Publ. RIMS Kyoto Univ. **47** (2011), 141–151
DOI 10.2977/PRIMS/33

Asymptotic Equivariant Index of Toeplitz Operators on the Sphere

This article is dedicated to M. Sato, whose work has always been a great source of inspiration

by

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Abstract

We illustrate the equivariant asymptotic index described in [6, 8] in the case of spheres $\mathbb{S}^{2N-1} \subset \mathbb{C}^N$, equipped with a unitary action of a compact group, for which this theory is more explicit. The article is mostly a review article, except for the last section (§5) in which we describe conjecturally some very natural generators of the relevant K-theory for a torus action on a sphere, generalizing in our Toeplitz operator context the generators proposed by M. F. Atiyah [2] for the transversally elliptic pseudodifferential theory.

2010 Mathematics Subject Classification: 19L47, 32A25, 53D10, 58J20, 58J40.
Keywords: Toeplitz operators, index, equivariant K-theory, contact manifolds.

§1. Toeplitz operators on the sphere

Let $X = \mathbb{S}^{2N-1}$ denote the unit sphere, bounding the unit ball $\mathbb{B}$ in $\mathbb{C}^N$.

We will denote by $\mathbb{H}$ (or $\mathbb{H}_X$) the kernel of the tangent Cauchy–Riemann system $\bar\partial_b$, i.e. the space of boundary values of holomorphic functions on $\mathbb{B}$ that are of moderate growth near the boundary (these are distributions on X).

We denote by S (or S_X) the *Szegö projector*, i.e. the orthogonal projector of $L^2(X)$ on $\mathbb{H} \cap L^2$:

$$Sf(z) = \int_X S(z,\bar w) f(w)\, d\sigma(w) \quad \text{with} \quad S(z,\bar w) = \frac{1}{v}(1 - z\cdot \bar w)^{-N}$$

where $v = 2\pi^N/(N-1)!$ is the volume of X.

This is a contribution to the special issue "The golden jubilee of algebraic analysis".

Communicated by M. Kashiwara. Received December 20, 2009.

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S is a Fourier integral operator (FIO) with complex phase (cf. [9]); it acts continuously on all Sobolev spaces $H^s(X)$ with range $\mathbb{H}^s = \mathbb{H} \cap H^s(X)$.

A *Toeplitz operator* of degree m is an operator of the form $f \in \mathbb{H} \mapsto T_P(f) = S(P(f))$ where P is a pseudodifferential operator (ψDO) of degree m on X; it is continuous $\mathbb{H}^s \to \mathbb{H}^{s-m}$ for all s. In particular if ϕ is a smooth function on X, it defines a Toeplitz operator T_ϕ ($f \mapsto S(\phi f)$). The operators initially introduced by Toeplitz are the T_ϕ on the circle ($N = 1$).

Toeplitz operators give rise to a symbolic calculus identical to the pseudodifferential calculus: the sphere X is a contact manifold with contact form induced by $\lambda = \mathrm{Im}(\bar{z} \cdot dz)$. The associated symplectic cone is the set $\Sigma \subset T^\bullet X$ of positive multiples of λ;[1] this is one half of the line bundle, the set of real characteristics of $\bar{\partial}_b$, which carries the microsingularities of functions in $\mathbb{H}$ or of S. The symbol $\sigma_m(A)$ of a Toeplitz operator A of degree $\leq m$ is a homogeneous function on Σ of degree m, the restriction to Σ of the symbol of P if $A = T_P$, and we have

$$\sigma_{m+m'}(P \circ Q) = \sigma_m(P)\sigma_{m'}(Q), \qquad \sigma_{m+m'-1}([P,Q]) = \frac{1}{i}\{\sigma_m(P), \sigma_{m'}(Q)\}.$$

Modulo operators of degree $-\infty$, Toeplitz operators are local; they define a sheaf of algebras on X, denoted $\mu\mathbb{H}$, acting on the sheaf of holomorphic microfunctions ($\ker \bar{\partial}_b$ mod C^∞); this is locally isomorphic to the sheaf of pseudodifferential operators (mod C^∞) in N real variables acting on microfunctions, on the cosphere $S^*\mathbb{R}^N$.

Example. If $P = P(z, \partial_z)$ is a holomorphic differential operator (defined in a neighborhood of $\mathbb{B}$), it obviously acts on $\mathbb{H}$ and defines a Toeplitz operator T_P (or P if this does not lead to confusion).

