Publ. RIMS Kyoto Univ. **46** (2010), 849–897
DOI 10.2977/PRIMS/28

Tempered Fundamental Group and Metric Graph of a Mumford Curve

by

Emmanuel Lepage

Abstract

The aim of this paper is to give some general results on the tempered fundamental group of p-adic smooth algebraic varieties (which is a sort of analog of the topological fundamental group of complex algebraic varieties in the p-adic world). The main result asserts that one can recover the metric structure of the graph of the stable model of a Mumford curve from the tempered fundamental group of the curve. We also prove the birational invariance, invariance under algebraically closed extensions and a Künneth formula for the tempered fundamental group. We describe the tempered fundamental group of an abelian variety and link the tempered fundamental group of a curve to the tempered fundamental group of its Jacobian variety.

2010 Mathematics Subject Classification: 14G22, 14H30, 14G20.
Keywords: p-adic fundamental groups, Berkovich geometry.

Contents

Communicated by S. Mochizuki. Received January 5, 2009. Revised April 12, 2010, and May 25, 2010.

E. Lepage: DMA, École Normale Supérieure, 45 rue d'Ulm, Paris, France;
e-mail: emmanuel.lepage@ens.fr

Introduction

This paper is an attempt to give some general results on the tempered fundamental group of p-adic smooth algebraic varieties.

The tempered fundamental group was introduced in André's [2, part III] as a sort of analog of the topological fundamental group of complex algebraic varieties; its profinite completion coincides with Grothendieck's algebraic fundamental group, but it has many infinite discrete quotients in general.

Since the analytification (in the sense of Berkovich or of rigid geometry) of a finite étale covering of a p-adic variety is not necessarily a topological covering, André had to use a slightly wider notion of covering. He defines tempered coverings, which are étale coverings in the sense of de Jong (that is, locally in the Berkovich topology, direct sums of finite coverings) such that, after pulling back by some surjective finite étale covering, they become topological coverings (for the Berkovich topology). Then the tempered fundamental group is a prodiscrete group that classifies those tempered coverings. To give a more handy description, if one has a sequence $((S_i, s_i))_{i\in\mathbf{N}}$ of pointed Galois finite coverings such that the corresponding pointed pro-covering of (X, x) is the universal pro-covering of (X, x), and if (S_i^∞, s_i^∞) is a universal topological covering of S_i, then the tempered fundamental group of X can be seen as $\pi_1^{\mathrm{temp}}(X, x) = \varprojlim_i \mathrm{Gal}(S_i^\infty/X)$. Therefore, to understand the tempered fundamental group of a variety, one mainly has to understand the topological behavior of the finite étale coverings of this variety.

There are many differences between the tempered fundamental group in the p-adic case and the topological fundamental group in the complex case. First, the tempered fundamental group is not discrete in general, even not locally compact. It is also much more difficult to describe explicitly (such a description is available for elliptic curves, or more generally abelian varieties, as will be shown in this article). As proved in [16] (and recalled in Section 1.2), the tempered fundamental group of a curve depends heavily on the combinatorial structure of its stable reduction (this suggests a geometric anabelian behavior of the tempered fundamental group which has no algebraic or complex counterpart). On the other hand, the tempered fundamental group, like the algebraic fundamental group, is also defined over non-algebraically closed nonarchimedean fields, and thus it interacts interestingly with Galois theory (which gives it a number-theoretical interest). However, in this article, we will only study the geometric tempered fundamental group.

This paper is divided into two main parts that are independent of each other and can be read separately:

- In the first one (§2 and §3), we will prove some results on the tempered fundamental group which are classical results for the profinite fundamental group or the topological fundamental group of complex varieties: we will link the abelianized tempered fundamental group of a curve to the tempered fundamental group of its Jacobian variety and we will prove a Künneth formula for a product of manifolds.
- In the second one (§4), probably the most interesting, we will show that one can recover the metric structure of the graph of the stable model of a Mumford curve (i.e. a curve such that all the normalized irreducible components of the stable reduction are projective lines) from the tempered fundamental group, thus illustrating that the tempered fundamental group depends much more on the variety itself than the profinite or the complex fundamental group does.

After recalling the basic results on the tempered fundamental group given in [2] (and deducing from them the birational invariance of the tempered fundamental group in Proposition 1.6), we will recall the results of Mochizuki concerning the tempered fundamental group of a curve and the stable reduction of this curve: whereas the profinite fundamental group or the topological fundamental group of a smooth curve of type (g,n) only depends in the complex case on g and n, the tempered fundamental group of a curve depends much more on the curve itself. How much is an unsolved problem, which is the leading thread of the present paper.

Mochizuki proved in [16] that one can recover from the tempered fundamental group of a curve the graph of its stable reduction (Theorem 1.7), even in the pro-(p') case (we will denote by $\pi_1^{\mathrm{temp}}(X)^{(p')}$ the group classifying the coverings that become topological after a base change by a finite Galois covering of order prime to p).

More precisely, the vertices of the graph correspond to the conjugacy classes of maximal subgroups of $\pi_1^{\mathrm{temp}}(X)^{(p')}$ (called therefore verticial subgroups) and the edges to classes of nontrivial intersections of verticial subgroups.

We will then study the tempered fundamental group of an abelian variety A in the geometric case. We will be able to give an explicit description of the tempered fundamental group, as was already done for elliptic curves in [2, III.2.3.2]. The point is that finite étale coverings are well understood (they are abelian varieties, and are dominated by multiplication by n on A for some n), as also are topological coverings, thanks to the p-adic uniformization of abelian varieties.

We will deduce from this that $\pi_1^{\text{temp}}(A)$ is isomorphic to $\widehat{\mathbf{Z}}^{2g-d} \times \mathbf{Z}^d$ (see Section 2.1), where g is the dimension of A and d is the rank of the topological fundamental group of A (which is also the dimension of the toric part of the semistable reduction).

One can then prove that, just as in the profinite case or the topological case, the abelianized tempered fundamental group of a curve X is canonically isomorphic to the tempered fundamental group of its Jacobian variety A (Theorem 2.1). The proof combines the birational morphism $X^g/\mathfrak{S}_g \to A$, where $\mathfrak{S}_g$ is the symmetric group of degree g acting on X^g by permutation of factors, the birational invariance of the tempered fundamental group, and the Künneth formula that will be proved in the next part.

For more general cases, one does not have such nice descriptions of the topological behavior of a smooth variety, not to mention the topological behavior of all its finite etale coverings. Nevertheless, thanks to de Jong's alteration results (see [7] and [8]) and to Berkovich's work (see [4], recalled in Section 3.1) on the topological structure of the generic fiber of a polystable formal scheme over the integer ring of a complete nonarchimedean field, one can prove that some Zariski open dense subset U of the smooth scheme X is homotopically equivalent to a certain polysimplicial set given by the combinatorial structure of the special fiber of some alteration of the variety.

One can then deduce from the results of [4] that for any isometric embedding of K in an algebraically closed field K', $|U_{K'}| \to |U|$ is a homotopy equivalence. Knowing this, and using a proposition of André which says that $U \to X$ induces an isomorphism on topological groups, we will be able to prove the invariance of the tempered fundamental group of a smooth scheme under algebraically closed extensions (Proposition 3.8).

In the same way, if V is a Zariski dense open subset of another smooth scheme Y satisfying the same properties as U, one can deduce from the results of [4] that $|U \times V| \to |U| \times |V|$ is a homotopy equivalence. Thanks to this, we will prove a Künneth formula for the tempered fundamental group of smooth schemes (Proposition 3.10).

In the last section, we will prove (Theorem 4.13) that, in the case of a Mumford curve, one may in fact recover the lengths of the edges from $\pi_1^{\text{temp}}(X)$ (obviously, one needs the whole $\pi_1^{\text{temp}}(X)$ and not only the pro-(p') version which only depends on the pro-(p') graph of groups of the stable reduction; thus this can only be done when $p \neq 0$, in contrast to Mochizuki's result).

Questions about the recovery of the length of the edges from fundamental groups have already been considered by Mochizuki in the profinite absolute situation over p-adic local fields (see [14, §6] and [15, Th. 2.7]).

We will deduce from Mochizuki's results about the recovery of the graph of the stable reduction from the tempered fundamental group that one can decide from the tempered fundamental group of a curve whether a finite covering is split over some vertex of the curve (and in a similar way, one can decide whether it is ramified over some cusp).

Knowing this, we will start by proving this for a punctured projective line and a punctured elliptic curve, to give an insight into the method in simpler cases, where the coverings we will use are described explicitly: for a punctured projective line, we will concentrate on coverings like $(\ )^{p^e} : \mathbf{G}_m \xrightarrow{z \mapsto z^{p^e}} \mathbf{G}_m$ (in this case one can easily calculate explicitly the number of preimages of a point of the Berkovich projective line), and for elliptic curves on coverings obtained by patching together those types of coverings.

In the more general case of a Mumford curve $X = \Omega/\Gamma$, we will study coverings of the topological uniformization Ω that descend to some finite topological covering of X that behave like $(\ )^{p^e}$ over big affinoids of Ω. More precisely, every covering of Ω which descends to X (in some nonunique way) is the pullback of $(\ )^{p^e}$ along a Γ-equivariant invertible section of Ω, and conversely. The equivariance prevents us from simply considering a homography, but we can consider a section of Ω which is arbitrarily close, on a given affinoid subset of Ω, to a homography. This will be enough to ensure that this covering is split over the same vertices of a big affinoid subset of Ω as the one obtained by pulling back $(\ )^{p^e}$ along the homography.

Using such constructions, one can recover from the tempered fundamental group the length of any loop of any topological covering of X. In order to get the whole metric structure of the graph of the stable model of X, we will end this article by proving a purely combinatorial result that shows that, if one knows the length of every loop of every covering of a graph whose edges have valency ≥ 3, one knows the length of every edge (Proposition A.1).

The recovering of the metric graph of the stable model from the tempered fundamental group can be quite easily extended to the case of punctured Mumford cases, but the proof does not seem to extend easily to more general curves, as one cannot find easily coverings of order divisible by p where one can say much about the graph of the stable reduction.

The proofs here also only consider some very simple coverings (for which we do not even describe completely the graph of the stable model), and it is hard to imagine what can be recovered from the whole tempered fundamental group.

§1. Tempered fundamental group

§1.1. Definition

Let K be a complete nonarchimedean field. Following [1, §4], a K-*manifold* will be a connected smooth paracompact strictly K-analytic space in the sense of Berkovich. For example, if X is a connected smooth algebraic K-variety, X^{an} is a K-manifold (and in fact, we will mainly be interested in those spaces). By [4], any K-manifold is locally contractible (we will explain in more detail the results of [4] in Section 3.1). In particular, it has a universal covering.

One can associate to any point x of a Berkovich space X a *residue field* denoted by $\mathcal{H}(x)$: if X is an affinoid space, then x corresponds to a seminorm $|\ |_x$ of its affinoid algebra A and $\mathcal{H}(x)$ is the completion of $A/\mathrm{Ker}|\ |_x$ for the norm induced by $|\ |_x$; in the general case, the residue field of x in X is just the residue field of x in any affinoid domain of X that contains x.

A morphism $f : S' \to S$ is said to be an *étale covering* if S is covered by open subsets U such that $f^{-1}(U) = \coprod V_j$ and $V_j \to U$ is étale finite ([6]).

For example, finite étale coverings, also called *algebraic coverings*, and coverings in the usual topological sense for the Berkovich topology, also called *topological coverings*, are étale coverings.

Then André defines tempered coverings as follows:

Definition 1.1 ([2, Def. 2.1.1]). An étale covering $S' \to S$ is *tempered* if it is a quotient of the composition of a topological covering $T' \to T$ and a finite étale covering $T \to S$. This is equivalent to saying that it becomes a topological covering after pullback by some finite étale covering.

We denote by $\mathrm{Cov}^{\mathrm{temp}}(X)$ (resp. $\mathrm{Cov}^{\mathrm{alg}}(X)$, $\mathrm{Cov}^{\mathrm{top}}(X)$) the category of tempered coverings (resp. algebraic coverings, topological coverings) of X (with the obvious morphisms).

A *geometric point* of a K-manifold X is a morphism of Berkovich spaces $\mathcal{M}(\Omega) \to X$ where Ω is an algebraically closed complete isometric extension of K.

Let $\bar{x}$ be a geometric point of X. Then one has a functor

$$F_{\bar{x}} : \mathrm{Cov}^{\mathrm{temp}}(X) \to \mathrm{Set}$$

which maps a covering $S \to X$ to the set $S_{\bar{x}}$. If $\bar{x}$ and $\bar{x}'$ are two geometric points, then $F_{\bar{x}}$ and $F_{\bar{x}'}$ are (noncanonically) isomorphic ([6, Prop. 2.9]).

The *tempered fundamental group* of X based at $\bar{x}$ is

$$\pi_1^{\mathrm{temp}}(X, \bar{x}) = \mathrm{Aut}\, F_{\bar{x}}.$$

When X is a smooth algebraic K-variety, $\mathrm{Cov}^{\mathrm{temp}}(X^{\mathrm{an}})$ and $\pi_1^{\mathrm{temp}}(X^{\mathrm{an}},\bar{x})$ will also be denoted simply by $\mathrm{Cov}^{\mathrm{temp}}(X)$ and $\pi_1^{\mathrm{temp}}(X,\bar{x})$.

By considering the stabilizers $(\mathrm{Stab}_{F(S)}(s))_{S\in\mathrm{Cov}^{\mathrm{temp}}(X),\, s\in F_{\bar{x}}(S)}$ as a basis of open subgroups of $\pi_1^{\mathrm{temp}}(X,\bar{x})$, $\pi_1^{\mathrm{temp}}(X,\bar{x})$ becomes a topological group. It is a prodiscrete topological group.

When X is algebraic, and K is of characteristic zero and has only countably many finite extensions in a fixed algebraic closure $\overline{K}$, $\pi_1^{\mathrm{temp}}(X,\bar{x})$ has a countable fundamental system of neighborhoods of 1 and all its discrete quotient groups are finitely generated ([2, Prop. 2.1.7]).

If $\bar{x}$ and $\bar{x}'$ are two geometric points, then $F_{\bar{x}}$ and $F_{\bar{x}'}$ are (noncanonically) isomorphic ([6, Prop. 2.9]). Thus, as usual, the tempered fundamental group depends on the basepoint only up to inner automorphism (this topological group, considered up to conjugation, will sometimes be denoted simply by $\pi_1^{\mathrm{temp}}(X)$).

The full subcategory of tempered coverings S for which $F_{\bar{x}}(S)$ is finite is equivalent to $\mathrm{Cov}^{\mathrm{alg}}(S)$, hence

$$\widehat{\pi_1^{\mathrm{temp}}(X},\bar{x}) = \pi_1^{\mathrm{alg}}(X,\bar{x})$$

(where $\widehat{\ }$ denotes, here and below, the profinite completion).

For any morphism $X \to Y$, the pullback defines a functor $\mathrm{Cov}^{\mathrm{temp}}(Y) \to \mathrm{Cov}^{\mathrm{temp}}(X)$. If $\bar{x}$ is a geometric point of X with image $\bar{y}$ in Y, this gives rise to a continuous homomorphism

$$\pi_1^{\mathrm{temp}}(X,\bar{x}) \to \pi_1^{\mathrm{temp}}(Y,\bar{y})$$

(hence an outer morphism $\pi_1^{\mathrm{temp}}(X) \to \pi_1^{\mathrm{temp}}(Y)$).

One has the analog of the usual Galois correspondence:

Theorem 1.2 ([2, Th. 1.4.5]). *$F_{\bar{x}}$ induces an equivalence of categories between the category of direct sums of tempered coverings of X and the category $\pi_1^{\mathrm{temp}}(X,\bar{x})$-*Set *of discrete sets endowed with a continuous left action of $\pi_1^{\mathrm{temp}}(X,\bar{x})$.*

If S is a finite Galois covering of X, its universal topological covering S^∞ is still Galois and every connected tempered covering is dominated by such a Galois tempered covering.

If $((S_i,\bar{s}_i))_{i\in\mathbf{N}}$ is a cofinal projective system (with morphisms $f_{ij} : S_i \to S_j$ which map $\bar{s}_i$ to $\bar{s}_j$ for $i \geq j$) of geometrically pointed Galois finite étale coverings of $(X,\bar{x})$, let $((S_i^\infty,\bar{s}_i^\infty))_{i\in\mathbf{N}}$ be the projective system, with morphisms f_{ij}^∞ for $i \geq j$, of its pointed universal topological coverings. Then $F_{\bar{x}}(S_i^\infty) = \pi_1^{\mathrm{temp}}(X,\bar{x})/\mathrm{Stab}_{F(S_i^\infty)}(\bar{s}_i^\infty)$ is naturally a quotient group G of $\pi_1^{\mathrm{temp}}(X,\bar{x})$ for which s_i^∞ is the neutral element. Moreover G acts by G-automorphisms on $F_{\bar{x}}(S_i^\infty)$ by right translations (and thus on S_i^∞ thanks to the Galois correspondence (Theorem 1.2)). Thus one gets a morphism $\pi_1^{\mathrm{temp}}(X,\bar{x}) \to \mathrm{Gal}(S_i^\infty/X)$. As $f_{ij}^\infty(s_i^\infty) = s_j^\infty$, these morphisms are compatible with

$$\mathrm{Gal}(S_i^\infty/X) \to \mathrm{Gal}(S_j^\infty/X).$$

Then, thanks to [2, Lem. III.2.1.5], we have

Proposition 1.3.

$$\pi_1^{\mathrm{temp}}(X,\bar{x}) \to \varprojlim \mathrm{Gal}(S_i^\infty/X)$$

is an isomorphism.

On the other hand, we will use the following results by André (they are not needed for §4):

Proposition 1.4 ([2, Prop. III.1.1.4]). *Let $\overline{S}$ be a manifold, and let Z be a Zariski closed nowhere dense reduced analytic subset. Then any topological covering of $S := \overline{S} \setminus Z$ extends uniquely to a topological covering of $\overline{S}$. Thus $\pi_1^{\mathrm{top}}(S,s) \to \pi_1^{\mathrm{top}}(\overline{S},s)$ is an isomorphism.*

Proposition 1.5 ([2, Th. III.2.1.11, Prop. III.2.1.13]). *Assume K is algebraically closed and of characteristic 0. Let $\overline{S}$ be a manifold, and let Z be a Zariski closed nowhere dense reduced analytic subset. Then the functor from tempered coverings of $\overline{S}$ to tempered coverings of $S = \overline{S} \setminus Z$ is fully faithful. If Z is of codimension ≥ 2, this functor is an equivalence of categories.*

Let K be an algebraically closed complete nonarchimedean field of characteristic 0. We will follow the proof of [11, Cor. X.3.4] in the algebraic case to get the birational invariance of the tempered fundamental group of smooth and proper K-schemes (see [12, Prop. 2.2.1] for a proof without any assumption on the characteristic of K).

Let $f : X \to Y$ be a dominant rational map between smooth and proper K-schemes. If f is defined on a Zariski open subset U of X (denote by f_U the morphism $U \to Y$ and by i_U the immersion $U \to X$) whose complement is of codimension ≥ 2 in X, one gets a functor from $\mathrm{Cov}^{\mathrm{temp}}(Y)$ to $\mathrm{Cov}^{\mathrm{temp}}(U)$ and one can compose it with a quasi-inverse of $\mathrm{Cov}^{\mathrm{temp}}(X) \to \mathrm{Cov}^{\mathrm{temp}}(U)$; one thus gets a functor $f_{(U)}^* : \mathrm{Cov}^{\mathrm{temp}}(Y) \to \mathrm{Cov}^{\mathrm{temp}}(X)$ such that $i_U^* f_{(U)}^*$ is isomorphic to f_U^*. If one takes another Zariski open subset U' of X with the same properties, one finds that $i_{U\cap U'}^* f_{(U)}^*$ and $i_{U\cap U'}^* f_{(U')}^*$ are both isomorphic to $f_{U\cap U'}^*$, and thus $f_{(U)}^*$ and $f_{(U')}^*$ are isomorphic, since $X \setminus U \cap U'$ is also of codimension ≥ 2 in X. Thus one gets an outer homomorphism of topological groups $f_* : \pi_1^{\mathrm{temp}}(X) \to \pi_1^{\mathrm{temp}}(Y)$, which does not depend on U. In particular if f is a morphism of schemes, one can choose $U = X$ and thus f_* is the usual outer morphism $\pi_1^{\mathrm{temp}}(X) \to \pi_1^{\mathrm{temp}}(Y)$.

