

## Regular Holonomic $\mathcal{D}[[\hbar]]$ -modules

*Dedicated to Professor Mikio Sato on the occasion of his 80th birthday with our deep admiration and warmest regards*

by

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### Abstract

We describe the category of regular holonomic modules over the ring  $\mathcal{D}[[\hbar]]$  of linear differential operators with a formal parameter  $\hbar$ . In particular, we establish the Riemann–Hilbert correspondence and discuss the additional  $t$ -structure related to  $\hbar$ -torsion.

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### Introduction

On a complex manifold  $X$ , we will be interested in the study of holonomic modules over the ring  $\mathcal{D}_X[[\hbar]]$  of differential operators with a formal parameter  $\hbar$ . Such modules naturally appear when studying deformation quantization modules (DQ-modules) along a smooth Lagrangian submanifold of a complex symplectic manifold (see [13, Chapter 7]).

In this paper, after recalling the tools from [13] that we shall use, we explain some basic notions of  $\mathcal{D}_X[[\hbar]]$ -modules theory. For example, it follows easily from general results on modules over  $\mathbb{C}[[\hbar]]$ -algebras that given two holonomic  $\mathcal{D}_X[[\hbar]]$ -

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modules  $\mathcal{M}$  and  $\mathcal{N}$ , the complex  $\mathrm{R}\mathcal{H}om_{\mathcal{D}_X[[\hbar]]}(\mathcal{M}, \mathcal{N})$  is constructible over  $\mathbb{C}[[\hbar]]$  and that the microsupport of the solution complex  $\mathrm{R}\mathcal{H}om_{\mathcal{D}_X[[\hbar]]}(\mathcal{M}, \mathcal{O}_X[[\hbar]])$  coincides with the characteristic variety of  $\mathcal{M}$ .

Then we establish our main result, the Riemann–Hilbert correspondence for regular holonomic  $\mathcal{D}_X[[\hbar]]$ -modules, an  $\hbar$ -variant of Kashiwara’s classical theorem. In other words, we show that the solution functor with values in  $\mathcal{O}_X[[\hbar]]$  induces an equivalence between the derived category of regular holonomic  $\mathcal{D}_X[[\hbar]]$ -modules and that of constructible sheaves over  $\mathbb{C}[[\hbar]]$ . A quasi-inverse is obtained by constructing the “sheaf” of holomorphic functions with temperate growth and a formal parameter  $\hbar$  in the subanalytic site. This needs some care since the literature on this subject is written in the framework of sheaves over a field and does not immediately apply to the ring  $\mathbb{C}[[\hbar]]$ .

We also discuss the  $t$ -structure related to  $\hbar$ -torsion. Indeed, as we work over the ring  $\mathbb{C}[[\hbar]]$  and not over a field, the derived category of holonomic  $\mathcal{D}_X[[\hbar]]$ -modules (or, equivalently, that of constructible sheaves over  $\mathbb{C}[[\hbar]]$ ) has an additional  $t$ -structure related to  $\hbar$ -torsion. We will show how the duality functor interchanges it with the natural  $t$ -structure.

We end this paper by describing some natural links between the ring  $\mathcal{D}_X[[\hbar]]$  and deformation quantization algebras, as mentioned above.

**Historical remark.** As is well-known, holonomic modules play an essential role in mathematics. They appeared independently in the work of M. Kashiwara [4] and J. Bernstein [1], but they were first invented by Mikio Sato in a series of (unfortunately unpublished) lectures at Tokyo University in the 60’s. (See [17] for a more detailed history.)

### Notation and conventions

We shall mainly follow the notation of [12]. In particular, if  $\mathcal{C}$  is an abelian category, we denote by  $\mathrm{D}(\mathcal{C})$  the derived category of  $\mathcal{C}$  and by  $\mathrm{D}^*(\mathcal{C})$  ( $*$  = +, −, b) the full triangulated subcategory consisting of objects with cohomology bounded from below (resp. bounded from above, resp. bounded).

For a sheaf  $\mathcal{R}$  of rings on a topological space  $X$ , or more generally on a site, we denote by  $\mathrm{Mod}(\mathcal{R})$  the category of left  $\mathcal{R}$ -modules and we write  $\mathrm{D}^*(\mathcal{R})$  instead of  $\mathrm{D}^*(\mathrm{Mod}(\mathcal{R}))$  ( $*$  =  $\emptyset$ , +, −, b). We denote by  $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{R})$  the full abelian subcategory of  $\mathrm{Mod}(\mathcal{R})$  of coherent objects, and by  $\mathrm{D}_{\mathrm{coh}}^b(\mathcal{R})$  the full triangulated subcategory of  $\mathrm{D}^b(\mathcal{R})$  of objects with coherent cohomology groups.

If  $R$  is a ring (a sheaf of rings over a point), we write for short  $\mathrm{D}_f^b(R)$  instead of  $\mathrm{D}_{\mathrm{coh}}^b(R)$ .

**§1. Formal deformations (after [13])**

We review here some definitions and results from [13] that we shall use in this paper.

**Modules over  $\mathbb{Z}[[\hbar]]$ -algebras.** Let  $X$  be a topological space. One says that a sheaf of  $\mathbb{Z}_X[[\hbar]]$ -modules  $\mathcal{M}$  has no  $\hbar$ -torsion if  $\hbar: \mathcal{M} \rightarrow \mathcal{M}$  is injective; and one says that  $\mathcal{M}$  is  $\hbar$ -complete if  $\mathcal{M} \rightarrow \varprojlim_n \mathcal{M}/\hbar^n \mathcal{M}$  is an isomorphism.

Let  $\mathcal{R}$  be a sheaf of  $\mathbb{Z}_X[[\hbar]]$ -algebras, and assume that  $\mathcal{R}$  has no  $\hbar$ -torsion. Set

$$\mathcal{R}^{\text{loc}} := \mathbb{Z}[[\hbar, \hbar^{-1}]] \otimes_{\mathbb{Z}[[\hbar]]} \mathcal{R}, \quad \mathcal{R}_0 := \mathcal{R}/\hbar \mathcal{R},$$

and consider the functors

$$\begin{aligned} (\bullet)^{\text{loc}}: \text{Mod}(\mathcal{R}) &\rightarrow \text{Mod}(\mathcal{R}^{\text{loc}}), & \mathcal{M} &\mapsto \mathcal{M}^{\text{loc}} := \mathcal{R}^{\text{loc}} \otimes_{\mathcal{R}} \mathcal{M}, \\ \text{gr}_{\hbar}: \text{D}(\mathcal{R}) &\rightarrow \text{D}(\mathcal{R}_0), & \mathcal{M} &\mapsto \text{gr}_{\hbar}(\mathcal{M}) := \mathcal{R}_0 \overset{\text{L}}{\otimes}_{\mathcal{R}} \mathcal{M}. \end{aligned}$$

Note that  $(\bullet)^{\text{loc}}$  is exact and that for  $\mathcal{M}, \mathcal{N} \in \text{D}^{\text{b}}(\mathcal{R})$  and  $\mathcal{P} \in \text{D}^{\text{b}}(\mathcal{R}^{\text{op}})$  one has isomorphisms

$$(1.1) \quad \text{gr}_{\hbar}(\mathcal{P} \overset{\text{L}}{\otimes}_{\mathcal{R}} \mathcal{M}) \simeq \text{gr}_{\hbar} \mathcal{P} \overset{\text{L}}{\otimes}_{\mathcal{R}_0} \text{gr}_{\hbar} \mathcal{M},$$

$$(1.2) \quad \text{gr}_{\hbar}(\text{RHom}_{\mathcal{R}}(\mathcal{M}, \mathcal{N})) \simeq \text{RHom}_{\mathcal{R}_0}(\text{gr}_{\hbar}(\mathcal{M}), \text{gr}_{\hbar}(\mathcal{N})).$$

Here, the functor  $\text{gr}_{\hbar}$  on the left hand side acts on  $\mathbb{Z}_X[[\hbar]]$ -modules.

**Cohomologically  $\hbar$ -complete sheaves**

**Definition 1.1.** One says that an object  $\mathcal{M}$  of  $\text{D}(\mathcal{R})$  is *cohomologically  $\hbar$ -complete* if  $\text{RHom}_{\mathcal{R}}(\mathcal{R}^{\text{loc}}, \mathcal{M}) = 0$ .

Hence, the full subcategory of cohomologically  $\hbar$ -complete objects is triangulated. In fact, it is the right orthogonal complement to the full subcategory  $\text{D}(\mathcal{R}^{\text{loc}})$  of  $\text{D}(\mathcal{R})$ .

Remark that  $\mathcal{M} \in \text{D}(\mathcal{R})$  is cohomologically  $\hbar$ -complete if and only if its image in  $\text{D}(\mathbb{Z}_X[[\hbar]])$  is cohomologically  $\hbar$ -complete.

**Proposition 1.2.** *Let  $\mathcal{M} \in \text{D}(\mathcal{R})$ . Then  $\mathcal{M}$  is cohomologically  $\hbar$ -complete if and only if*

$$\varinjlim_{U \ni x} \text{Ext}_{\mathbb{Z}[[\hbar]]}^j(\mathbb{Z}[[\hbar, \hbar^{-1}]], H^i(U; \mathcal{M})) = 0$$

for any  $x \in X$ , any integer  $i \in \mathbb{Z}$  and any  $j = 0, 1$ . Here,  $U$  ranges over an open neighborhood system of  $x$ .

