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Segal–Bargmann Transform and Paley–Wiener Theorems on Motion Groups

by

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Abstract

We study the Segal–Bargmann transform on a motion group $\mathbb{R}^n \ltimes K$, where K is a compact subgroup of $SO(n)$. A characterization of the Poisson integrals associated to the Laplacian on $\mathbb{R}^n \ltimes K$ is given. We also establish a Paley–Wiener type theorem using complexified representations.

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§1. Introduction

The Segal–Bargmann transform, also called the coherent state transform, was developed independently in the early 1960's by Segal in the infinite-dimensional context of scalar quantum field theories and by Bargmann in the finite-dimensional context of quantum mechanics on $\mathbb{R}^n$. We consider the following equivalent form of Bargmann's original result.

A function $f \in L^2(\mathbb{R}^n)$ admits a factorization $f(x) = g * p_t(x)$ where $g \in L^2(\mathbb{R}^n)$ and $p_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$ (the heat kernel on $\mathbb{R}^n$) if and only if f extends as an entire function to $\mathbb{C}^n$ and $(2\pi t)^{-n/2} \int_{\mathbb{C}^n} |f(z)|^2 e^{-|y|^2/(2t)}\, dx\, dy < \infty$ ($z = x + iy$). In this case we also have

$$\|g\|_2^2 = \frac{1}{(2\pi t)^{n/2}} \int_{\mathbb{C}^n} |f(z)|^2 e^{-|y|^2/(2t)}\, dx\, dy.$$

The mapping $g \mapsto g * p_t$ is called the *Segal–Bargmann transform* and the above says that the Segal–Bargmann transform is a unitary map from $L^2(\mathbb{R}^n)$ onto $\mathcal{O}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, \mu)$, where $d\mu(z) = (2\pi t)^{-n/2} e^{-|y|^2/(2t)}\, dx\, dy$ and $\mathcal{O}(\mathbb{C}^n)$ denotes the space of entire functions on $\mathbb{C}^n$.

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In the paper [4], B. C. Hall introduced a generalization of the Segal–Bargmann transform on a compact Lie group. If K is such a group, this coherent state transform maps $L^2(K)$ isometrically onto the space of holomorphic functions in $L^2(G,\mu_t)$, where G is the complexification of K and μ_t is an appropriate heat kernel measure on G. The generalized coherent state transform is defined in terms of the heat kernel on the compact group K and its analytic continuation to the complex group G. Similar results have been proved by various authors. See [12], [6], [5], [8] and [7].

Next, consider the following result on $\mathbb{R}$ due to Paley and Wiener. A function $f \in L^2(\mathbb{R})$ admits a holomorphic extension to the strip $\{x+iy : |y| < t\}$ such that

$$\sup_{|y|\le s} \int_{\mathbb{R}} |f(x+iy)|^2 \, dx < \infty \quad \forall s < t$$

if and only if

$$\int_{\mathbb{R}} e^{s|\xi|} |\widetilde{f}(\xi)|^2 \, d\xi < \infty \quad \forall s < t \tag{1.1}$$

where $\widetilde{f}$ denotes the Fourier transform of f.

The condition (1.1) is the same as

$$\int_{\mathbb{R}} |\widetilde{e^{s\Delta^{1/2}} f}(\xi)|^2 \, d\xi < \infty \quad \forall s < t$$

where Δ is the Laplacian on $\mathbb{R}$. This point of view was explored by R. Goodman in Theorem 2.1 of [2].

The condition (1.1) is also equivalent to

$$\int_{\mathbb{R}} |e^{i(x+iy)\xi}|^2 |\widetilde{f}(\xi)|^2 \, d\xi < \infty \quad \forall |y| < t.$$

Here $\xi \mapsto e^{i(x+iy)\xi}$ may be seen as the complexification of the parameters of the unitary irreducible representations $\xi \mapsto e^{ix\xi}$ of $\mathbb{R}$. This point of view was also further developed by R. Goodman (see Theorem 3.1 from [3]). Similar results were established for the Euclidean motion group $M(2)$ of the plane $\mathbb{R}^2$ in [11]. The aim of this paper is to prove corresponding results in the context of general motion groups.

The plan of this paper is as follows: In the following section we recall the representation theory and Plancherel theorem of the motion group M. We also describe the Laplacian on M. In the next section we prove the unitarity of the Segal–Bargmann transform on M and we study the generalized Segal–Bargmann transform, obtaining analogues of Theorems 8 and 10 in [4]. The fourth section is devoted to a study of Poisson integrals on M via a Gutzmer-type formula on M

which is proved by using a Gutzmer formula for compact Lie groups established by Lassalle in 1978 (see [9]). This section is modelled after the work of Goodman [2]. In the final section we prove another characterization of functions extending holomorphically to the complexification of M, which is an analogue of Theorem 3.1 of [3].

§2. Preliminaries

Let K be a compact, connected Lie group which acts as a linear group on a finite-dimensional real vector space V. Let M be the semidirect product of V and K with the group law

$$(x_1, k_1) \cdot (x_2, k_2) = (x_1 + k_1 x_2, k_1 k_2) \quad \text{where } x_1, x_2 \in V,\ k_1, k_2 \in K.$$

Then M is called the *motion group*. Since K is compact, there exists a K-invariant inner product on V. Hence, we can assume that K is a connected subgroup of $SO(n)$, where $n = \dim V$. When $K = \{1\}$, $M = V \cong \mathbb{R}^n$ and if $K = SO(n)$, M is the Euclidean motion group. Henceforth we shall identify V with $\mathbb{R}^n$ and K with a subgroup of $SO(n)$.

The group M may be identified with a matrix subgroup of $GL(n+1, \mathbb{R})$ via the map

$$(x, k) \mapsto \begin{pmatrix} k & x \\ 0 & 1 \end{pmatrix}$$

where $x \in \mathbb{R}^n$ and $k \in K \subseteq SO(n)$.

We normalize the Haar measure dm on M in such a way that $dm = dx\, dk$, where $dx = (2\pi)^{-n/2}\, dx_1 \cdots dx_n$ and dk is the normalized Haar measure on K. Let $\mathcal{H} = L^2(K)$ be the Hilbert space of all square integrable functions on K. Denote by $\langle \cdot, \cdot \rangle$ the Euclidean inner product on $\mathbb{R}^n$. Let $\widehat{V}$ be the dual space of V. Then we can identify $\widehat{V}$ with $\mathbb{R}^n$ so that K acts on $\widehat{V}$ naturally by $\langle k \cdot \xi, x \rangle = \langle \xi, k^{-1} \cdot x \rangle$ where $\xi \in \widehat{V},\ x \in V,\ k \in K$.

For any $\xi \in \widehat{V}$ let U^ξ denote the induced representation of M by the unitary representation $x \mapsto e^{i\langle \xi, x \rangle}$ of V. Then for $F \in \mathcal{H}$ and $(x, k) \in M$,

$$U^\xi_{(x,k)} F(u) = e^{i\langle x, u \cdot \xi \rangle} F(k^{-1} u).$$

The representation U^ξ is not irreducible. Any irreducible unitary representation of M is, however, contained in U^ξ for some $\xi \in \widehat{V}$ as an irreducible component.

Let K_ξ be the isotropy subgroup of $\xi \in \widehat{V}$, i.e. $K_\xi = \{k \in K : k \cdot \xi = \xi\}$. Consider $\sigma \in \widehat{K_\xi}$, the unitary dual of K_ξ. Denote by χ_σ, d_σ and σ_{ij} the character, degree and matrix coefficients of σ respectively. Let R be the right regular representation of K. Define

$$P^\sigma = d_\sigma \int_{K_\xi} \overline{\chi_\sigma(w)} R_w \, dw \quad \text{and} \quad P^\sigma_\gamma = d_\sigma \int_{K_\xi} \overline{\sigma_{\gamma\gamma}(w)} R_w \, dw$$

where dw is the normalized Haar measure on K_ξ. Then P^σ and P^σ_γ are both orthogonal projections on $\mathcal{H}$. Let $\mathcal{H}^\sigma = P^\sigma\mathcal{H}$ and $\mathcal{H}^\sigma_\gamma = P^\sigma_\gamma\mathcal{H}$. The subspaces $\mathcal{H}^\sigma_\gamma$ are invariant under U^ξ for $1 \le \gamma \le d_\sigma$ and the representations of M induced on $\mathcal{H}^\sigma_\gamma$ under U^ξ are equivalent for all $1 \le \gamma \le d_\sigma$. We fix one of them and denote it by $U^{\xi,\sigma}$. Two representations $U^{\xi,\sigma}$ and $U^{\xi',\sigma'}$ are equivalent if and only if there exists an element $k \in K$ such that $\xi = k \cdot \xi'$ and σ' is equivalent to σ^k where $\sigma^k(w) = \sigma(kwk^{-1})$ for $w \in K_\xi$.