Let $g : Y \to Z$ be another dominant rational map between smooth and proper K-schemes. It is defined on a Zariski open subset V of Y whose complement is of codimension ≥ 2 in Y, and $gf : X \to Z$ is also a dominant rational map between smooth and proper schemes, so it is also defined over a Zariski open subset W of X whose complement is of codimension ≥ 2. Let $U_0 = U \cap f_U^{-1}(V) \cap W$ (note that $X \setminus U_0$ may be of codimension < 2). There are morphisms $U_0 \to V$ and $V \to Z$ representing f and g such that the composed morphism $(gf)_{U_0} : U_0 \to Z$ represents gf. One then sees that $i^*_{U_0} f^*_{(U)} g^*_{(V)}$ and $i^*_{U_0}(gf)^*_{(W)}$ are both isomorphic to $(gf)^*_{U_0}$. Since $i^*_{U_0}$ is fully faithful, $f^*_{(U)} g^*_{(V)}$ and $(gf)^*_{(W)}$ are isomorphic (and $g_* f_* = (gf)_*$). Thus one gets a functor from the category of smooth and proper K-schemes with dominant rational maps to the category of groups with outer homomorphisms.

In particular,

Proposition 1.6. *Let $X \to Y$ be a birational map between smooth and proper K-schemes. Then*

$$\pi_1^{\mathrm{temp}}(X) \to \pi_1^{\mathrm{temp}}(Y)$$

is an isomorphism.

§1.2. Mochizuki's results on the pro-(p') tempered group of a curve

Mochizuki [16] links the tempered fundamental group of an algebraic curve X to the graph of the stable reduction of X, giving a more combinatorial description of a pro-(p') version of the tempered fundamental group. In particular, he proves that one can recover the graph of the stable reduction of X from $\pi_1^{\mathrm{temp}}(X, \bar{x})$. We will explain here the main results of [16] (they will only be used in §4).

If one considers the full subcategory $\mathrm{Cov}^{\mathrm{temp}}(X)^{(p')}$ of tempered coverings that become topological after pullback by a finite Galois covering of order prime to p where p is the residual characteristic of K, one gets in the same fashion a pro-(p') version $\pi_1^{\mathrm{temp}}(X, \bar{x})^{(p')}$ of the tempered fundamental group (see [16, Rem. 3.10.1] in the case of a curve).

Following [15, Appendix], a *semigraph* $\mathbb{G}$ is given by a set $\mathcal{V}$ of vertices, a set $\mathcal{E}$ of edges and, for every $e \in \mathcal{E}$, a set of cardinality ≤ 2 of branches $\mathcal{B}_e$ with a map $\zeta_e : \mathcal{B}_e \to \mathcal{V}$.

We will say that a branch b of e *ends at* v if $\zeta_e(b) = v$. We will say that a semigraph is a *graph* if for every $e \in \mathcal{E}$, e has exactly two branches.

Recall from [16, Def. 2.1] that if one has a semigraph $\mathbb{G}$, the structure of a *semigraph of anabelioids* $\mathcal{G}$ on this graph corresponds to the following data:

- for each vertex or edge x, a Galois category $\mathcal{G}_x$ (also named connected anabelioid in [16]),

- for each branch b of an edge e ending at a vertex v, a morphism of anabelioids (i.e. an exact functor) $b_* : \mathcal{G}_e \to \mathcal{G}_v$.

Semigraphs of anabelioids form a 2-category. Recall also that working with Galois categories is equivalent to working with profinite groups up to inner automorphism.

Recall that if $\mathcal{C}$ is a Galois category whose fundamental group is Π, then Ind-$\mathcal{C}$ is equivalent to the topos Π-Set.

A *covering* S of a semigraph $\mathcal{G}$ of anabelioids consists of:

- for every vertex v of $\mathbb{G}$, an object S_v of Ind-$\mathcal{G}_v$,
- for every edge e with branches b_1 and b_2 ending at v_1 and v_2, an isomorphism ϕ_e between $b_{1*}S_{v_1}$ and $b_{2*}S_{v_2}$.

One has a natural notion of a morphism of such coverings, so that one gets a category $\mathcal{B}^{\mathrm{cov}}(\mathcal{G})$. Mochizuki also defines a 2-functor from the category of coverings of $\mathcal{G}$ to the 2-category of semigraphs of anabelioids above $\mathcal{G}$.

An object of $\mathcal{B}^{\mathrm{cov}}(\mathcal{G})$ is finite if each S_v is in $\mathcal{G}_v$, *topological* if for each v, S_v is a constant object of Ind-$\mathcal{G}_v$, and *tempered* if it becomes topological after pulling back along some finite covering. The full subcategory of tempered (resp. finite, topological) coverings is then denoted by $\mathcal{B}^{\mathrm{temp}}(\mathcal{G})$ (resp. $\mathcal{B}^{\mathrm{alg}}(\mathcal{G})$, $\mathcal{B}^{\mathrm{top}}(\mathcal{G})$).

If $\mathbb{G}$ is connected, $\mathcal{B}^{\mathrm{alg}}(\mathcal{G})$ is a Galois category whose fundamental group is denoted by $\pi_1^{\mathrm{alg}}(\mathcal{G})$.

If v is a vertex of $\mathbb{G}$ and F is an exact and conservative functor $\mathcal{G}_v \to \mathrm{Set}$ (such a functor is called a fundamental functor in [11, Section 5]; it extends to a point of Ind-$\mathcal{G}_v$, also denoted by F), one can define a functor $F_{(v,F)} : \mathcal{B}^{\mathrm{cov}}(\mathcal{G}) \to \mathrm{Set}$ which maps S to $F(S_v)$ (if one changes the base point (v,F), one gets an isomorphic functor). Let $F^{\mathrm{temp}}_{(v,F)}$ be its restriction to $\mathcal{B}^{\mathrm{temp}}(\mathcal{G})$. Then one defines

$$\pi_1^{\mathrm{temp}}(\mathcal{G},(v,F)) := \mathrm{Aut}(F^{\mathrm{temp}}_{(v,F)}).$$

Let us illustrate these definitions by associating to a curve a semigraph of anabelioids.

Let us assume that K is discretely valued, and let $\overline{K}$ be the completion of an algebraic closure of K. Let $(\overline{X}, D)$ be a smooth n-pointed hyperbolic curve of type g over K, let $X = \overline{X} \setminus D$, let $(\overline{\mathcal{X}}, \mathcal{D})$ be a semistable model over $O_{\overline{K}}$, and let $\mathcal{X} = \overline{\mathcal{X}} \setminus \mathcal{D}$. The semigraph of $\mathcal{X}_s$ is defined as follows: the vertices are the irreducible components of $\mathcal{X}_s$, the edges are the nodes and the marked points. A node e has two branches that end at the irreducible components that contain e; a marked point e has only one branch that ends at the irreducible component containing the marked point.

When $\mathcal{X}$ is the stable model of X, this semigraph will be denoted by $\mathbb{G}^c_X$ (or $\mathbb{G}^c$ when there is no risk of confusion).

One can endow the semigraph $\mathbb{G}^c$ with the structure of a "semigraph of anabelioids" $\mathcal{G}^c$. Indeed for a vertex v_i corresponding to an irreducible component C_i of $\mathcal{X}_s$, let $\overline{C}_i$ be the closure of C_i in $\mathcal{X}_s$, let $\overline{C}_i'$ be the normalization of $\overline{C}_i$ and let U_i be the open subset of $\overline{C}_i'$ which is the complement of the marked points and of the preimages of the double points of $\mathcal{X}_s$ (the points of $\overline{C}_i' - U_i$ thus correspond exactly to branches ending at v_i). Then the group $\Pi_{(v_i)}$ is the tame fundamental group $\pi_1^{\mathrm{t}}(U_i)$ of U_i in $\overline{C}_i'$. The group of an edge is $\widehat{\mathbf{Z}(1)}^{(p')} = \varprojlim_{(n,p)=1} \mu_n$ ($\simeq \widehat{\mathbf{Z}}^{(p')}$) (as usual, the superscript (p') indicates the pro-prime-to-p maximal quotient), which is canonically isomorphic to the monodromy subgroup in $\pi_1^{\mathrm{t}}(U_i)$ of a point in $\overline{C}_i' - U_i$. The morphism corresponding to a branch is the embedding of the monodromy group of the corresponding point of $\overline{C}_i' - U_i$ (which is defined up to conjugation), whereas, for an edge with two branches, one identifies the two $\widehat{\mathbf{Z}(1)}^{(p')}$ by $x \mapsto x^{-1}$.

If $\mathcal{G}^{(p')}$ denotes the semigraph of anabelioids obtained from $\mathcal{G}$ by replacing each profinite group by its pro-(p') completion, and if $\pi_1^{\mathrm{temp}}(X_{\overline{K}})^{(p')}$ is the pro-(p') version of the tempered fundamental group of $X_{\overline{K}}$, then

$$\pi_1^{\mathrm{temp}}(\mathcal{G}_X^{c,(p')}) = \pi_1^{\mathrm{temp}}(X_{\overline{K}})^{(p')} \quad ([16, \text{Ex. } 3.10]).$$

Assume now that K is a discretely valued field of characteristic 0 and of residual characteristic $p > 0$. Mochizuki then shows:

Theorem 1.7 ([16, Cor. 3.11]). *If X_α and X_β are two curves, then every isomorphism $\gamma : \pi_1^{\mathrm{temp}}(X_{\alpha,\overline{K}}) \simeq \pi_1^{\mathrm{temp}}(X_{\beta,\overline{K}})$ determines, functorially in γ up to 2-isomorphism, an isomorphism of semigraphs of anabelioids $\gamma' : \mathcal{G}^c_{X_\alpha} \simeq \mathcal{G}^c_{X_\beta}$.*

More precisely, the following induced diagram of topological groups is commutative:

$$\begin{array}{ccc} \pi_1^{\mathrm{temp}}(\mathcal{G}^{c,(p')}_{X_\alpha}) & \xrightarrow{\pi_1^{\mathrm{temp}}(\gamma'^{(p')})} & \pi_1^{\mathrm{temp}}(\mathcal{G}^{c,(p')}_{X_\beta}) \\ \| & & \| \\ \pi_1^{\mathrm{temp}}(X_{\alpha,\overline{K}})^{(p')} & \xrightarrow{\gamma^{(p')}} & \pi_1^{\mathrm{temp}}(X_{\beta,\overline{K}})^{(p')} \end{array}$$

The vertices of the graph then correspond to the conjugacy classes of maximal compact subgroups of $\pi_1^{\mathrm{temp}}(X)^{(p')}$ (therefore called the *verticial* subgroups of $\pi_1^{\mathrm{temp}}(X)^{(p')}$), and the edges to the conjugacy classes of nontrivial intersections of verticial subgroups ([16, Th. 3.7]).

§2. Abelianized tempered fundamental group of a curve

Here we study the tempered fundamental group of an abelian variety and the abelianized tempered fundamental group of a curve.

We first prove that if A is an abelian variety over an algebraically closed complete nonarchimedean field of characteristic 0, then the tempered fundamental group is abelian and fits in a split exact sequence

$$0 \to T \to \pi_1^{\mathrm{temp}}(A) \to \pi_1^{\mathrm{top}}(A) \to 0$$

where T is profinite (and more precisely isomorphic to $\widehat{\mathbf{Z}}^n$ for some n).

Next we will prove that (as in the case of algebraic fundamental groups or complex topological fundamental groups) if C is a curve and A its Jacobian variety, the natural morphism $\pi_1^{\mathrm{temp}}(C)^{\mathrm{ab}} \to \pi_1^{\mathrm{temp}}(A)$ is an isomorphism.

§2.1. Tempered fundamental group of an abelian variety

Let K be an algebraically closed complete nonarchimedean field of characteristic 0. Let A be an abelian variety over K, and let g be its dimension.

Recall the basics of p-adic uniformization of abelian varieties. By [10], there is a commutative algebraic group G (more precisely a semiabelian variety) and a surjective analytic morphism $u : G^{\mathrm{an}} \to A^{\mathrm{an}}$ which is the universal topological covering of A and $\ker u$ is a discrete free abelian group Λ of rank d. Moreover, we know that $(A^{(n)} \to A)_{n\in\mathbf{N}}$ ($\mathbf{N}$ being ordered by divisibility), with $A^{(n)}$ a copy of A and $A^{(n)} \to A$ multiplication by n, is a cofinal family of finite Galois coverings of A (by [11, lecture XI]).

Let $G^{(n)}$ be the universal topological covering of $A^{(n)}$ (which is isomorphic to G since $A^{(n)}$ is isomorphic to A); one has

$$\pi_1^{\mathrm{temp}}(A) = \varprojlim_n \mathrm{Gal}(G^{(n)}/A).$$

Recall that a topological group is said to be *residually finite* if the intersection of all open subgroups of finite index is $\{1\}$. Since $\pi_1^{\mathrm{top}}(A^{(n)}) = \pi_1^{\mathrm{top}}(A) = \Lambda \simeq \mathbf{Z}^d$ is residually finite and is a subgroup of finite index of $\mathrm{Gal}(G^{(n)}/A)$, $\mathrm{Gal}(G^{(n)}/A)$ is residually finite for every n, so $\pi_1^{\mathrm{temp}}(A)$ is also residually finite as a projective limit of residually finite groups. Thus $\pi_1^{\mathrm{temp}}(A) \to \pi_1^{\mathrm{alg}}(A)$ is injective and, since $\pi_1^{\mathrm{alg}}(A)$ is abelian, $\pi_1^{\mathrm{temp}}(A)$ is also an abelian group.

If $n \mid m$, one has the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Gal}(G^{(m)}/G) \arrow[r] \arrow[d] & \mathrm{Gal}(G^{(m)}/A) \arrow[r] \arrow[d] & \mathrm{Gal}(G/A) = \Lambda \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Gal}(G^{(n)}/G) \arrow[r] & \mathrm{Gal}(G^{(n)}/A) \arrow[r] & \mathrm{Gal}(G/A) = \Lambda \arrow[r] & 0
\end{tikzcd}$$

Let us write $T(G) = \varprojlim \mathrm{Gal}(G^{(n)}/G)$. This is a profinite abelian group, so it splits (canonically) as the product of its pro-p-Sylow subgroups $T_l(G)$. By taking the projective limit in the previous commutative diagram, one gets the following exact sequence (it is right exact because the $\mathrm{Gal}(G^{(n)}/G)$ are finite):

$$0 \to T(G) \to \pi_1^{\mathrm{temp}}(A) \to \Lambda \to 0,$$

with $\pi_1^{\mathrm{temp}}(A)$ abelian. Thus it is an exact sequence of abelian groups. But Λ is a free abelian group, so the exact sequence splits. One thus gets a noncanonical isomorphism

$$\pi_1^{\mathrm{temp}}(A) \simeq \Lambda \times T(G).$$

By taking the pro-l completion of the isomorphism above, one gets

$$\mathbf{Z}_l^{2g} \simeq \pi_1^{\mathrm{alg}}(A)^l \simeq \widehat{\pi_1^{\mathrm{temp}}(A)}^l \simeq \mathbf{Z}_l^d \times T_l(G),$$

and so $T_l(G) \simeq \mathbf{Z}_l^{2g-d}$. We finally obtain a noncanonical isomorphism

$$\pi_1^{\mathrm{temp}}(A) \simeq \mathbf{Z}^d \times \widehat{\mathbf{Z}}^{2g-d}.$$

§2.2. Jacobian variety and $(\pi_1^{\mathrm{temp}})^{\mathrm{ab}}$ of a curve

If G is a topological prodiscrete group with a countable basis of neighborhoods of 1, then G^{ab} is the topological group $G/\overline{D(G)}$ (where $\overline{D(G)}$ is the closure of the derived subgroup of G); it is also a prodiscrete group and $G \to G^{\mathrm{ab}}$ makes G^{ab}-Set a full subcategory of G-Set.

Let K be a complete discrete valuation field of characteristic 0, $\bar{K}$ the completion of its algebraic closure. Let C be a curve over K and let A be the Jacobian variety of $C_{\bar{K}}$. Let P be a closed point of $C_{\bar{K}}$. Consider the morphism $C_{\bar{K}} \to A$ that maps x to the divisor $[x] - [P]$. One gets a homomorphism $\pi_1^{\mathrm{temp}}(C_{\bar{K}}, P) \to \pi_1^{\mathrm{temp}}(A, 0)$ that factorizes through $\pi_1^{\mathrm{temp}}(C_{\bar{K}}, P)^{\mathrm{ab}}$ since $\pi_1^{\mathrm{temp}}(A, 0)$ is abelian.

Theorem 2.1. *The morphism $\pi_1^{\mathrm{temp}}(C_{\bar{K}}, P)^{\mathrm{ab}} \to \pi_1^{\mathrm{temp}}(A, 0)$ is an isomorphism.*

Proof. We have a morphism $C_{\bar{K}}^g \to A$ which maps $(x_1, \ldots, x_g)$ to the divisor $[x_1] + \cdots + [x_g] - g[P]$ of $C_{\bar{K}}$. This morphism is invariant under the action of $\mathfrak{S}_g$ on $C_{\bar{K}}^g$ and thus factorizes through a morphism $C_{\bar{K}}^{(g)} := C_{\bar{K}}^g/\mathfrak{S}_g \to A$. Recall that this is a birational morphism and that $C_{\bar{K}}^{(g)}$ is smooth over $\bar{K}$ (see [13, Th. 5.1(a), Prop. 3.2]).

We thus get a sequence of morphisms

$$C_{\bar{K}} \to C_{\bar{K}}^g \to C_{\bar{K}}^{(g)} \to A,$$

where the left morphism maps x to $(x, P, \ldots, P)$ and the composed morphism maps x to $[x] - [P]$.

Since $C_{\bar{K}}^{(g)} \to A$ is a birational morphism of proper smooth $\bar{K}$-varieties, $\pi_1^{\mathrm{temp}}(C_{\bar{K}}^{(g)}, (P,\dots,P)) \to \pi_1^{\mathrm{temp}}(A,0)$ is an isomorphism according to Proposition 1.6. Thus $\pi_1^{\mathrm{temp}}(C_{\bar{K}}^{(g)}, (P\dots,P))$ is abelian and residually finite, and $\pi_1^{\mathrm{temp}}(C_{\bar{K}}, P) \to \pi_1^{\mathrm{temp}}(C_{\bar{K}}^{(g)}, (P,\dots,P))$ factorizes through $\phi : \pi_1^{\mathrm{temp}}(C_{\bar{K}}, P)^{\mathrm{ab}} \to \pi_1^{\mathrm{temp}}(C_{\bar{K}}^{(g)}, (P,\dots,P))$.

As $\pi_1^{\mathrm{temp}}(C_{\bar{K}}, P)^{\mathrm{ab}}$ is a projective limit of abelian groups of finite type (and thus residually finite groups), it must be residually finite too. The commutative diagram

$$\begin{tikzcd}
\pi_1^{\mathrm{temp}}(C_{\bar{K}}, P)^{\mathrm{ab}} \arrow[r, "\phi"] \arrow[d] & \pi_1^{\mathrm{temp}}(C_{\bar{K}}^{(g)}, (P,\dots,P)) \arrow[d] \\
\pi_1^{\mathrm{alg}}(C_{\bar{K}}, P)^{\mathrm{ab}} \arrow[r, "\simeq"] & \pi_1^{\mathrm{alg}}(C_{\bar{K}}^{(g)}, (P,\dots,P))
\end{tikzcd}$$

whose vertical arrows are injective, shows that ϕ is injective.

Since $C_{\bar{K}}^{(g)} = C_{\bar{K}}^{g}/\mathfrak{S}_g$, one may also put an orbifold structure on $C_{\bar{K}}^{(g)}$ in the sense of [2, §III.4] such that $C_{\bar{K}}^{g} \to C_{\bar{K}}^{(g)}$ is an orbifold uniformization (then $\pi_1^{\mathrm{orb}}(C_{\bar{K}}^{(g)})$ denotes the corresponding extension of $\mathfrak{S}_g$ by $\pi_1^{\mathrm{temp}}(C_{\bar{K}}^{g})$, as in [2, Prop. III.4.5.8]). One then has the following commutative diagram whose row is exact:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1^{\mathrm{temp}}(C_{\bar{K}}^{g}, (P,\dots,P)) \arrow[r, "i"] \arrow[ddr, "\alpha"'] & \pi_1^{\mathrm{orb}}(C_{\bar{K}}^{(g)}, (P,\dots,P)) \arrow[r] \arrow[d, "\pi_1"] & \mathfrak{S}_g \arrow[r] & 1 \\
& & \pi_1^{\mathrm{orb}}(C_{\bar{K}}^{(g)}, (P,\dots,P))^{\mathrm{ab}} \arrow[d, "\pi_2"] & & \\
& & \pi_1^{\mathrm{temp}}(C_{\bar{K}}^{(g)}, (P,\dots,P)) & &
\end{tikzcd}$$

The morphisms i, π_1 and π_2 are open, thus $\alpha = \pi_2\pi_1 i$ is open: $\operatorname{Im}\alpha$ is an open subgroup of $\pi_1^{\mathrm{temp}}(C_{\bar{K}}^{(g)}, (P,\dots,P))$.