**Corollary 1.3.** *Let  $\mathcal{M} \in \text{Mod}(\mathcal{R})$ . Assume that  $\mathcal{M}$  has no  $\hbar$ -torsion, is  $\hbar$ -complete and there exists a base  $\mathfrak{B}$  of open subsets such that  $H^i(U; \mathcal{M}) = 0$  for any  $i > 0$  and any  $U \in \mathfrak{B}$ . Then  $\mathcal{M}$  is cohomologically  $\hbar$ -complete.*

The functor  $\text{gr}_{\hbar}$  is conservative on the category of cohomologically  $\hbar$ -complete objects:

**Proposition 1.4.** *Let  $\mathcal{M} \in \text{D}(\mathcal{R})$  be a cohomologically  $\hbar$ -complete object. If  $\text{gr}_{\hbar}(\mathcal{M}) = 0$ , then  $\mathcal{M} = 0$ .*

**Proposition 1.5.** *If  $\mathcal{M} \in \text{D}(\mathcal{R})$  is cohomologically  $\hbar$ -complete, then the object  $\text{R}\mathcal{H}om_{\mathcal{R}}(\mathcal{N}, \mathcal{M}) \in \text{D}(\mathbb{Z}_X[\hbar])$  is cohomologically  $\hbar$ -complete for any  $\mathcal{N} \in \text{D}(\mathcal{R})$ .*

**Proposition 1.6.** *Let  $f: X \rightarrow Y$  be a continuous map, and  $\mathcal{M} \in \text{D}(\mathbb{Z}_X[\hbar])$ . If  $\mathcal{M}$  is cohomologically  $\hbar$ -complete, then so is  $\text{R}f_*\mathcal{M}$ .*

**Reductions to  $\hbar = 0$ .** Now we assume that  $X$  is a Hausdorff locally compact topological space.

By a *basis  $\mathfrak{B}$  of compact subsets* of  $X$ , we mean a family of compact subsets such that for any  $x \in X$  and any open neighborhood  $U$  of  $x$ , there exists  $K \in \mathfrak{B}$  such that  $x \in \text{Int}(K) \subset K \subset U$ .

Let  $\mathcal{A}$  be a  $\mathbb{Z}[\hbar]$ -algebra, and recall that we set  $\mathcal{A}_0 = \mathcal{A}/\hbar\mathcal{A}$ . Consider the following conditions:

- (i)  $\mathcal{A}$  has no  $\hbar$ -torsion and is  $\hbar$ -complete,
- (ii)  $\mathcal{A}_0$  is a left Noetherian ring,
- (iii) there exists a basis  $\mathfrak{B}$  of compact subsets of  $X$  and a prestack  $U \mapsto \text{Mod}_{\text{good}}(\mathcal{A}_0|_U)$  ( $U$  open in  $X$ ) such that
  - (a) for any  $K \in \mathfrak{B}$  and any open subset  $U$  such that  $K \subset U$ , there exists  $K' \in \mathfrak{B}$  such that  $K \subset \text{Int}(K') \subset K' \subset U$ ,
  - (b)  $U \mapsto \text{Mod}_{\text{good}}(\mathcal{A}_0|_U)$  is a full subprestack of  $U \mapsto \text{Mod}_{\text{coh}}(\mathcal{A}_0|_U)$ ,
  - (c) for any  $K \in \mathfrak{B}$ , any open set  $U$  containing  $K$ , any  $j > 0$  and any  $\mathcal{M} \in \text{Mod}_{\text{good}}(\mathcal{A}_0|_U)$ , one has  $H^j(K; \mathcal{M}) = 0$ ,
  - (d) for any open subset  $U$  and any  $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{A}_0|_U)$ , if  $\mathcal{M}|_V$  belongs to  $\text{Mod}_{\text{good}}(\mathcal{A}_0|_V)$  for any relatively compact open subset  $V$  of  $U$ , then  $\mathcal{M}$  belongs to  $\text{Mod}_{\text{good}}(\mathcal{A}_0|_U)$ ,
  - (e) for any  $U$  open in  $X$ ,  $\text{Mod}_{\text{good}}(\mathcal{A}_0|_U)$  is stable under subobjects, quotients and extensions in  $\text{Mod}_{\text{coh}}(\mathcal{A}_0|_U)$ ,
  - (f) for any  $U$  open in  $X$  and any  $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{A}_0|_U)$ , there exists an open covering  $U = \bigcup_i U_i$  such that  $\mathcal{M}|_{U_i} \in \text{Mod}_{\text{good}}(\mathcal{A}_0|_{U_i})$ ,
  - (g)  $\mathcal{A}_0 \in \text{Mod}_{\text{good}}(\mathcal{A}_0)$ ,

(iii)' there exists a basis  $\mathfrak{B}$  of open subsets of  $X$  such that for any  $U \in \mathfrak{B}$ , any  $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{A}_0|_U)$  and any  $j > 0$ , one has  $H^j(U; \mathcal{M}) = 0$ .

We will suppose that  $\mathcal{A}$  and  $\mathcal{A}_0$  satisfy either Assumption 1.7 or Assumption 1.8 below.

**Assumption 1.7.**  $\mathcal{A}$  and  $\mathcal{A}_0$  satisfy conditions (i)–(iii) above.

**Assumption 1.8.**  $\mathcal{A}$  and  $\mathcal{A}_0$  satisfy conditions (i), (ii) and (iii)' above.

**Theorem 1.9.**

- (i)  $\mathcal{A}$  is a left Noetherian ring.
- (ii) Any coherent  $\mathcal{A}$ -module  $\mathcal{M}$  is  $\hbar$ -complete.
- (iii) Let  $\mathcal{M} \in \text{D}_{\text{coh}}^b(\mathcal{A})$ . Then  $\mathcal{M}$  is cohomologically  $\hbar$ -complete.

**Corollary 1.10.** The functor  $\text{gr}_{\hbar}: \text{D}_{\text{coh}}^b(\mathcal{A}) \rightarrow \text{D}_{\text{coh}}^b(\mathcal{A}_0)$  is conservative.

**Theorem 1.11.** Let  $\mathcal{M} \in \text{D}^+(\mathcal{A})$  and assume:

- (a)  $\mathcal{M}$  is cohomologically  $\hbar$ -complete,
- (b)  $\text{gr}_{\hbar}(\mathcal{M}) \in \text{D}_{\text{coh}}^+(\mathcal{A}_0)$ .

Then  $\mathcal{M} \in \text{D}_{\text{coh}}^+(\mathcal{A})$  and for all  $i \in \mathbb{Z}$  we have the isomorphism

$$H^i(\mathcal{M}) \xrightarrow{\sim} \varprojlim_n H^i(\mathcal{A}/\hbar^n \mathcal{A} \overset{\text{L}}{\otimes}_{\mathcal{A}} \mathcal{M}).$$

**Theorem 1.12.** Assume that  $\mathcal{A}_0^{\text{op}} = \mathcal{A}^{\text{op}}/\hbar \mathcal{A}^{\text{op}}$  is a Noetherian ring and the flabby dimension of  $X$  is finite. Let  $\mathcal{M}$  be an  $\mathcal{A}$ -module. Assume the following conditions:

- (a)  $\mathcal{M}$  has no  $\hbar$ -torsion,
- (b)  $\mathcal{M}$  is cohomologically  $\hbar$ -complete,
- (c)  $\mathcal{M}/\hbar \mathcal{M}$  is a flat  $\mathcal{A}_0$ -module.

Then  $\mathcal{M}$  is a flat  $\mathcal{A}$ -module.

If moreover  $\mathcal{M}/\hbar \mathcal{M}$  is a faithfully flat  $\mathcal{A}_0$ -module, then  $\mathcal{M}$  is a faithfully flat  $\mathcal{A}$ -module.

**Theorem 1.13.** Let  $d \in \mathbb{N}$ . Assume that  $\mathcal{A}_0$  is  $d$ -syzygic, i.e., any coherent  $\mathcal{A}_0$ -module locally admits a projective resolution of length  $\leq d$  by free  $\mathcal{A}_0$ -modules of finite rank. Then

- (a)  $\mathcal{A}$  is  $(d + 1)$ -syzygic.

- (b) Let  $\mathcal{M}^\bullet$  be a complex of  $\mathcal{A}$ -modules concentrated in degrees  $[a, b]$  and with coherent cohomology groups. Then locally there exists a quasi-isomorphism  $\mathcal{L}^\bullet \rightarrow \mathcal{M}^\bullet$  where  $\mathcal{L}^\bullet$  is a complex of free  $\mathcal{A}$ -modules of finite rank concentrated in degrees  $[a - d - 1, b]$ .

**Proposition 1.14.** *Let  $\mathcal{M} \in D_{\text{coh}}^b(\mathcal{A})$  and let  $a \in \mathbb{Z}$ . The conditions below are equivalent:*

- (i)  $H^a(\text{gr}_{\hbar}(\mathcal{M})) \simeq 0$ ,
- (ii)  $H^a(\mathcal{M}) \simeq 0$  and  $H^{a+1}(\mathcal{M})$  has no  $\hbar$ -torsion.