The Mackey theory [10] shows that under certain conditions on K (for details refer to Section 6.6 of [1]), each $U^{\xi,\sigma}$ is irreducible and every infinite-dimensional irreducible unitary representation is equivalent to $U^{\xi,\sigma}$ for some $\xi \in \mathbb{R}^n$ and $\sigma \in \widehat{K_\xi}$. Since $\mathcal{H} = \bigoplus_{\sigma\in\widehat{K_\xi}} \mathcal{H}^\sigma$ and $\mathcal{H}^\sigma = \bigoplus_{\gamma=1}^{d_\sigma} \mathcal{H}^\sigma_\gamma$, we have

$$U^\xi \cong \bigoplus_{\sigma\in\widehat{K_\xi}} d_\sigma U^{\xi,\sigma}.$$

For any $f \in L^1(M)$ define the Fourier transform of f by

$$\widehat{f}(\xi,\sigma) = \int_M f(m) U^{\xi,\sigma}_m \, dm.$$

Then the Plancherel formula gives

$$\int_M |f(m)|^2 \, dm = \int_{\mathbb{R}^n} \sum_{\sigma\in\widehat{K_\xi}} d_\sigma \|\widehat{f}(\xi,\sigma)\|^2_{\mathrm{HS}} \, d\xi$$

where $\|\cdot\|_{\mathrm{HS}}$ is the Hilbert–Schmidt norm of an operator. We will be working with the generalized Fourier transform defined by

$$\widehat{f}(\xi) = \int_M f(m) U^\xi_m \, dm.$$

Then we also have

$$\int_M |f(m)|^2 \, dm = \int_{\mathbb{R}^n} \|\widehat{f}(\xi)\|^2_{\mathrm{HS}} \, d\xi.$$

Let $\underline{k}$ and $\underline{m}$ be the Lie algebras of K and M respectively. Then

$$\underline{m} = \left\{ \begin{pmatrix} K & X \\ 0 & 0 \end{pmatrix} : X \in \mathbb{R}^n,\ K \in \underline{k} \right\}.$$

Let $K_1, \ldots, K_N$ and $X_1, \ldots, X_n$ be orthonormal bases of $\underline{k}$ and $\mathbb{R}^n$ respectively. Define

$$M_i = \begin{cases} \begin{pmatrix} K_i & 0 \\ 0 & 0 \end{pmatrix} & \text{for } 1 \le i \le N, \\ \begin{pmatrix} 0 & X_i \\ 0 & 0 \end{pmatrix} & \text{for } N+1 \le i \le N+n. \end{cases}$$

Then it is easy to see that $\{M_1, \ldots, M_{N+n}\}$ forms a basis for $\underline{m}$. The Laplacian $\Delta_M = \Delta$ is defined by

$$\Delta = -(M_1^2 + \cdots + M_{N+n}^2).$$

A simple computation using the fact that $K \subseteq SO(n)$ shows that

$$\Delta = -\Delta_{\mathbb{R}^n} - \Delta_K$$

where $\Delta_{\mathbb{R}^n}$ and Δ_K are the Laplacians on $\mathbb{R}^n$ and K respectively given by $\Delta_{\mathbb{R}^n} = X_1^2 + \cdots + X_n^2$ and $\Delta_K = K_1^2 + \cdots + K_N^2$.

§3. Segal–Bargmann transform and its generalization

Since $\Delta_{\mathbb{R}^n}$ and Δ_K commute, it follows that the heat kernel ψ_t associated to Δ is given by the product of the heat kernels p_t on $\mathbb{R}^n$ and q_t on K. In other words

$$\psi_t(x,k) = p_t(x)q_t(k) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)} \sum_{\pi \in \widehat{K}} d_\pi e^{-\lambda_\pi t/2} \chi_\pi(k).$$

Here, for each unitary, irreducible representation π of K, d_π is the degree of π, λ_π is such that $\pi(\Delta_K) = -\lambda_\pi I$, and $\chi_\pi(k) = \mathrm{Tr}(\pi(k))$ is the character of π.

Denote by G the complexification of K. Let κ_t be the fundamental solution at the identity of the following equation on G:

$$\frac{du}{dt} = \frac{1}{4}\Delta_G u$$

where Δ_G is the Laplacian on G (for details see [4]). It should be noted that κ_t is the real, positive heat kernel on G which is not the same as the analytic continuation of q_t on K.

Let $\mathcal{H}(\mathbb{C}^n \times G)$ be the Hilbert space of holomorphic functions on $\mathbb{C}^n \times G$ which are square integrable with respect to $\mu \otimes \nu(z,g)$ where

$$d\mu(z) = \frac{1}{(2\pi t)^{n/2}} e^{-|y|^2/(2t)}\, dx\, dy \quad \text{on } \mathbb{C}^n$$

and

$$d\nu(g) = \int_K \kappa_t(xg)\, dx \quad \text{on } G.$$

Then we have the following theorem:

Theorem 3.1. *If $f \in L^2(M)$, then $f * \psi_t$ extends holomorphically to $\mathbb{C}^n \times G$. Moreover, the map $C_t : f \mapsto f * \psi_t$ is a unitary map from $L^2(M)$ onto $\mathcal{H}(\mathbb{C}^n \times G)$.*

Proof. Let $f \in L^2(M)$. Expanding f in the K-variable using the Peter–Weyl theorem we obtain

$$f(x,k) = \sum_{\pi \in \widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} f_{ij}^\pi(x) \phi_{ij}^\pi(k)$$

where for each $\pi \in \widehat{K}$, d_π is the degree of π, ϕ_{ij}^π's are the matrix coefficients of π and $f_{ij}^\pi(x) = \int_K f(x,k)\overline{\phi_{ij}^\pi(k)}\, dk$. Here, the convergence is understood in the L^2-sense. Moreover, by the universal property of the complexification of a compact Lie group (see Section 3 of [4]), all the representations of K, and hence all the matrix entries, extend to G holomorphically.

Since ψ_t is K-invariant (as a function on $\mathbb{R}^n$) a simple computation shows that

$$f * \psi_t(x,k) = \sum_{\pi \in \widehat{K}} d_\pi e^{-\lambda_\pi t/2} \sum_{i,j=1}^{d_\pi} f_{ij}^\pi * p_t(x) \phi_{ij}^\pi(k).$$

It is easily seen that $f_{ij}^\pi \in L^2(\mathbb{R}^n)$ for every $\pi \in \widehat{K}$ and $1 \le i,j \le d_\pi$. Hence $f_{ij}^\pi * p_t$ extends to a holomorphic function on $\mathbb{C}^n$ and by the unitarity of the Segal–Bargmann transform in $\mathbb{R}^n$ we have, for $z = x+iy \in \mathbb{C}^n$,

$$\int_{\mathbb{C}^n} |f_{ij}^\pi * p_t(z)|^2 \mu(y)\, dx\, dy = \int_{\mathbb{R}^n} |f_{ij}^\pi(x)|^2\, dx. \tag{3.1}$$

The analytic continuation of $f * \psi_t$ to $\mathbb{C}^n \times G$ is given by

$$f * \psi_t(z,g) = \sum_{\pi \in \widehat{K}} d_\pi e^{-\lambda_\pi t/2} \sum_{i,j=1}^{d_\pi} f_{ij}^\pi * p_t(z) \phi_{ij}^\pi(g).$$

We claim that the above series converges uniformly on compact subsets of $\mathbb{C}^n \times G$ so that $f * \psi_t$ extends to a holomorphic function on $\mathbb{C}^n \times G$. We know from [4, Section 4, Proposition 1] that the holomorphic extension of the heat kernel q_t on K is given by

$$q_t(g) = \sum_{\pi \in \widehat{K}} d_\pi e^{-\lambda_\pi t/2} \chi_\pi(g).$$

For each $g \in G$, define $q_t^g(k) = q_t(gk)$. Then q_t^g is a smooth function on K and is given by

$$q_t^g(k) = \sum_{\pi \in \widehat{K}} d_\pi e^{-\lambda_\pi t/2} \chi_\pi(gk) = \sum_{\pi \in \widehat{K}} d_\pi e^{-\lambda_\pi t/2} \sum_{i,j=1}^{d_\pi} \phi_{ij}^\pi(g) \phi_{ji}^\pi(k).$$