We will use in particular the operator ρ, which is an elliptic Toeplitz operator of degree 1:

$$\rho = z \cdot \partial_z = \sum z_j \partial_j, \quad f = \sum f_\alpha z^\alpha \mapsto \sum |\alpha| f_\alpha z^\alpha. \tag{1.1}$$

Any Toeplitz operator is equivalent (mod C^∞) to an asymptotic sum: $T_P \sim \sum_{m \leq m_0} \rho^m T_{\phi_m}$; the "coefficients" $\phi_m \in C^\infty(X)$ are smooth functions on X and are uniquely determined.[2]

The following relation was pointed out in [6]:

Lemma 1. *On the sphere $X = \mathbb{S}^{2N-1}$ of $\mathbb{C}^N$ we have*

$$\partial_j = (\rho + N) T_{\bar{z}_j} \quad \textit{with as above} \quad \rho = \sum_{j=1}^N z_j \partial_j. \tag{1.2}$$

[1] $T^\bullet X$ denotes the cotangent bundle T^*X deprived of its zero section.

[2] This is only asymptotic (mod C^∞); the series usually does not converge. This does not really give rise to a practical "total" symbolic calculus.

It is sometimes practical to identify Σ with $\mathbb{C}^N - \{0\}$, with Liouville form $\lambda = \mathrm{Im}(\bar z \cdot dz)$ and symplectic form $\omega = \mathrm{Im}(d\bar z \cdot dz)$ (twice the canonical one), the coordinate functions $z_j, \bar z_j$ being homogeneous of degree $\frac{1}{2}$ (however this can lead to confusion: in this identification the symbol of T_{z_j}, resp. T_{∂_j}, is $\frac{1}{|z|} z_j$, resp. $|z| \bar z_j$).

§2. Equivariant index

Let G be a compact Lie group, with a linear unitary action on $\mathbb{C}^N$. Then G preserves holomorphic functions and the Szegö kernel (which is invariant under the action of $U(N)$), and we may consider invariant Toeplitz operators.

We denote by $\mathfrak{g}$ the Lie algebra of G. Any $\gamma \in \mathfrak{g}$ produces a vector field tangent to the sphere X, whose holomorphic part defines a Toeplitz operator of the form

$$L_\gamma = i \sum a_{pq} z_q \partial_p$$

where (a_{pq}) is a hermitian matrix. The symbol of $\frac{1}{i} L_\gamma$, when we identify Σ with $\mathbb{C}^N - \{0\}$ as above, is the hermitian form $q(z) = \sum a_{pq} z_q \bar z_p$ (symbol map $\sim$ moment map).

Definition 2. We denote by $\operatorname{char} \mathfrak{g} \subset \Sigma$ the set of points of Σ where all symbols $\sigma(\frac{1}{i} L_\gamma) = \langle A_\gamma z | z \rangle$, $\gamma \in \mathfrak{g}$, vanish.

The *base* $Z \subset X$ is the set of points where all vector fields L_γ ($\gamma \in \mathfrak{g}$) are orthogonal to the contact form λ (this still makes sense on any contact G-manifold: if X is the cosphere bundle of a G-manifold M, we have $\operatorname{char} \mathfrak{g} = T^*_G M$, the set of covectors orthogonal to the orbits of G).

If G is a torus acting diagonally: $gz = (e^{2i\pi \mu_k(\gamma)} z_k)$ if $g = \gamma \bmod \mathbb{Z}^n$ ($\mu_k \in \mathbb{Z}^n \subset \mathfrak{g}^*$), Z is the inverse image under the map $z \mapsto (\lambda_k = |z_k|^2)$ of the convex rational polyhedron $\{\sum \lambda_k \mu_k = 0\}$ in $\mathbb{R}^N_+$ ($\lambda_k \geq 0, \sum \lambda_k = 1$).

For index theory it is also necessary to consider Szegö projectors on the space of sections of G-vector bundles (for which the definition and existence of equivariant generalized Szegö projectors works as well, cf. [3, 7]). In the case where X is a sphere it is enough to consider "trivial" bundles, of the form $X \times E$, where E is some finite representation of G;[3] for these the holomorphic Szegö projector is well defined ($S \otimes \mathrm{Id}_E$).