**Cohomologically  $\hbar$ -complete sheaves on real manifolds.** Let now  $X$  be a real analytic manifold. Recall from [9] that the microsupport of  $F \in D^b(\mathbb{Z}_X)$  is a closed involutive subset of the cotangent bundle  $T^*X$  denoted by  $\text{SS}(F)$ . The microsupport is additive on  $D^b(\mathbb{Z}_X)$  (cf. Definition 3.3(ii) below). Considering the distinguished triangle  $F \xrightarrow{\hbar} F \rightarrow \text{gr}_{\hbar} F \xrightarrow{+1}$ , one gets

$$(1.3) \quad \text{SS}(\text{gr}_{\hbar}(F)) \subset \text{SS}(F).$$

**Proposition 1.15.** *Let  $F \in D^b(\mathbb{Z}_X[\hbar])$  and assume that  $F$  is cohomologically  $\hbar$ -complete. Then*

$$(1.4) \quad \text{SS}(F) = \text{SS}(\text{gr}_{\hbar}(F)).$$

*Proof.* It is enough to show that  $\text{SS}(F) \subset \text{SS}(\text{gr}_{\hbar}(F))$ . For  $V \subset U$  open subsets, consider the distinguished triangle

$$\text{R}\Gamma(U; F) \rightarrow \text{R}\Gamma(V; F) \rightarrow G \xrightarrow{+1}.$$

By Proposition 1.6,  $\text{R}\Gamma(U; F)$  and  $\text{R}\Gamma(V; F)$  are cohomologically  $\hbar$ -complete, and thus so is  $G$ . One has the distinguished triangle

$$\text{R}\Gamma(U; \text{gr}_{\hbar} F) \rightarrow \text{R}\Gamma(V; \text{gr}_{\hbar} F) \rightarrow \text{gr}_{\hbar} G \xrightarrow{+1}.$$

By the definition of microsupport, it is enough to prove that  $\text{gr}_{\hbar} G = 0$  implies  $G = 0$ . This follows from Proposition 1.4. □

For  $\mathbb{K}$  a commutative unital Noetherian ring, one denotes by  $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X)$  the full subcategory of  $\text{Mod}(\mathbb{K}_X)$  consisting of  $\mathbb{R}$ -constructible sheaves and by  $D_{\mathbb{R}\text{-c}}^b(\mathbb{K}_X)$  the full triangulated subcategory of  $D^b(\mathbb{K}_X)$  consisting of objects with  $\mathbb{R}$ -constructible cohomology (see [9, §8.4]). In this paper, we shall mainly be interested in the case where  $\mathbb{K}$  is either  $\mathbb{C}$  or the ring of formal power series in an indeterminate  $\hbar$ , which we denote by

$$\mathbb{C}^{\hbar} := \mathbb{C}[[\hbar]].$$

**Proposition 1.16.** *Let  $F \in \mathbf{D}_{\mathbb{R}\text{-c}}^b(\mathbb{C}_X^\hbar)$ . Then  $F$  is cohomologically  $\hbar$ -complete.*

*Proof.* This follows from Proposition 1.2 since for any  $x \in X$  one has  $\mathrm{R}\Gamma(U; F) \xrightarrow{\sim} F_x$  for  $U$  in a fundamental system of neighborhoods of  $x$ .  $\square$

**Corollary 1.17.** *The functor  $\mathrm{gr}_\hbar: \mathbf{D}_{\mathbb{R}\text{-c}}^b(\mathbb{C}_X^\hbar) \rightarrow \mathbf{D}_{\mathbb{R}\text{-c}}^b(\mathbb{C}_X)$  is conservative.*

**Corollary 1.18.** *For  $F \in \mathbf{D}_{\mathbb{R}\text{-c}}^b(\mathbb{C}_X^\hbar)$ , one has the equality*

$$\mathrm{SS}(\mathrm{gr}_\hbar(F)) = \mathrm{SS}(F).$$

**Proposition 1.19.** *For  $F \in \mathbf{D}_{\mathbb{R}\text{-c}}^b(\mathbb{C}_X^\hbar)$  and  $i \in \mathbb{Z}$  one has  $\mathrm{supp} H^i(F) \subset \mathrm{supp} H^i(\mathrm{gr}_\hbar F)$ . In particular if  $H^i(\mathrm{gr}_\hbar F) = 0$  then  $H^i(F) = 0$ .*

*Proof.* We apply Proposition 1.14 to  $F_x$  for any  $x \in X$ .  $\square$

## §2. Formal extension

Let  $X$  be a topological space, or more generally a site, and let  $\mathcal{R}_0$  be a sheaf of rings on  $X$ . In this section, we let

$$\mathcal{R} := \mathcal{R}_0[[\hbar]] = \prod_{n \geq 0} \mathcal{R}_0 \hbar^n$$

be the formal extension of  $\mathcal{R}_0$ , whose sections on an open subset  $U$  are formal series  $r = \sum_{n=0}^{\infty} r_n \hbar^n$ , with  $r_n \in \Gamma(U; \mathcal{R}_0)$ . Consider the associated functor

$$(2.1) \quad (\cdot)^\hbar: \mathrm{Mod}(\mathcal{R}_0) \rightarrow \mathrm{Mod}(\mathcal{R}), \quad \mathcal{N} \mapsto \mathcal{N}[[\hbar]] = \varprojlim_n (\mathcal{R}_n \otimes_{\mathcal{R}_0} \mathcal{N}),$$

where  $\mathcal{R}_n := \mathcal{R}/\hbar^{n+1}\mathcal{R}$  is regarded as an  $(\mathcal{R}, \mathcal{R}_0)$ -bimodule. Since  $\mathcal{R}_n$  is free of finite rank over  $\mathcal{R}_0$ , the functor  $(\cdot)^\hbar$  is left exact. We denote by  $(\cdot)^{\mathrm{R}\hbar}$  its right derived functor.

**Proposition 2.1.** *For  $\mathcal{N} \in \mathbf{D}^b(\mathcal{R}_0)$  one has*

$$\mathcal{N}^{\mathrm{R}\hbar} \simeq \mathrm{R}\mathcal{H}om_{\mathcal{R}_0}(\mathcal{R}^{\mathrm{loc}}/\hbar\mathcal{R}, \mathcal{N}),$$

where  $\mathcal{R}^{\mathrm{loc}}/\hbar\mathcal{R}$  is regarded as an  $(\mathcal{R}_0, \mathcal{R})$ -bimodule.

*Proof.* It is enough to prove that for  $\mathcal{N} \in \mathrm{Mod}(\mathcal{R}_0)$  one has

$$\mathcal{N}^\hbar \simeq \mathcal{H}om_{\mathcal{R}_0}(\mathcal{R}^{\mathrm{loc}}/\hbar\mathcal{R}, \mathcal{N}).$$

Using the right  $\mathcal{R}_0$ -module structure of  $\mathcal{R}_n$ , set  $\mathcal{R}_n^* = \mathcal{H}om_{\mathcal{R}_0}(\mathcal{R}_n, \mathcal{R}_0)$ . Then  $\mathcal{R}_n^*$  is an  $(\mathcal{R}_0, \mathcal{R})$ -bimodule, and

$$\mathcal{N}^\hbar = \varprojlim_n (\mathcal{R}_n \otimes_{\mathcal{R}_0} \mathcal{N}) \simeq \mathcal{H}om_{\mathcal{R}_0}(\varinjlim_n \mathcal{R}_n^*, \mathcal{N}).$$

Since

$$\mathcal{R}^{\text{loc}}/\hbar\mathcal{R} \simeq \varinjlim_n (\hbar^{-n}\mathcal{R}/\hbar\mathcal{R}),$$

it is enough to prove that there is an isomorphism of  $(\mathcal{R}_0, \mathcal{R})$ -bimodules

$$\mathcal{H}om_{\mathcal{R}_0}(\mathcal{R}_n, \mathcal{R}_0) \simeq \hbar^{-n}\mathcal{R}/\hbar\mathcal{R}.$$

Recalling that  $\mathcal{R}_n = \mathcal{R}/\hbar^{n+1}\mathcal{R}$ , this follows from the pairing

$$(\mathcal{R}/\hbar^{n+1}\mathcal{R}) \otimes_{\mathcal{R}_0} (\hbar^{-n}\mathcal{R}/\hbar\mathcal{R}) \rightarrow \mathcal{R}_0, \quad f \otimes g \mapsto \text{Res}_{\hbar=0}(fg d\hbar/\hbar). \quad \square$$

Note that the isomorphism of  $(\mathcal{R}, \mathcal{R}_0)$ -bimodules

$$\mathcal{R} \simeq (\mathcal{R}_0)^{\hbar} = \mathcal{H}om_{\mathcal{R}_0}(\mathcal{R}^{\text{loc}}/\hbar\mathcal{R}, \mathcal{R}_0)$$

induces a natural morphism

$$(2.2) \quad \mathcal{R} \otimes_{\mathcal{R}_0}^{\mathbb{L}} \mathcal{N} \rightarrow \mathcal{N}^{\text{R}\hbar} \quad \text{for } \mathcal{N} \in \mathbb{D}^b(\mathcal{R}_0).$$

**Proposition 2.2.** *For  $\mathcal{N} \in \mathbb{D}^b(\mathcal{R}_0)$ , the formal extension  $\mathcal{N}^{\text{R}\hbar}$  is cohomologically  $\hbar$ -complete.*

*Proof.* The statement follows from  $(\mathcal{R}^{\text{loc}}/\hbar\mathcal{R}) \otimes_{\mathcal{R}}^{\mathbb{L}} \mathcal{R}^{\text{loc}} \simeq 0$  and from the isomorphism

$$\mathbb{R}\mathcal{H}om_{\mathcal{R}}(\mathcal{R}^{\text{loc}}, \mathcal{N}^{\text{R}\hbar}) \simeq \mathbb{R}\mathcal{H}om_{\mathcal{R}_0}((\mathcal{R}^{\text{loc}}/\hbar\mathcal{R}) \otimes_{\mathcal{R}}^{\mathbb{L}} \mathcal{R}^{\text{loc}}, \mathcal{N}). \quad \square$$