If $\operatorname{Im}\alpha$ is a strict subgroup, $\pi_1^{\mathrm{temp}}(C_{\bar{K}}^{(g)}, (P,\dots,P))/\operatorname{Im}\alpha$ is a nontrivial abelian group of finite type and thus has a nontrivial finite quotient, which corresponds to an open subgroup of finite index of $\pi_1^{\mathrm{temp}}(C_{\bar{K}}^{(g)}, (P,\dots,P))$ which contains $\operatorname{Im}\alpha$. But the profinite completion $\hat{\alpha} : \pi_1^{\mathrm{alg}}(C_{\bar{K}}^{g}, (P,\dots,P)) \to \pi_1^{\mathrm{alg}}(C_{\bar{K}}^{(g)}, (P,\dots,P))$ of α is surjective (since $\pi_1^{\mathrm{alg}}(C_{\bar{K}}) \to \pi_1^{\mathrm{alg}}(C_{\bar{K}}^{(g)})$, which factorizes through $\hat{\alpha}$, is surjective), and thus one gets a contradiction: α is also surjective.

Let us now consider the homomorphism

$$\delta : \pi_1^{\mathrm{temp}}(C_{\bar{K}}, P) \to \pi_1^{\mathrm{temp}}(C_{\bar{K}}^{g}, (P,\dots,P))$$

induced by the morphism $C_{\bar{K}} \to C_{\bar{K}}^g$ that maps x to $(x, P, \dots, P)$. By identifying

$$\pi_1^{\text{temp}}(C_{\bar{K}}^g, (P, \dots, P)) \quad \text{and} \quad \pi_1^{\text{temp}}(C_{\bar{K}}, (P, \dots, P))^g$$

thanks to Proposition 3.10 that we will prove in the next section, δ may be identified to $\pi_1^{\text{temp}}(C_{\bar{K}}, P) \to \pi_1^{\text{temp}}(C_{\bar{K}}, P)^g : g \mapsto (g, 1, \dots, 1)$. Thus $\pi_1^{\text{temp}}(C_{\bar{K}}^g, (P, \dots, P))$ is generated by the family $(\sigma \circ \delta(\pi_1^{\text{temp}}(C_{\bar{K}}, P)))_{\sigma \in \mathfrak{S}_g}$. Since α is invariant under $\mathfrak{S}_g$, $\alpha \circ \delta$ is surjective, so

$$\pi_1^{\text{temp}}(C_{\bar{K}}, P)^{\text{ab}} \to \pi_1^{\text{temp}}(C_{\bar{K}}^{(g)}, (P, \dots, P))$$

is also surjective, thus it is bijective.

If U is an open subgroup of $\pi_1^{\text{temp}}(C_{\bar{K}}, P)$, the group generated by $\sigma(\delta(U))$ is an open subgroup of $\pi_1^{\text{temp}}(C_{\bar{K}}^g, (P, \dots, P))$, thus, as α is open and $\mathfrak{S}_g$-invariant, $\alpha \circ \delta$ is open, thus $\pi_1^{\text{temp}}(C_{\bar{K}}, P)^{\text{ab}} \to \pi_1^{\text{temp}}(C_{\bar{K}}^{(g)}, (P, \dots, P))$ is also open, so it is an isomorphism. □

§3. Alterations and tempered fundamental group

In this section we will describe two applications of de Jong's alteration theorems to the tempered fundamental group. Indeed, Berkovich already showed in [4, §9] how these theorems help to build a skeleton, which is homeomorphic to the geometric realization of a polysimplicial set, onto which a Zariski open subset of the variety retracts.

Relying on the results of Berkovich, we will prove in characteristic zero that the tempered fundamental group of a smooth algebraic variety is invariant under base change of algebraically closed complete fields, and that the tempered fundamental group of the product of two smooth varieties (over an algebraically closed base field) is canonically isomorphic to the product of the tempered fundamental groups of the factors.

§3.1. Preliminaries about the skeleton of a Berkovich space

In this subsection, we recall the description of the topology of manifolds with nice enough reduction (for example, semistable) given by Berkovich in [4] and [5]. We also recall how to use de Jong's alteration theorems to get information on the topology of any manifold.

Let K be a complete nonarchimedean field and let O_K be its ring of integers. If $\mathfrak{X}$ is a locally finitely presented formal scheme over O_K, $\mathfrak{X}_\eta$ will denote the generic fiber of $\mathfrak{X}$ in the sense of Berkovich ([3, Section 1]).

Recall the definition of a polystable morphism of formal schemes:

Definition 3.1 ([4, Def. 1.2], [5, Section 4.1]). Let $\phi : \mathfrak{Y} \to \mathfrak{X}$ be a locally finitely presented morphism of formal schemes over O_K.

(i) ϕ is said to be *strictly polystable* if, for every point $y \in \mathfrak{Y}$, there exists an open affine neighborhood $\mathfrak{X}' = \mathrm{Spf}(A)$ of $\phi(y)$ and an open neighborhood $\mathfrak{Y}' \subset \phi^{-1}(\mathfrak{X}')$ of y such that the induced morphism $\mathfrak{Y}' \to \mathfrak{X}'$ goes through an étale morphism $\mathfrak{Y}' \to \mathrm{Spf}(B_0) \times_{\mathfrak{X}'} \cdots \times_{\mathfrak{X}'} \mathrm{Spf}(B_p)$ where each B_i is of the form $A\{T_0, \dots, T_{n_i}\}/(T_0 \cdots T_{n_i} - a_i)$ with $a_i \in A$ and $n_i \geq 0$. It is said to be *nondegenerate* if one can choose $\mathfrak{X}'$, $\mathfrak{Y}'$ and (B_i, a_i) such that $\{x \in (\mathrm{Spf}(A))_\eta \mid a_i(x) = 0\}$ is nowhere dense.

(ii) ϕ is said to be *polystable* if there exists a surjective étale morphism $\mathfrak{Y}' \to \mathfrak{Y}$ such that $\mathfrak{Y}' \to \mathfrak{X}$ is strictly polystable. It is said to be *nondegenerate* if one can choose $\mathfrak{Y}'$ such that $\mathfrak{Y}' \to \mathfrak{X}$ is nondegenerate.

If $\mathfrak{S}$ is a locally finitely presented formal scheme over O_K, then a (*nondegenerate*) *polystable fibration* of length l over $\mathfrak{S}$ is a sequence of (nondegenerate) polystable morphisms $\underline{\mathfrak{X}} = (\mathfrak{X}_l \to \cdots \to \mathfrak{X}_1 \to \mathfrak{S})$.

Then K-$\mathcal{P}stf_l^{\text{ét}}$ will denote the category of polystable fibrations of length l over O_K, where a morphism $\underline{\mathfrak{X}} \to \underline{\mathfrak{Y}}$ is a collection of étale morphisms $(\mathfrak{X}_i \to \mathfrak{Y}_i)_{1 \leq i \leq l}$ which satisfies natural commutation assumptions.

$Pstf_l^{\text{ét}}$ will denote the category of couples $(\underline{\mathfrak{X}}, K_1)$ where K_1 is a complete non-archimedean field and $\underline{\mathfrak{X}}$ is a polystable fibration over O_{K_1}, and a morphism $(\underline{\mathfrak{X}}, K_1) \to (\underline{\mathfrak{Y}}, K_2)$ is a couple (ϕ, ψ) where ϕ is an isometric extension $K_2 \to K_1$ and ψ is a morphism $\underline{\mathfrak{X}} \to \underline{\mathfrak{Y}} \otimes_{O_{K_2}} O_{K_1}$ in K_1-$\mathcal{P}stf_l^{\text{ét}}$.

Berkovich defines polysimplicial sets in [4, Section 3] as follows. For an integer n, denote $[n] = \{0, 1, \dots, n\}$. For a tuple $\mathbf{n} = (n_0, \dots, n_p)$ with either $p = n_0 = 0$ or $n_i \geq 1$ for all i, let $[\mathbf{n}]$ denote the set $[n_0] \times \cdots \times [n_p]$ and $w(\mathbf{n})$ denote the number p. Berkovich defines a category $\mathbf{\Lambda}$ whose objects are $[\mathbf{n}]$ and morphisms are maps $[\mathbf{m}] \to [\mathbf{n}]$ associated with triples (J, f, α), where:

- J is a subset of $[w(\mathbf{m})]$ assumed to be empty if $[\mathbf{m}] = [0]$,
- f is an injective map $J \to [w(\mathbf{n})]$,
- α is a collection $\{\alpha_l\}_{0 \leq l \leq p}$, where α_l is an injective map $[m_{f^{-1}(l)}] \to [n_l]$ if $l \in \mathrm{Im}(f)$, and a map $[0] \to [n_l]$ otherwise.

The map $\gamma : [\mathbf{m}] \to [\mathbf{n}]$ associated with (J, f, α) takes $\mathbf{j} = (j_0, \dots, j_{w(\mathbf{m})}) \in [\mathbf{m}]$ to $\mathbf{i} = (i_0, \dots, i_{w(\mathbf{n})})$ with $i_l = \alpha_l(j_{f^{-1}(l)})$ for $l \in \mathrm{Im}(f)$, and $i_l = \alpha_l(0)$ otherwise. Then a *polysimplicial set* is a functor $\mathbf{\Lambda}^{\mathrm{op}} \to \mathrm{Set}$; they form a category denoted by $\mathbf{\Lambda}^\circ\,\mathrm{Set}$.

Let Δ be the strict simplicial category of integers with increasing maps. One has a functor from the category of strict simplicial sets to the category of polysimplicial sets by extending to $\Delta^\circ\,\mathrm{Set} \to \mathbf{\Lambda}^\circ\,\mathrm{Set}$ the functor $\Delta \to \mathbf{\Lambda}$ that maps $[n]$ to itself, in such a way that it commutes with direct limits.

Berkovich then considers a functor $\Sigma : \mathbf{\Lambda} \to \mathcal{K}e$ to the category of *Kelley spaces*, i.e. topological spaces X such that a subset of X is closed whenever its intersection with any compact subset of X is closed. This functor takes $[\mathbf{n}]$ to $\Sigma_{\mathbf{n}} = \{(u_{il})_{0\le i\le p, 0\le l\le n_i} \in [0,1]^{[\mathbf{n}]} \mid \sum_l u_{il} = 1\}$, and takes a map γ associated to (J, f, α) to $\Sigma(\gamma)$ that maps $\mathbf{u} = (u_{jk})$ to $\mathbf{u'} = (u'_{il})$ defined as follows: if $[\mathbf{m}] \neq [0]$ and $i \notin \mathrm{Im}(f)$, or $[\mathbf{m}] = [0]$, then $u'_{il} = 1$ for $l = \alpha_i(0)$ and $u'_{il} = 0$ otherwise; if $[\mathbf{m}] \neq [0]$ and $i \in \mathrm{Im}(f)$, then $u'_{il} = u_{f^{-1}(i), \alpha_i^{-1}(l)}$ for $l \in \mathrm{Im}(\alpha_i)$ and $u'_{il} = 0$ otherwise. This induces a functor, the *geometric realization*, $|\ | : \mathbf{\Lambda}^\circ\,\mathrm{Set} \to \mathcal{K}e$ (by extending Σ in such a way that it commutes with direct limits).

Berkovich attaches to a polystable fibration $\underline{\mathfrak{X}} = (\mathfrak{X}_l \to \mathfrak{X}_{l-1} \to \cdots \to \mathrm{Spf}(O_K))$ a polysimplicial set $\mathbf{C}_l(\mathfrak{X}_s)$ (which only depends on the special fiber) and a subset of the generic fiber $\mathfrak{X}_{l,\eta}$ of $\mathfrak{X}_l$, the *skeleton* $S(\underline{\mathfrak{X}})$ of $\underline{\mathfrak{X}}$, which is canonically homeomorphic to $|\mathbf{C}_l(\mathfrak{X}_s)|$ (see [4, Th. 8.2]), and such that $\mathfrak{X}_{l,\eta}$ retracts by a proper strong deformation onto $S(\underline{\mathfrak{X}})$.

In fact, when $\underline{\mathfrak{X}}$ is nondegenerate—for example generically smooth (we will only use the results of Berkovich for such polystable fibrations)—the skeleton of $\underline{\mathfrak{X}}$ depends only on $\mathfrak{X}_l$ according to [5, Prop. 4.3.1(ii)]; such a formal scheme that fits into a polystable fibration will be called a *pluristable morphism*, and we will denote by $S(\mathfrak{X}_l)$ this skeleton.

In this case, [5, Prop. 4.3.1(ii)] gives a description of $S(\mathfrak{X}_l)$, which is independent of the retraction. For any $x, y \in \mathfrak{X}_{l,\eta}$, we write $x \preceq y$ if for every étale morphism $\mathfrak{X}' \to \mathfrak{X}_l$ and any x' over x, there exists y' over y such that for any $f \in O(\mathfrak{X}'_\eta)$, $|f(x')| \le |f(y')|$ ($\preceq$ is a partial order on $\mathfrak{X}_{l,\eta}$). Then $S(\mathfrak{X}_l)$ is just the set of maximal points of $\mathfrak{X}_{l,\eta}$ for $\preceq$.

We will not give the construction of $\mathbf{C}_l(\mathfrak{X}_s)$ in full detail, as we will not need it. Rather, let us give some examples of polystable fibrations of length $l = 1$ ([4, Section 3]).

If $\mathfrak{X}$ is just $\mathrm{Spf}(B_0) \times_{\mathrm{Spf}\, O_K} \cdots \times_{\mathrm{Spf}\, O_K} \mathrm{Spf}(B_{p+1})$ where each B_i is of the form $O_K\{T_0, \ldots, T_{n_i}\}/(T_0 \cdots T_{n_i} - a_i)$ with $a_i \in O_K$, $|a_i| < 1$ for $i \le p$ and $|a_{p+1}| = 1$, then $\mathbf{C}_1(\mathfrak{X} \to \mathrm{Spf}(O_K))$ is just $[\mathbf{n}]$ with $\mathbf{n} = (n_0, \ldots, n_p)$.

If $\mathfrak{X}$ is a semistable formal scheme over $\mathrm{Spf}(O_K)$, then $\mathbf{C}_1(\mathfrak{X} \to \mathrm{Spf}(O_K))$ is a strict simplicial set.

For example, if $\mathcal{X}$ is a semistable model of a complete curve (which is clearly polystable over O_K), the polysimplicial set defined by Berkovich is just the graph of the stable reduction defined in the previous section (a graph can be considered as a simplicial set of dimension 1, and thus as a polysimplicial set).

If $(\overline{\mathcal{X}}, \mathcal{D})$ is a semistable model of a curve (not necessarily complete), the graph of $(\overline{\mathcal{X}})$ is the semigraph of $\mathcal{X} = \overline{\mathcal{X}} \setminus \mathcal{D}$ in which one deletes all the edges with only one branch. The retraction of $(\overline{\mathcal{X}}_\eta)^{\mathrm{an}}$ to the geometric realization of this graph restricts to a retraction of $(\mathcal{X}_\eta)^{\mathrm{an}}$ thanks to [4, Cor. 8.4].

The retraction to $S(\underline{\mathfrak{X}})$ commutes with étale morphisms:

Theorem 3.2 ([4, Th. 8.1]). *One can construct, for every polystable fibration $\underline{\mathfrak{X}} = (\mathfrak{X}_l \overset{f_{l-1}}{\to} \cdots \overset{f_1}{\to} \mathfrak{X}_1 \to \mathrm{Spf}(O_K))$, a proper strong deformation retraction $\Phi^l : \mathfrak{X}_{l,\eta} \times [0,l] \to \mathfrak{X}_{l,\eta}$ of $\mathfrak{X}_{l,\eta}$ onto the skeleton $S(\underline{\mathfrak{X}})$ of $\underline{\mathfrak{X}}$ such that:*

(i) $S(\underline{\mathfrak{X}}) = \bigcup_{x \in S(\underline{\mathfrak{X}}_{l-1})} S(\mathfrak{X}_{l,x})$ *(set-theoretic disjoint union), where* $\underline{\mathfrak{X}}_{l-1} = (\mathfrak{X}_{l-1} \to \cdots \to \mathrm{Spf}(O_K))$*;*

(ii) *if $\phi : \underline{\mathfrak{Y}} \to \underline{\mathfrak{X}}$ is a morphism of fibrations in $\mathcal{P}stf_l^{\acute{e}t}$, one has*

$$\phi_{l,\eta}(\Phi^l(y,t)) = \Phi^l(\phi_{l,\eta}(y), t)$$

for every $y \in \mathfrak{Y}_{l,\eta}$ and $t \in [0,l]$.

We will simply write x_t for $\Phi^l(x,t)$. We will now assume for simplicity that K is algebraically closed. Berkovich deduces from (ii) the following corollary:

Corollary 3.3 ([4, Cor. 8.5]). *Let $\underline{\mathfrak{X}}$ be a polystable fibration over O_K with a normal generic fiber $\mathfrak{X}_{l,\eta}$. Suppose we are given an action of a finite group G on $\underline{\mathfrak{X}}$ over O_K and a G-invariant Zariski open dense subset U of $\mathfrak{X}_{l,\eta}$. Then there is a strong deformation retraction of the Berkovich space $G \setminus U$ to a closed subset homeomorphic to $G \setminus |\mathbf{C}^l(\underline{\mathfrak{X}})|$.*

More precisely, in this corollary, the closed subset in question is the image of $S(\underline{\mathfrak{X}})$ (which is G-equivariant and contained in U) under $U \to G \setminus U$.

Theorem 3.2 also implies that the skeleton is functorial with respect to pluristable morphisms:

Proposition 3.4 ([5, Prop. 4.3.2(i)]). *If $\phi : \mathfrak{X} \to \mathfrak{Y}$ is a pluristable morphism between nondegenerate pluristable formal schemes over O_K, then $\phi_\eta(S(\mathfrak{X})) \subset S(\mathfrak{Y})$.*

In fact, more precisely, from the construction of S, $S(\mathfrak{Y}) = \bigcup_{x \in S(\mathfrak{X})} S(\mathfrak{Y}_x)$. Recall also that if the residue field k of K is algebraically closed, then the polysimplicial complex of a polystable fibration commutes with base change:

Proposition 3.5 ([4, Prop. 6.10]). *If $\underline{\mathfrak{X}}$ is a polystable fibration over O_K, then for any isometric extension $K \to K'$, $\mathbf{C}((\underline{\mathfrak{X}} \otimes O_{K'})_s) \to \mathbf{C}(\underline{\mathfrak{X}}_s)$ is an isomorphism.*

In order to use the previous description of the Berkovich space of a scheme with a model over O_K which admits a polystable fibration for understanding the topology of a smooth scheme over K, we will need de Jong's result about the existence of alterations by such pluristable schemes over O_K.

More precisely we will use the following consequence of de Jong's theory given by Berkovich (as we will work over valuation fields of characteristic 0, we give here only a version restricted to this case):

Lemma 3.6 ([4, Lem. 9.2]). *Assume K has characteristic 0, and let X be an integral scheme proper flat and of finite presentation over O_K, with an irreducible generic fiber of dimension l. Then there are:*

(a) *a polystable fibration $\underline{X}' = (X'_l \to \cdots \to X'_0 = \operatorname{Spec} O_K)$, where every morphism is projective of relative dimension 1 with smooth geometrically irreducible generic fibers,*

(b) *an action of a finite group G on $\underline{X}'$ over O_K,*

(c) *a dominant G-equivariant morphism $\phi : X'_l \to X$ over O_K,*

such that the generic fiber is generically étale with Galois group G.

§3.2. Invariance of π_1^{temp} under change of algebraically closed fields

Let X be a smooth connected algebraic variety over an algebraically closed complete nonarchimedean field K of characteristic 0. Let K'/K be an isometric extension of complete valued fields.

