end this note with some comments on the extension of analytic functions associated to polynomial ideals.

The concept of holomorphy type was introduced by Nachbin in [23] (see also [11]). The most natural holomorphy types can be seen as sequences of polynomial ideals $\mathcal{Q} = \{\mathcal{Q}_k\}_k$ ($\mathcal{Q}_k$ is an ideal of polynomials of degree k, $k = 1, 2, \dots$), where some kind of affinity between ideals of different degrees is necessary [3, 6]. In [5], given such a sequence of polynomial ideals, an associated Fréchet space of entire functions is defined. In [22], the corresponding definition for analytic functions defined on the unit ball of a Banach is given:

Definition 3.9. Let $\mathcal{Q} = \{\mathcal{Q}_k\}_k$ be a sequence of polynomial ideals and E be a Banach space. The space of $\mathcal{Q}$-*holomorphic functions of bounded type* on B_E is defined as

$$H_{b\mathcal{Q}}(B_E) = \left\{ f \in H(B_E) : \frac{d^k f(0)}{k!} \in \mathcal{Q}_k(E) \text{ and } \lim_{k \to \infty} \left\| \frac{d^k f(0)}{k!} \right\|_{\mathcal{Q}_k(E)}^{1/k} < 1 \right\}.$$

An example of this kind of space is that of boundedly-integral holomorphic functions in the ball H_{bI}, studied in [10].

An immediate consequence of our results is the following: let $\mathcal{Q} = \{\mathcal{Q}_k\}_k$ be a sequence of polynomial ideals, each $\mathcal{Q}_k$ being either maximal or minimal. If E is a Banach space, then a holomorphic function f belongs to $H_{b\mathcal{Q}}(B_E)$ if and only if its Aron–Berner extension belongs to $H_{b\mathcal{Q}}(B_{E''})$. Note that no coherence between ideals of different degrees is needed for this to hold.

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