**Lemma 2.3.** *Assume that  $\mathcal{R}_0$  is an  $\mathcal{S}_0$ -algebra, for  $\mathcal{S}_0$  a commutative sheaf of rings, and let  $\mathcal{S} = \mathcal{S}_0[[\hbar]]$ . For  $\mathcal{M}, \mathcal{N} \in \mathbb{D}^b(\mathcal{R}_0)$  we have an isomorphism in  $\mathbb{D}^b(\mathcal{S})$*

$$\mathbb{R}\mathcal{H}om_{\mathcal{R}_0}(\mathcal{M}, \mathcal{N})^{\text{R}\hbar} \simeq \mathbb{R}\mathcal{H}om_{\mathcal{R}_0}(\mathcal{M}, \mathcal{N}^{\text{R}\hbar}).$$

*Proof.* Note the isomorphisms

$$\mathcal{R}^{\text{loc}}/\hbar\mathcal{R} \simeq \mathcal{R}_0 \otimes_{\mathcal{S}_0} (\mathcal{S}^{\text{loc}}/\hbar\mathcal{S}) \simeq \mathcal{R}_0 \otimes_{\mathcal{S}_0}^{\mathbb{L}} (\mathcal{S}^{\text{loc}}/\hbar\mathcal{S})$$

as  $(\mathcal{R}_0, \mathcal{S})$ -bimodules. Then one has

$$\begin{aligned} \mathbb{R}\mathcal{H}om_{\mathcal{R}_0}(\mathcal{M}, \mathcal{N})^{\text{R}\hbar} &= \mathbb{R}\mathcal{H}om_{\mathcal{S}_0}(\mathcal{S}^{\text{loc}}/\hbar\mathcal{S}, \mathbb{R}\mathcal{H}om_{\mathcal{R}_0}(\mathcal{M}, \mathcal{N})) \\ &\simeq \mathbb{R}\mathcal{H}om_{\mathcal{R}_0}(\mathcal{M}, \mathbb{R}\mathcal{H}om_{\mathcal{S}_0}(\mathcal{S}^{\text{loc}}/\hbar\mathcal{S}, \mathcal{N})) \\ &\simeq \mathbb{R}\mathcal{H}om_{\mathcal{R}_0}(\mathcal{M}, \mathbb{R}\mathcal{H}om_{\mathcal{R}_0}(\mathcal{R}^{\text{loc}}/\hbar\mathcal{R}, \mathcal{N})) \\ &= \mathbb{R}\mathcal{H}om_{\mathcal{R}_0}(\mathcal{M}, \mathcal{N}^{\text{R}\hbar}). \end{aligned} \quad \square$$

**Lemma 2.4.** *Let  $f: X \rightarrow Y$  be a morphism of sites, and assume that  $(f^{-1}\mathcal{R}_0)^\hbar \simeq f^{-1}\mathcal{R}$ . Then the functors  $Rf_*$  and  $(\bullet)^{R\hbar}$  commute, that is, for  $\mathcal{P} \in D^b(f^{-1}\mathcal{R}_0)$  we have  $(Rf_*\mathcal{P})^{R\hbar} \simeq Rf_*(\mathcal{P}^{R\hbar})$  in  $D^b(\mathcal{R})$ .*

*Proof.* One has the isomorphism

$$\begin{aligned} Rf_*(\mathcal{P}^{R\hbar}) &= Rf_*R\mathcal{H}om_{f^{-1}\mathcal{R}_0}(f^{-1}(\mathcal{R}^{\text{loc}}/\hbar\mathcal{R}), \mathcal{P}) \\ &\simeq R\mathcal{H}om_{\mathcal{R}_0}(\mathcal{R}^{\text{loc}}/\hbar\mathcal{R}, Rf_*\mathcal{P}) = Rf_*(\mathcal{P})^{R\hbar}. \quad \square \end{aligned}$$

**Proposition 2.5.** *Let  $\mathcal{T}$  be either a basis of open subsets of the site  $X$  or, assuming that  $X$  is a locally compact topological space, a basis of compact subsets. Denote by  $J_{\mathcal{T}}$  the full subcategory of  $\text{Mod}(\mathcal{R}_0)$  consisting of  $\mathcal{T}$ -acyclic objects, i.e., sheaves  $\mathcal{N}$  for which  $H^k(S; \mathcal{N}) = 0$  for all  $k > 0$  and all  $S \in \mathcal{T}$ . Then  $J_{\mathcal{T}}$  is injective with respect to the functor  $(\bullet)^\hbar$ . In particular, for  $\mathcal{N} \in J_{\mathcal{T}}$ , we have  $\mathcal{N}^\hbar \simeq \mathcal{N}^{R\hbar}$ .*

*Proof.* (i) Since injective sheaves are  $\mathcal{T}$ -acyclic,  $J_{\mathcal{T}}$  is cogenerating.

(ii) Consider an exact sequence  $0 \rightarrow \mathcal{N}' \rightarrow \mathcal{N} \rightarrow \mathcal{N}'' \rightarrow 0$  in  $\text{Mod}(\mathcal{R}_0)$ . Clearly, if both  $\mathcal{N}'$  and  $\mathcal{N}$  belong to  $J_{\mathcal{T}}$ , then so does  $\mathcal{N}''$ .

(iii) Consider an exact sequence as in (ii) and assume that  $\mathcal{N}' \in J_{\mathcal{T}}$ . We have to prove that  $0 \rightarrow \mathcal{N}'^\hbar \rightarrow \mathcal{N}^\hbar \rightarrow \mathcal{N}''^\hbar \rightarrow 0$  is exact. Since  $(\bullet)^\hbar$  is left exact, it is enough to prove that  $\mathcal{N}^\hbar \rightarrow \mathcal{N}''^\hbar$  is surjective. Noticing that  $\mathcal{N}^\hbar \simeq \prod_{\mathbb{N}} \mathcal{N}$  as  $\mathcal{R}_0$ -modules, it is enough to prove that  $\prod_{\mathbb{N}} \mathcal{N} \rightarrow \prod_{\mathbb{N}} \mathcal{N}''$  is surjective.

(iii)-(a) Assume that  $\mathcal{T}$  is a basis of open subsets. Any open subset  $U \subset X$  has a cover  $\{U_i\}_{i \in I}$  by elements  $U_i \in \mathcal{T}$ . For any  $i \in I$ , the morphism  $\mathcal{N}(U_i) \rightarrow \mathcal{N}''(U_i)$  is surjective. The result follows taking the product over  $\mathbb{N}$ .

(iii)-(b) Assume that  $\mathcal{T}$  is a basis of compact subsets. For any  $K \in \mathcal{T}$ , the morphism  $\mathcal{N}(K) \rightarrow \mathcal{N}''(K)$  is surjective. Hence, there exists a basis  $\mathcal{V}$  of open subsets such that for any  $x \in X$  and any  $V \ni x$  in  $\mathcal{V}$ , there exists  $V' \in \mathcal{V}$  with  $x \in V' \subset V$  and the image of  $\mathcal{N}(V') \rightarrow \mathcal{N}''(V')$  contains the image of  $\mathcal{N}''(V)$  in  $\mathcal{N}''(V')$ . The result follows as in (iii)-(a) by taking the product over  $\mathbb{N}$ .  $\square$

**Corollary 2.6.** *The following sheaves are acyclic for the functor  $(\bullet)^\hbar$ :*

- (i)  $\mathbb{R}$ -constructible sheaves of  $\mathbb{C}$ -vector spaces on a real analytic manifold  $X$ ,
- (ii) coherent modules over the ring  $\mathcal{O}_X$  of holomorphic functions on a complex analytic manifold  $X$ ,
- (iii) coherent modules over the ring  $\mathcal{D}_X$  of linear differential operators on a complex analytic manifold  $X$ .

*Proof.* The statements follow by applying Proposition 2.5 for the following choices of  $\mathcal{T}$ .

(i) Let  $F$  be an  $\mathbb{R}$ -constructible sheaf. Then for any  $x \in X$  one has  $F_x \xleftarrow{\sim} \mathrm{R}\Gamma(U_x; F)$  for  $U_x$  in a fundamental system of open neighborhoods of  $x$ . Take for  $\mathcal{T}$  the union of these fundamental systems.

(ii) Take for  $\mathcal{T}$  the family of open Stein subsets.