Lemma 3.7. *The functor $\mathrm{Cov}^{\mathrm{top}}(X) \to \mathrm{Cov}^{\mathrm{top}}(X_{K'})$ is an equivalence of categories. Thus, if x' is a geometric point of $X_{K'}$ with image x in X, then*

$$\pi_1^{\mathrm{top}}(X_{K'}, x') \to \pi_1^{\mathrm{top}}(X, x)$$

is an isomorphism.

Proof. Let us embed X in an integral scheme $\overline{X}$ which is proper, of finite presentation and flat over O_K. Then by Lemma 3.6 there is a generically smooth polystable fibration X' over O_K endowed with an action of a group G such that (X', G) is a Galois alteration of $\overline{X}$.

Let U be a dense Zariski open subset of $\overline{X}$ included in X such that $U' \to U$ is finite (where U' is the pullback of U in X'). Then $G\backslash S(X'_s)$ is a strong deformation retract of U^{an} by Corollary 3.3. Moreover $X'_{O_{K'}}$ is a polystable fibration endowed with an action of G, and $X'_{O_{K'}}$ is also a Galois alteration of $\overline{X}_{K'}$, finite over $U_{K'}$. Thus, as in the previous case, $U^{\mathrm{an}}_{K'}$ retracts to the closed subset $G\backslash S(X'_{K'})$ and the natural morphism $U^{\mathrm{an}}_{K'} \to U^{\mathrm{an}}$ maps $S(X'_{K'})$ to $S(X')$. But $\mathbf{C}(X'_{K',s}) \to \mathbf{C}(X'_s)$ is an isomorphism according to Proposition 3.5. The morphism $U^{\mathrm{an}}_{K'} \to U^{\mathrm{an}}$ is thus a homotopy equivalence.

One has the following 2-commutative diagram:

$$\begin{tikzcd} \mathrm{Cov}^{\mathrm{top}}(U_{K'}) & \mathrm{Cov}^{\mathrm{top}}(U) \arrow[l] \\ \mathrm{Cov}^{\mathrm{top}}(X_{K'}) \arrow[u] & \mathrm{Cov}^{\mathrm{top}}(X) \arrow[l] \arrow[u] \end{tikzcd}$$

The vertical arrows are equivalences by Proposition 1.4, and the top arrow is also an equivalence by what has just been shown. Thus $\pi_1^{\mathrm{top}}(X_{K'}, x') \to \pi_1^{\mathrm{top}}(X, x)$ is an isomorphism. □

Let us assume now that K' is also algebraically closed.

Proposition 3.8. *If x' is a geometric point of $X_{K'}$ with image x in X, then the morphism $\pi_1^{\mathrm{temp}}(X_{K'}, x') \to \pi_1^{\mathrm{temp}}(X, x)$ is an isomorphism.*

Proof. Let us consider a countable cofinal projective system of geometrically pointed (the points being defined in a large enough valuation field Ω) Galois coverings $(X_i, x_i)_{i\in\mathbf{N}}$ of X. Then $(X_{i,K'}, x'_i)_{i\in\mathbf{N}}$, where x'_i is some point of $X_{i,K'}$ over x_i, is also a cofinal projective system of Galois covering of $X_{K'}$ by [11, lecture XIII].

If X_i^∞ is the universal topological covering of X_i then $X_{i,K'}^\infty := (X_i^\infty)_{K'}$ is the universal topological covering of $X_{i,K'}$ by Lemma 3.7.

Since $\mathrm{Gal}(X_{i,K'}^\infty/X_{K'}) = \mathrm{Gal}(X_i^\infty/X)$, by taking the projective limit over $i \in \mathbf{N}$, one gets the desired result. □

§3.3. Products and tempered fundamental group

Let X, Y be smooth connected algebraic varieties over an algebraically closed complete nonarchimedean field K of characteristic 0.

Lemma 3.9. *If x and y are geometric points of X and Y (with values in the same field), then*

$$\pi_1^{\mathrm{top}}(X \times Y, (x, y)) \to \pi_1^{\mathrm{top}}(X, x) \times \pi_1^{\mathrm{top}}(Y, y)$$

is an isomorphism.

Proof. Let $\overline{X}$ (resp. $\overline{Y}$) be a scheme which is proper, of finite presentation and flat over O_K, in which X (resp. Y) is embedded as an open subvariety, and let $(X', G) \to \overline{X}$ (resp. $(Y', H) \to \overline{Y}$) be a Galois alteration such that $X' \to O_K$ (resp. $Y' \to O_K$) is pluristable. Let also $U \subset X$ (resp. $V \subset Y$) be a dense Zariski open embedding such that $U' \to U$ (resp. $V' \to V$) is finite.

Since the fact that $\pi_1^{\mathrm{top}}(X \times Y, (x, y)) \to \pi_1^{\mathrm{top}}(X, x) \times \pi_1^{\mathrm{top}}(Y, y)$ is an isomorphism does not depend on x and y, one can assume that $x \in U$ and $y \in V$.

Then, as in the proof of Lemma 3.7, $G \setminus S(X')$ (resp. $H \setminus S(Y')$) is a strong deformation retract of U^{an} (resp. V^{an}).

One gets the same results for $X \times Y$, that is, $X' \times Y' \to X \times Y$ is a Galois alteration of group $G \times H$, and $U \times V$ retracts to $(G \times H) \setminus S(X' \times Y')$. But the pluristable maps $X' \times Y' \to X'$ and $X' \times Y' \to Y'$ send $S(X' \times Y')$ into $S(X')$ and $S(Y')$ respectively. This yields a map $f : S(X' \times Y') \to S(X') \times S(Y')$ (which is compatible with the action of $G \times H$). But as

$$S(X' \times Y') = \bigcup_{x \in S(X')} S((X' \times Y')_x) = \bigcup_{x \in S(X')} S(Y' \otimes \mathcal{H}(x))$$

and since $S(Y' \otimes \mathcal{H}(x)) \to S(Y')$ is a homeomorphism by Proposition 3.5, f is bijective, hence a homeomorphism since $S(X' \times Y')$ is compact.

Thus $(G \times H) \setminus S(X' \times Y') \to G \setminus S(X') \times H \setminus S(Y')$ is a homeomorphism. Therefore $(U \times V)^{\mathrm{an}} \to U^{\mathrm{an}} \times V^{\mathrm{an}}$ is a homotopy equivalence (the product on the right is the product of topological spaces) and thus $\pi_1^{\mathrm{top}}(U \times V, (x, y)) \to \pi_1^{\mathrm{top}}(U, x) \times \pi_1^{\mathrm{top}}(V, y)$ is an isomorphism.

By applying Proposition 1.4 to $U \subset X$, $V \subset Y$ and $U \times V \subset X \times Y$, one finds that

$$\pi_1^{\mathrm{top}}(X \times Y, (x, y)) \to \pi_1^{\mathrm{top}}(X, x) \times \pi_1^{\mathrm{top}}(Y, y)$$

is an isomorphism. □

Proposition 3.10. *If x and y are geometric points of X and Y (with values in the same field), then*

$$\pi_1^{\mathrm{temp}}(X \times Y, (x, y)) \to \pi_1^{\mathrm{temp}}(X, x) \times \pi_1^{\mathrm{temp}}(Y, y)$$

is an isomorphism.

Proof. Let $(X_i, x_i)_i$ and $(Y_j, y_j)_j$ be countable cofinal projective systems of connected geometrically pointed Galois coverings of X and Y. Then by [11, lecture XIII], $(X_i \times Y_j, (x_i, y_j))_{(i,j)}$ is a cofinal projective system of connected Galois coverings of $X \times Y$. By Lemma 3.9, $(X_i \times Y_j)^\infty = X_i^\infty \times Y_j^\infty$ and

$$\mathrm{Gal}((X_i \times Y_j)^\infty / (X \times Y)) = \mathrm{Gal}(X_i^\infty / X) \times \mathrm{Gal}(Y_j^\infty / Y).$$

Thus, by taking the projective limit over (i, j) in the previous isomorphism, one gets the desired result. □

§4. Metric structure of the graph of the stable model of a curve

The main goal of this section is to prove that the metric structure on the graph of the stable model of a Mumford curve (or equivalently, the graph structure of the skeleton of the curve considered as a Berkovich space, cf. 3.1) can be recovered from the tempered fundamental group of this curve.

We will only consider the case of mixed characteristic (in the case of equal characteristic 0, this is surely false, as the whole tempered fundamental group can be recovered only from the graph of groups associated with the stable reduction of the curve according to [16, Ex. 3.10]).

Definition 4.1. A *metric structure* on a graph $\mathbb{G}$ is a function

$$d : \{\text{edges of } \mathbb{G}\} \to \mathbf{R}_{>0}.$$

For any e, $d(e)$ is called the *length* of the edge e with respect to the metric structure d. A graph endowed with a metric structure is called a *metric graph.*

For a curve X over an algebraically closed complete nonarchimedean field K with a semistable model $\mathcal{X}$, let e be an edge of the graph of this semistable reduction (that is, a node of $\mathcal{X}_s$). Then, locally for the étale topology at this node, $\mathcal{X}$ is étale over $O_K[X_0, X_1]/(X_0X_1 - a)$, with $a \in O_K$. According to [17, Cor. 2.2.18], $|a|$ does not depend on any choice. Then we set

$$d(e) = -\log_p(|a|),$$

which defines a natural metric structure on the graph of the stable reduction of X.

For example, if $\mathcal{X}$ is the stable model of $\mathbf{P}^1 \setminus \{0, 1, \infty, \lambda\}$ with $|\lambda| < 1$, then the graph of $\mathcal{X}$ has a single edge of length $-\log_p(|\lambda|)$.

We already know, from Theorem 1.7, that one can recover the graph of the stable model from the tempered fundamental group. In fact we will deduce from Mochizuki's study that one can decide, for every finite index open subgroup of the tempered fundamental group and every vertex of the skeleton of the curve, whether the corresponding covering of the curve is split over this vertex (an étale covering $X' \to X$ of manifolds is said to be *split* over a point $x \in X$ if $\mathcal{H}(x) \to \mathcal{H}(x')$ is an isomorphism for every $x' \in X'$ over x; a covering $X' \to X$ of order n is split over x if and only if the fiber of x has cardinality n, which amounts to the fact that locally in a neighborhood of x, $X' \to X$ pulls back to a topological covering [2, III.1.2.1]).

This suggests asking whether finite étale coverings of a Mumford curve are split over vertex points. Studying simple coverings may be enough to see that the metric structure of the skeleton must play a role in the structure of the tempered fundamental group.

Let us begin with an elementary situation.

Lemma 4.2. *The covering* $\mathbf{G}_m \xrightarrow{z \mapsto z^{p^h}} \mathbf{G}_m$ *is split over a Berkovich point* $B(1,r)$ *corresponding to the ball of center* 1 *and radius* r *with* $r < 1$ *if and only if* $r < p^{-h-1/(p-1)}$.

More precisely, $B(1,r)$ *has* p^i *preimages if:*

- $i = 0$ *and* $r \in]p^{-p/(p-1)}, 1]$;
- $1 \le i \le h-1$ *and* $r \in]p^{-i-p/(p-1)}, p^{-i-1/(p-1)}]$;
- $i = h$ *and* $r \in [0, p^{-h-1/(p-1)}]$.

Proof. Let $g : z \mapsto z^p$, and let us calculate $g(B(z_1, r))$ with $|z_1| = 1$ and $r < 1$. Let $f_{z_1} : z \mapsto (z+z_1)^p - z_1^p = \sum_{i=1}^p a_i z^i$ with $|a_p| = 1$ and $|a_i| = p^{-1}$ if $1 \le i \le p-1$.

Then $g(B(z_1, r)) = B(z_1^p, r')$ with

$$r' = \max_{|z|<r} |f(z)| = \max |a_i| r^i = \max\{r^p, rp^{-1}\} = \begin{cases} rp^{-1} & \text{if } r \le p^{-1/(p-1)}, \\ r^p & \text{if } r \ge p^{-1/(p-1)}. \end{cases}$$

Moreover let z_0 be of norm 1, let $z_0^{1/p}$ be a pth root of z_0, and let $r' < 1$. Then $|z_1^p - z_0| = \prod_{\zeta \in \mu_p} |z_1 - \zeta z_0^{1/p}| \le r'$ implies that there exists $\zeta_0 \in \mu_p$ such that $|z_1 - \zeta_0 z_0^{1/p}| \le r'^{1/p}$ (i.e. $B(z_1, r'^{1/p}) = B(z_0^{1/p}, r'^{1/p})$).

Suppose now $|\zeta - \zeta'| = p^{-1/(p-1)}$. Since $|\zeta - \zeta_0| = p^{-1/(p-1)}$ if $\zeta \in \mu_p \setminus \{\zeta_0\}$, one has $|z_1 - z_0^{1/p}\zeta| = p^{-1/(p-1)}$ if $\zeta \in \mu_p \setminus \{\zeta_0\}$ and thus $|z_1 - z_0^{1/p}\zeta_0| = |z_1^p - z_0| / \prod_{\zeta \in \mu_p \setminus \{\zeta_0\}} |z_1 - z_0^{1/p}\zeta| \le pr'$ (i.e. $B(h', pr') = B(\zeta_0 z_0^{1/p}, pr')$). Thus

$$g^{-1}(B(z_0, r')) = \begin{cases} \{B(\zeta z_0^{1/p}, pr')\}_{\zeta \in \mu_p} & \text{if } r' \le p^{-p/(p-1)}, \\ \{B(\zeta z_0^{1/p}, r'^{1/p})\}_{\zeta \in \mu_p} & \text{if } r' \ge p^{-p/(p-1)}. \end{cases}$$

Since $|\zeta - \zeta'| = p^{-1/(p-1)}$ if $\zeta \ne \zeta' \in \mu_p$, one finds that $g^{-1}(B(z_0, r'))$ has a single element if $r' \ge p^{-p/(p-1)}$ and p elements otherwise. Thus one gets the desired result when $h = 1$.

In the general case, one uses induction on h by decomposing $z \mapsto z^{p^h}$ into $z \mapsto z^{p^{h-1}} \mapsto z^{p^h}$. □

We will first study two examples where one can use Lemma 4.2 in a direct way:

- a punctured line (Theorem 4.6),
- a punctured elliptic curve, for which we will cut and paste the kind of coverings considered in Lemma 4.2 (Theorem 4.7).

No direct reference to the proofs of these examples will be made in the proof of the general case of Mumford curves, so the reader can skip the examples if he prefers (the case of punctured lines and elliptic curves can also be recovered from the case of proper hyperbolic Mumford curve).

For the more general case of a Mumford curve X, we will also study the structure of $\mathbf{Z}/p^h\mathbf{Z}$-torsors. The theory of theta functions as can be found in [19] and [18] tells us that the pullback of such a torsor to the universal topological covering Ω is in fact the pullback of $\mathbf{G}_m \xrightarrow{z\mapsto z^{p^h}} \mathbf{G}_m$ along some theta function $\Omega \to \mathbf{G}_m$, which in turn corresponds to some equivariant current over the tree $\mathbb{T}(\Omega)$ of Ω. Therefore, we will begin our study by proving that if two currents coincide over a sufficiently large part of $\mathbb{T}(\Omega)$, then the quotient of the two associated invertible functions is nearly constant over some smaller part of $\mathbb{T}(\Omega)$ and thus the two corresponding $\mathbf{Z}/p^h\mathbf{Z}$-torsors are split over the same vertices in this part of $\mathbb{T}(\Omega)$. Thus, we will consider some currents which, over some "large" part of Ω, coincide with the one corresponding to a homography invertible over Ω and which is equivariant under some subgroup of finite index of $\mathrm{Gal}(\Omega/X)$. Then we consider the corresponding $\mathbf{Z}/p^h\mathbf{Z}$-torsor over some finite étale covering of X which will behave like $\mathbf{G}_m \xrightarrow{z\mapsto z^{p^h}} \mathbf{G}_m$ over some part of $\mathbb{T}(\Omega)$.

We will deduce from this that the length of every loop of every finite topological covering of the graph of the stable model of X can be recovered from the tempered fundamental group. A final combinatorial consideration will give us what we wanted.

§4.1. Preliminaries

The recovering of the graph of the stable reduction from the tempered fundamental group is functorial with respect to (p')-finite coverings, in the sense that one can recover the morphism of graphs induced by the covering from the morphism of the tempered fundamental groups. A general finite covering does not induce an actual morphism of graphs. However, as we prove here, one can still recover some combinatorial data from the morphism of the tempered fundamental groups.

Let K be a complete dicrete valuation field of characteristic 0, O_K its integer ring, k its residue field, assumed of characteristic $p > 0$, and let $\overline{K}$ be the completion of the algebraic closure of K (with integer ring $O_{\overline{K}}$).

From now on, we assume that we have chosen a compatible system of roots of 1 in $\overline{K}$, so that we may identify μ_n and $\mathbf{Z}/n\mathbf{Z}$. Thus we will often talk of $\mathbf{Z}/n\mathbf{Z}$-torsors over a curve over $\overline{K}$ when we should better talk of μ_n-torsors.

Let $X_{1,K}$, $X_{2,K}$ be smooth hyperbolic curves of type (g, n) over K. Let ϕ be an isomorphism $\pi_1^{\mathrm{temp}}(X_{1,\overline{K}}) \simeq \pi_1^{\mathrm{temp}}(X_{2,\overline{K}})$.

Let H_1 be an open subgroup of finite index in $\pi_1^{\mathrm{temp}}(X_{1,\overline{K}})$ and let $H_2 := \phi(H_1)$.

For $i = 1,2$, we will write $\Pi_i := \pi_1^{\mathrm{temp}}(X_{i,\overline{K}})$ to simplify notation. Let $Y_{i,\overline{K}} \to X_{i,\overline{K}}$ be the connected finite étale covering of $X_{i,\overline{K}}$ corresponding to H_i. Let X_i and Y_i be the stable models of $X_{i,\overline{K}}$ and $Y_{i,\overline{K}}$, and $\psi_i : Y_i \to X_i$ the unique morphism extending $Y_{i,\overline{K}} \to X_{i,\overline{K}}$. Furthermore, $\overline{X}_i$ and $\overline{Y}_i$ will denote the semistable compactfications of X_i and Y_i.

By Theorem 1.7, ϕ induces an isomorphism $\mathcal{G}_1^c \simeq \mathcal{G}_2^c$ between the semigraphs of anabelioids of X_1 and X_2. Likewise, $H_1 \simeq H_2$ induces an isomorphism $\mathcal{H}_1^c \to \mathcal{H}_2^c$ between the semigraphs of anabelioids of Y_1 and Y_2.

We are now interested in reconstructing from $\pi_1^{\mathrm{temp}}(Y_{i,K}) \to \pi_1^{\mathrm{temp}}(X_{i,K})$ what data can be recovered of the preimage of the cusps and vertices of the skeleton of $X_{i,K}$.

Lemma 4.3. *Let x_1 be a cusp of $X_{1,s}$, and x_2 the cusp of $X_{2,s}$ corresponding to x_1 via $\mathbb{G}_1^c \simeq \mathbb{G}_2^c$. Let x_i' be the corresponding cusp of the generic fiber $X_{i,\eta}$. Then $\psi_{1,\eta}^{-1}(x_1')$ and $\psi_{2,\eta}^{-1}(x_2')$ have the same number of elements.*

Proof. Let y_i be a cusp of $Y_{i,s}$ (corresponding to a cusp y_i' of $Y_{i,\eta}$) and let z_i be its image in $X_{i,s}$ (corresponding to a cusp z_i' of $X_{i,\eta}$). Assume that y_1 and y_2 correspond to each other under $\mathbb{G}_1^c \simeq \mathbb{G}_2^c$.

Let $H_i^{(p')}$ (resp. $\Pi_i^{(p')}$) be the inverse limit of the discrete quotients of H_i (resp. Π_i) that are extensions of a (p')-finite group by a torsionfree group. Thus $H_i^{(p')} \simeq \pi_1^{\mathrm{temp}}(Y_{i,\overline{K}})^{(p')}$ and $\Pi_i^{(p')} \simeq \pi_1^{\mathrm{temp}}(X_{i,\overline{K}})^{(p')}$. Moreover, ϕ induces isomorphisms $\Pi_1^{(p')} \simeq \Pi_2^{(p')}$ and $H_1^{(p')} \simeq H_2^{(p')}$.