If P is an equivariant Toeplitz operator acting on holomorphic sections of G-vector bundles, and $P \sim 0$ near $\operatorname{char} \mathfrak{g}$, then the G-trace $\mathrm{Tr}_G(P)$ is defined as

[3]Because any G-bundle is stably trivial, i.e. there exist trivial bundles E', F such that $E \oplus F \sim E'$.

in [2]. It is a distribution on G; if P is of trace class (i.e. of degree $< -N$) it is the continuous function $g \mapsto \mathrm{Tr}_G(P)(g) = \mathrm{tr}\,(g \circ P)$. In the general case, let Δ be a biinvariant elliptic operator of degree > 0 (e.g. the Casimir operator of some faithful representation), Δ_X the Toeplitz operator it defines on X (which is elliptic outside of $\mathrm{char}\,\mathfrak{g}$); dividing repeatedly we get $P = \Delta_X^m Q + R$ where R is of degree $-\infty$, and for large m the quotient Q is of trace class; we have $\mathrm{Tr}_G(P) = \Delta^m\,\mathrm{Tr}_G(Q) + \mathrm{Tr}_G(R)$. The G-trace makes sense for any operator $P \sim 0$ near $\mathrm{char}\,\mathfrak{g}$; its restriction to equivariant operators is a trace, i.e. $\mathrm{Tr}_G(AB) = \mathrm{Tr}_G(BA)$ if A, B are both equivariant and one of them is ~ 0 near $\mathrm{char}\,\mathfrak{g}$.

We will say that P is *G-elliptic*[4] if it is elliptic on $\mathrm{char}\,\mathfrak{g}$, i.e. its principal symbol is invertible there (thus also near $\mathrm{char}\,\mathfrak{g}$). The definition of M. F. Atiyah [2] can be reproduced:

Definition 3. If P is equivariant and G-elliptic, its *G-index* is defined as the virtual representation $\ker P - \mathrm{coker}\,P$. Its *character* is the distribution $\mathrm{Tr}_G(1-QP) - \mathrm{Tr}_G(1-PQ)$ where Q is any G-parametrix of P (i.e. Q is equivariant, $QP \sim 1$, $PQ \sim 1$ near $\mathrm{char}\,\mathfrak{g}$).

$\ker P$ and $\mathrm{coker}\,P$ are both tempered representations of G, and their characters are well defined as distributions. The index is the difference of their characters: we have

$$\mathrm{Ind}_G(P) = \sum \frac{I_\alpha}{d_\alpha} \chi_\alpha \tag{2.1}$$

where α ranges over the set of irreducible representations, with character χ_α and dimension d_α, and I_α is the index of the restriction of P to the isotypic components of type α (I_α/d_α is integral).

These definitions extend to equivariant complexes of Toeplitz operators.

The symbol of an equivariant G-elliptic system P defines a K-theoretical element $[P]^{\mathrm{abs}} \in K_0^G(\mathrm{char}\,\mathfrak{g} \cup \{0\})$; the G-index of P only depends on this (because it is additive and deformation invariant).[5]

§3. Asymptotic index

Szegö projectors and Toeplitz operators are defined more generally on any compact oriented contact manifold X (whose associated symplectic cone is again the half-line subcone of $T^\bullet X$ consisting of positive multiples of the contact form).

Let us recall from [7, 4, 5] that if Σ is a symplectic subcone of $T^\bullet M$, with M a smooth compact manifold, a generalized Szegö projector associated to Σ is a

[4] "Transversally elliptic" in [2], but "transversally" is not suitable in the Toeplitz context.

[5] This would in general not make sense for contact manifolds other than spheres.

self-adjoint elliptic Fourier integral projector S of degree 0 ($S = S^* = S^2$), whose complex canonical relation $\mathcal{C}$ is $\gg 0$, with real part the diagonal diag Σ (elliptic means that the principal symbol of S does not vanish on Σ, $\gg 0$ refers to the work of A. Melin and J. Sjöstrand [13, 14]).

Such a projector always exists; its range $\mathbb{H}$ is the associated "Toeplitz space" (following the terminology of [7]). As before, the Toeplitz operators corresponding to S are the operators on $\mathbb{H}$ of the form $u \in \mathbb{H} \mapsto T_P(u) = SPS(u)$ with P a pseudodifferential operator on M. They form an algebra (i.e. sums and products of Toeplitz operators are Toeplitz operators). Modulo smoothing operators, they form a sheaf acting on $\mu\mathbb{H}$, locally isomorphic to the sheaf of pseudodifferential operators acting on the sheaf of microfunctions in p variables if $\dim \Sigma = 2p$.