Since q_t^g is a smooth function on K, we have for each $g \in G$,

$$\int_K |q_t^g(k)|^2\,dk = \sum_{\pi\in\widehat{K}} d_\pi e^{-\lambda_\pi t} \sum_{i,j=1}^{d_\pi} |\phi_{ij}^\pi(g)|^2 < \infty. \tag{3.2}$$

Let L be a compact set in $\mathbb{C}^n \times G$. For $(z,g) \in L$ we have

$$|f * \psi_t(z,g)| \le \sum_{\pi\in\widehat{K}} d_\pi e^{-x\lambda_\pi t/2} \sum_{i,j=1}^{d_\pi} |f_{ij}^\pi * p_t(z)|\,|\phi_{ij}^\pi(g)|. \tag{3.3}$$

By the Fourier inversion

$$f_{ij}^\pi * p_t(z) = \int_{\mathbb{R}^n} \widetilde{f_{ij}^\pi}(\xi) e^{-t|\xi|^2} e^{i\xi\cdot(x+iy)}\,d\xi$$

where $z = x+iy \in \mathbb{C}^n$ and $\widetilde{f_{ij}^\pi}$ is the Fourier transform of f_{ij}^π. Hence, if z varies in a compact subset of $\mathbb{C}^n$ (namely, the projection of L in $\mathbb{C}^n$), we have

$$|f_{ij}^\pi * p_t(z)| \le \|f_{ij}^\pi\|_2 \int_{\mathbb{R}^n} e^{-2(t|\xi|^2+y\cdot\xi)}\,d\xi \le C_L \|f_{ij}^\pi\|_2.$$

Using the above in (3.3) and applying the Cauchy–Schwarz inequality we get

$$\begin{aligned}
|f * \psi_t(z,g)| &\le C_L \sum_{\pi\in\widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \|f_{ij}^\pi\|_2 e^{-\lambda_\pi t/2} |\phi_{ij}^\pi(g)| \\
&\le C_L \Bigg(\sum_{\pi\in\widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \int_{\mathbb{R}^n} |f_{ij}^\pi(x)|^2\,dx \Bigg)^{1/2} \Bigg(\sum_{\pi\in\widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} e^{-\lambda_\pi t} |\phi_{ij}^\pi(g)|^2 \Bigg)^{1/2}.
\end{aligned}$$

Noting that

$$\|f\|_2^2 = \sum_{\pi\in\widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \int_{\mathbb{R}^n} |f_{ij}^\pi(x)|^2\,dx$$

and q_t is a smooth function on G we prove the claim using (3.2). Applying Theorem 2 in [4] we get

$$\int_G |f * \psi_t(z,g)|^2\,d\nu(g) = \sum_{\pi\in\widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} |f_{ij}^\pi * p_t(z)|^2.$$

Integrating the above against $\mu(y)\,dx\,dy$ on $\mathbb{C}^n$ and using (3.1) we conclude that C_t is isometric,

$$\int_{\mathbb{C}^n} \int_G |f * \psi_t(z,g)|^2 \mu(y)\,dx\,dy\,d\nu(g) = \|f\|_2^2.$$

To prove that the map C_t is surjective it suffices to prove that the range of C_t is dense in $\mathcal{H}(\mathbb{C}^n \times G)$. For this, consider functions of the form $f(x,k) = h_1(x)h_2(k) \in L^2(M)$ where $h_1 \in L^2(\mathbb{R}^n)$, $h_2 \in L^2(K)$. Then a simple computation shows that

$$f * \psi_t(z,g) = h_1 * p_t(z) h_2 * q_t(g) \quad \text{for } (z,g) \in \mathbb{C}^n \times G.$$

Suppose $F \in \mathcal{H}(\mathbb{C}^n \times G)$ is such that

$$\int_{\mathbb{C}^n \times G} F(z,g)\overline{h_1 * p_t(z) h_2 * q_t(g)}\mu(y)\,dx\,dy\,d\nu(g) = 0 \tag{3.4}$$

for all $h_1 \in L^2(\mathbb{R}^n)$ and $h_2 \in L^2(K)$. From (3.4) we have

$$\int_G \left(\int_{\mathbb{C}^n} F(z,g)\overline{h_1 * p_t(z)}\,d\mu(z)\right)\overline{h_2 * q_t(g)}\,d\nu(g) = 0,$$

which by Theorem 2 of [4] implies that

$$\int_{\mathbb{C}^n} F(z,g)\overline{h_1 * p_t(z)}\,d\mu(z) = 0.$$

Finally, an application of the surjectivity of the Segal–Bargmann transform on $\mathbb{R}^n$ shows that $F \equiv 0$, as desired. □

In [4] Brian C. Hall proved the following generalizations of the Segal–Bargmann transforms for $\mathbb{R}^n$ and compact Lie groups:

Theorem 3.2. (I) *Let μ be any measurable function on $\mathbb{R}^n$ such that*

- *μ is strictly positive and locally bounded away from zero,*
- *for all $x \in \mathbb{R}^n$, $\sigma(x) = \int_{\mathbb{R}^n} e^{2x\cdot y}\mu(y)\,dy < \infty$.*

Define, for $z \in \mathbb{C}^n$,

$$\psi(z) = \int_{\mathbb{R}^n} \frac{e^{ia(y)}}{\sqrt{\sigma(y)}} e^{-iy\cdot z}\,dy,$$

where a is a real-valued measurable function on $\mathbb{R}^n$. Then the mapping C_ψ : $L^2(\mathbb{R}^n) \to \mathcal{O}(\mathbb{C}^n)$ defined by

$$C_\psi(z) = \int_{\mathbb{R}^n} f(x)\psi(z-x)\,dx$$

is an isometric isomorphism of $L^2(\mathbb{R}^n)$ onto $\mathcal{O}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, dx\,\mu(y)\,dy)$.

(II) *Let K be a compact Lie group and G be its complexification. Let ν be a measure on G such that*

- *ν is bi-K-invariant,*
- *ν is given by a positive density which is locally bounded away from zero,*
- *for each irreducible representation π of K, analytically continued to G,*

$$\delta(\pi) = \frac{1}{\dim V_\pi} \int_G \|\pi(g^{-1})\|^2 \, d\nu(g) < \infty.$$

Define

$$\tau(g) = \sum_{\pi \in \widehat{K}} \frac{d_\pi}{\sqrt{\delta(\pi)}} \operatorname{Tr}(\pi(g^{-1}) U_\pi)$$

where $g \in G$ and U_π's are arbitrary unitary matrices. Then the mapping

$$C_\tau f(g) = \int_K f(k) \tau(k^{-1} g) \, dk$$

is an isometric isomorphism of $L^2(K)$ onto $\mathcal{O}(G) \cap L^2(G, d\nu(w))$.

A similar result holds for M. Let μ be any real-valued K-invariant function on $\mathbb{R}^n$ that satisfies the conditions of Theorem 3.2(I). Define, for $z \in \mathbb{C}^n$,

$$\psi(z) = \int_{\mathbb{R}^n} \frac{e^{ia(y)}}{\sqrt{\sigma(y)}} e^{-iy\cdot z} \, dy$$

where a is a real valued measurable K-invariant function on $\mathbb{R}^n$. Next, let ν, δ and τ be as in Theorem 3.2(II). Also define $\phi(z, g) = \psi(z)\tau(g)$ for $z \in \mathbb{C}^n$, $g \in G$. It is easy to see that $\phi(z, w)$ is a holomorphic function on $\mathbb{C}^n \times G$. Then it is easy to prove the following analogue of Theorem 3.2 for M.

Theorem 3.3. *The mapping*

$$C_\phi f(z, g) = \int_M f(\xi, k) \phi((\xi, k)^{-1}(z, g)) \, d\xi \, dk$$

is an isometric isomorphism of $L^2(M)$ onto

$$\mathcal{O}(\mathbb{C}^n \times G) \cap L^2(\mathbb{C}^n \times G, \mu(y) \, dx \, dy \, d\nu(g)).$$

§4. Gutzmer's formula and Poisson integrals

In this section we first briefly recall Gutzmer's formula on compact, connected Lie groups given by Lassalle in [9]. Let $\underline{k}$ and $\underline{g}$ be the Lie algebras of a compact, connected Lie group K and its complexification G. Then we can write $\underline{g} = \underline{k} + \underline{p}$

where $\underline{p} = i\underline{k}$ and any element $g \in G$ can be written in the form $g = k \exp iH$ for some $k \in K$ and $H \in \underline{k}$. If $\underline{h}$ is a maximal abelian subalgebra of $\underline{k}$ and $\underline{a} = i\underline{h}$ then every element of $\underline{p}$ is conjugate under K to an element of $\underline{a}$. Thus each $g \in G$ can be written (non-uniquely) in the form $g = k_1 \exp(iH)k_2$ for $k_1, k_2 \in K$ and $H \in \underline{h}$. If $k_1 \exp(iH_1)k_1' = k_2 \exp(iH_2)k_2'$, then there exists $w \in W$, the Weyl group with respect to $\underline{h}$, such that $H_1 = w \cdot H_2$ where $\cdot$ denotes the action of the Weyl group on $\underline{h}$. Since K is compact, there exists an $\operatorname{Ad} K$-invariant inner product on $\underline{k}$, and hence on $\underline{g}$. Let $|\cdot|$ denote the norm with respect to that inner product. Then we have the following Gutzmer formula by Lassalle.