Let $I_i \subset H_i^{(p')}$ be an inertia group of y_i. The image of I_i in $\Pi_i^{(p')}$ is an open subgroup (and thus it is nontrivial) of an inertia group of z_i. Since the intersection of inertia groups of two different cusps is $\{1\}$, the image of I_i is not contained in any other inertia group of a cusp of $X_{i,s}$, thus z_i is characterized by the morphism $H_i \to \Pi_i$ as being the only cusp of $X_{i,s}$ such that inertia groups of y_i map by $H_i \to \Pi_i$ to inertia groups of z_i. Since $\Pi_1 \to \Pi_2$ (resp. $H_1 \simeq H_2$) sends the inertia groups of a cusp of $X_{1,s}$ (resp. $Y_{1,s}$) to inertia groups of the corresponding cusp of $X_{2,s}$ (resp. $Y_{2,s}$), z_2 is the cusp corresponding to z_1 by $H_1 \to H_2$. Thus the following diagram commutes:

$$\begin{array}{ccc} \text{cusps of } Y_{1,s} & \simeq & \text{cusps of } Y_{2,s} \\ \downarrow & & \downarrow \\ \text{cusps of } X_{1,s} & \simeq & \text{cusps of } X_{2,s} \end{array}$$

This gives the result. $\square$

In particular, the morphism $Y_{1,\eta} \to X_{1,\eta}$ is ramified at x_1' if and only if $Y_{2,\eta} \to X_{2,\eta}$ is ramified at x_2'.

Lemma 4.4. *Let $\widetilde{X}_{i,\eta}$ be a Zariski open subset of $\overline{X}_{i,\eta}$ containing $X_{i,\eta}$. Assume that the cusps of $\widetilde{X}_{1,\eta}$ correspond to the cusps of $\widetilde{X}_{2,\eta}$ under the bijection* $\mathrm{cusps}(X_{1,\eta}) \simeq \mathrm{cusps}(X_{2,\eta})$*. Then ϕ induces an isomorphism $\pi_1^{\mathrm{temp}}(\widetilde{X}_{1,\eta}) \to \pi_1^{\mathrm{temp}}(\widetilde{X}_{2,\eta})$.*

Proof. Let S_i^∞ be the universal topological covering of a Galois finite covering S_i of $X_{i,\eta}$. Then the corresponding discrete quotient of Π_i is an extension G_i of a free group by a finite quotient $G_{1,i}$ of Π_i, which corresponds to the finite covering S_i of $X_{i,\eta}$.

On the other hand, if G_i' is a quotient of Π_i (corresponding to a tempered covering S_i'') such that G_i' is an extension of a free group by a finite quotient $G_{1,i}'$ (which corresponds to a finite covering S_i'), then $S_i'' \to S_i'$ is a topological covering ([2, Th. III.2.1.9.a]).

Then, by Proposition 1.3, $\pi_1^{\mathrm{temp}}(\widetilde{X}_{i,\eta}) = \varprojlim_j \Pi_i/H_{i,j}$, where $(\Pi_i/H_{i,j})_j$ are the discrete quotients of $\pi_1^{\mathrm{temp}}(X_{i,\eta})$ which are extensions of a free group by a finite quotient of $\pi_1^{\mathrm{temp}}(X_{i,\eta})$ corresponding to a finite covering which is unramified above $\widetilde{X}_{i,\eta}$. By Lemma 4.3 the family $(H_{2,j})_j$ is just $(\phi(H_{1,j}))_j)$. This gives the desired isomorphism. □

Lemma 4.5. *Let x_1 be the generic point of an irreducible component of $X_{1,s}$, and x_2 the generic point of the corresponding irreducible component of $X_{2,s}$ under $\mathbb{G}_1^c \simeq \mathbb{G}_2^c$. Let x_i' be the corresponding point of the skeleton of the generic fiber $X_{i,\eta}$. Then $\psi_{1,\eta}^{-1}(x_1')$ and $\psi_{2,\eta}^{-1}(x_2')$ have the same number of elements.*

Proof. If $X_{i,0}$ is an irreducible component of $X_{i,s}$, let $Y_{i,0}$ be an irreducible component of $Y_{i,s}$ which maps surjectively to $X_{i,0}$. Then the morphism between the components of the graphs of groups $\Pi_{Y_{i,0}}^{(p')} \to \Pi_{X_{i,0}}^{(p')}$ is open (in particular, its image is noncommutative) since it embeds in the commutative diagram

$$\begin{tikzcd}
\mathrm{Gal}\left(\overline{K(Y_{i,0})}/K(Y_{i,0})\right) \arrow[r, hook] \arrow[d] & \mathrm{Gal}\left(\overline{K(X_{i,0})}/K(X_{i,0})\right) \arrow[d] \\
\Pi_{Y_{i,0}}^{(p')} \arrow[r] & \Pi_{X_{i,0}}^{(p')}
\end{tikzcd}$$

where the upper arrow is an open embedding and the vertical arrows are projections.

Since $\Pi_{X_{i,0}}^{(p')} \to \Pi_i^{(p')}$ (defined up to conjugation) is injective, the image of $\Pi_{Y_{i,0}}^{(p')}$ in $\Pi_i^{(p')}$ (defined up to conjugation) is noncommutative, and thus $\Pi_{X_{i,0}}^{(p')}$ is the only verticial subgroup of $\Pi_i^{(p')}$ which contains the image of $\Pi_{Y_{i,0}}^{(p')}$.

Moreover, if $Y_{i,0}$ is an irreducible component of $Y_{i,s}$ which does not map surjectively onto an irreducible component of $X_{i,s}$, the image of $\Pi^{(p')}_{Y_{i,0}}$ in $\Pi^{(p')}_i$ is commutative, so the embedding $H_i \to \Pi_i$ decides which components of $Y_{i,s}$ map surjectively onto which components of $X_{i,s}$.

In particular, if x_1 and x_2 are generic points of connected components of $X_{i,s}$ corresponding to each other, the number of preimages of x_i under $Y_{i,s} \to X_{i,s}$ is independent of i.

Let us now write π for the continuous map from the generic analytic fiber to the special fiber. Then $x_i = \pi(x'_i)$ is the generic point of an irreducible component of X_η, every preimage y_i of x_i under ψ_i is a generic point of an irreducible component of Y_η, and $\pi^{-1}(y_i)$ is reduced to a single element by [4, Cor. 1.7], which must map to x'_i because $\pi^{-1}(x_i) = \{x'_i\}$ by [4, Cor. 1.7]. Thus $\psi^{-1}_{i,\eta}(x'_i)$ is in natural bijection with $\psi_i^{-1}(x_i)$. □

§4.2. Case of $\mathbf{P}^1 \setminus \{z_1, \ldots, z_n\}$

Let $i = 1$ or 2. Let $z_{i,1}, \ldots, z_{i,n} \in \mathbf{Q}_p^{\mathrm{nr}}$, with $n \geq 4$, where $\mathbf{Q}_p^{\mathrm{nr}}$ denotes a maximal unramified extension of $\mathbf{Q}_p$. Write $X_i = \mathbf{P}^1 \setminus \{z_{i,1}, \ldots, z_{i,n}\}$. Let $\Pi_i = \pi_1^{\mathrm{temp}}(\mathbf{P}^1 \setminus \{z_{i,1}, \ldots, z_{i,n}\})$. We already know that an isomorphism $\phi : \Pi_1 \simeq \Pi_2$ induces an isomorphism between the semigraphs of the stable reductions of $\mathbf{P}^1 \setminus \{z_{1,1}, \ldots, z_{1,n}\}$. After reordering the $z_{2,j}$, we may assume that this morphism of semigraphs identifies the inertia subgroup (defined up to conjugation) of the cusp $z_{1,j}$ with the inertia subgroup of the cusp $z_{2,j}$.

Theorem 4.6. *The isomorphism of graphs thus defined by ϕ between the skeletons of $(\mathbf{P}^1 \setminus \{z_{1,1}, \ldots, z_{1,n}\})^{\mathrm{an}}$ and $(\mathbf{P}^1 \setminus \{z_{2,1}, \ldots, z_{2,n}\})^{\mathrm{an}}$ preserves the lengths of edges (i.e. it induces an isomorphism of metric graphs).*

Equivalently, for every (j_1, j_2, j_3, j_4), one gets the equality of cross-ratios' norms:

$$|(z_{1,j_1}, z_{1,j_2}, z_{1,j_3}, z_{1,j_4})| = |(z_{2,j_1}, z_{2,j_2}, z_{2,j_3}, z_{2,j_4})|.$$

In fact, we will be able to prove this result without assuming $z_{i,1}, \ldots, z_{i,n} \in \mathbf{Q}_p^{\mathrm{nr}}$ after studying the case of an elliptic curve (this will give the result for $p \neq 2$) and without any assumption after studying the case of a Mumford curve.

Proof of Theorem 4.6. By Lemma 4.4, one can assume that $n = 4$. One can also assume that the cusps are $0, 1, \infty$ and λ_i (since the length is invariant under automorphisms of $\mathbf{P}^1$). Moreover, we may assume that X_i does not have good reduction (in which case there is nothing to prove), and thus $v_p(\lambda_i - 1) > 0$ (it is an integer since $\lambda \in \mathbf{Q}_p^{\mathrm{nr}}$) after permuting $0, 1$ and ∞ by another automorphism of $\mathbf{P}^1$. We now have to prove that $v_p(\lambda_1 - 1) = v_p(\lambda_2 - 1)$.

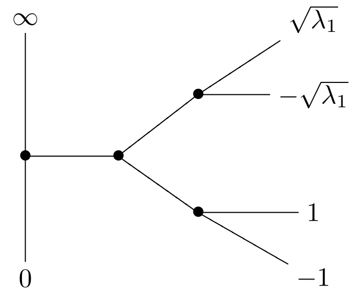
Assume, ab absurdo, that $v_p(\lambda_1 - 1) < v_p(\lambda_2 - 1)$. Let $h := v_p(\lambda_2 - 1) - 1 \geq v_p(\lambda_1 - 1)$ and let H_i be the subgroup of Π_i of index p^h corresponding to the unique connected covering $Y_i \to X_i$ of degree p^h unramified outside 0 and ∞ (this is the morphism $z \mapsto z^{p^h}$ from $\mathbf{P}^1$ to $\mathbf{P}^1$). One has $\phi(H_1) = H_2$ by Lemma 4.3.

But, according to Lemma 4.2, $B(1, r)$ (the point of the Berkovich space of $\mathbf{P}^1$ corresponding to the ball of radius r and center 1) has p^{h-1} preimages if $p^{-(h+1/(p-1))} \leq r < p^{-(h-1+1/(p-1))}$ and has p^h preimages if $r < p^{-(h+1/(p-1))}$.

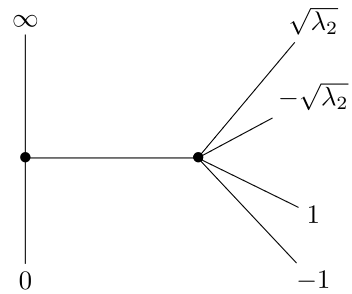
Thus if $p \neq 2$, $B(1, |\lambda_1 - 1|)$ has p^{h-1} preimages in Y_1 and $B(1, |\lambda_2 - 1|)$ has p^h preimages in Y_2, which contradicts Lemma 4.5.

In the case $p = 2$ and $h \geq 2$, $B(1, |\lambda_1 - 1|)$ has 2^{h-2} preimages in Y_1 and $B(1, |\lambda_2 - 1|)$ has 2^{h-1} preimages in Y_2, which contradicts Lemma 4.5.

If $h = 1$, and therefore $v_2(\lambda_1) = 1$ and $v_2(\lambda_2) = 2$, the semigraph of the reduction of $Y_1 \to X_1$ is (we marked the different cusps of Y_1, and $\sqrt{\lambda_i}$ is a square root of λ_i):



The semigraph of the stable reduction of Y_2 is



They are not isomorphic, and thus one also gets a contradiction. $\square$

§4.3. Case of a punctured elliptic curve

Let $i = 1$ or 2. Let X_i be two punctured Tate curves $\mathbf{C}_p^*/q_i^{\mathbf{Z}} - \{1\}$ with $q_i \in \overline{\mathbf{Q}}_p$ and $|q_i| < 1$. Let $\Pi_i = \pi_1^{\text{temp}}(X_i)$ and let $\phi : \Pi_1 \simeq \Pi_2$ be an isomorphism.

Theorem 4.7. $|q_1| = |q_2|$.

The main idea of the proof will be to consider a $\mathbf{Z}/p\mathbf{Z}$-torsor of the universal topological covering $\mathbf{P}^1 \setminus \{0,\infty\}$ of the elliptic curve which is ramified at only two cusps (and which is not ramified at 0 and ∞). As the covering becomes split near 0 and ∞, one can patch parts of this covering to make it periodic (thus it will descent to a $\mathbf{Z}/p\mathbf{Z}$-torsor of a finite topological covering). We will then use Lemma 4.2 as in the case of a punctured curve.

To verify that the torsors we have just constructed must (almost) correspond to each other by ϕ, we will have to study the $\mathbf{F}_p$-vector space of torsors of the elliptic curve ramified at only two points; this is a 3-dimensional vector space. A basis will be obtained by considering a topological covering, the torsor we have just constructed and another one which we will construct in a similar way. To give a better description in terms of currents (which will not be used in the proof), if the torsor we constructed corresponds to a current following a path linking the two cusps, the other torsor will correspond to a current linking the cusp following the other path linking the two cusps.

We do not have to assume $q_i \in \mathbf{Q}_p^{\mathrm{nr}}$ as in the case of a punctured line, because, by taking an unramified covering of the elliptic curve, we get as much vertices on the skeleton of the curve as we want.

Proof of Theorem 4.7. Let us choose integers n, l and m such that:

- n is prime to p and $n \geq \dfrac{v_p(q_2)v_p(q_1)(p-1)}{|v_p(q_2)-v_p(q_1)|p}$,
- $l \geq 1 + \dfrac{2np}{(p-1)v_p(q_i)}$,
- $m \geq 2l/n$.

Let $H_{0,i} = [\Pi_i, \Pi_i]\Pi_i^n$ be the preimage in Π_i of the image under multiplication by n in the abelianized group of Π_i. Then ϕ induces an isomorphism $H_{0,1} \to H_{0,2}$. The covering $Y_{0,i}$ of X_i corresponding to H_i is the multiplication $\overline{X}_i \xrightarrow{\times n} \overline{X}_i$ by n on the elliptic curve $\overline{X}_i$.

Let now $H_{1,i}$ be the subgroup of $H_{0,i}$ corresponding to the unique connected topological covering $Y_{1,i}$ of degree n of $Y_{0,i}$. Then $Y_{1,i} \simeq \mathbf{C}_p^*/q^{m\mathbf{Z}} - \{q^{a/n}\zeta^b\}_{(a,b)\in\mathbf{Z}^2}$ where $q^{1/n}$ is an n-th root of q and ζ is an n-th root of 1. The semigraph of the stable reduction of $Y_{1,i}$ has mn vertices joined in a circle (the distance of two such successive vertices is $v_p(q_i)/n$), and n cusps end at each vertex.

The map ϕ induces an isomorphism $H_{1,1} \simeq H_{1,2}$ which itself induces an isomorphism between the semigraphs of the stable reduction of $Y_{1,i}$. Let us number from 0 to $mn-1$ the vertices of the graphs by following the circle compatibly with the isomorphism induced by ϕ (let us write $x_{i,0}, \ldots, x_{i,mn-1}$ for the corresponding vertices of the skeleton of $Y_{1,i}$).

Let $z_{1,0}$ and $z_{1,l}$ be two cusps of $Y_{1,1}$ ending at the vertices of the graph numbered 0 and l respectively. Let $z_{2,0}$ and $z_{2,l}$ be the corresponding cusps of $Y_{1,2}$.

Let us now focus on $\mathbf{Z}_p$-torsors over $\overline{Y}_{1,i}$ which are unramified outside $z_{i,0}$ and $z_{i,l}$. They are the elements of an $\mathbf{F}_p$-vector space V_i of dimension 3.

Recall that we chose m, n and l so that

$$\frac{l-1}{n}v_p(q_i) > \frac{2p}{p-1} \quad \text{and} \quad \frac{mn-l}{n}v_p(q_i) > \frac{l}{n}v_p(q_i).$$

We will now describe a basis of this $\mathbf{F}_p$-vector space.

Let S_i be the universal topological covering of $\overline{Y}_{1,i}$; we identify it with $\mathbf{P}^1 \setminus \{0,\infty\} \subset \mathbf{P}^1$. Let $s_{i,0}$ and $s_{i,l}$ be the unique preimages in S_i of $z_{i,0}$ and $z_{i,l}$ of norm 1 and $|q|^{l/mn}$ and let $U_{1,i} \subset S_i$ be the open annulus $\{|q_i|^{(l-mn)/2n} > |z| > |q_i|^{(l+mn)/2n}\}$ and $U_{2,i} \subset S_i$ the open annulus $\{|q_i|^{l/n}p^{-p/(p-1)} > |z| > |q_i|^m p^{p/(p-1)}\}$. Maps from $U_{1,i}$ and from $U_{2,i}$ to $\overline{Y}_{1,i}$ are still open embeddings which, together, cover $\overline{Y}_{1,i}$.

Let $T_{1,i}$ be the restriction to $U_{1,i}$ of the ramified (only over $s_{i,0}$ and $s_{i,l}$) covering $\mathbf{P}^1 \to \mathbf{P}^1 : z \mapsto s_{i,0}z^p + s_{i,l}/(z^p+1)$, which is Galois with Galois group isomorphic to $\mathbf{Z}/p\mathbf{Z}$, and choose such an isomorphism to get a $\mathbf{Z}/p\mathbf{Z}$-torsor. Let $T_{2,i}$ be the trivial $\mathbf{Z}/p\mathbf{Z}$-torsor over $U_{2,i}$ and let $T_{3,i} = T_{1,i} \amalg T_{2,i} \to U_{3,i} = U_{1,i} \amalg U_{2,i}$.

Over $U_{1,i} \times_{\overline{Y}_{1,i}} U_{2,i}$ (which has two connected components), $T_{1,i}$ is trivial. Choosing a trivialization, one may now descend $T_{3,i} \to U_{3,i}$ to a $\mathbf{Z}/p\mathbf{Z}$-torsor $T_i \to \overline{Y}_{1,i}$, which is only ramified over $z_{i,0}$ and $z_{i,l}$.

According to Lemma 4.2, $T_i \to \overline{Y}_{1,i}$ is split over x_j (with $j \in [0, mn-1]$) if and only if $|q_i|^{j/n} \in [|q_i|^m p^{p/(p-1)}, |q_i|^{l/n}p^{-p/(p-1)}]$, i.e.

$$j \in I_{1,i} := \left[l + \frac{np}{v_p(q_i)(p-1)}, mn - \frac{np}{v_p(q_i)(p-1)}\right].$$

There is such an integer thanks to the assumption about l, m and n because

$$\lg(I_{1,i}) = mn - l - 2\frac{np}{v_p(q_i)(p-1)} \geq 1$$

(where lg denotes the length of an interval).

Likewise, let $s'_{i,l}$ be a preimage of $z_{i,l}$ of norm $|q|^{(l-mn)/n}$, let $U'_{1,i}$ be the annulus $\{|q_i|^{(l-2mn)/2n} > |z| > |q_i|^{l/2n}\}$, and $U'_{2,i}$ the annulus $\{p^{-p/(p-1)} > |z| > |q_i|^{(mn-l)/n}p^{p/(p-1)}\}$. These are open subsets of $\overline{Y}_{1,i}$ and cover it.

Let $T'_{1,i}$ be a $\mathbf{Z}/p\mathbf{Z}$-torsor over U'_1 obtained as the restriction of a $\mathbf{Z}/p\mathbf{Z}$-torsor over $\mathbf{P}^1$ ramified only above $z_{i,0}$ and $z'_{i,l}$. It is trivial over $U'_{1,i} \cap U'_{2,i}$, and, by choosing such a trivialization, one gets by descent a $\mathbf{Z}/p\mathbf{Z}$-torsor T'_i over $\overline{Y}_{1,i}$, ramified only over $z_{i,0}$ and $z_{i,l}$.