The generalized Szegö projectors associated to Σ are "essentially unique": if S, S' are two generalized Szegö projectors, S' induces a Fredholm map $\mathbb{H} \to \mathbb{H}'$. More generally if u, u' are two embeddings of Σ in cotangent bundles $T^\bullet M, T^\bullet M'$ and $\mathbb{H}$, $\mathbb{H}'$ the corresponding "Toeplitz spaces", there always exists a FIO F, whose (complex) canonical relation is $\gg 0$ with real part the graph of $u' \circ u^{-1}$, inducing a Fredholm map $\mathbb{H} \to \mathbb{H}'$ ("adapted" in the terminology of [7], or [4]).

The pair $(\mathcal{A}_\Sigma, \mu\mathbb{H})$ consisting of the sheaf of micro Toeplitz operators (i.e. mod smoothing operators) acting on $\mu\mathbb{H}$ is well defined, up to (non-unique) isomorphism: it only depends on the symplectic cone Σ, not on the embedding.

All the constructions above allow a compact group action (cf. [7]), i.e. S can be constructed invariant; if u, u' above are equivariant, F can be constructed equivariant, etc.[6]

If X is a contact G-manifold and we have chosen an invariant Szegö projector, Atiyah's theory of transversally elliptic indices can be reproduced to some extent: the characteristic set char $\mathfrak{g}$ is defined exactly as above, its base Z is again the set of points of X where the infinitesimal generators L_γ of the action of G are orthogonal to the contact form. Again there is a notion of elliptic system (or complex) of Toeplitz operators (i.e. elliptic along char $\mathfrak{g}$), and such a system has a G-index, i.e. a trace class virtual representation, whose character is a distribution on G.[7]

However because from the contact data alone the Szegö projector is at best known up to a compact operator, $\mathbb{H}$ is at best known modulo a finite-dimensional space and there is no hope to compute the index by local or differential formulae

[6] In these constructions one uses means repeatedly, and the compactness condition on the group cannot be relaxed.

[7] We recall that a virtual representation is of trace class if its character $\sum m_\alpha \chi_\alpha$ converges in distribution sense, i.e. the coefficients m_α (which are integers) are bounded by some power of the eigenvalues of a faithful Casimir operator (or any biinvariant elliptic operator).

involving the principal symbol alone, without further data. (This is even worse for G-vector bundles, even in the case of the sphere if a "trivial" or at least holomorphic structure is not given. There is a pseudodifferential analogue: if E is a vector bundle on a manifold M and $p \in C^\infty(S^*M, L(E))$ a projector defined on the cosphere, there always exists a pseudodifferential projector P with symbol p (compatible with the action of G if there is one), but this is obviously not unique, any two such differ by a compact operator; the projection from the range of one to the other is a Fredholm operator, on the index of which there can be no control from the topological data alone.)

A remedy to this, used in [8], is to introduce the asymptotic G-index, which is well defined for equivariant Toeplitz operators, only depends on their symbols, and still encodes pertinent information if G is not finite:

Definition 4. The *asymptotic index* is the absolute index modulo a finite virtual representation; its *character* is the singularity (distribution mod C^∞) of the distribution character of the "absolute" G-index.

(Note that a series $\sum m_\alpha \chi_\alpha$ has a C^∞ sum iff the sequence of coefficients m_α is rapidly decreasing; if the m_α are integers it is a finite sum.)

If u is an isomorphism from $\Sigma \subset T^\bullet M$ to $\Sigma' \subset T^\bullet M'$ (equivalently a contact isomorphism on the bases), and $F : \mathbb{H} \to \mathbb{H}'$ an adapted FIO as above, it can be used to transport G-elliptic systems and preserves the asymptotic G-index (not the absolute index).