(iii) Let  $\mathcal{M}$  be a coherent  $\mathcal{D}_X$ -module. The problem being local, we may assume that  $\mathcal{M}$  is endowed with a good filtration. Then take for  $\mathcal{T}$  the family of compact Stein subsets.  $\square$

**Example 2.7.** Let  $X = \mathbb{R}$ ,  $\mathcal{R}_0 = \mathbb{C}_X$ ,  $Z = \{1/n : n = 1, 2, \dots\} \cup \{0\}$  and  $U = X \setminus Z$ . One has the isomorphisms  $(\mathbb{C}^{\hbar})_X \simeq (\mathbb{C}_X)^{\hbar} \simeq (\mathbb{C}_X)^{\mathrm{R}\hbar}$  and  $(\mathbb{C}^{\hbar})_U \simeq (\mathbb{C}_U)^{\hbar}$ . Considering the exact sequences

$$\begin{aligned} 0 \rightarrow (\mathbb{C}^{\hbar})_U \rightarrow (\mathbb{C}^{\hbar})_X \rightarrow (\mathbb{C}^{\hbar})_Z \rightarrow 0, \\ 0 \rightarrow (\mathbb{C}_U)^{\hbar} \rightarrow (\mathbb{C}_X)^{\hbar} \rightarrow (\mathbb{C}_Z)^{\hbar} \rightarrow H^1(\mathbb{C}_U)^{\mathrm{R}\hbar} \rightarrow 0, \end{aligned}$$

we get  $H^1(\mathbb{C}_U)^{\mathrm{R}\hbar} \simeq (\mathbb{C}_Z)^{\hbar}/(\mathbb{C}^{\hbar})_Z$ , whose stalk at the origin does not vanish. Hence  $\mathbb{C}_U$  is not acyclic for the functor  $(\cdot)^{\hbar}$ .

Assume now that

$$\mathcal{A}_0 = \mathcal{R}_0 \quad \text{and} \quad \mathcal{A} = \mathcal{R}_0[[\hbar]]$$

satisfy either Assumption 1.7 or Assumption 1.8 (where condition (i) is clear) and that  $\mathcal{A}_0$  is syzygic. Note that by Proposition 2.5 one has  $\mathcal{A} \simeq (\mathcal{A}_0)^{\mathrm{R}\hbar}$ .

**Proposition 2.8.** For  $\mathcal{N} \in \mathrm{D}_{\mathrm{coh}}^{\mathrm{b}}(\mathcal{A}_0)$ :

- (i) there is an isomorphism  $\mathcal{N}^{\mathrm{R}\hbar} \xrightarrow{\sim} \mathcal{A} \otimes_{\mathcal{A}_0}^{\mathrm{L}} \mathcal{N}$  induced by (2.2),
- (ii) there is an isomorphism  $\mathrm{gr}_{\hbar}(\mathcal{N}^{\mathrm{R}\hbar}) \simeq \mathcal{N}$ .

*Proof.* Since  $\mathcal{A}_0$  is syzygic, we may locally represent  $\mathcal{N}$  by a bounded complex  $\mathcal{L}^{\bullet}$  of free  $\mathcal{A}_0$ -modules of finite rank. Then (i) is obvious. As for (ii), both complexes are isomorphic to the mapping cone of  $\hbar: (\mathcal{L}^{\bullet})^{\hbar} \rightarrow (\mathcal{L}^{\bullet})^{\hbar}$ .  $\square$

In particular, the functor  $(\cdot)^{\hbar}$  is exact on  $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{A}_0)$  and preserves coherence. One thus gets a functor  $(\cdot)^{\mathrm{R}\hbar}: \mathrm{D}_{\mathrm{coh}}^{\mathrm{b}}(\mathcal{A}_0) \rightarrow \mathrm{D}_{\mathrm{coh}}^{\mathrm{b}}(\mathcal{A})$ .

**The subanalytic site.** The subanalytic site associated to an analytic manifold  $X$  has been introduced and studied in [11, Chapter 7] (see also [15] for a detailed and systematic study as well as for complementary results). Denote by  $\mathrm{Op}_X$  the category of open subsets of  $X$ , the morphisms being the inclusion morphisms, and by  $\mathrm{Op}_{X_{\mathrm{sa}}}$  the full subcategory consisting of relatively compact subanalytic open subsets of  $X$ . The site  $X_{\mathrm{sa}}$  is the presite  $\mathrm{Op}_{X_{\mathrm{sa}}}$  endowed with the Grothendieck

topology for which the coverings are those admitting a finite subcover. One calls  $X_{\text{sa}}$  the subanalytic site associated to  $X$ . Denote by  $\rho: X \rightarrow X_{\text{sa}}$  the natural morphism of sites. Recall that the inverse image functor  $\rho^{-1}$ , besides the usual right adjoint given by the direct image functor  $\rho_*$ , admits a left adjoint denoted  $\rho!$ . Consider the diagram

$$\begin{array}{ccc} \mathrm{D}^b(\mathbb{C}_X) & \begin{array}{c} \xrightarrow{R\rho_*} \\ \xleftarrow{\rho^{-1}} \end{array} & \mathrm{D}^b(\mathbb{C}_{X_{\text{sa}}}) \\ (\cdot)^{\mathrm{R}\hbar} \downarrow & & \downarrow (\cdot)^{\mathrm{R}\hbar} \\ \mathrm{D}^b(\mathbb{C}_X^{\hbar}) & \begin{array}{c} \xrightarrow{R\rho_*} \\ \xleftarrow{\rho^{-1}} \end{array} & \mathrm{D}^b(\mathbb{C}_{X_{\text{sa}}}^{\hbar}) \end{array}$$

- Lemma 2.9.** (i) *The functors  $\rho^{-1}$  and  $(\cdot)^{\mathrm{R}\hbar}$  commute, that is, for  $G \in \mathrm{D}^b(\mathbb{C}_{X_{\text{sa}}})$  we have  $(\rho^{-1}G)^{\mathrm{R}\hbar} \simeq \rho^{-1}(G^{\mathrm{R}\hbar})$  in  $\mathrm{D}^b(\mathbb{C}_X^{\hbar})$ .*  
(ii) *The functors  $R\rho_*$  and  $(\cdot)^{\mathrm{R}\hbar}$  commute, that is, for  $F \in \mathrm{D}^b(\mathbb{C}_X)$  we have  $(R\rho_*F)^{\mathrm{R}\hbar} \simeq R\rho_*(F^{\mathrm{R}\hbar})$  in  $\mathrm{D}^b(\mathbb{C}_{X_{\text{sa}}}^{\hbar})$ .*

*Proof.* (i) Since it admits a left adjoint, the functor  $\rho^{-1}$  commutes with projective limits. It follows that for  $G \in \mathrm{Mod}(\mathbb{C}_{X_{\text{sa}}})$  one has an isomorphism

$$\rho^{-1}(G^{\hbar}) \rightarrow (\rho^{-1}G)^{\hbar}.$$

To conclude, it remains to show that  $(\rho^{-1}(\cdot))^{\mathrm{R}\hbar}$  is the derived functor of  $(\rho^{-1}(\cdot))^{\hbar}$ . Recall that an object  $G$  of  $\mathrm{Mod}(\mathbb{C}_{X_{\text{sa}}})$  is quasi-injective if the functor  $\mathrm{Hom}_{\mathbb{C}_{X_{\text{sa}}}}(\cdot, G)$  is exact on the category  $\mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ . By a result of [15], if  $G \in \mathrm{Mod}(\mathbb{C}_{X_{\text{sa}}})$  is quasi-injective, then  $\rho^{-1}G$  is soft. Hence,  $\rho^{-1}G$  is injective for the functor  $(\cdot)^{\hbar}$  by Proposition 2.5.

(ii) By (i) we can apply Lemma 2.4. □

### §3. $\mathcal{D}[[\hbar]]$ -modules and propagation

Let now  $X$  be a complex analytic manifold of complex dimension  $d_X$ . As usual, denote by  $\mathbb{C}_X$  the constant sheaf with stalk  $\mathbb{C}$ , by  $\mathcal{O}_X$  the structure sheaf and by  $\mathcal{D}_X$  the ring of linear differential operators on  $X$ . We will use the notation

$$\begin{aligned} \mathrm{D}' : \mathrm{D}^b(\mathbb{C}_X)^{\mathrm{op}} &\rightarrow \mathrm{D}^b(\mathbb{C}_X), & F &\mapsto \mathrm{R}\mathcal{H}om_{\mathbb{C}_X}(F, \mathbb{C}_X), \\ \mathbb{D} : \mathrm{D}_{\mathrm{coh}}^b(\mathcal{D}_X)^{\mathrm{op}} &\rightarrow \mathrm{D}_{\mathrm{coh}}^b(\mathcal{D}_X), & \mathcal{M} &\mapsto \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})[d_X], \\ \mathrm{Sol} : \mathrm{D}_{\mathrm{coh}}^b(\mathcal{D}_X)^{\mathrm{op}} &\rightarrow \mathrm{D}^b(\mathbb{C}_X), & \mathcal{M} &\mapsto \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X), \\ \mathrm{DR} : \mathrm{D}_{\mathrm{coh}}^b(\mathcal{D}_X) &\rightarrow \mathrm{D}^b(\mathbb{C}_X), & \mathcal{M} &\mapsto \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M}), \end{aligned}$$

where  $\Omega_X$  denotes the line bundle of holomorphic forms of maximal degree and  $\Omega_X^{\otimes -1}$  the dual bundle.

As shown in Corollary 2.6, the sheaves  $\mathbb{C}_X$ ,  $\mathcal{O}_X$  and  $\mathcal{D}_X$  are all acyclic for the functor  $(\bullet)^{\hbar}$ . We will be interested in the formal extensions

$$\mathbb{C}_X^{\hbar} = \mathbb{C}_X[[\hbar]], \quad \mathcal{O}_X^{\hbar} = \mathcal{O}_X[[\hbar]], \quad \mathcal{D}_X^{\hbar} = \mathcal{D}_X[[\hbar]].$$

In the following, we shall treat left  $\mathcal{D}_X^{\hbar}$ -modules, but all results apply to right modules since the categories  $\text{Mod}(\mathcal{D}_X^{\hbar})$  and  $\text{Mod}(\mathcal{D}_X^{\hbar, \text{op}})$  are equivalent.