The covering $T_i \to \overline{Y}_{1,i}$ is split over x_j (with $j \in [0, mn-1]$) if and only if $|q_i|^{j/n} \in [|q_i|^{j/n} p^{p/(p-1)}, p^{-p/(p-1)}]$, i.e.

$$j \in I_{2,i} := \left[\frac{np}{v_p(q_i)(p-1)}, l - \frac{np}{v_p(q_i)(p-1)}\right].$$

There is such an integer thanks to the assumption about l, m and n because

$$\lg(I_{2,i}) = l - 2\frac{np}{v_p(q_i)(p-1)} \geq 1.$$

Let finally T''_i be the essentially unique connected topological covering of degree p of $\overline{Y}_i$ and let us choose an isomorphism from $\mathbf{Z}/p\mathbf{Z}$ onto its Galois group so that it becomes a $\mathbf{Z}/p\mathbf{Z}$-torsor. Let us then show that T_i, T'_i, and T''_i constitute a basis of V_i.

Let i be an integer in $I_{1,i}$. As $I_{1,i} \cap I_{2,i} = \emptyset$ and as T'' is everywhere split, if $aT_i + bT'_i + cT''_i$ (the linear combination is in the sense of the structure of the vector space V_i) is split over x_j, then $c = 0$. By the same argument with $I_{2,i}$, if $aT_i + bT'_i + cT''_i = 0$, one gets $b = 0$, but as T_i is not trivial, one indeed finds that T_i, T'_i and T''_i constitute a basis of V_i.

Assume now ab absurdo that $|q_1| > |q_2|$; then $I_{1,1} \subset I_{1,2}$. Let $T_{0,2} = aT_2 + bT'_2 + cT''_2$ be the image of T_1 under $(\phi^{-1})^* : V_1 \to V_2$ (indeed, ϕ induces such a $(\phi^{-1})^*$ according to Lemma 4.3).

Let $j \in I_{1,1}$. Then T_1 is split over $x_{1,j}$, so, according to Section 4.5, $T_{0,2}$ is split over $x_{2,j}$, and thus $c = 0$. Thus, $T_{0,2}$ is split over every x_j if $b = 0$ or exactly over the x_j with $j \in I_{1,2}$ otherwise.

Yet, according to Section 4.5, $T_{0,2}$ must be split exactly over the $x_{2,j}$ such that T_1 is split over $x_{1,j}$, i.e. $j \in I_{1,1}$.

Thus, one cannot be in the case $b = 0$ and $I_{1,2}$ and $I_{1,1}$ must contain exactly the same integers.

But this cannot be if $n \geq \frac{v_p(q_2)v_p(q_1)(p-1)}{(v_p(q_2)-v_p(q_1))p}$, because

$$\lg(I_{1,2}) - \lg(I_{1,1}) = 2\frac{np}{v_p(q_1)(p-1)} - 2\frac{np}{v_p(q_2)(p-1)} \geq 2. \qquad \square$$

Remark. Assume $p \neq 2$. Let $\{z_{i,1}, \ldots, z_{i,4}\}$ be four elements of $\overline{\mathbf{Q}}_p$, and let ϕ be an isomorphism between the $\pi_1^{\mathrm{temp}}(\mathbf{P}^1 \backslash \{z_{i,1}, \ldots, z_{i,4}\})$, which identifies $z_{1,j}$ to $z_{2,j}$ by the identification of the cusps of the graphs of the stable reduction (we assume that the curves have bad reduction, and that the length of the edge of the graph is l_i). Let E_i be the unique $\mathbf{Z}/2\mathbf{Z}$-covering of $\mathbf{P}^1$ ramified over $\{z_{i,1}, \ldots, z_{i,4}\}$ and only over those points (the subgroups of index 2 corresponding to E_i thus map to one another). E_i is a Tate curve with $|q_i| = 2l_i$. By Theorem 4.7, $|q_1| = |q_2|$ so $l_1 = l_2$, which again proves Theorem 4.6 without assuming that the points are in $\mathbf{Q}_p^{\mathrm{nr}}$.

If $p = 2$ and $l_i > 4$, then E_i is also a Tate curve, $|q_i| = 2l_i - 8$ and the previous argument works as well.

§4.4. Case of a Mumford curve

4.4.1. Reminder on Mumford curves and currents. Let X be a Mumford curve of genus $g \geq 2$ over $\overline{K}$, let $\Omega \subset \mathbf{P}^1$ be its universal topological covering, and $\Gamma = \mathrm{Gal}(\Omega/X)$, so that $X = \Omega/\Gamma$. Let $O(\Omega)$ be the ring of analytic functions on Ω. Let Φ be the retraction of Ω, as a Berkovich space, onto its skeleton $\mathbb{T} = \mathbb{T}(\Omega)$. The graph of the stable reduction of X is $\mathbb{G} = \mathbb{T}/\Gamma$.

For $z, z' \in \Omega$, set

$$d(z, z') = \sup_{x_1, x_2 \in \mathbf{P}^1 \setminus \Omega} \left| v_p \left(\frac{z' - x_1}{z - x_1} \frac{z - x_2}{z' - x_2} \right) \right|,$$

which is invariant under homographies stabilizing Ω. Moreover $d(z, z')$ depends only on $\Phi(z)$ and $\Phi(z')$, and it is nothing other than the distance between $\Phi(z)$ and $\Phi(z')$ for the usual metric structure on the tree of Ω (that we will also denote by d, so that $d(z, z') = d(\Phi(z), \Phi(z')))$.

If $z \in \Omega$ and $\lambda > 0$, let $U_{z,\lambda}$ denote $\{z \in \Omega \mid d(z, z') \leq \lambda\}$. It is an affinoid subspace of Ω.

Let $\mathcal{L} = \mathbf{P}^1 \setminus \Omega$, a compact subset of $\mathbf{P}^1$. In [9, 1.8.9], Fresnel and van der Put define a *measure* on a profinite topological space Z (i.e. a totally disconnected compact topological space) to be a function

$$\mu : \{\text{compact open subsets of } Z\} \to \mathbf{Z}$$

such that $\mu(U_1 \cup U_2) + \mu(U_1 \cap U_2) = \mu(U_1) + \mu(U_2)$ for any compact open subsets U_1, U_2 of Z and $\mu(\emptyset) = 0$. The group of measures on Z such that $\mu(Z) = 0$ is then denoted by $M_0(Z)$.

One can then associate to $f \in O(\Omega)^*$ a measure μ_f on $\mathcal{L}$. Following [9], we define a *hole* of an affinoid subspace U of $\mathbf{P}^1$ to be any connected component of the complement of this affinoid; we denote by $t(U)$ the set of all holes.

One gets the following exact sequence of groups ([9, Prop. 1.8.9]):

$$1 \to \overline{K}^* \to O(\Omega)^* \to M_0(\mathcal{L}) \to 0.$$

More precisely, according to [9, 1.8.10, Ex. β], if $a, b \in \mathcal{L}$ and $f : z \mapsto \frac{z-a}{z-b}$, then $\mu_f = \delta_a - \delta_b$ where δ_x is the Dirac probability measure with support at x (i.e. $\delta_x(U) = 1$ if $x \in U$ and $\delta_x(U) = 0$ if $x \notin U$).

For a general $f \in O(\Omega)^*$ and $\mu = \mu_f$, μ is a weak limit of a sequence of $\mathbf{Z}$-linear combinations $(\mu_k)_{k \in \mathbf{N}}$ of Dirac measures. If $\mu_k = \sum n_i(k) \delta_{a_i(k)}$, let $f_k =$

$\prod(1-a_i(k)/z)^{n_i(k)}$. Then $\mu_{f_k} = \mu_k$, and (f_k) tends uniformly on every affinoid to an invertible function g. Then $\mu_g = \mu$ and thus $g = \lambda f$ with $\lambda \in \overline{K}^*$.

If $\mathbb{G}_0$ is a locally finite graph and A is an abelian group, a *current* on $\mathbb{G}_0$ with coefficients in A is a function $\mathcal{C} : \{\text{oriented edges of } \mathbb{G}_0\} \to A$ such that:

- $\mathcal{C}(e) = -\mathcal{C}(e')$ if e and e' are the same edge but with reversed orientation;
- if v is a vertex of $\mathbb{G}_0$, $\sum_{e \text{ ending at } v} \mathcal{C}(e) = 0$.

The group of currents on $\mathbb{G}_0$ with coefficients in A will be denoted $C(\mathbb{G}_0, A)$. We will simply write $C(\mathbb{G}_0)$ for $C(\mathbb{G}_0, \mathbf{Z})$.

According to [19, Prop. 1.1], one also has the following exact sequence:

$$1 \to \overline{K}^* \to O(\Omega)^* \to C(\mathbb{T}) \to 0$$

which together with the previous one gives an isomorphism $M_0(\mathcal{L}) \to C(\mathbb{T})$.

One can describe this isomorphism in the following way. Let $\mu \in M_0(\mathcal{L})$ and let $\mathcal{C}$ be the image of μ under this isomorphism. If e is an oriented edge of $\mathbb{T}$, then $\Phi^{-1}(\mathbb{T} \setminus e)$ has two connected components, and $\Phi^{-1}(e) \cap \mathcal{L} = \emptyset$, so one gets a partition of $\mathcal{L}$ into two open subsets $\mathcal{L}_1(e)$ at the beginning of e and $\mathcal{L}_2(e)$ at the end of e (for $\mu \in M_0(\mathcal{L})$ one has $\mu(\mathcal{L}_1(e)) = -\mu(\mathcal{L}_2(e))$. Then $\mathcal{C}(e) = \mu(\mathcal{L}_2(e))$.

More generally, if $\mathbb{K}$ is a finite connected subgraph of $\mathbb{T}$ (containing at least one edge), then $\Phi^{-1}(\mathbb{K})$ is an affinoid contained in Ω. There is a natural bijection between $t(\Phi^{-1}(\mathbb{K}))$ and the set of edges of $\mathbb{T} \setminus \mathbb{K}$ that have one end in $\mathbb{K}$. If $\mu \in M_0(\mathcal{L})$ and $C \in t(\Phi^{-1}(\mathbb{K}))$, and e is the corresponding oriented edge of $\mathbb{T} \setminus \mathbb{K}$ starting in $\mathbb{K}$, then $C \cap \mathcal{L} = \mathcal{L}_2(e)$ and thus $\mathcal{C}(e) = \mu(C \cap \mathcal{L})$. In particular $\mu(C \cap \mathcal{L}) = 0$ for every $C \in t(\Phi^{-1}(\mathbb{K}))$ if and only if $\mathcal{C}(e) = 0$ for every edge of $\mathbb{T} \setminus \mathbb{K}$ having an end in $\mathbb{K}$, if and only if $\mathcal{C}$ is zero on the star of $\mathbb{K}$ (the *star* of $\mathbb{K}$ is by definition the set of edges of $\mathbb{T}$ having at least one end point in $\mathbb{K}$).

Let Θ denote the group of *theta functions* of X, that is, the group of $f \in O(\Omega)^*$ such that for every $\gamma \in \Gamma$, $z \mapsto f(\gamma z)/f(z)$ is a constant function (this means that the corresponding current is Γ-equivariant). Then one has the exact sequence

$$1 \to \overline{K}^* \to \Theta \to C(\mathbb{G}) \to 0$$

and thus $\Theta/\overline{K}^*$ is a free $\mathbf{Z}$-module of rank g. One deduces from [18, Th. 2.1] the following:

Proposition 4.8. *For every $n \geq 2$ and every $\mathbf{Z}/n\mathbf{Z}$-torsor Y over X, there exists an element θ in Θ, unique modulo $\overline{K}^*\Theta^n$, such that $Y \times_X \Omega = \Omega[f]/(f^n = \theta)$ where $\Omega[f]/(f^n = \theta)$ denotes the pullback of the $\mathbf{Z}/n\mathbf{Z}$-torsor $\mathbf{G}_m \xrightarrow{z \mapsto z^n} \mathbf{G}_m$ along $\Omega \xrightarrow{\theta} \mathbf{G}_m$. Conversely, for every θ in Θ, there exists a $\mathbf{Z}/n\mathbf{Z}$-torsor Y over X such that $Y \times_X \Omega = \Omega[f]/(f^n = \theta)$.*

Proof. Following the notation of [18, Section 2], let Ω_* be a connected component of $Y \times_X \Omega$.

Suppose first that $Y \times_X \Omega$ is connected, so that $\Omega_* = Y \times_X \Omega$. Then, according to [18, Prop. 2.1] there is a unique lattice T in $(\Theta/\overline{K}^*) \otimes \mathbf{Q}$ containing $\Theta/\overline{K}^*$ such that, as an X-covering, $\Omega_* = \Omega(T) := \Omega \times_{(\operatorname{Spec} \overline{K}[\Theta/\overline{K}^*])^{\mathrm{an}}} (\operatorname{Spec} \overline{K}[T])^{\mathrm{an}}$ (where the morphism $\Omega \to (\operatorname{Spec} K[\Theta/\overline{K}^*])^{\mathrm{an}}$ is given by some retraction of $\Theta \to \Theta/\overline{K}^*$; $\Omega(T)$ does not depend on the choice of such a retraction). Then $T/(\Theta/\overline{K}^*)$ is isomorphic to $\mathbf{Z}/n\mathbf{Z}$, and choosing a generator $\bar{f}$ of $T/(\Theta/\overline{K}^*)$ amounts to choosing a $\mathbf{Z}/n\mathbf{Z}$-torsor structure on Ω_*.

If one takes $\bar{f}^i$ (with $i \in (\mathbf{Z}/n\mathbf{Z})^*$) as another generator, the corresponding torsor is $i \cdot \Omega_*$ (for the $\mathbf{Z}/n\mathbf{Z}$-module structure on the set of $\mathbf{Z}/n\mathbf{Z}$-torsors). Thus by changing the generator, one can get all the different $\mathbf{Z}/n\mathbf{Z}$-torsor structures on the covering Ω_* of Ω.

If $\bar{f}$ is the generator corresponding to the $\mathbf{Z}/n\mathbf{Z}$-torsor structure on $Y \times_X \Omega$, then θ is any lifting of $\bar{f}^n \in T^n/(\Theta/\overline{K}^*)^n$.

In the general case, Ω_* acquires the structure of a $\mathbf{Z}/m\mathbf{Z}$-torsor over Ω with $m \mid n$, and as before one can find a unique θ_0 modulo $\overline{K}^*\Theta^m$ such that $\Omega_* = \Omega[f]/(f^m = \theta_0)$. Then $Y \times_X \Omega = \operatorname{Ind}_{\mathbf{Z}/m\mathbf{Z}}^{\mathbf{Z}/n\mathbf{Z}} \Omega_*$, and thus $Y \times_X \Omega = \Omega[f]/(f^n = \theta_0^{m/n})$, and $\theta = \theta_0^{m/n}$.

The second statement comes from the fact that if Ω_* is a connected component of $\Omega[f]/(f^n = \theta)$, $\operatorname{Gal}(\Omega_*/X)$ is (noncanonically) isomorphic to the direct product of $\operatorname{Gal}(\Omega_*/\Omega)$ and Γ (according to [18, Section 2, intro.]). Thus Ω_* can descend (noncanonically) to X by considering $Y_0 = \Omega_*/N$ where N is some complement of $\operatorname{Gal}(\Omega_*/\Omega)$ and $\operatorname{Gal}(\Omega_*/X)$ (and thus $\Omega[f]/(f^n = \theta)$ by taking a direct sum of Y_0's). □

Remark. One could also show the same by considering $\widetilde{J} = \mathcal{H}om(\Theta/\overline{K}^*, \mathbf{G}_m) \to J$ where J is the Jacobian variety of X and $\widetilde{J}$ is its universal topological covering, and by showing that $\pi_1^{\mathrm{alg}}(\widetilde{J})$ is a direct summand of $\pi_1^{\mathrm{alg}}(J)$.

4.4.2. Preliminary results on ramifications of torsors corresponding to currents. One can associate to a current on $\mathbb{G}$ a $\mathbf{Z}/n\mathbf{Z}$-torsor on Ω. Here we study how the torsor associated to a current splits.

Recall that $U_{z,\lambda}$ is the affinoid subset $\{z' \mid d(z,z') \leq \lambda\}$ of Ω.

Proposition 4.9. *Let $z \in \Omega$ and $\lambda > 0$. Let $f \in O(\Omega)^*$ be such that $f(z) = 1$. Let μ be the measure on $\mathcal{L}$ corresponding to f and assume $\mu(C \cap \mathcal{L}) = 0$ for every hole C of $U_{z,\lambda}$. Then $|f(z') - 1| \leq p^{d(z,z')-\lambda}$ for all $z' \in U_{z,\lambda}$.*

Proof. To simplify, assume $z = \infty$. According to [9, 1.8.10, Ex. β] and [9, Prop. 1.8.9(i)], $f = \lim f_k$ uniformly on every affinoid of Ω (in particular over $U_{z,\lambda}$) where f_k is of the form

$$f_k(z') = \prod_{i=1}^{s_k} \left(1 - \frac{x_{i,k}}{z'}\right)^{n_{i,k}}.$$

Then the measure on $\mathcal{L}$ corresponding to f_k is $\mu_k = \sum_i n_{i,k}\delta_{x_{i,k}}$ and μ_k tends weakly to μ.

For k large enough, $\mu_k(C \cap \mathcal{L}) = 0$ for every hole C of $U_{z,\lambda}$ and $|f(z') - f_k(z')| \leq p - \lambda$ for every $z' \in U_{z,\lambda}$. We thus only have to prove the result for f_k which is a product of functions such as

$$g : z' \mapsto z'' = \left(1 - \frac{x_1}{z'}\right)\left(1 - \frac{x_2}{z'}\right)^{-1} = \frac{z' - x_1}{z' - x_2}$$

with x_1 and x_2 in the same hole C of $U_{z,\lambda}$. We thus only have to prove the result for g, which is easily seen. $\square$

If x and x' are two points of the geometric realization $|\mathbb{T}_0|$ of a tree $\mathbb{T}_0$, there is a smallest connected subset of $|\mathbb{T}_0|$ containing x and x'. It is denoted $[x, x']$.

Corollary 4.10. *Let $z, z' \in \Omega$. Let $f \in O(\Omega)^*$ be such that $f(z) = 1$. Let U be an affinoid of Ω such that $\mu(C \cap \mathcal{L}) = 0$ for every hole C of U. Assume that $U_{z'',\lambda} \subset U$ for all $z'' \in \Phi^{-1}([\Phi(z), \Phi(z')])$. Then $|f(z') - 1| \leq p^{-\lambda}$.*

Proof. Let $\epsilon > 0$. Let $(z_i)_{i=0}^n$ be such that $z_0 = z$, $z_n = z'$, $\Phi(z_i) \in [\Phi(z), \Phi(z')]$ and $d(\Phi(z_i), \Phi(z_{i+1})) \leq \epsilon$. Then, according to the previous proposition, $|f(z_{i+1})/f(z_i) - 1| \leq p^{\epsilon-\lambda}$. Thus $|f(z'') - 1| \leq \sup |f(z_{i+1}) - f(z_i)| \leq p^{\epsilon-\lambda}$. One gets the result by letting ϵ tend to 0. $\square$

Corollary 4.11. *Let f be as previously, and U be such that $\mu(C \cap \mathcal{L}) = 0$ for every hole C of U. Let e be a positive integer. Let $\lambda > e + \frac{1}{p-1}$, and let $Y \to \Omega$ be the finite covering obtained by pulling back $\mathbf{G}_m \xrightarrow{z \mapsto z^{p^e}} \mathbf{G}_m$ along $f : \Omega \to \mathbf{G}_m$. Let $V \subset U$ be such that $U_{z,\lambda} \subset U$ for all $z \in V$. Then Y is split over V.*

Proof. We may assume V is connected, because we only have to prove the result for every connected component of V. Let $z \in V$. Multiplying f by a constant, which does not change Y, we may assume that $f(z) = 1$. From the previous corollary, $f(V) \subset D(1, p^{-\lambda})$. But, according to Lemma 4.2, $\mathbf{G}_m \xrightarrow{z \mapsto z^{p^e}} \mathbf{G}_m$ is split over $D(1, p^{-\lambda})$, which ends the proof. $\square$

Proposition 4.12. *Let $\mathcal{C}$ be a current on $\mathbb{T}(\Omega)$ corresponding to an invertible function f on Ω. Let a be a vertex of $\mathbb{T}$ such that the restriction $\mathcal{C}_a$ of $\mathcal{C}$ to the star* *of a is not zero modulo n. If $Y \to \Omega$ is the finite covering obtained by pullback of $\mathbf{G}_m \xrightarrow{z\mapsto z^n} \mathbf{G}_m$ along $f : \Omega \to \mathbf{G}_m$, then $Y \to \Omega$ is not split over a considered as a Berkovich point of Ω.*

Proof. $Y \to \Omega$ is split over a if and only if there exists $f_1 \in \mathcal{O}_{\Omega,a}$ such that $f_1^n = f_{|\mathcal{O}_{\Omega,a}}$. By multiplying f by a constant, we may assume $|f|_a = 1$.