Theorem 4.1. *Let f be holomorphic in $K \exp(i\Omega_r)K \subseteq G$ where $\Omega_r = \{H \in \underline{k} : |H| < r\}$. Then*

$$\int_K \int_K |f(k_1 \exp(iH)k_2)|^2 \, dk_1 \, dk_2 = \sum_{\pi \in \widehat{K}} \|\widehat{f}(\pi)\|_{\mathrm{HS}}^2 \chi_\pi(\exp 2iH)$$

where $H \in \Omega_r$ and $\widehat{f}(\pi)$ is the operator-valued Fourier transform of f at π defined by $\widehat{f}(\pi) = \int_K f(k)\pi(k^{-1}) \, dk$.

For the proof see [9]. We prove a Gutzmer-type result on M using Lassalle's theorem above. Define $\Omega_{t,r} = \{(z, g) \in \mathbb{C}^n \times G : |\operatorname{Im} z| < t,\ |H| < r$ where $g = k_1 \exp(iH)k_2,\ k_1, k_2 \in K,\ H \in \underline{h}\}$. Notice that the domain $\Omega_{t,r}$ is well defined since $|\cdot|$ is invariant under the Weyl group action.

Lemma 4.2. *Let $f \in L^2(M)$ extend holomorphically to the domain $\Omega_{t,r}$ and*

$$\sup_{\{|y|<s,\, |H|<q\}} \int_{\mathbb{R}^n} \int_K \int_K |f(x + iy, k_1 \exp(iH)k_2)|^2 \, dk_1 \, dk_2 \, dx < \infty$$

for all $s < t$ and $q < r$. Then

$$\int_{\mathbb{R}^n} \int_K \int_K |f(x + iy, k_1 \exp(iH)k_2)|^2 \, dk_1 \, dk_2 \, dx = \sum_{\pi \in \widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \left(\int_{\mathbb{R}^n} |\widetilde{f_{ij}^\pi}(\xi)|^2 e^{-2\xi \cdot y} \, d\xi \right) \chi_\pi(\exp 2iH)$$

provided $|y| < t$ and $|H| < r$. Conversely, if

$$\sup_{\{|y|<s,\, |H|<q\}} \sum_{\pi \in \widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \left(\int_{\mathbb{R}^n} |\widetilde{f_{ij}^\pi}(\xi)|^2 e^{-2\xi \cdot y} \, d\xi \right) \chi_\pi(\exp 2iH) < \infty$$

for all $s < t$ and $q < r$ then f extends holomorphically to the domain $\Omega_{t,r}$ and

$$\sup_{\{|y|<s,\, |H|<q\}} \int_{\mathbb{R}^n} \int_K \int_K |f(x+iy, k_1 \exp(iH)k_2)|^2 \, dk_1 \, dk_2 \, dx < \infty$$

for all $s < t$ and $q < r$.

Proof. Notice that $f_{ij}^\pi(x) = \int_K f(x,k)\overline{\phi_{ij}^\pi(k)}\, dk$. It follows that f_{ij}^π has a holomorphic extension to $\{z \in \mathbb{C}^n : |\mathrm{Im}\, z| < t\}$ and

$$\sup_{|y|<s} \int_{\mathbb{R}^n} |f_{ij}^\pi(x+iy)|^2 \, dx < \infty \qquad \forall s < t.$$

Consequently,

$$\int_{\mathbb{R}^n} |f_{ij}^\pi(x+iy)|^2 \, dx = \int_{\mathbb{R}^n} |\widetilde{f_{ij}^\pi}(\xi)|^2 e^{-2\xi\cdot y} \, d\xi \qquad \text{for } |y| < s \qquad \text{and all } s < t.$$

Now, for each fixed $z \in \mathbb{C}^n$ with $|\mathrm{Im}\, z| < s$ the function $g \mapsto f(z,g)$ is holomorphic in the domain $\{g \in G : |H| < r$ where $g = k_1 \exp(iH)k_2,\ k_1, k_2 \in K,\ H \in \underline{h}\}$ for all $s < t$ and $q < r$ and so admits a holomorphic Fourier series (as in [4])

$$f(z,g) = \sum_{\pi \in \widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} a_{ij}^\pi(z) \phi_{ij}^\pi(g).$$

It follows that $a_{ij}^\pi(z) = f_{ij}^\pi(z)$ for every $\pi \in \widehat{K}$ and $1 \le i,j \le d_\pi$. Hence by using Theorem 4.1 we have, for $(z,g) \in \Omega_{t,r}$,

$$\begin{aligned} \int_K \int_K |f(x+iy, k_1 \exp(iH)k_2)|^2 \, dk_1 \, dk_2 &= \sum_{\pi \in \widehat{K}} \|\widehat{f}_z(\pi)\|_{\mathrm{HS}}^2 \chi_\pi(\exp 2iH) \\ &= \sum_{\pi \in \widehat{K}} \sum_{i,j=1}^{d_\pi} |f_{ij}^\pi(z)|^2 \chi_\pi(\exp 2iH) \end{aligned}$$

where $f_z(g) = f(z,g)$. Integrating over $\mathbb{R}^n$ we get

$$\begin{aligned} &\int_{\mathbb{R}^n} \int_K \int_K |f(x+iy, k_1 \exp(iH)k_2)|^2 \, dk_1 \, dk_2 \, dx \\ &\qquad = \sum_{\pi \in \widehat{K}} \sum_{i,j=1}^{d_\pi} \int_{\mathbb{R}^n} |f_{ij}^\pi(x+iy)|^2 \, dx \, \chi_\pi(\exp 2iH) \\ &\qquad = \sum_{\pi \in \widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \left(\int_{\mathbb{R}^n} |\widetilde{f_{ij}^\pi}(\xi)|^2 e^{-2\xi\cdot y} \, d\xi \right) \chi_\pi(\exp 2iH). \end{aligned}$$

Hence the first part of the lemma is proved. The converse can also be proved similarly. □

Recall that the Laplacian Δ on M is given by $\Delta = -\Delta_{\mathbb{R}^n} - \Delta_K$. If $f \in L^2(M)$ it is easy to see that

$$e^{-t\Delta^{1/2}} f(x,k) = \sum_{\pi \in \widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \left(\int_{\mathbb{R}^n} e^{-t(|\xi|^2+\lambda_\pi)^{1/2}} \widetilde{f_{ij}^\pi}(\xi) e^{i\xi\cdot x}\, d\xi \right) \phi_{ij}^\pi(k).$$

We have the following (almost) characterization of the Poisson integrals. Let $\Omega_{t,r}$ denote the domain defined prior to Lemma 4.2.

Theorem 4.3. *Let $f \in L^2(M)$. Then there exists a constant N such that $g = e^{-t\Delta^{1/2}} f$ extends to a holomorphic function on the domain $\Omega_{t/\sqrt{2},t\sqrt{2}/N}$ and*

$$\sup_{\{|y|<t/\sqrt{2},\,|H|\le t\sqrt{2}/N\}} \int_{\mathbb{R}^n} \int_K \int_K |g(x+iy, k_1 \exp(iH) k_2)|^2\, dk_1\, dk_2\, dx < \infty.$$

Conversely, there exists a fixed constant C such that whenever g is a holomorphic function on $\Omega_{t,2t/C}$ and

$$\sup_{\{|y|<s,\,|H|<2s/C\}} \int_{\mathbb{R}^n} \int_K \int_K |g(x+iy, k_1 \exp(iH) k_2)|^2\, dk_1\, dk_2\, dx < \infty$$

for $s < t$, then for all $s < t$ there exists $f \in L^2(M)$ such that $e^{-s\Delta^{1/2}} f = g$.