**Proposition 3.1.** *Assumption 1.7 is satisfied by the  $\mathbb{C}^{\hbar}$ -algebras  $\mathcal{D}_X^{\hbar}$  and  $\mathcal{D}_X^{\hbar, \text{op}}$ .*

*Proof.* Assumption 1.7 holds for  $\mathcal{A} = \mathcal{D}_X^{\hbar}$ ,  $\mathcal{A}_0 = \mathcal{D}_X$ ,  $\text{Mod}_{\text{good}}(\mathcal{A}_0|_U)$  the category of good  $\mathcal{D}_U$ -modules (see [7]) and for  $\mathfrak{B}$  the family of Stein compact subsets of  $X$ . □

In particular, by Theorem 1.9,  $\mathcal{D}_X^{\hbar}$  is right and left Noetherian (and thus coherent). Moreover, by Theorem 1.13 any object of  $\text{D}_{\text{coh}}^{\text{b}}(\mathcal{D}_X^{\hbar})$  can be locally represented by a bounded complex of free  $\mathcal{D}_X^{\hbar}$ -modules of finite rank.

We will use the notation

$$\begin{aligned} \text{D}'_{\hbar} &: \text{D}^{\text{b}}(\mathbb{C}_X^{\hbar})^{\text{op}} \rightarrow \text{D}^{\text{b}}(\mathbb{C}_X^{\hbar}), & F &\mapsto \text{R}\mathcal{H}om_{\mathbb{C}_X^{\hbar}}(F, \mathbb{C}_X^{\hbar}), \\ \mathbb{D}_{\hbar} &: \text{D}_{\text{coh}}^{\text{b}}(\mathcal{D}_X^{\hbar})^{\text{op}} \rightarrow \text{D}_{\text{coh}}^{\text{b}}(\mathcal{D}_X^{\hbar}), & \mathcal{M} &\mapsto \text{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}, \mathcal{D}_X^{\hbar} \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})[d_X], \\ \text{Sol}_{\hbar} &: \text{D}_{\text{coh}}^{\text{b}}(\mathcal{D}_X^{\hbar})^{\text{op}} \rightarrow \text{D}^{\text{b}}(\mathbb{C}^{\hbar}), & \mathcal{M} &\mapsto \text{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}, \mathcal{O}_X^{\hbar}), \\ \text{DR}_{\hbar} &: \text{D}_{\text{coh}}^{\text{b}}(\mathcal{D}_X^{\hbar}) \rightarrow \text{D}^{\text{b}}(\mathbb{C}^{\hbar}), & \mathcal{M} &\mapsto \text{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{O}_X^{\hbar}, \mathcal{M}). \end{aligned}$$

By Proposition 2.8 and Lemma 2.3, for  $\mathcal{N} \in \text{D}_{\text{coh}}^{\text{b}}(\mathcal{D}_X)$  one has

$$(3.1) \quad \mathcal{N}^{\text{R}\hbar} \simeq \mathcal{D}_X^{\hbar} \overset{\text{L}}{\otimes}_{\mathcal{D}_X} \mathcal{N},$$

$$(3.2) \quad \text{gr}_{\hbar}(\mathcal{N}^{\text{R}\hbar}) \simeq \mathcal{N},$$

$$(3.3) \quad \text{Sol}_{\hbar}(\mathcal{N}^{\text{R}\hbar}) \simeq \text{Sol}(\mathcal{N})^{\text{R}\hbar}.$$

**Definition 3.2.** For  $\mathcal{M} \in \text{Mod}(\mathcal{D}_X^{\hbar})$ , denote by  $\mathcal{M}_{\hbar\text{-tor}}$  its submodule consisting of sections locally annihilated by some power of  $\hbar$  and set  $\mathcal{M}_{\hbar\text{-tf}} = \mathcal{M} / \mathcal{M}_{\hbar\text{-tor}}$ . We say that  $\mathcal{M} \in \text{Mod}(\mathcal{D}_X^{\hbar})$  is an  $\hbar$ -torsion module if  $\mathcal{M}_{\hbar\text{-tor}} \xrightarrow{\sim} \mathcal{M}$  and that  $\mathcal{M}$  has no  $\hbar$ -torsion (or is  $\hbar$ -torsion free) if  $\mathcal{M} \xrightarrow{\sim} \mathcal{M}_{\hbar\text{-tf}}$ .

Denote by  ${}_n\mathcal{M}$  the kernel of  $\hbar^{n+1}: \mathcal{M} \rightarrow \mathcal{M}$ . Then  $\mathcal{M}_{\hbar\text{-tor}}$  is the sheaf associated with the increasing union of the  ${}_n\mathcal{M}$ 's. Hence, if  $\mathcal{M}$  is coherent, the increasing family  $\{{}_n\mathcal{M}\}_n$  is locally stationary and  $\mathcal{M}_{\hbar\text{-tor}}$  as well as  $\mathcal{M}_{\hbar\text{-tf}}$  are coherent.

**Characteristic variety.** Recall the following definition.

- Definition 3.3.** (i) For  $\mathcal{C}$  an abelian category, a function  $c: \text{Ob}(\mathcal{C}) \rightarrow \text{Set}$  is called *additive* if  $c(M) = c(M') \cup c(M'')$  for any short exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ .
- (ii) For  $\mathcal{T}$  a triangulated category, a function  $c: \text{Ob}(\mathcal{T}) \rightarrow \text{Set}$  is called *additive* if  $c(M) = c(M[1])$  and  $c(M) \subset c(M') \cup c(M'')$  for any distinguished triangle  $M' \rightarrow M \rightarrow M'' \xrightarrow{+1}$ .

Note that an additive function  $c$  on  $\mathcal{C}$  naturally extends to the derived category  $D(\mathcal{C})$  by setting  $c(M) = \bigcup_i c(H^i(M))$ .

For  $\mathcal{N}$  a coherent  $\mathcal{D}_X$ -module, denote by  $\text{char}(\mathcal{N})$  its characteristic variety, a closed involutive subvariety of the cotangent bundle  $T^*X$ . The characteristic variety is additive on  $\text{Mod}_{\text{coh}}(\mathcal{D}_X)$ . For  $\mathcal{N} \in D_{\text{coh}}^b(\mathcal{D}_X)$  one sets  $\text{char}(\mathcal{N}) = \bigcup_i \text{char}(H^i(\mathcal{N}))$ .

**Definition 3.4.** The *characteristic variety* of  $\mathcal{M} \in D_{\text{coh}}^b(\mathcal{D}_X^{\hbar})$  is defined by

$$\text{char}_{\hbar}(\mathcal{M}) = \text{char}(\text{gr}_{\hbar}(\mathcal{M})).$$

To  $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_X^{\hbar})$  one associates the coherent  $\mathcal{D}_X$ -modules

$$(3.4) \quad {}_0\mathcal{M} = \text{Ker}(\hbar: \mathcal{M} \rightarrow \mathcal{M}) = H^{-1}(\text{gr}_{\hbar} \mathcal{M}),$$

$$(3.5) \quad \mathcal{M}_0 = \text{Coker}(\hbar: \mathcal{M} \rightarrow \mathcal{M}) = H^0(\text{gr}_{\hbar} \mathcal{M}).$$

**Lemma 3.5.** For  $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_X^{\hbar})$  an  $\hbar$ -torsion module, one has

$$\text{char}_{\hbar}(\mathcal{M}) = \text{char}(\mathcal{M}_0) = \text{char}({}_0\mathcal{M}).$$

*Proof.* By definition,  $\text{char}_{\hbar}(\mathcal{M}) = \text{char}(\mathcal{M}_0) \cup \text{char}({}_0\mathcal{M})$ . It is thus enough to prove the equality  $\text{char}(\mathcal{M}_0) = \text{char}({}_0\mathcal{M})$ .

Since the statement is local we may assume that  $\hbar^N \mathcal{M} = 0$  for some  $N \in \mathbb{N}$ . We proceed by induction on  $N$ .

For  $N = 1$  we have  $\mathcal{M} \simeq \mathcal{M}_0 \simeq {}_0\mathcal{M}$ , and the statement is obvious.

Assume that the statement has been proved for  $N - 1$ . The short exact sequence

$$(3.6) \quad 0 \rightarrow \hbar\mathcal{M} \rightarrow \mathcal{M} \rightarrow \mathcal{M}_0 \rightarrow 0$$

induces the distinguished triangle

$$\text{gr}_{\hbar} \hbar\mathcal{M} \rightarrow \text{gr}_{\hbar} \mathcal{M} \rightarrow \text{gr}_{\hbar} \mathcal{M}_0 \xrightarrow{+1}.$$

Noticing that  $\mathcal{M}_0 \simeq (\mathcal{M}_0)_0 \simeq {}_0(\mathcal{M}_0)$ , the associated long exact cohomology sequence gives

$$0 \rightarrow {}_0(\hbar\mathcal{M}) \rightarrow {}_0\mathcal{M} \rightarrow \mathcal{M}_0 \rightarrow (\hbar\mathcal{M})_0 \rightarrow 0.$$

By inductive hypothesis we have  $\text{char}({}_0(\hbar\mathcal{M})) = \text{char}((\hbar\mathcal{M})_0)$ , and we deduce  $\text{char}(\mathcal{M}_0) = \text{char}(\mathcal{M})$  by additivity of  $\text{char}$ .  $\square$

**Proposition 3.6.** (i) For  $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_X^{\hbar})$  one has

$$\text{char}_{\hbar}(\mathcal{M}) = \text{char}(\mathcal{M}_0).$$

(ii) The characteristic variety  $\text{char}_{\hbar}$  is additive both on  $\text{Mod}_{\text{coh}}(\mathcal{D}_X^{\hbar})$  and on  $\text{D}_{\text{coh}}^b(\mathcal{D}_X^{\hbar})$ .