The asymptotic index of an equivariant G-elliptic Toeplitz system P obviously only depends on its top order symbol, which defines an element $[P] \in K^G(X - Z)$, where $K^G(X-Z)$ is the group of virtual G-bundles with compact support in $X-Z$; its elements are stable equivalent classes of equivariant G-bundle homomorphisms $E \to F$ on the sphere, invertible on Z (the base of $\operatorname{char} \mathfrak{g}$). Since the asymptotic index is also additive and deformation invariant, we get (cf. [6]):

Proposition 5. *The asymptotic index of an equivariant G-elliptic system P only depends on the K-theoretic element $[P] \in K^G(X - Z)$ it defines. The index defines a linear map*

$$\operatorname{asInd} : K^G(X - Z) \to \operatorname{Sing}(G).$$

(The K-theoretic element $[P] \in K^G(X - Z)$ is well defined for any contact manifold; in the case of the sphere, it is the image of $[P]^{\text{abs}}$ mentioned above.[8])

[8] An equivariant map $u : E \to F$ invertible on Z defines the 0-element iff u can be stably deformed into an invertible homomorphism on the whole sphere X, i.e. it lies in the image of the obvious map $K_0^G(\mathbb{B}) \to K_0^G(\operatorname{char} \mathfrak{g})$.

§4. Embedding

It is possible to transfer the index from a G-sphere to a larger one. Let $X = \mathbb{S}^{2N-1}$ be the unit sphere of $V = \mathbb{C}^N$, with a unitary action of G, as in §1. Let V', V'' be two orthogonal invariant subspaces and $Y = V' \cap X$, an invariant subsphere.

Definition 6. We denote by $d_{V''}$ (or d'') the partial holomorphic de Rham complex, acting on z''-differential forms with holomorphic coefficients in all variables (z', z''):

$$(4.1) \qquad (\mathbb{E}^k, d'') \quad \text{with } \mathbb{E}^k = \mathbb{H} \otimes \bigwedge{}^k V''^*, \; d''\omega(z', z'') = \sum dz''_j \, \frac{\partial \omega}{\partial z''_j}$$

(V''^* denotes the dual of V''). This is obviously equivariant (if V', V'' are invariant), and its cohomology is $\mathbb{H}'$ (0 in positive degree). The symbol of d'' is, except for the factor $a = \sigma(z \cdot \partial_z) > 0$, the exterior multiplication by $\sum \bar{z}_j dz_j$.

Formula (1.2) shows that the symbol of d'' is, up to a positive factor ($\sigma(\rho)$), adjoint to the symbol of the Koszul complex $k_{V''}$, which is the complex (in negative degrees)

$$(\mathbb{E}_{-k}\,, I_{z''}) \quad \text{with } I_{z''} \text{ the interior product } \omega(z', z'') \mapsto \sum z''_j \partial_{z''_j} \lrcorner\, \omega.$$

The K-theoretical element $[k_{V'}] = [d_{V''}] \in K^G_Y(X)$ is precisely the Bott element, defining the Bott isomorphism $E \mapsto E \otimes [k_{V'}]$ of $K^G(Y)$ to $K^G_Y(X)$ (the corresponding assignment of operators is $P \mapsto P \otimes d_{V''}$).

Thus this "obvious" transfer takes a G-elliptic system P on Y to a G-elliptic system P' on X which has the same asymptotic G-index, and whose associated K-theoretical element $[P']$ is the Bott image of that of P.[9]

More generally, let Y, X be two compact contact G-manifolds and $u : Y \to X$ an invariant contact embedding (i.e. the contact form λ_Y is a positive multiple of $u^*\lambda_X$, equivalently u is the base map of a symplectic embedding). Let Y' be the image submanifold, with symplectic cones $\Sigma_Y \subset \Sigma_X$ and G-invariant Toeplitz spaces $\mathbb{H}_Y \subset \mathbb{H}_X$ (e.g. Σ_Y is realized as a symplectic subbundle of $T^\bullet M$, $Y = S^*M$, $\mathbb{H}_X = L^2(M)$).

Then (cf. [8]) $\mathbb{H}_Y$ is the set of solutions of a complex $D_{Y\to X}$ of Toeplitz operators mimicking $\bar{\partial}_b$.

This can be used to transfer the computation of the asymptotic on Y to that on X. The K-theoretic element $[k_{Y\to X}] \in K^G_Y(X)$ defined by the symbol of $D_{Y\to X}$

[9] In the case of spheres, there is also a result for the absolute index, provided we work with trivial bundles, or at least with holomorphic bundles, so that the range of the Szegö projector is unambiguously defined; the K-theoretical element $[P]$ is then an element of $K^G(\operatorname{char} \mathfrak{g})$, the K-theory with compact support of the cone $\operatorname{char} \mathfrak{g}$ (including the origin), rather than $K^G(X - Z)$.

is precisely the element which defines the Bott isomorphism $K^G(Y) \to K^G_Y(X)$. If a K-theoretical element $u \in K^G(X - Z)$ is supported by Y, it belongs to the range of the canonical homomorphism $K^G_Y(X - Z) \to K^G(X - Z)$: it is the image via the Bott homomorphism of an element of $K^G(X - Z)$ and the corresponding Toeplitz system has the same equivariant index.