Proof. We know that if $f \in L^2(M)$ then

$$g(x,k) = e^{-t\Delta^{1/2}} f(x,k) = \sum_{\pi \in \widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \left(\int_{\mathbb{R}^n} e^{-t(|\xi|^2+\lambda_\pi)^{1/2}} \widetilde{f_{ij}^\pi}(\xi) e^{i\xi\cdot x}\, d\xi \right) \phi_{ij}^\pi(k).$$

Also,

$$g(x,k) = \sum_{\pi \in \widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} g_{ij}^\pi(x) \phi_{ij}^\pi(k) \quad \text{with} \quad \widetilde{g_{ij}^\pi}(\xi) = \widetilde{f_{ij}^\pi}(\xi) e^{-t(|\xi|^2+\lambda_\pi)^{1/2}}.$$

From Lemmas 6 and 7 of [4] we know that there exist constants a, M such that $\lambda_\pi \ge a|\mu|^2$ and $|\chi_\pi(\exp iY)| \le d_\pi e^{M|Y|\,|\mu|}$ where μ is the highest weight of π. Hence we have

$$|\chi_\pi(\exp 2iH)| \le d_\pi e^{2M|H|\,|\mu|} \le d_\pi e^{N|H|\sqrt{\lambda_\pi}}$$

where $N = 2M/\sqrt{a}$. If $s \le t\sqrt{2}/N$ it is easy to show that

$$\sup_{\{\xi\in\mathbb{R}^n,\,\lambda_\pi\ge 0\}} e^{-2t(|\xi|^2+\lambda_\pi)^{1/2}} e^{2|\xi|\,|y|} e^{N|\sqrt{\lambda_\pi}|s} \le C < \infty \qquad \text{for } |y| \le t/\sqrt{2}.$$

It follows that

$$\sup_{\{|y|<t/\sqrt{2},\,|H|\le t\sqrt{2}/N\}} \sum_{\pi\in\widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \left(\int_{\mathbb{R}^n} |\widetilde{g_{ij}^{\pi}}(\xi)|^2 e^{-2\xi\cdot y}\, d\xi\right) e^{N\sqrt{\lambda_\pi}|H|} < \infty.$$

So we have

$$\sup_{\{|y|<t/\sqrt{2},\,|H|\le t\sqrt{2}/N\}} \sum_{\pi\in\widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} \left(\int_{\mathbb{R}^n} |\widetilde{g_{ij}^{\pi}}(\xi)|^2 e^{-2\xi\cdot y}\, d\xi\right) \chi_\pi(\exp 2iH) < \infty.$$

Hence by Lemma 4.2 we have proved the first part of the theorem.

To prove the converse, we first show that there exist constants A, C such that

$$\int_{|H|=r} \chi_\pi(\exp 2iH)\, d\sigma_r(H) \ge d_\pi A e^{Cr\sqrt{\lambda_\pi}} \tag{4.1}$$

where $d\sigma_r(H)$ is the normalized surface measure on the sphere $\{H \in \underline{h} : |H| = r\} \subseteq \mathbb{R}^m$ where $m = \dim \underline{h}$. If $H \in \underline{a}$, then there exists a non-singular matrix Q and pure-imaginary-valued linear forms $\nu_1, \ldots, \nu_{d_\pi}$ on $\underline{a}$ such that

$$Q\pi(H)Q^{-1} = \operatorname{diag}(\nu_1(H), \ldots, \nu_{d_\pi}(H))$$

where $\operatorname{diag}(a_1, \ldots, a_k)$ denotes the $k \times k$ diagonal matrix with diagonal entries $a_1, \ldots, a_k$. Now, $\nu(H) = i\langle \nu, H\rangle$ where ν is a weight of π. Then

$$\exp(2iQ\pi(H)Q^{-1}) = Q\exp(2i\pi(H))Q^{-1} = \operatorname{diag}(e^{2i\nu_1(H)}, \ldots, e^{2i\nu_{d_\pi}(H)}).$$

Hence

$$\begin{aligned}\chi_\pi(\exp 2iH) &= \operatorname{Tr}(Q\exp(2i\pi(H))Q^{-1})\\ &= e^{-2\langle\nu_1,H\rangle} + \cdots + e^{-2\langle\nu_{d_\pi},H\rangle} \ge e^{-2\langle\mu,H\rangle}\end{aligned}$$

where μ is the highest weight corresponding to π. Integrating the above over $|H| = r$ we get

$$\begin{aligned}\int_{|H|=r} \chi_\pi(\exp 2iH)\, d\sigma_r(H) &\ge \int_{|H|=r} e^{-2\langle\mu,H\rangle}\, d\sigma_r(H)\\ &= \frac{J_{m/2-1}(2ir|\mu|)}{(2ir|\mu|)^{m/2-1}} \ge Be^{r|\mu|}\end{aligned}$$

where $J_{m/2-1}$ is the Bessel function of order $m/2-1$. By Weyl's dimension formula we know that d_π can be written as a polynomial in μ and $\lambda_\pi \approx |\mu|^2$. Hence

$$\int_{|H|=r} \chi_\pi(\exp 2iH)\, d\sigma_r(H) \ge Ad_\pi e^{Cr\sqrt{\lambda_\pi}}$$

for some C. Consider the domain $\Omega_{t,2t/C}$ for that C. Let g be a holomorphic function on $\Omega_{t,2t/C}$ and

$$\sup_{\{|y|<s,\,|H|<2s/C\}} \int_{\mathbb{R}^n}\int_K\int_K |g(x+iy,k_1\exp(iH)k_2)|^2\,dk_1\,dk_2\,dx<\infty$$

for $s<t$. By Lemma 4.2 we have

$$\sup_{\{|y|<s,\,|H|<2s/C\}} \sum_{\pi\in\widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi}\left(\int_{\mathbb{R}^n}|\widetilde{g^\pi_{ij}}(\xi)|^2 e^{-2\xi\cdot y}\,d\xi\right)\chi_\pi(\exp 2iH)<\infty$$

for all $s<t$. Integrating the above over $|H|=r=2s/C$ and $|y|=s<t$ we obtain

$$\sum_{\pi\in\widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi}\left(\int_{\mathbb{R}^n}|\widetilde{g^\pi_{ij}}(\xi)|^2 \frac{J_{n/2-1}(2is|\xi|)}{(2is|\xi|)^{n/2-1}}\,d\xi\right)\int_{|H|=r}\chi_\pi(\exp 2iH)\,d\sigma_r(H)<\infty.$$

Noting that $\dfrac{J_{n/2-1}(2is|\xi|)}{(2is|\xi|)^{n/2-1}}\sim e^{2s|\xi|}$ for large $|\xi|$ and using (4.1) we obtain

$$\sum_{\pi\in\widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi}\int_{\mathbb{R}^n}|\widetilde{g^\pi_{ij}}(\xi)|^2 e^{2s|\xi|}e^{2s\sqrt{\lambda_\pi}}\,d\xi<\infty \quad \text{for } s<t.$$

This surely implies that

$$\sum_{\pi\in\widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi}\int_{\mathbb{R}^n}|\widetilde{g^\pi_{ij}}(\xi)|^2 e^{2s(|\xi|^2+\lambda_\pi)^{1/2}}\,d\xi<\infty \quad \text{for } s<t.$$

Defining $\widetilde{f^\pi_{ij}}(\xi)=\widetilde{g^\pi_{ij}}(\xi)e^{s(|\xi|^2+\lambda_\pi)^{1/2}}$ we obtain

$$f(x,k)=\sum_{\pi\in\widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} f^\pi_{ij}(x)\phi^\pi_{ij}(k)\in L^2(M)$$

and $g=e^{-s\Delta^{1/2}}f$. □

§5. Complexified representations and Paley–Wiener type theorems

Recall the representations U^ξ and the generalized Fourier transform $\widehat{f}(\xi)$ from the introduction where

$$\widehat{f}(\xi)=\int_M f(m)U^\xi_m\,dm.$$

For $(x,k)\in M$ and matrix coefficients ϕ^π_{ij} of π we have

$$(U^\xi_{(x,k)}\phi^\pi_{ij})(u)=e^{i\langle x,u\cdot\xi\rangle}\phi^\pi_{ij}(k^{-1}u).$$

This action of $U^{\xi}_{(x,k)}$ on ϕ^{π}_{ij} can clearly be analytically continued to $\mathbb{C}^n \times G$ and we obtain

$$(U^{\xi}_{(z,g)}\phi^{\pi}_{ij})(u) = e^{i\langle x, u\cdot\xi\rangle} e^{-\langle y, u\cdot\xi\rangle}\phi^{\pi}_{ij}(e^{-iH}k^{-1}u)$$

where $(z,g) \in \mathbb{C}^n \times G$ and $z = x + iy \in \mathbb{C}^n$ and $g = ke^{iH} \in G$.