*Proof.* (i) As  $\text{char}(\text{gr}_{\hbar}\mathcal{M}) = \text{char}(\mathcal{M}_0) \cup \text{char}({}_0\mathcal{M})$ , it is enough to prove the inclusion

$$(3.7) \quad \text{char}({}_0\mathcal{M}) \subset \text{char}(\mathcal{M}_0).$$

Consider the short exact sequence  $0 \rightarrow \mathcal{M}_{\hbar\text{-tor}} \rightarrow \mathcal{M} \rightarrow \mathcal{M}_{\hbar\text{-tf}} \rightarrow 0$ . Since  $\mathcal{M}_{\hbar\text{-tf}}$  has no  $\hbar$ -torsion,  ${}_0(\mathcal{M}_{\hbar\text{-tf}}) = 0$ . The associated long exact cohomology sequence thus gives

$${}_0(\mathcal{M}_{\hbar\text{-tor}}) \simeq {}_0\mathcal{M}, \quad 0 \rightarrow (\mathcal{M}_{\hbar\text{-tor}})_0 \rightarrow \mathcal{M}_0 \rightarrow (\mathcal{M}_{\hbar\text{-tf}})_0 \rightarrow 0.$$

We deduce

$$\text{char}({}_0\mathcal{M}) = \text{char}({}_0(\mathcal{M}_{\hbar\text{-tor}})) = \text{char}((\mathcal{M}_{\hbar\text{-tor}})_0) \subset \text{char}(\mathcal{M}_0),$$

where the second equality follows from Lemma 3.5.

(ii) It is enough to prove the additivity on  $\text{Mod}_{\text{coh}}(\mathcal{D}_X^{\hbar})$ , i.e. the equality

$$\text{char}_{\hbar}(\mathcal{M}) = \text{char}_{\hbar}(\mathcal{M}') \cup \text{char}_{\hbar}(\mathcal{M}'')$$

for  $0 \rightarrow \mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}'' \rightarrow 0$  a short exact sequence of coherent  $\mathcal{D}_X^{\hbar}$ -modules.

The associated distinguished triangle  $\text{gr}_{\hbar}\mathcal{M}' \rightarrow \text{gr}_{\hbar}\mathcal{M} \rightarrow \text{gr}_{\hbar}\mathcal{M}'' \xrightarrow{+1}$  induces the long exact cohomology sequence

$${}_0(\mathcal{M}'') \rightarrow (\mathcal{M}')_0 \rightarrow \mathcal{M}_0 \rightarrow (\mathcal{M}'')_0 \rightarrow 0.$$

By additivity of  $\text{char}(\bullet)$ , the exactness of this sequence at the first, second and third term from the right, respectively, gives

$$\begin{aligned} \text{char}_{\hbar}(\mathcal{M}'') &\subset \text{char}_{\hbar}(\mathcal{M}), \\ \text{char}_{\hbar}(\mathcal{M}) &\subset \text{char}_{\hbar}(\mathcal{M}') \cup \text{char}_{\hbar}(\mathcal{M}''), \\ \text{char}_{\hbar}(\mathcal{M}') &\subset \text{char}({}_0(\mathcal{M}'')) \cup \text{char}_{\hbar}(\mathcal{M}). \end{aligned}$$

Finally, note that  $\text{char}({}_0(\mathcal{M}'')) \subset \text{char}_{\hbar}(\mathcal{M}'') \subset \text{char}_{\hbar}(\mathcal{M})$ .  $\square$

In view of Proposition 3.6 (i), in order to define the characteristic variety of a coherent  $\mathcal{D}_X^\hbar$ -module  $\mathcal{M}$  one could avoid derived categories considering  $\text{char}(\mathcal{M}_0)$  instead of  $\text{char}(\text{gr}_\hbar \mathcal{M})$ . The next lemma shows that these definitions are still compatible for  $\mathcal{M} \in \text{D}_{\text{coh}}^b(\mathcal{D}_X^\hbar)$ .

**Lemma 3.7.** *For  $\mathcal{M} \in \text{D}_{\text{coh}}^b(\mathcal{D}_X^\hbar)$  one has*

$$\bigcup_i \text{char}(H^i(\text{gr}_\hbar \mathcal{M})) = \bigcup_i \text{char}((H^i \mathcal{M})_0).$$

*Proof.* By additivity of char, the short exact sequence

$$(3.8) \quad 0 \rightarrow (H^i \mathcal{M})_0 \rightarrow H^i(\text{gr}_\hbar \mathcal{M}) \rightarrow {}_0(H^{i+1} \mathcal{M}) \rightarrow 0$$

from [13, Lemma 1.4.2] induces the relations

$$\begin{aligned} \text{char}((H^i \mathcal{M})_0) &\subset \text{char}(H^i(\text{gr}_\hbar \mathcal{M})), \\ \text{char}(H^i(\text{gr}_\hbar \mathcal{M})) &= \text{char}((H^i \mathcal{M})_0) \cup \text{char}({}_0(H^{i+1} \mathcal{M})). \end{aligned}$$

One concludes by noticing that (3.7) gives

$$\text{char}({}_0(H^{i+1} \mathcal{M})) \subset \text{char}((H^{i+1} \mathcal{M})_0). \quad \square$$

**Proposition 3.8.** *Let  $\mathcal{M} \in \text{Mod}(\mathcal{D}_X^\hbar)$  be an  $\hbar$ -torsion module. Then  $\mathcal{M}$  is coherent as a  $\mathcal{D}_X^\hbar$ -module if and only if it is coherent as a  $\mathcal{D}_X$ -module, and in this case one has  $\text{char}_\hbar(\mathcal{M}) = \text{char}(\mathcal{M})$ .*

*Proof.* As in the proof of Lemma 3.5 we assume that  $\hbar^N \mathcal{M} = 0$  for some  $N \in \mathbb{N}$ . Since coherence is preserved by extension and since the characteristic varieties of  $\mathcal{D}_X^\hbar$ -modules and  $\mathcal{D}_X$ -modules are additive, we can argue by induction on  $N$  using the exact sequence (3.6). We are thus reduced to the case  $N = 1$ , where  $\mathcal{M} = \mathcal{M}_0$  and the statement becomes obvious.  $\square$

From (3.2) we obtain

**Proposition 3.9.** *For  $\mathcal{N} \in \text{D}_{\text{coh}}^b(\mathcal{D}_X)$  one has  $\text{char}_\hbar(\mathcal{N}^\hbar) = \text{char}(\mathcal{N})$ .*

**Holonomic modules.** Recall that a coherent  $\mathcal{D}_X$ -module (or an object of the derived category) is called *holonomic* if its characteristic variety is isotropic. We refer e.g. to [7, Chapter 5] for the notion of regularity.

**Definition 3.10.** We say that  $\mathcal{M} \in \text{D}_{\text{coh}}^b(\mathcal{D}_X^\hbar)$  is holonomic, or regular holonomic, if so is  $\text{gr}_\hbar(\mathcal{M})$ . We denote by  $\text{D}_{\text{hol}}^b(\mathcal{D}_X^\hbar)$  the full triangulated subcategory of  $\text{D}_{\text{coh}}^b(\mathcal{D}_X^\hbar)$  of holonomic objects and by  $\text{D}_{\text{rh}}^b(\mathcal{D}_X^\hbar)$  the full triangulated subcategory of regular holonomic objects.

Note that a coherent  $\mathcal{D}_X^{\hbar}$ -module is holonomic if and only if its characteristic variety is isotropic.

**Example 3.11.** Let  $\mathcal{N}$  be a regular holonomic  $\mathcal{D}_X$ -module. Then

- (i)  $\mathcal{N}$  itself, considered as a  $\mathcal{D}_X^{\hbar}$ -module, is regular holonomic, as follows from the isomorphism  $\mathrm{gr}_{\hbar} \mathcal{N} \simeq \mathcal{N} \oplus \mathcal{N}[1]$ ;
- (ii)  $\mathcal{N}^{\hbar}$  is a regular holonomic  $\mathcal{D}_X^{\hbar}$ -module, as follows from the isomorphism  $\mathrm{gr}_{\hbar} \mathcal{N}^{\hbar} \simeq \mathcal{N}$ . In particular,  $\mathcal{O}_X^{\hbar}$  is regular holonomic.

**Remark 3.12.** We denote by  $\mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)$  the category of regular holonomic  $\mathcal{D}_X$ -modules and by  $\mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X^{\hbar})$  the subcategory of  $\mathrm{Mod}(\mathcal{D}_X^{\hbar})$  of regular holonomic objects in the above sense. The proofs of Lemma 3.5 and Proposition 3.6 adapt to the notion of regular holonomy and give the following results:

- (i) For  $\mathcal{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X^{\hbar})$  an  $\hbar$ -torsion module,

$$\mathcal{M} \in \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X^{\hbar}) \Leftrightarrow \mathcal{M}_0 \in \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X) \Leftrightarrow {}_0\mathcal{M} \in \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X).$$

- (ii) For  $\mathcal{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X^{\hbar})$ ,

$$\mathcal{M} \in \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X^{\hbar}) \Leftrightarrow \mathcal{M}_0 \in \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X).$$

In this case,  ${}_0\mathcal{M} \in \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)$ .