Since any compact contact G-manifold can be embedded in a contact sphere with a linear unitary action of G, this theoretically reduces the index problem to the problem on the sphere, essentially studied by M. F. Atiyah [2] (at least when G is a torus with a symmetric action as indicated below).

§5. Generators

If $V_0 \subset V$ is the fixed subspace of G, it is immediate that any G-elliptic system can be deformed equivariantly into a G-elliptic system also elliptic outside of the orthogonal complement of V_0, whose K-theoretical element is then supported by $X \cap V_0^{\perp}$. Thus for the computation of the index, we are reduced to the case where V has no trivial component.

Definition 7. We will say that the action of G is *elliptic* if char $\mathfrak{g}$ is empty[10] (the base Z is empty).

For the cotangent sphere of a G-manifold M, this would just mean that M is a finite union of open orbits; for a sphere or a contact manifold, it is less trivial.

If the action of G is elliptic, any equivariant bundle homomorphism $u : E \to F$ is G-elliptic; its G-index is $\operatorname{Ind}_G E - \operatorname{Ind}_G F$ (where for short we have denoted by $\operatorname{Ind}_G E$ the G-index of the zero map $E \to 0$).

From now on we will suppose that the base is a sphere $X \subset V$ as in §1, and that G has no fixed point. As mentioned above any G-bundle on the sphere is stably isomorphic to a trivial (product) bundle. If the action of G is elliptic, any equivariant bundle homomorphism is G-elliptic. The index of the trivial bundle of rank 1, with space of holomorphic sections $\mathbb{H}_V$, is the polynomial algebra:

$$\operatorname{Ind}_G([1]) = \sum S^n V^{*n} \ (= \operatorname{Ind}_G(\mathbb{H}_V)).$$

The G-index of the trivial bundle $X \times E$ is $E \otimes \operatorname{Ind}(\mathbb{H}_V)$.

We now further suppose that G is a torus $\mathbb{R}^n/\mathbb{Z}^n$. We may suppose that the action on $X \subset V = \mathbb{C}^N$ is diagonal, and that there is no fixed point ($\neq 0$):

$$gz = (e^{2i\pi\mu_k(\gamma)} z_k) \quad \text{if } g = \gamma \bmod \mathbb{Z}^n$$

where the μ_k ($1 \le k \le N$) are integral linear forms ($\mu_k \in \mathbb{Z}^n \subset \mathfrak{g}^* \simeq \mathbb{R}^n$).

[10] Or reduced to the origin if we extend it as a closed subset of $\mathbb{C}^N$.

Lemma 8. *The action of G is elliptic iff the characters μ_k generate a strictly convex cone in $\mathfrak{g}^*$.*

Indeed, strict convexity is equivalent to the fact that $\sum \lambda_k \mu_k = 0$, $\lambda_k \geq 0$ ($\lambda_k = z_k \bar{z}_k$) implies $\lambda_k = 0$ for all k.

The K-theoretic element defined by the Koszul (or de Rham) complex is

$$[k_V] = \prod (1 - \mu_k)$$

and we get

$$\mathrm{Ind}_G(\mathbb{H}_V) = [k_V]^{-1} = \prod (1 - \mu_k)^{-1}$$

where each factor $(1 - \mu_k)^{-1}$ is expanded in positive powers of the μ_k (this is well defined if the μ_k generate a strictly convex cone); $\mathrm{Ind}_G(\mathbb{H}_V)$ is the partition function, given by a Hilbert–Samuel series. The G-index or asymptotic G-index of any equivariant Toeplitz system is then a multiple of $\chi(\mathbb{H}_V)$ in $\widehat{R}_G$ (resp. $\widehat{R}_G$ mod R_G).

The map $K^G(\mathbb{B}) \to K^G(X)$ always induces an isomorphism $R_G/R_G \cdot [k_V] \to K^G(X)$ (for any action of G), so in the elliptic case, the equivariant asymptotic index is an isomorphism from $K^G(X)$ to $R_G \cdot [k_V]^{-1}/R_G$.