If f_1 exists, by looking at the residue field $\overline{\mathcal{H}(a)}$ of $\mathcal{H}(a)$, we have $\overline{f_1}^n = \overline{f}$ (where $\overline{f}$ is the image of f in $\overline{\mathcal{H}(a)} \simeq k(X)$). If $\overline{f_1}(z) = \lambda \prod(z - a_i)$, then $\overline{f} = \lambda^n \prod(z - a_i)^n$, and so all the poles and zeros are of order a multiple of n, which ends the proof. □

4.4.3. The metric graph of the reduction and the tempered fundamental group. Let $i = 1$ or 2. Assume now that X_1 and X_2 are two Mumford curves over $\overline{K}$, but are pullbacks of curves over K (so that we may use [16, Ex. 3.10]), and that there is an isomorphism

$$\phi : \pi_1^{\mathrm{temp}}(X_1) \xrightarrow{\sim} \pi_1^{\mathrm{temp}}(X_2),$$

which thus induces an isomorphism of graphs

$$\mathbb{G}_1 \xrightarrow{\sim} \mathbb{G}_2,$$

hence an isomorphism $\mathbb{T}(\Omega_1) \xrightarrow{\sim} \mathbb{T}(\Omega_2)$.

Theorem 4.13. *The isomorphism $\mathbb{G}_1 \to \mathbb{G}_2$ of graphs is in fact an isomorphism of metric graphs.*

Remark. Suppose X_1 and X_2 are projective lines minus four points a_i, b_i, c_i, d_i with $v_p(a_i, b_i, c_i, d_i) = h_i > 0$ with an isomorphism ϕ between their tempered fundamental groups, and let $l \geq 3$ be prime to p. One can consider a covering X_1' of order l of X_1 such that the restriction of the map between the stable models of X_1' and X_1 to each irreducible component of the stable reduction of X_1 is connected but it is split over the double point of the stable reduction of X_1. For example, let $f : X_1 \to \mathbf{G}_m$ map x to $\frac{x-a_1}{x-c_1}\frac{x-b_1}{x-d_1}$. Let $\bar{a}_1$ (resp. $\bar{b}_1$, $\bar{c}_1$, $\bar{d}_1$) be the image of a_1 (resp. b_1, c_1, d_1) in the stable reduction of X_1. One identifies $\bar{a}_1, \bar{b}_1, \bar{c}_1$ and $\bar{d}_1$ with elements of $\mathbf{P}_k^1$ after choosing an isomorphism of each irreducible component of the stable reduction with $\mathbf{P}_k^1$. One can then choose X_1' to be the pullback of $\mathbf{G}_m \xrightarrow{z\mapsto z^l} \mathbf{G}_m$ along f. Indeed, the induced covering of the irreducible component of the stable reduction containing $\bar{a}_1$ and $\bar{c}_1$ (resp. $\bar{b}_1$ and $\bar{d}_1$) is the pullback of $\mathbf{G}_m \xrightarrow{z\mapsto z^l} \mathbf{G}_m$ along $z \mapsto \frac{z-\bar{a}_1}{z-\bar{c}_1}$ (resp. $z \mapsto \frac{z-\bar{b}_1}{z-\bar{d}_1}$), and thus is étale at the node. Let X_2' be the covering of X_2 corresponding to X_1' under ϕ; it has the same properties. Then the compactification $\overline{X_i'}$ of X_i' is a Mumford curve (the corresponding covering of each irreducible component of the stable reduction of X_i is ramified at only two points, so it is a covering by a projective line) whose tree has l edges, each of length h_i. Lemma 4.4 gives an isomorphism between the tempered fundamental groups of $\overline{X_1'}$ and $\overline{X_2'}$. Thus, by Theorem 4.13, $h_1 = h_2$, which ends the proof for punctured lines.

One can in fact do quite the same for more general punctured Mumford curves, by considering, for an edge e of the graph, a tamely ramified covering by a Mumford curve such that there is an edge e' over e which is actually an edge of the graph of the stable reduction of the compactification of this covering (see [12, Cor. 3.4.7]).

Let us start with a sketch of the proof of Theorem 4.13. Let Φ_i be the usual retraction $\Omega_i \to |\mathbb{T}|$ where $\mathbb{T} := \mathbb{T}(\Omega_1) = \mathbb{T}(\Omega_2)$. Fix two terminal points of $\mathbb{T}$. We assume that the corresponding points of $\mathbf{P}^1 \setminus \Omega_i$ are 0 and ∞ (and let $\widetilde{L}$ be the path joining them). Choose a finite subtree $\mathbb{K}_0$ of $\mathbb{T}$. We will define a $\mathbf{Z}/p^h\mathbf{Z}$-torsor $\widetilde{X}_1''$ on Ω_1 such that:

(i) it induces by descent a torsor on some finite topological covering of X_1;
(ii) its restriction to $\Phi_1^{-1}(|\mathbb{K}_0|)$ is isomorphic to the restriction of

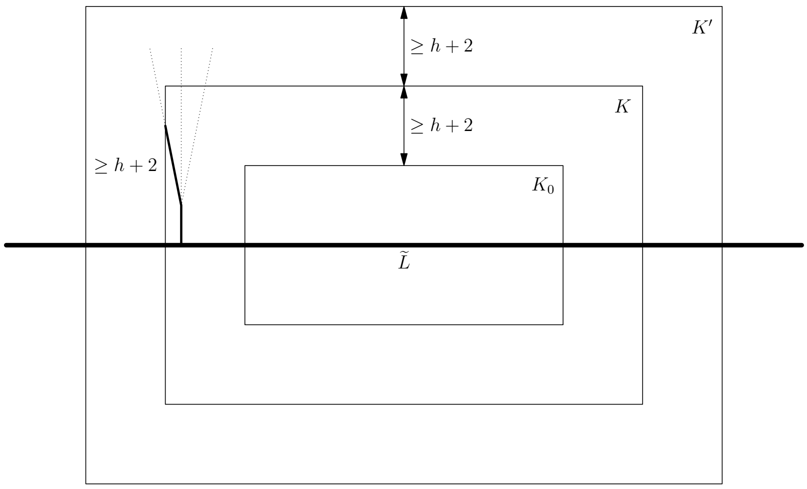
$$\mathbf{G}_m \xrightarrow{z \mapsto z^{p^h}} \mathbf{G}_m;$$

(iii) the torsor $\widetilde{X}_2''$ on Ω_2 corresponding to $\widetilde{X}_1''$ under ϕ is also isomorphic to $\mathbf{G}_m \xrightarrow{z \mapsto z^{p^h}} \mathbf{G}_m$ on $\Phi_2^{-1}(|\mathbb{K}_0|)$ up to a constant in $(\mathbf{Z}/p^h\mathbf{Z})^*$ (for the $\mathbf{Z}/p^h\mathbf{Z}$-module structure on the set of $\mathbf{Z}/p^h\mathbf{Z}$-torsors).

By choosing $\mathbb{K}_0$ and h large enough and applying Lemmata 4.5 and 4.2, one finds that the difference between the two distances from a vertex of $\mathbb{T}$ to the path joining 0 and ∞ for the two different metric structures on $\mathbb{T}$ is bounded. A purely combinatorial result (Proposition A.1) will end the proof.

To construct $\widetilde{X}_1''$, consider the current on $\mathbb{T}$ following the path from 0 to ∞ and make it equivariant for the action of a finite index subgroup Γ' of $\Gamma = \mathrm{Gal}(\mathbb{T}/\mathbb{G})$, so that the corresponding torsor $\widetilde{X}_1''$ on Ω_1 satisfies (i). We choose Γ' so that the current $\mathcal{C}_1$ thus defined is equal to $\mathcal{C}_0$ on $\Phi_1^{-1}(|\mathbb{K}'|)$ for a subgraph $\mathbb{K}'$ of $\mathbb{T}$ large enough compared to $\mathbb{K}_0$ (hence (ii) holds).

According to the lemmata of the previous subsection, $\widetilde{X}_1''$ is isomorphic to $\mathbf{G}_m \xrightarrow{z \mapsto z^{p^h}} \mathbf{G}_m$ on $\Phi_1^{-1}(|\mathbb{K}|)$ for a subgraph $\mathbb{K}$ of $\mathbb{K}'$, but still big enough compared to $\mathbb{K}_0$. Lemmata 4.2 and 4.5 tell us over which vertices of $\mathbb{K}_0$, $\widetilde{X}_1''$ and $\widetilde{X}_2''$ are split. In particular, one can choose $\mathbb{K}$ and $\mathbb{K}'$ so that $\widetilde{X}_i''$ is split over every vertex of the boundary of $\mathbb{K}$ except those on the path from 0 to ∞. The current $\mathcal{C}_2$ is then trivial around those points of the boundary, and thus is equal to $\mathcal{C}_0$ on $\mathbb{K}$ up to a constant in $(\mathbf{Z}/p^h\mathbf{Z})^*$. If $\mathbb{K}$ was chosen big enough compared to $\mathbb{K}_0$, one gets (iii).



Proof of Theorem 4.13. A *loop* of a graph is a cyclic sequence of oriented edges of the graph such that the end of an edge is the beginning of the following edge, and which never goes through the same vertex or (unoriented) edge twice.

We identify $\mathbb{G}_1$ and $\mathbb{G}_2$ thanks to the isomorphism induced by ϕ, and we simply call this graph $\mathbb{G}$. The two metrics on $\mathbb{G}$ will be denoted by d_1 and d_2. For $i = 1$ or 2, the usual retraction $\Omega_i \to \mathbb{T}$ will be denoted by Φ_i.

Let C be a loop of $\mathbb{G}$, and denote by $\lg_i(C)$ the length of C with respect to the metric d_i on $\mathbb{G}$. Let $\widetilde{C}$ be the universal covering of C, let $\widetilde{C} \to \mathbb{T}$ be a lifting of $\widetilde{C} \to \mathbb{G}$ and let z_0 be a vertex of $\mathbb{T}$ which belongs to $\widetilde{C}$. Let us then label $(z_j)_{j\in\mathbf{Z}}$ the vertices of $\widetilde{C}$ with the same image that z_0 has in $\mathbb{G}$. Let L be another loop (we choose an orientation on it) of $\mathbb{G}$ (there must be another loop since $g > 1$), let $\widetilde{L}$ be a lifting of L to $\mathbb{T}$, let $r_i = d_i(\widetilde{L}, z_0)$ (we may assume, by changing the numbering of the z_j that $d_i(\widetilde{L}, z_n) = r_i + n\lg_i(C)$ for $n \geq 0$) and let z_0' be the point of $\widetilde{L}$ nearest to z_0 (this does not depend on i).

For z a vertex of $\mathbb{T}$, let F_z denote the connected component of $\Omega \setminus \{\text{open edges of } \widetilde{L}\}$ which contains z.

Let $h \geq 1$ be an integer. Let $\mathbb{K}_0$ be a connected finite subgraph of $\mathbb{T}$ containing z_1 (so that $\Phi^{-1}(|\mathbb{K}_0|)$ is compact by properness of Φ). Let $\mathbb{K}$ be a connected finite subgraph of $\mathbb{T}$ such that (for $i = 1$ and $i = 2$):

- for every $z \in \Phi_i^{-1}(|\mathbb{K}_0|)$, $U_{z,h+2} \subset \Phi_i^{-1}(|\mathbb{K}|)$;

- for every vertex z of $\widetilde{L}\cap\mathbb{K}$, $\{z'\in F_z \mid d_i(z',z)\leq h+2\}\subset\Phi_i^{-1}(|\mathbb{K}|)$ (in particular, if z' is a vertex of the boundary of $\mathbb{K}$ in $\mathbb{T}$ which is not one of the end points of the segment $\widetilde{L}\cap\mathbb{K}$, then $d_i(z',\widetilde{L})\geq h+2$).

Let $\mathbb{K}'$ be a compact subgraph of $\mathbb{T}$ such that, for $i=1,2$, $\Phi_i^{-1}(|\mathbb{K}'|)$ contains $U_{z,h+2}$ for every z in $\Phi_i^{-1}(|\mathbb{K}|)$.

Let $\Gamma=\mathrm{Gal}(\mathbb{T}/\mathbb{G})$, let $H=\mathrm{Stab}(\widetilde{L})$ ($\simeq\mathbf{Z}$) and let Γ' be a subgroup of finite index of Γ such that, for every $g\neq 1\in\Gamma'$, $\Phi_i^{-1}(|\mathbb{K}'|)\cap g\cdot\Phi_i^{-1}(|\mathbb{K}'|)=\emptyset$ and for every $g\in\Gamma'\setminus H$, $d_i(g\cdot\widetilde{L},\widetilde{L})>\mathrm{diam}_i(|\mathbb{K}'|)$.

Such a Γ' exists. Indeed, $A:=\{g\neq 1\in\Gamma\mid\Phi_i^{-1}(|\mathbb{K}'|)\cap g\cdot\Phi_i^{-1}(|\mathbb{K}'|)\neq\emptyset\}$ is finite by compactness of $\Phi_i^{-1}(|\mathbb{K}'|)$. So, as Γ is residually finite, there exists Γ_1' of finite index in Γ which does not intersect A. The set $B:=\{g\in\Gamma/H-H\mid d_i(g\cdot\widetilde{L},\widetilde{L})\leq\mathrm{diam}_i(|\mathbb{K}'|)\}$ is also finite, and as $\overline{H}\cap\Gamma=H$ where $\overline{H}$ denotes the closure of H in the profinite completion of Γ, there exists Γ_2' of finite index in Γ containing H such that $\Gamma_2'\cap B\cdot H=\emptyset$. We may then choose $\Gamma'=\Gamma_1'\cap\Gamma_2'$.

Let $H'=H\cap\Gamma'$. Let $X_i'=\Omega_i/\Gamma'$; it is a finite topological covering of X_i, and the isomorphism $\phi:\pi_1^{\mathrm{temp}}(X_1)\simeq\pi_1^{\mathrm{temp}}(X_2)$ induces an isomorphism ϕ' : $\pi_1^{\mathrm{temp}}(X_1')\simeq\pi_1^{\mathrm{temp}}(X_2')$.

Let $\mathcal{C}_0$ be the current on $\mathbb{T}$ with $\mathcal{C}_0(e)=+1$ if e is an edge of $\widetilde{L}$ (and e has the same orientation as $\widetilde{L}$) and 0 otherwise (except if e is an edge of $\widetilde{L}$ with the opposite orientation, in which case $\mathcal{C}_0(e)=-1$); this current is invariant under H. Let $\mathcal{C}_1=\sum_{g\in\Gamma'/H'}g\cdot\mathcal{C}_0$. It is a current on $\mathbb{T}$, invariant under Γ' and which coincides with $\mathcal{C}_0$ on $\mathbb{K}'$.

Let $f_1\in O(\Omega_1)^*$ be the corresponding invertible function, and let X_1'' be a $\mathbf{Z}/p^h\mathbf{Z}$-torsor of X_1' corresponding to that current, that is, such that its pullback $\widetilde{X}_1''$ to Ω_1 is isomorphic to $\Omega_1\times_{\mathbf{G}_m}\mathbf{G}_m\to\Omega_1$ where the fiber product is taken, on the left side, along f_1 and on the right side along $z\mapsto z^{p^h}$. Let $X_2''=\phi'^*X_1''$ (do not forget that X_2'' has no reason to correspond to the current $\mathcal{C}_1$).

Let also $f_{0,i}\in O(\Omega_i)^*$ be the invertible function corresponding to the current $\mathcal{C}_0$ and let $\widetilde{X}_{0,i}$ be the corresponding $\mathbf{Z}/p^h\mathbf{Z}$-torsor on Ω_i. Recall that, due to Corollary 4.2, this torsor is split over a point $z\in\mathbb{T}(\Omega_i)$ of Ω_i if and only if $d_i(z,\widetilde{L})>h+1/(p-1)$.

According to Corollary 4.11 applied to $U=\Phi^{-1}(|\mathbb{K}'|)$ and $V=\Phi^{-1}(|\mathbb{K}|)$, the torsor $\widetilde{X}_1''-\widetilde{X}_{0,1}$ on Ω_1, which corresponds to the current $\mathcal{C}_1-\mathcal{C}_0$ which is zero over $\mathbb{K}'$, is split over $\Phi^{-1}(|\mathbb{K}|)$ since for every $z\in\Phi^{-1}(|\mathbb{K}|)$, $U_{z,h+2}\in\Phi^{-1}(|\mathbb{K}'|)$. Thus, for $z\in\mathbb{K}$, $\widetilde{X}_1''$ is split if and only if $\widetilde{X}_{0,1}$ is, if and only if $d_1(z,\widetilde{L})>h+1/(p-1)$.

In particular $\widetilde{X}_1''$ is split over the vertices of the boundary of $\mathbb{K}$ which are not the end points of $\mathbb{K}\cap\widetilde{L}$. Thus, according to Lemma 4.5 applied to X_1'' and X_2''

(as $\Omega_i \to X'_i$ is a topological covering, $\widetilde{X}''_i \to \Omega_i$ is split over a point if and only if $X''_i \to X'_i$ is split over the image of that point), $\widetilde{X}''_{0,2}$ is also split over the vertices of the boundary of $\mathbb{K}$ which are not the end points of $\mathbb{K} \cap \widetilde{L}$.

Let $\mathcal{C}_2$ be a current on $\mathbb{T}(\Omega_2)$ corresponding to the $\mathbf{Z}/p^h\mathbf{Z}$-torsor $\widetilde{X}''_2$ (the current corresponding to $\widetilde{X}''_2$ is well defined only modulo p^h). According to Proposition 4.12, the restriction of $\mathcal{C}_2$ to the star of a vertex of the boundary of $\mathbb{K}$ which is not an end point of $\mathbb{K} \cap \widetilde{L}$ is zero modulo p^h. One deduces from this that, modulo p^h, the restriction of $\mathcal{C}_2$ to the star of $\mathbb{K}$ must be congruent to the restriction of $a\mathcal{C}_0$ for some integer a. By adding to $\mathcal{C}_2$ a current which is a multiple of p^h, we may assume that $\mathcal{C}_2 - a\mathcal{C}_0$ is zero on the star of $\mathbb{K}$ (because every current with boundary on the star of $\mathbb{K}$, that is, that respects Kirchhoff's law at every vertex of $\mathbb{K}$ but with no condition on the boundary of the star of $\mathbb{K}$, can be extended to a current on the whole $\mathbb{T}$).

Thus, by applying Corollary 4.11 to $U = \Phi^{-1}(|\mathbb{K}|)$ and $V = \Phi^{-1}(|\mathbb{K}_0|)$, one may deduce that $\widetilde{X}''_2 - a\widetilde{X}_{0,2}$ is split over $|\mathbb{K}_0|$, so if z is a vertex of $\mathbb{K}_0$, $\widetilde{X}''_2$ is split over z if and only if $\widetilde{X}_{0,2}$ is (a is necessarily nonzero modulo p^h because $\widetilde{X}''_2$ cannot be split over z'_0), if and only if $d_2(z, \widetilde{L}) > h + \frac{1}{p-1}$, according to Lemma 4.2.

Therefore, $d_2(z, \widetilde{L}) > h + \frac{1}{p-1}$ if and only if $d_1(z, \widetilde{L}) > h + \frac{1}{p-1}$ for every vertex $z \in \mathbb{K}_0$. As one may choose $\mathbb{K}_0$ as large as one wants, one may deduce that for every z of $\mathbb{T}$, $d_1(z, \widetilde{L}) > h + \frac{1}{p-1}$ if and only if $d_2(z, \widetilde{L}) > h + \frac{1}{p-1}$, and this for every integer $h \geq 1$. Thus

$$\max\left(1, \left\lceil d_2(z, \widetilde{L}) - \frac{1}{p-1} \right\rceil\right) = \max\left(1, \left\lceil d_1(z, \widetilde{L}) - \frac{1}{p-1} \right\rceil\right).$$

By applying it to $(z_j)_{j \geq 0}$, one gets that for every $j \geq 0$,

$$\max\left(1, \left\lceil j \lg_1(C) + r_1 - \frac{1}{p-1} \right\rceil\right) = \max\left(1, \left\lceil j \lg_2(C) + r_2 - \frac{1}{p-1} \right\rceil\right).$$

Hence for every loop C of $\mathbb{G}$ (and of every topological covering of $\mathbb{G}$),

$$\lg_1(C) = \lg_2(C).$$

One concludes the proof with the help of Proposition A.1. $\square$

Appendix A. A combinatorial result

To end the proof of Theorem 4.13, we need to prove that one can recover from the length of all the loops of every finite covering of $\mathbb{G}$ the length of every edge of $\mathbb{G}$. We thus have to prove a general result on graphs with every edge of valency at least 3.