We also note that the action of $K \subseteq SO(n)$ on $\mathbb{R}^n$ naturally extends to an action of $G \subseteq SO(n,\mathbb{C})$ on $\mathbb{C}^n$. Then we have the following theorem:

Theorem 5.1. *Let $f \in L^2(M)$. Then f extends holomorphically to $\mathbb{C}^n \times G$ with*

$$\int_{|y|=r}\int_K\int_{\mathbb{R}^n} |f(e^{-iH}(x+iy), e^{-iH}k)|^2\, dx\, dk\, d\mu_r(y) < \infty$$

for all $H \in \underline{k}$ (where μ_r is the normalized surface area measure on the sphere $\{|y| = r\} \subseteq \mathbb{R}^n$) iff

$$\int_{\mathbb{R}^n}\int_{|y|=r} \|U^{\xi}_{(z,g)}\widehat{f}(\xi)\|^2_{\mathrm{HS}}\, d\mu_r(y)\, d\xi < \infty$$

where $z = x + iy \in \mathbb{C}^n$ and $g = ke^{iH} \in G$. In this case we also have

$$\int_{\mathbb{R}^n}\int_{|y|=r} \|U^{\xi}_{(z,g)}\widehat{f}(\xi)\|^2_{\mathrm{HS}}\, d\mu_r(y)\, d\xi = \int_{|y|=r}\int_K\int_{\mathbb{R}^n} |f(e^{-iH}(x+iy), e^{-iH}k)|^2\, dx\, dk\, d\mu_r(y).$$

We know that any $f \in L^2(M)$ can be expanded in the K variable using the Peter–Weyl theorem to obtain

$$f(x,k) = \sum_{\pi\in\widehat{K}} d_\pi \sum_{i,j=1}^{d_\pi} f^{\pi}_{ij}(x)\overline{\phi^{\pi}_{ij}(k)} \tag{5.1}$$

where for each $\pi \in \widehat{K}$, d_π is the degree of π, ϕ^{π}_{ij}'s are the matrix coefficients of π and $f^{\pi}_{ij}(x) = \int_K f(x,k)\phi^{\pi}_{ij}(k)\, dk$.

Now, for $F \in L^2(\mathbb{R}^n)$, consider the decomposition of the function $k \mapsto F(k\cdot x)$ in terms of the irreducible unitary representations of K given by

$$F(k\cdot x) = \sum_{\lambda\in\widehat{K}} d_\lambda \sum_{l,m=1}^{d_\lambda} F^{lm}_{\lambda}(x)\phi^{\lambda}_{lm}(k)$$

where $F^{lm}_{\lambda}(x) = \int_K F(k\cdot x)\overline{\phi^{\lambda}_{lm}(k)}\, dk$. Putting $k = e$, the identity element of K, we obtain

$$F(x) = \sum_{\lambda\in\widehat{K}} d_\lambda \sum_{l=1}^{d_\lambda} F^{ll}_{\lambda}(x).$$

Then it is easy to see that for $u \in K$,

$$F_\lambda^{ll}(u \cdot x) = \sum_{m=1}^{d_\lambda} F_\lambda^{lm}(x)\phi_{lm}^\lambda(u). \tag{5.2}$$

It also follows that the Euclidean Fourier transform $\widetilde{F_\lambda^{lm}}$ of F_λ^{lm} satisfies

$$\widetilde{F_\lambda^{ll}}(u \cdot x) = \sum_{m=1}^{d_\lambda} \phi_{lm}^\lambda(u)\widetilde{F_\lambda^{lm}}(x) \qquad \forall u \in K. \tag{5.3}$$

From the above and the fact that $f_{ij}^\pi \in L^2(\mathbb{R}^n)$ for every $\pi \in \widehat{K}$ and $1 \le i, j \le d_\pi$ it follows that any $f \in L^2(M)$ can be written as

$$f(x,k) = \sum_{\pi \in \widehat{K}} d_\pi \sum_{\lambda \in \widehat{K}} d_\lambda \sum_{i,j=1}^{d_\pi} \sum_{l=1}^{d_\lambda} (f_{ij}^\pi)_\lambda^{ll}(x)\overline{\phi_{ij}^\pi(k)}.$$

We need the following lemma to prove Theorem 5.1:

Lemma 5.2. *For fixed $\pi, \lambda \in \widehat{K}$, the conclusion of the theorem is true for functions of the form*

$$f(x,k) = \sum_{i,j=1}^{d_\pi} \sum_{l=1}^{d_\lambda} f_{ij}^{ll}(x)\overline{\phi_{ij}^\pi(k)}$$

where for simplicity we write $(f_{ij}^\pi)_\lambda^{ll}$ as f_{ij}^{ll}.

Proof. For $\xi \in \mathbb{R}^n$, $u \in K$, $\gamma \in \widehat{K}$ and $1 \le p, q \le d_\gamma$ we have

$$\begin{aligned}
(\widehat{f}(\xi)\overline{\phi_{pq}^\gamma})(u) &= \int_{\mathbb{R}^n}\int_K \sum_{i,j=1}^{d_\pi}\sum_{l=1}^{d_\lambda} f_{ij}^{ll}(x)\overline{\phi_{ij}^\pi(k)} e^{i\langle x, u\cdot\xi\rangle}\overline{\phi_{pq}^\gamma(k^{-1}u)}\, dk\, dx \\
&= \sum_{i,j=1}^{d_\pi}\sum_{l=1}^{d_\lambda} \widetilde{f_{ij}^{ll}}(u\cdot\xi) \sum_{t=1}^{d_\gamma} \phi_{qt}^\gamma(u^{-1})\langle \phi_{ij}^\pi, \phi_{tp}^\gamma\rangle_{L^2(K)} \\
&= \frac{\delta_{\gamma\pi}}{d_\pi} \sum_{i=1}^{d_\pi}\sum_{l,m=1}^{d_\lambda} \widetilde{f_{ip}^{lm}}(\xi)\phi_{lm}^\lambda(u)\phi_{qi}^\pi(u^{-1})
\end{aligned}$$

by (5.3) and Schur's orthogonality relations where $\delta_{\gamma\pi}$ is the Kronecker delta in the sense of equivalence of unitary representations. Then we have

$$\begin{aligned}
&\big(U^\xi_{(x+iy,ke^{iH})}\widehat{f}(\xi)\overline{\phi_{pq}^\gamma}\big)(u) \\
&\qquad = \frac{\delta_{\gamma\pi}}{d_\pi} e^{i\langle x+iy, u\cdot\xi\rangle} \sum_{i=1}^{d_\pi}\sum_{l,m=1}^{d_\lambda} \widetilde{f_{ip}^{lm}}(\xi)\phi_{lm}^\lambda(e^{-iH}k^{-1}u)\phi_{qi}^\pi(u^{-1}ke^{iH}).
\end{aligned}$$

Hence

$$\|U^\xi_{(x+iy,ke^{iH})}\widehat{f}(\xi)\|^2_{\mathrm{HS}} = \frac{1}{d_\pi}\sum_{p,q=1}^{d_\pi}\int_K e^{-2\langle y,u\cdot\xi\rangle}\Big|\sum_{i=1}^{d_\pi}\sum_{l,m=1}^{d_\lambda}\widetilde{f^{lm}_{ip}}(\xi)\phi^\lambda_{lm}(e^{-iH}k^{-1}u)\phi^\pi_{qi}(u^{-1}ke^{iH})\Big|^2\,du.$$

Integrating the above over $|y|=r$, we obtain

$$\begin{aligned}(5.4)\qquad \int_{|y|=r}\|U^\xi_{(x+iy,ke^{iH})}\widehat{f}(\xi)\|^2_{\mathrm{HS}}\,d\mu_r(y) &= \frac{1}{d_\pi}\frac{J_{n/2-1}(2ir|\xi|)}{(2ir|\xi|)^{n/2-1}}\\ &\quad\times\sum_{p,q=1}^{d_\pi}\int_K\Big|\sum_{i=1}^{d_\pi}\sum_{l,m=1}^{d_\lambda}\widetilde{f^{lm}_{ip}}(\xi)\phi^\lambda_{lm}(e^{-iH}u)\phi^\pi_{qi}(u^{-1}e^{iH})\Big|^2\,du\end{aligned}$$

where $J_{n/2-1}$ is the Bessel function of order $n/2-1$ and μ_r is the normalized surface area measure on the sphere $\{|y|=r\}\subseteq\mathbb{R}^n$.