In the general case (G still a torus, but not an elliptic action), there may be several invariant elliptic subspheres $X_J = X \cap V_J$, corresponding to the subsets $J \subset [1, N]$ such that the μ_j, $j \in J$, generate a strictly convex cone (i.e. there exists $v \in \mathfrak{g}$ such that $\mu_j(v) > 0$ for $j \in J$). Each of them defines a Bott map $K^G(X_J) \to K^G(X - Z)$; an element of $K^G(X - Z)$ is in the range of this Bott map iff it is supported by X_J (i.e. can be defined by an equivariant symbol u invertible outside of X_j); the corresponding G-indices are the multiples of the inverse of the Koszul symbol $[k_{V_J}]^{-1}$ (mod R_G) ($[k_{V_J}] = [d''_{V_J}]$ with the notations above; and $[k_{V_J}]^{-1}$ should be expanded as a power series of the μ_j, $j \in J$, as above).

Conjecture. *Any element of $K^G(X - Z)$ is a sum of elements supported by elliptic subspheres X_J. In other words $K^G(X - Z)$ is generated by the K-theoretic elliptic elements $[k_{V_J}]$ associated to invariant elliptic subspheres (equivalently: by the images of the corresponding Bott maps).*

Accordingly asymptotic indices are of the form $\sum P_J [k_{V_J}]^{-1}$, where $[k_{V_J}]$ denotes the K-theoretical element corresponding to the Koszul complex of elliptic subspheres (the inverse is understood as a series as above), and the P_J are (finite) elements of R_G. More precisely: the asymptotic index map should be an isomorphism of $K^G(X - Z)$ to the R_G-submodule of $\widehat{R}_G/R_G$ generated by the $\mathbb{H}_{V_J} = [k_{V_J}]^{-1}$ for elliptic $V_J \subset V$.

The assertion is easily seen to be true if $G = U(1)$ is the circle group ($n = 1$): in this case Z is a hypersurface of real codimension 1; it cuts $K^G(X - Z)$ (just as X) in two components, each of which is supported by a maximal positive (resp. negative) subsphere. The corresponding indices are those of the form

$$P(\tau^{\pm 1})(k_{\pm})^{-1}$$

where τ denotes the generator of R_G (the tautological representation), P is a polynomial, and

$$k_{\pm} = \prod_{\pm n_k > 0} (1 - \tau^{n_k})$$

where the n_k are the exponents of the action of $G = U(1)$ ($e^{2i\pi\theta} \cdot z = (e^{2i\pi n_k \theta} z_k)$); as above the inverse is expanded as a series of positive powers of $\tau^{\pm 1}$.

The conjecture is also true (G a torus) if Z is reduced to one 'interior' N-torus, i.e. the equations $\sum \lambda_k \mu_k = 0$ have exactly one solution $\lambda_k > 0$ up to a constant factor; equivalently: all hyperplanes $z_k = 0$ are elliptic. In that case G acts through an $N-1$-dimensional torus, and Z/G is a one-dimensional circle. Let us first examine the case of an elliptic symbol $u : E \to F$ where E, F are of rank one: then u can be deformed into a monomial z^α (the characters of E, F being linked by $\chi_E - \chi_F = \sum \alpha_j \chi_k$). Since a composition $v \circ v'$ is stably homotopic to $v \oplus v'$, u is stably homotopic to a $\oplus u_k$ where u_k is multiplication by one coordinate power $z_k^{\alpha_k}$ (resp. $\bar{z}_k^{-\alpha_k}$ if $\alpha_k < 0$), and the corresponding element $[u_k]$ is supported by the subsphere ($z_k = 0$).

In the general case, because the quotient Z/G is a circle, one sees easily that any equivariant homomorphism $u : E \to F$ can be deformed into a sum $\oplus u_\alpha : X \times E_\alpha \to X \times F_\alpha$ where the E_α, F_α are representations of degree 1, and our assertion follows.

The conjecture is finally also true if the set of characters μ_k is symmetric, i.e. $[1, N]$ is a union of disjoint pairs (k, k') with $\mu_{k'} = -\mu_k$. In this case it is not hard to see that our index situation can be deformed to the spherical case studied by M. F. Atiyah in [2]. The 'maximal complex structures' of Atiyah, which provide the K-theoretical generators, exactly correspond to the maximal elliptic subspheres. For a very elegant analysis of this, see also [10, 11]. Since any G-sphere can be embedded in a symmetric one, this at least shows that any index is a sum of indices supported by some convex cone (coming from some possibly larger elliptic sphere).

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