To prove this, we will work by induction on the number of edges of the graph. However, by removing an edge of our graph, one does not in general get a graph with every edge of valency at least three, and we may have to concatenate two edges or remove another edge to apply our induction assumption (but there will only be a few edges whose length cannot be recovered directly by the induction assumption). To exhibit enough loops that go through some edges whose length we do not know yet from the induction assumption, we will have to distinguish many cases depending on the number of connected components of the subgraph of all the edges whose length is already known.

Proposition A.1. *Let $\mathbb{G}$ be a finite graph with every vertex of valency is at least* 3*. Let $f : \{edges\ of\ \mathbb{G}\} \to \mathbf{R}$ be any function. Denote also by f the induced function on the set of edges of a (topological) covering of $\mathbb{G}$. For any loop C of a covering of $\mathbb{G}$, set*

$$f(C) = \sum_{x \in \{edges\ of\ C\}} f(x).$$

If $f(C) = 0$ for every loop C of every covering of $\mathbb{G}$, then $f = 0$.

Proof. Remark that if $\mathbb{G}$ is a finite graph with every vertex of valency at least 3 and if $\mathbb{H}$ is a connected subgraph such that the number of half-edges of $\mathbb{G} \setminus \mathbb{H}$ which end in $\mathbb{H}$ is less than 3, then $\mathbb{H}$ is not a tree (if $\mathbb{H}$ is a tree with at least one edge, then it has at least two vertices of valency 1, and thus one already has four half-edges of $\mathbb{G} \setminus \mathbb{H}$ which must end in one of those two vertices; if $\mathbb{H}$ is only a vertex, this is equally obvious).

We will proceed by induction on the number of edges of $\mathbb{G}$. Thus let $(\mathbb{G}, f)$ be a graph with $n \geq 1$ edges and a function f on the set of edges of $\mathbb{G}$ which satisfy the hypotheses of the proposition, and assume the proposition is true if $\mathbb{G}$ has fewer than n edges. We may assume $\mathbb{G}$ is connected (otherwise we apply the induction hypothesis to the connected components).

Let e be an edge of $\mathbb{G}$, and start from

Case 1: the two end points of e are different. Let (m, n) be the valencies of the end points of e in $\mathbb{G} \setminus \{e\}$. By our assumption about $\mathbb{G}$, we have $m, n \geq 2$.

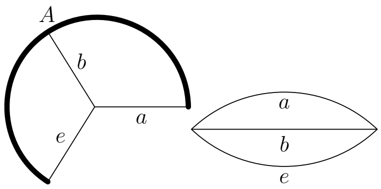
(a) If $m \geq 3$ and $n \geq 3$, then $\mathbb{G}' = \mathbb{G} \setminus \{e\}$ still has the valency of every vertex at least 3, thus one may apply our induction hypothesis to $\mathbb{G}'$ and to f to deduce that $f(x) = 0$ for every edge of $\mathbb{G}$ other than e.

   i. If $\mathbb{G}'$ is connected, then one may find a loop C of $\mathbb{G}$ that goes through e, and then $f(e) = f(C) = 0$, which implies the result.

   ii. If $\mathbb{G}'$ has two connected components A and B (they cannot be trees according to the remark at the beginning of the proof), one may consider two coverings A' and B' of A and B respectively, both of order 2, that one patches together into a covering $\mathbb{G}'$ of $\mathbb{G}$ of order 2. Then there exists a loop C of $\mathbb{G}'$ passing through the two edges over e. Hence $2f(e) = f(C) = 0$, which gives the desired result.

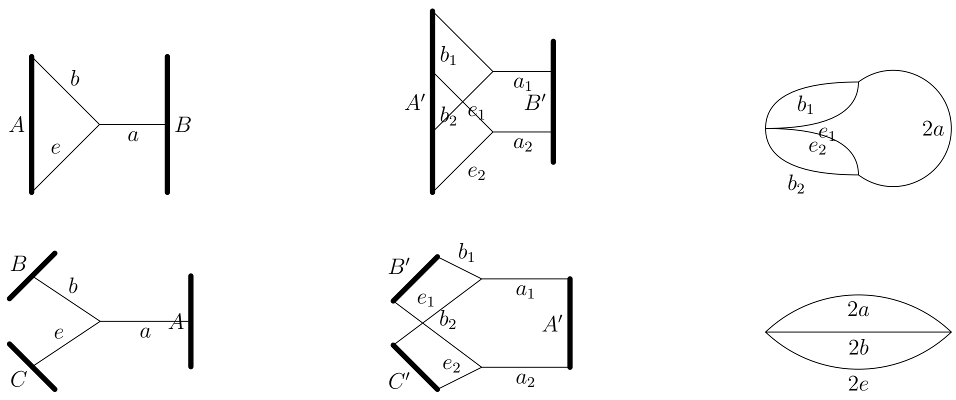
(b) If $m \geq 3$ and $n = 2$ (or the other way round), then let a and b be the two edges starting at the second end point of e (if a and b are in fact the two half-edges of a single edge, then the graph has the same structure as in case 2(c).i below; thus we will assume here that a and b are two different edges). Let $\mathbb{G}'$ be the graph obtained from $\mathbb{G}$ by removing e and concatenating a and b into a single edge that we denote $a+b$ (and we will define $f(a+b) = f(a) + f(b)$). Then $(\mathbb{G}', f)$ satisfies the required conditions (because every covering of $\mathbb{G}'$ extends to a covering of $\mathbb{G}$; thus every loop C of a covering of $\mathbb{G}'$ is also a loop of a covering of $\mathbb{G}$, hence $f(C) = 0$) and so $f(x) = 0$ for every edge of $\mathbb{G}$ other than a, b and e. Depending on the number of connected components of $\mathbb{G}'' = \mathbb{G} \setminus \{a, b, e\}$, we distinguish several cases:

i. If $\mathbb{G}''$ has only one connected component, then one may contract $\mathbb{G}''$ to a point to get a graph $\mathbb{G}_1$ with three edges (indeed, every loop C_1 of $\mathbb{G}_1$ may be lifted to a loop C of $\mathbb{G}$ as $\mathbb{G}''$ is connected, and $f(C_1) = f(C)$ since $f = 0$ over $\mathbb{G}''$), and thus $f(a) + f(b) = 0$, $f(a) + f(e) = 0$ and $f(b) + f(e) = 0$, which gives the desired result.



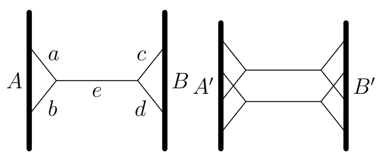
ii. If $\mathbb{G}''$ has two connected components A and B as in the picture (now a, b and e play the same role and the proof is the same if they are exchanged), then we start by considering a connected covering $\mathbb{G}_1$ of order 2 of $\mathbb{G}$ whose restrictions A' and B' to A and to B are connected (there exists such a covering as A and B cannot be trees according to the remark at the beginning of the proof). Then one may contract A' and B' to a graph $\mathbb{G}_2$. One gets $f(b) + f(e) = 0$, $2f(a) + 2f(b) = 0$, $2f(a) + 2f(e) = 0$, which implies what we wanted.

iii. If $\mathbb{G}''$ has three connected components A, B and C, then we start by considering a connected covering $\mathbb{G}_1$ of order 2 of $\mathbb{G}$ whose restrictions A', B' and C' to A, B and C are connected; then one may contract them. One may deduce that $2f(a) + 2f(b) = 0$, $2f(b) + 2f(e) = 0$, $2f(e) + 2f(a) = 0$, which implies the result.



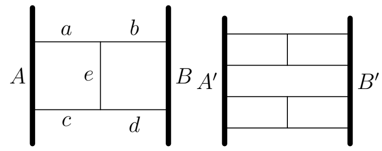
(c) If $m = n = 2$, then let a, b be the two (half-)edges ending at one end point of e, and c, d the (half-)edges ending at the other end point of e. Let $\mathbb{G}'$ be the graph obtained from $\mathbb{G}$ by removing e and concatenating a and b into $a+b$, and c and d into $c+d$. If one defines $f(a+b) = f(a)+f(b)$ and $f(c+d) = f(c)+f(d)$, then $(\mathbb{G}', f)$ satisfies the assumptions of the proposition, and so, by the induction hypothesis, $f(x) = 0$ for every edge x of $\mathbb{G}$ other than a, b, c, d and e. Let $\mathbb{G}'' = \mathbb{G} \setminus \{a, b, c, d, e\}$. Depending on the number of connected components of $\mathbb{G}''$, we distinguish several cases:

i. If $\mathbb{G}''$ has two connected components, A containing one end point of a and of b, and B containing one end point of c and of d (if (a, b) or (c, d) are the two half-edges of a single edge, the graph has the same structure as in case 2(c).ii.B below; if (a, b) and (c, d) both make single edges, then the structure is the one of the degenerate case of 2(c).ii), then start by considering a connected covering $\mathbb{G}_1$ of order 2 of $\mathbb{G}$ whose restrictions A' and B' to A and B are connected, and then contract A' and B'. One finds, for example, that $f(c)+f(d)$, $f(a)+f(b)$, $2(f(a)+f(e)+f(d))$, $2(f(a)+f(e)+f(c)$, and $2(f(b)+f(e)+f(d))$ are zero, which implies the result.

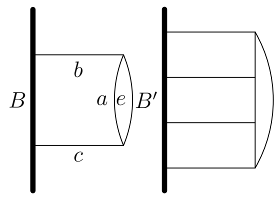


ii. If $\mathbb{G}''$ has two connected components, A containing an end point of a and c, and B containing an end point of b and d, then consider as usual a connected covering $\mathbb{G}_1$ of order 2 of $\mathbb{G}$ whose restrictions A' and B' to A and B are connected, and contract A' and B'. One gets for example

$f(a)+f(b)+f(c)+f(d) = 2(f(a)+f(e)+f(d)) = 2(f(b)+f(e)+f(c)) = 2(f(a)+f(c)) = f(b)+f(c)+f(e) = 0$, which implies the result.

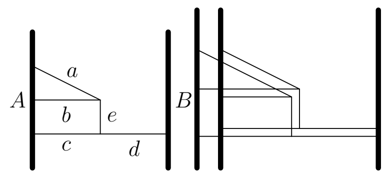


If (a,c) (or symmetrically (b,d)) are the two half-edges of a single edge a, consider a connected covering $\mathbb{G}_1$ of order 2 of $\mathbb{G}$ such that its restrictions B' to B and $(a\cup e)'$ to $a\cup e$ are connected. One gets $f(a)+f(b)+f(c) = f(e)+f(b)+f(c) = 2(f(a)+f(e)) = f(a)+f(e)+2f(b) = 0$, which implies the result.



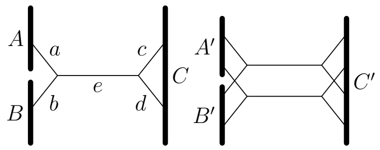
If (a,c) and (b,d) degenerate into two edges, then $\mathbb{G}$ only has two vertices and three arrows joining them, and the result is obvious.

iii. If $\mathbb{G}''$ has two connected components, A containing an end point of a, b and of c, and B containing an end point of d, then we start by considering a connected covering $\mathbb{G}_1$ of order 2 such that its restriction B' to B is connected, the restriction to A is disconnected but the restriction to $A\cup a\cup b$ is connected. Now, one sees that $f(a)+f(b) = f(b)+f(c)+f(e) = f(a)+f(c)+f(e) = f(a)+f(b)+2(f(c)+f(d)) = f(a)+f(b)+2(f(e)+f(d)) = 0$, which implies the result.



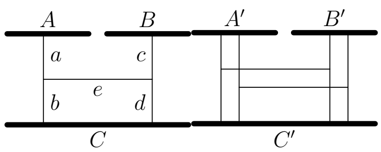
iv. If $\mathbb{G}''$ has three connected components, A containing an end point of a, B containing an end point of b, and C containing an end point of c and of d, then consider as usual a connected covering $\mathbb{G}_1$ of order 2 of $\mathbb{G}$ whose restrictions A', B' and C' to A, B and C are connected, and contract A', B' and C'. One gets for example $f(c)+f(d) = 2(f(a)+f(b)) =$

$2(f(a)+f(e)+f(c)) = 2(f(b)+f(e)+f(c)) = 2f(a)+2f(e)+f(c)+f(d) = 0$, which implies the result.

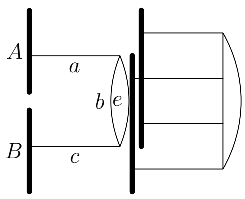


If (c, d) degenerates to a single edge c, the graph has the same structure as in case 2(c).ii.A.

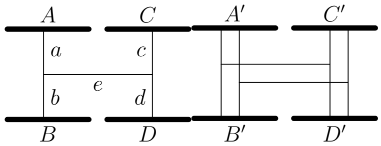
v. If $\mathbb{G}''$ has three connected components, A containing an end point of a, B containing an end point of c, and C containing an end point of b and d, then consider a connected covering $\mathbb{G}_1$ of order 2 of $\mathbb{G}$ whose restrictions A', B' and C' to A, B and C are connected, and contract A', B' and C'. One gets for example $f(b) + f(d) + f(e) = f(b) + f(d) + f(e) + 2f(a) = 2(f(a) + f(b)) = 2(f(d) + f(c)) = 2(f(c) + f(e) + f(a)) = 0$.



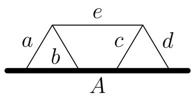
If (b, d) degenerates to a single edge b, consider a covering $\mathbb{G}_1$ of order 2 of $\mathbb{G}$ whose restrictions to A, B and $b \cap e$ are connected, and contract A' and B'. One gets $2f(a)+f(e)+f(b) = 2(f(b)+f(e)) = 2f(c)+f(b)+f(e) = 2(f(a) + f(b) + f(c)) = 0$.



vi. If $\mathbb{G}''$ has four connected components, A containing an end point of a, B containing an end point of c, C containing an end point of b, and D containing an end point of d, then consider a connected covering $\mathbb{G}_1$ of order 2 of $\mathbb{G}$ whose restrictions A', B', C' and D' to A, B, C and D are connected, and contract A', B', C' and D'. One finds for example that $2(f(b) + f(d) + f(e)) = f(b) + f(d) + f(e) + 2f(a) = f(b) + f(d) + f(e) + 2f(c) = 2(f(b) + f(a)) = 2(f(c) + f(d)) = 0$.



vii. If $\mathbb{G}''$ has a single connected component A, then contract it. One gets $f(a)+f(b)=f(c)+f(d)=f(a)+f(c)+f(e)=f(b)+f(c)+f(e)=f(a)+f(d)+f(e)=0$, which implies the result.

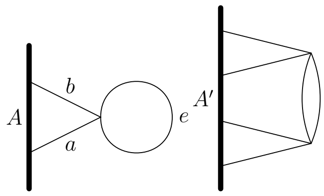


Case 2: the two end points of e are the same vertex; let m be its valency in $\mathbb{G}\setminus\{e\}$. One has $m\geq 1$.

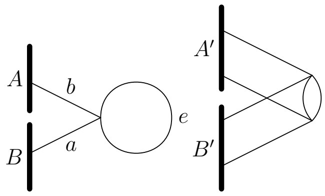
(a) Assume $m\geq 3$. Then $\mathbb{G}\setminus\{e\}$ satisfies the assumptions of the proposition and thus by induction hypothesis $f(x)=0$ for every edge of $\mathbb{G}$ other than e, and as e is already a loop, $f(e)=0$ too.

(b) Assume $m=2$, and let a and b be the two edges ending at the end point of e (if a and b are in fact only the two half-edges of a single edge, $\mathbb{G}$ is only the wedge of two loops, and the result is obvious). Let $\mathbb{G}'$ be the graph obtained from $\mathbb{G}$ by removing e and by concatenating a and b into an edge $a+b$, and define $f(a+b)=f(a)+f(b)$. Then $\mathbb{G}'$ satisfies the assumptions of the proposition and thus, by induction hypothesis, $f(x)=0$ for every edge x other than e, a or b. Depending on the number of connected components of $\mathbb{G}''=\mathbb{G}\setminus\{a,b,e\}$, we distinguish the following cases:

i. If $\mathbb{G}''$ has a single connected component A, then consider a covering $\mathbb{G}_1$ of order 2 of $\mathbb{G}$ whose restrictions to A and e are connected, and contract A'; this yields a graph $\mathbb{G}_2$. One gets $2f(a)+f(e)=2f(b)+f(e)=2f(e)=0$, which implies the result.

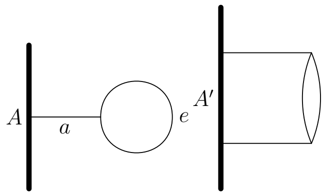


ii. If $\mathbb{G}''$ has two connected components, A containing the end point of a, and B containing the end point of b, consider a covering $\mathbb{G}_1$ of order 2 of $\mathbb{G}$ whose restrictions A' and B' to A and B are connected and whose restriction to e is connected too, and contract A' and B' to get a graph $\mathbb{G}_2$. One finds that $2f(a)+f(e)=2f(b)+f(e)=2f(e)=0$, which implies the result.



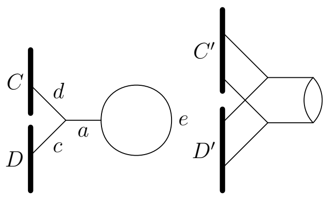
(c) Assume $m = 1$, and let a be the edge ending at the end point of e. Let n be the valency of the other end point of a. One must have $n \geq 2$.

i. Assume $n \geq 3$. Then $\mathbb{G}' = \mathbb{G} \setminus \{a, e\}$ satisfies the assumptions of the proposition, so, by induction hypothesis, $f = 0$ over $\mathbb{G}'$. Consider now a covering $\mathbb{G}_1$ of order 2 of $\mathbb{G}$ whose restrictions to $\mathbb{G}'$ and to e are connected, and contract the preimage of $\mathbb{G}'$. One gets $2f(e) = 2f(a) + f(e) = 0$, which implies the result.

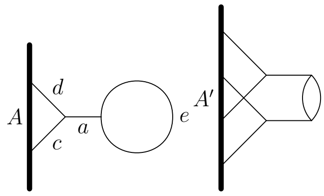


ii. Assume $n \geq 2$, and let c and d be those two edges (if they are only the two half-edges of a single edge, $\mathbb{G}$ is made of two loops joined by an edge; one shows the result for this particular graph by considering the covering of order 2 whose restrictions to the loops are connected). Let $\mathbb{G}''$ be the graph obtained from $\mathbb{G}'$ by concatenating c and d into $c + d$ and define $f(c+d) = f(c) + f(d)$. As $\mathbb{G}''$ satisfies the assumptions of the proposition, $f(x) = 0$ for every edge x of $\mathbb{G}$ other than a, e, c and d (and, in fact, $f(e) = 0$ too). Let $\mathbb{G}''' = \mathbb{G} \setminus \{a, c, d, e\}$, and distinguish the following cases depending on the number of connected components of $\mathbb{G}'''$:

A. If $\mathbb{G}'''$ has two connected components, C containing the end point of c and D containing the end point of d, consider a covering $\mathbb{G}_1$ of $\mathbb{G}$ whose restrictions C', D' and e' to C, D and e are connected, and contract C', D' and e'. One gets $2f(a) + 2f(c) = 2f(a) + 2f(d) = 2f(c) + 2f(d) = 0$, which implies the result.



B. If $\mathbb{G}'''$ has a single connected component A, consider a covering $\mathbb{G}_1$ of $\mathbb{G}$ whose restrictions A' and e' to A and e are connected, and contract A' and e'. One gets $2f(a)+2f(c) = 2f(a)+2f(d) = 2f(c)+2f(d) = 0$, which implies the result.



□

Acknowledgements

This work is part of a PhD thesis. I would like to thank my advisor, Yves André, for guiding my work, suggesting the main idea of some proofs presented here, and taking the time of reading and correcting this work.

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