Let $\mathcal{H}_\pi$ be the Hilbert space on which $\pi(k)$ acts for every $k\in K$, and $e_1,\ldots,e_{d_\pi}$ be a basis of $\mathcal{H}_\pi$. Then, for any $c_i, 1\le i\le d_\pi$,

$$\begin{aligned}\sum_{q=1}^{d_\pi}\Big|\sum_{i=1}^{d_\pi}c_i\phi^\pi_{qi}(u^{-1}e^{iH})\Big|^2 &= \sum_{q=1}^{d_\pi}\sum_{i=1}^{d_\pi}c_i\phi^\pi_{qi}(u^{-1}e^{iH})\sum_{a=1}^{d_\pi}\overline{c_a}\overline{\phi^\pi_{qa}(u^{-1}e^{iH})}\\ &= \sum_{i,a=1}^{d_\pi}c_i\overline{c_a}\sum_{q=1}^{d_\pi}\langle\pi(u^{-1}e^{iH})e_i,e_q\rangle\langle e_q,\pi(u^{-1}e^{iH})e_a\rangle\\ &= \sum_{i,a=1}^{d_\pi}c_i\overline{c_a}\langle\pi(u^{-1})\pi(e^{iH})e_i,\pi(u^{-1})\pi(e^{iH})e_a\rangle\\ &= \sum_{q=1}^{d_\pi}\Big|\sum_{i=1}^{d_\pi}c_i\phi^\pi_{qi}(e^{iH})\Big|^2,\end{aligned}$$

since π is a unitary representation of K. So, we have

$$\begin{aligned}&\sum_{q=1}^{d_\pi}\Big|\sum_{i=1}^{d_\pi}\sum_{l,m=1}^{d_\lambda}\widetilde{f^{lm}_{ip}}(\xi)\phi^\lambda_{lm}(e^{-iH}u)\phi^\pi_{qi}(u^{-1}e^{iH})\Big|^2\\ &\qquad= \sum_{q=1}^{d_\pi}\Big|\sum_{i=1}^{d_\pi}\sum_{l,m=1}^{d_\lambda}\widetilde{f^{lm}_{ip}}(\xi)\phi^\lambda_{lm}(e^{-iH}u)\phi^\pi_{qi}(e^{iH})\Big|^2.\end{aligned}$$

Hence from (5.4) we get

$$\int_{|y|=r} \|U^{\xi}_{(x+iy,ke^{iH})}\widehat{f}(\xi)\|^2_{\mathrm{HS}}\, d\mu_r(y)$$
$$= \frac{1}{d_\pi}\frac{J_{n/2-1}(2ir|\xi|)}{(2ir|\xi|)^{n/2-1}} \sum_{p,q=1}^{d_\pi} \int_K \Big| \sum_{i=1}^{d_\pi} \sum_{l,m,k=1}^{d_\lambda} \widetilde{f^{lm}_{ip}}(\xi)\phi^\lambda_{lk}(e^{-iH})\phi^\lambda_{km}(u)\phi^\pi_{qi}(e^{iH})\Big|^2\, du$$
$$= \frac{1}{d_\pi d_\lambda}\frac{J_{n/2-1}(2ir|\xi|)}{(2ir|\xi|)^{n/2-1}} \sum_{p,q=1}^{d_\pi} \sum_{m,k=1}^{d_\lambda} \Big| \sum_{i=1}^{d_\pi} \sum_{l=1}^{d_\lambda} \widetilde{f^{lm}_{ip}}(\xi)\phi^\lambda_{lk}(e^{-iH})\phi^\pi_{qi}(e^{iH})\Big|^2,$$

by Schur's orthogonality relations. The above can also be written as

$$\text{(5.5)}\qquad \int_{|y|=r} \|U^{\xi}_{(x+iy,ke^{iH})}\widehat{f}(\xi)\|^2_{\mathrm{HS}}\, d\mu_r(y)$$
$$= \frac{1}{d_\pi d_\lambda}\int_{|y|=r} e^{-2\langle y,\xi\rangle}\, d\mu_r(y) \sum_{p,q=1}^{d_\pi} \sum_{m,k=1}^{d_\lambda} \Big| \sum_{i=1}^{d_\pi} \sum_{l=1}^{d_\lambda} \widetilde{f^{lm}_{ip}}(\xi)\phi^\lambda_{lk}(e^{-iH})\phi^\pi_{qi}(e^{iH})\Big|^2.$$

We have obtained an expression for the left hand side of the desired formula. Now, looking at the right hand side, we have

$$f(u^{-1}\cdot x, u^{-1}k^{-1}) = \sum_{i,j=1}^{d_\pi} \sum_{l,m=1}^{d_\lambda} f^{lm}_{ij}(x)\phi^\lambda_{lm}(u^{-1})\phi^\pi_{ji}(ku).$$

So, if f is holomorphic on $\mathbb{C}^n \times G$, for $z = x+iy$ we get

$$f(e^{-iH}u^{-1}\cdot z, e^{-iH}u^{-1}k^{-1}) = \sum_{i,j,q=1}^{d_\pi} \sum_{l,m=1}^{d_\lambda} f^{lm}_{ij}(z)\phi^\lambda_{lm}(e^{-iH}u^{-1})\phi^\pi_{jq}(k)\phi^\pi_{qi}(ue^{iH}).$$

Again, by Schur's orthogonality relations and similar reasoning as before, we have

$$\int_K |f(e^{-iH}u^{-1}\cdot z, e^{-iH}u^{-1}k^{-1})|^2\, dk$$
$$= \frac{1}{d_\pi}\sum_{j,q=1}^{d_\pi} \Big| \sum_{i=1}^{d_\pi} \sum_{l,m=1}^{d_\lambda} f^{lm}_{ij}(z)\phi^\lambda_{lm}(e^{-iH}u^{-1})\phi^\pi_{qi}(ue^{iH})\Big|^2$$
$$= \frac{1}{d_\pi}\sum_{j,q=1}^{d_\pi} \Big| \sum_{i=1}^{d_\pi} \sum_{l,m=1}^{d_\lambda} f^{lm}_{ij}(z)\phi^\lambda_{lm}(e^{-iH}u^{-1})\phi^\pi_{qi}(e^{iH})\Big|^2.$$

Hence, by the invariance of Haar measure, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_K\int_K |f(e^{-iH}u^{-1}\cdot z, e^{-iH}u^{-1}k^{-1})|^2\,dk\,du\,dx \\
&\quad = \frac{1}{d_\pi}\sum_{j,q=1}^{d_\pi}\int_{\mathbb{R}^n}\int_K\Big|\sum_{i=1}^{d_\pi}\sum_{p,l,m=1}^{d_\lambda} f_{ij}^{lm}(z)\phi_{lp}^{\lambda}(e^{-iH})\phi_{pm}^{\lambda}(u^{-1})\phi_{qi}^{\pi}(e^{iH})\Big|^2\,du\,dx \\
&\quad = \frac{1}{d_\pi d_\lambda}\sum_{j,q=1}^{d_\pi}\sum_{p,m=1}^{d_\lambda}\int_{\mathbb{R}^n}\Big|\sum_{i=1}^{d_\pi}\sum_{l=1}^{d_\lambda} f_{ij}^{lm}(x+iy)\phi_{lp}^{\lambda}(e^{-iH})\phi_{qi}^{\pi}(e^{iH})\Big|^2\,dx \\
&\quad = \frac{1}{d_\pi d_\lambda}\sum_{j,q=1}^{d_\pi}\sum_{p,m=1}^{d_\lambda}\int_{\mathbb{R}^n}\Big|\sum_{i=1}^{d_\pi}\sum_{l=1}^{d_\lambda} \widetilde{f_{ij}^{lm}}(\xi)\phi_{lp}^{\lambda}(e^{-iH})\phi_{qi}^{\pi}(e^{iH})\Big|^2 e^{-2y\cdot\xi}\,d\xi.
\end{aligned}$$

Now by the invariance of Lebesgue measure under the K-action on $\mathbb{R}^n$ we get

$$\begin{aligned}
&\int_{|y|=r}\int_{\mathbb{R}^n}\int_K\int_K |f(e^{-iH}u^{-1}\cdot z, e^{-iH}u^{-1}k^{-1})|^2\,dk\,du\,dx\,d\mu_r(y) \\
&\qquad = \int_{|y|=r}\int_{\mathbb{R}^n}\int_K |f(e^{-iH}\cdot z, e^{-iH}k)|^2\,dk\,dx\,d\mu_r(y).
\end{aligned}$$

Hence the lemma follows from (5.5). □

Proof of Theorem 5.1. To prove the theorem, it is enough to prove the orthogonality of the components

$$f_\pi^\lambda(x,k) = \sum_{i,j=1}^{d_\pi}\sum_{l=1}^{d_\lambda} f_{ij}^{ll}(x)\overline{\phi_{ij}^{\pi}(k)}.$$

For $\pi,\lambda,\tau,\nu\in\widehat{K}$, we have

$$\begin{aligned}
&\langle U^{\xi}_{(x+iy,ke^{iH})}\widehat{f_\pi^\lambda}(\xi), U^{\xi}_{(x+iy,ke^{iH})}\widehat{f_\tau^\nu}(\xi)\rangle_{\mathrm{HS}} \\
&\quad = \sum_{\gamma\in\widehat{K}} d_\gamma \sum_{p,q=1}^{d_\gamma}\int_K \frac{\delta_{\gamma\pi}}{d_\pi} e^{i\langle x+iy,u\cdot\xi\rangle}\sum_{i=1}^{d_\pi}\sum_{l,m=1}^{d_\lambda}\widetilde{f_{ip}^{lm}}(\xi)\phi_{lm}^{\lambda}(e^{-iH}k^{-1}u)\phi_{qi}^{\pi}(u^{-1}ke^{iH}) \\
&\qquad \times \frac{\delta_{\gamma\tau}}{d_\tau}\overline{e^{i\langle x+iy,u\cdot\xi\rangle}}\sum_{a=1}^{d_\tau}\sum_{b,c=1}^{d_\nu}\overline{\widetilde{f_{ap}^{bc}}(\xi)\phi_{bc}^{\nu}(e^{-iH}k^{-1}u)\phi_{qa}^{\tau}(u^{-1}ke^{iH})}\,du \\
&\quad = 0 \quad \text{if } \pi\ncong\tau.
\end{aligned}$$

Assume $\pi \cong \tau$. Then

$$\int_{|y|=r} \langle U^{\xi}_{(x+iy,ke^{iH})} \widehat{f^{\lambda}_{\pi}}(\xi), U^{\xi}_{(x+iy,ke^{iH})} \widehat{f^{\nu}_{\pi}}(\xi) \rangle_{\mathrm{HS}}\, d\mu_r(y)$$
$$= \frac{1}{d_\pi} \frac{J_{n/2-1}(2ir|\xi|)}{(2ir|\xi|)^{n/2-1}} \sum_{a,i,p=1}^{d_\pi} \sum_{l,m=1}^{d_\lambda} \sum_{b,c=1}^{d_\nu} \widetilde{f^{lm}_{ip}}(\xi) \overline{\widetilde{f^{bc}_{ap}}(\xi)}$$
$$\cdot \int_K \Big(\sum_{q=1}^{d_\pi} \phi^{\pi}_{qi}(u^{-1}e^{iH}) \overline{\phi^{\pi}_{qa}(u^{-1}e^{iH})} \Big) \phi^{\lambda}_{lm}(e^{-iH}u) \overline{\phi^{\nu}_{bc}(e^{-iH}u)}\, du$$
$$= \frac{1}{d_\pi} \frac{J_{n/2-1}(2ir|\xi|)}{(2ir|\xi|)^{n/2-1}} \sum_{a,i,p,q=1}^{d_\pi} \sum_{l,m=1}^{d_\lambda} \sum_{b,c=1}^{d_\nu} \widetilde{f^{lm}_{ip}}(\xi) \overline{\widetilde{f^{bc}_{ap}}(\xi)} \phi^{\pi}_{qi}(e^{iH}) \overline{\phi^{\pi}_{qa}(e^{iH})}$$
$$\cdot \sum_{j=1}^{d_\lambda} \sum_{k=1}^{d_\nu} \phi^{\lambda}_{lj}(e^{-iH}) \overline{\phi^{\nu}_{bk}(e^{-iH})} \int_K \phi^{\lambda}_{jm}(u) \overline{\phi^{\nu}_{kc}(u)}\, du$$
$$= 0 \quad \text{if } \lambda \ncong \nu.$$

On the other hand, we have

$$\int_K f^{\lambda}_{\pi}(e^{-iH}u^{-1} \cdot z, e^{-iH}u^{-1}k^{-1}) \overline{f^{\nu}_{\tau}(e^{-iH}u^{-1} \cdot z, e^{-iH}u^{-1}k^{-1})}\, dk$$
$$= \sum_{i,j,q=1}^{d_\pi} \sum_{l,m=1}^{d_\lambda} \sum_{a,b,p=1}^{d_\tau} \sum_{s,t=1}^{d_\nu} f^{lm}_{ij}(z) \overline{f^{st}_{ab}(z)} \phi^{\lambda}_{lm}(e^{-iH}u^{-1}) \overline{\phi^{\nu}_{st}(e^{-iH}u^{-1})}$$
$$\cdot\, \phi^{\pi}_{qi}(ue^{iH}) \overline{\phi^{\tau}_{pa}(ue^{iH})} \int_K \phi^{\pi}_{jq}(k) \overline{\phi^{\tau}_{bp}(k)}\, dk$$
$$= 0 \quad \text{if } \pi \ncong \tau.$$

Assume $\pi \cong \tau$. Then we get

$$\int_K \int_K f^{\lambda}_{\pi}(e^{-iH}u^{-1} \cdot z, e^{-iH}u^{-1}k^{-1}) \overline{f^{\nu}_{\pi}(e^{-iH}u^{-1} \cdot z, e^{-iH}u^{-1}k^{-1})}\, dk\, du$$
$$= \sum_{i,a,j=1}^{d_\pi} \sum_{l,m=1}^{d_\lambda} \sum_{s,t=1}^{d_\nu} f^{lm}_{ij}(z) \overline{f^{st}_{aj}(z)} \Big(\sum_{q=1}^{d_\pi} \phi^{\pi}_{qi}(e^{iH}) \overline{\phi^{\tau}_{pa}(e^{iH})} \Big)$$
$$\cdot \sum_{\alpha=1}^{d_\lambda} \sum_{\beta=1}^{d_\nu} \phi^{\lambda}_{l\alpha}(e^{-iH}) \overline{\phi^{\nu}_{s\beta}(e^{-iH})} \int_K \phi^{\lambda}_{\alpha m}(u^{-1}) \overline{\phi^{\nu}_{\beta t}(u^{-1})}\, du$$
$$= 0 \quad \text{if } \lambda \ncong \nu.$$

This finishes the proof. □

It is easy to see that

$$\int_{\mathbb{R}^n} \|U^{\xi}_{(z,g)}\widehat{f}(\xi)\|^2_{\mathrm{HS}}\,d\xi = \int_{\mathbb{R}^n} \sum_{\sigma\in\widehat{K_\xi}} d_\sigma \|U^{\xi,\sigma}_{(z,g)}\widehat{f}(\xi,\sigma)\|^2_{\mathrm{HS}}\,d\xi.$$

Hence we have the following corollary:

Corollary 5.3. *For $f \in L^2(M)$, f extends holomorphically to $\mathbb{C}^n \times G$ with*

$$\int_{|y|=r}\int_K\int_{\mathbb{R}^n} |f(e^{-iH}(x+iy), e^{-iH}k)|^2\,dx\,dk\,d\mu_r(y) < \infty$$

(where μ_r is the normalized surface area measure on the sphere $\{|y| = r\} \subseteq \mathbb{R}^n$) iff

$$\int_{\mathbb{R}^n} \sum_{\sigma\in\widehat{K_\xi}} d_\sigma \int_{|y|=r} \|U^{\xi,\sigma}_{(z,g)}\widehat{f}(\xi,\sigma)\|^2_{\mathrm{HS}}\,d\mu_r(y)\,d\xi < \infty$$

where $z = x + iy \in \mathbb{C}^n$, $g \in G$ and we also have

$$\int_{\mathbb{R}^n} \sum_{\sigma\in\widehat{K_\xi}} d_\sigma \int_{|y|=r} \|U^{\xi,\sigma}_{(z,g)}\widehat{f}(\xi,\sigma)\|^2_{\mathrm{HS}}\,d\mu_r(y)\,d\xi = \int_{|y|=r}\int_K\int_{\mathbb{R}^n} |f(e^{-iH}(x+iy), e^{-iH}k)|^2\,dx\,dk\,d\mu_r(y).$$

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