Now for  $\mathcal{M} \in \mathrm{D}_{\mathrm{coh}}^{\mathrm{b}}(\mathcal{D}_X^{\hbar})$  the exact sequence (3.8) shows that, for any  $i$ ,

$$H^i(\mathrm{gr}_{\hbar} \mathcal{M}) \in \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X) \Leftrightarrow (H^i \mathcal{M})_{0,0} \in \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X).$$

By the above we deduce that  $\mathcal{M} \in \mathrm{D}_{\mathrm{rh}}^{\mathrm{b}}(\mathcal{D}_X^{\hbar})$  if and only if  $(H^i \mathcal{M})_0 \in \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X)$  for all  $i$ . This is again equivalent to  $H^i \mathcal{M} \in \mathrm{Mod}_{\mathrm{rh}}(\mathcal{D}_X^{\hbar})$  for all  $i$ .

**Propagation.** Denote by  $\mathrm{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(\mathbb{C}_X^{\hbar})$  the full triangulated subcategory of  $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_X^{\hbar})$  consisting of objects with  $\mathbb{C}$ -constructible cohomology over the ring  $\mathbb{C}^{\hbar}$ .

**Theorem 3.13.** Let  $\mathcal{M}, \mathcal{N} \in \mathrm{D}_{\mathrm{coh}}^{\mathrm{b}}(\mathcal{D}_X^{\hbar})$ . Then

$$\mathrm{SS}(\mathrm{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}, \mathcal{N})) = \mathrm{SS}(\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathrm{gr}_{\hbar}(\mathcal{M}), \mathrm{gr}_{\hbar}(\mathcal{N}))).$$

If moreover  $\mathcal{M}$  and  $\mathcal{N}$  are holonomic, then  $\mathrm{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}, \mathcal{N})$  is an object of  $\mathrm{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(\mathbb{C}_X^{\hbar})$ .

*Proof.* Set  $F = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}, \mathcal{N})$ . By Theorem 1.9 and Proposition 1.5,  $F$  is cohomologically  $\hbar$ -complete. Hence  $\mathrm{SS}(F) = \mathrm{SS}(\mathrm{gr}_{\hbar}(F))$  by Proposition 1.15. If moreover  $\mathcal{M}$  and  $\mathcal{N}$  are holonomic, then  $\mathrm{gr}_{\hbar} F$  is  $\mathbb{C}$ -constructible. The equality  $\mathrm{SS}(F) = \mathrm{SS}(\mathrm{gr}_{\hbar}(F))$  implies that  $F$  is weakly  $\mathbb{C}$ -constructible. Moreover, the

finiteness of the stalks  $\mathrm{gr}_{\hbar}(F)_x \simeq \mathrm{gr}_{\hbar}(F_x)$  over  $\mathbb{C}$  implies the finiteness of  $F_x$  over  $\mathbb{C}^{\hbar}$  by Theorem 1.11 applied with  $X = \{\mathrm{pt}\}$  and  $\mathcal{A} = \mathbb{C}^{\hbar}$ .  $\square$

Applying Theorem 3.13, and [9, Theorem 11.3.3], we get:

**Corollary 3.14.** *Let  $\mathcal{M} \in \mathrm{D}_{\mathrm{coh}}^{\mathrm{b}}(\mathcal{D}_X^{\hbar})$ . Then*

$$\mathrm{SS}(\mathrm{Sol}_{\hbar}(\mathcal{M})) = \mathrm{SS}(\mathrm{DR}_{\hbar}(\mathcal{M})) = \mathrm{char}_{\hbar}(\mathcal{M}).$$

*If moreover  $\mathcal{M}$  is holonomic, then  $\mathrm{Sol}_{\hbar}(\mathcal{M})$  and  $\mathrm{DR}_{\hbar}(\mathcal{M})$  belong to  $\mathrm{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(\mathbb{C}_X^{\hbar})$ .*

**Theorem 3.15.** *Let  $\mathcal{M} \in \mathrm{D}_{\mathrm{hol}}^{\mathrm{b}}(\mathcal{D}_X^{\hbar})$ . Then there is a natural isomorphism in  $\mathrm{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(\mathbb{C}_X^{\hbar})$*

$$(3.9) \quad \mathrm{Sol}_{\hbar}(\mathcal{M}) \simeq \mathrm{D}'_{\hbar}(\mathrm{DR}_{\hbar}(\mathcal{M})).$$

*Proof.* The natural  $\mathbb{C}^{\hbar}$ -linear morphism

$$\mathrm{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{O}_X^{\hbar}, \mathcal{M}) \otimes_{\mathbb{C}_X^{\hbar}}^{\mathrm{L}} \mathrm{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}, \mathcal{O}_X^{\hbar}) \rightarrow \mathrm{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{O}_X^{\hbar}, \mathcal{O}_X^{\hbar}) \simeq \mathbb{C}_X^{\hbar}$$

induces the morphism in  $\mathrm{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(\mathbb{C}_X^{\hbar})$

$$(3.10) \quad \alpha: \mathrm{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}, \mathcal{O}_X^{\hbar}) \rightarrow \mathrm{D}'_{\hbar}(\mathrm{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{O}_X^{\hbar}, \mathcal{M})).$$

(Note that, choosing  $\mathcal{M} = \mathcal{D}_X^{\hbar}$ , this morphism defines the morphism  $\mathcal{O}_X^{\hbar} \rightarrow \mathrm{D}'_{\hbar}(\Omega_X^{\hbar}[-d_X])$ .) The morphism (3.10) induces an isomorphism

$$\mathrm{gr}_{\hbar}(\alpha): \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathrm{gr}_{\hbar}(\mathcal{M}), \mathcal{O}_X) \rightarrow \mathrm{D}'(\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathrm{gr}_{\hbar}(\mathcal{M}))).$$

It is thus an isomorphism by Corollary 1.17.  $\square$

#### §4. Formal extension of tempered functions

Let us start by reviewing after [11, Chapter 7] the construction of the sheaves of tempered distributions and of  $C^{\infty}$ -functions with temperate growth on the subanalytic site.

Let  $X$  be a real analytic manifold, and  $U$  an open subset. One says that a function  $f \in \mathcal{C}_X^{\infty}(U)$  has *polynomial growth* at  $p \in X$  if, for a local coordinate system  $(x_1, \dots, x_n)$  around  $p$ , there exist a sufficiently small compact neighborhood  $K$  of  $p$  and a positive integer  $N$  such that

$$\sup_{x \in K \cap U} (\mathrm{dist}(x, K \setminus U))^N |f(x)| < \infty.$$

One says that  $f$  is *tempered* at  $p$  if all its derivatives are of polynomial growth at  $p$ . One says that  $f$  is tempered if it is tempered at any point of  $X$ . One denotes

by  $\mathcal{C}_X^{\infty,t}(U)$  the  $\mathbb{C}$ -vector subspace of  $\mathcal{C}^\infty(U)$  consisting of tempered functions. It then follows from a theorem of Łojasiewicz that  $U \mapsto \mathcal{C}_X^{\infty,t}(U)$  ( $U \in \text{Op}_{X_{\text{sa}}}$ ) is a sheaf on  $X_{\text{sa}}$ . We denote it by  $\mathcal{C}_{X_{\text{sa}}}^{\infty,t}$  or simply  $\mathcal{C}_X^{\infty,t}$  if there is no risk of confusion.

**Lemma 4.1.** *One has  $H^j(U; \mathcal{C}_X^{\infty,t}) = 0$  for any  $U \in \text{Op}_{X_{\text{sa}}}$  and any  $j > 0$ .*

This result is well-known (see [10, Chapter 1]), but we recall its proof for the reader's convenience.

*Proof.* Consider the full subcategory  $\mathcal{J}$  of  $\text{Mod}(\mathbb{C}_{X_{\text{sa}}})$  whose objects are sheaves  $F$  such that for any pair  $U, V \in \text{Op}_{X_{\text{sa}}}$ , the Mayer–Vietoris sequence

$$0 \rightarrow F(U \cup V) \rightarrow F(U) \oplus F(V) \rightarrow F(U \cap V) \rightarrow 0$$

is exact. Let us check that this category is injective with respect to the functor  $\Gamma(U; \bullet)$ . The only non-obvious fact is that if  $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$  is an exact sequence and that  $F'$  belongs to  $\mathcal{J}$ , then  $F(U) \rightarrow F''(U)$  is surjective. Let  $t \in F''(U)$ . There exist a finite covering  $U = \bigcup_{i \in I} U_i$  and  $s_i \in F(U_i)$  whose image in  $F''(U_i)$  is  $t|_{U_i}$ . Then the proof goes by induction on the cardinality of  $I$  using the property of  $F'$  and standard arguments. To conclude, note that  $\mathcal{C}_X^{\infty,t}$  belongs to  $\mathcal{J}$  thanks to Łojasiewicz's result (see [14]).  $\square$

Let  $\mathcal{D}b_X$  be the sheaf of distributions on  $X$ . For  $U \in \text{Op}_{X_{\text{sa}}}$ , denote by  $\mathcal{D}b_X^t(U)$  the space of tempered distributions on  $U$ , defined by the exact sequence

$$0 \rightarrow \Gamma_{X \setminus U}(X; \mathcal{D}b_X) \rightarrow \Gamma(X; \mathcal{D}b_X) \rightarrow \mathcal{D}b_X^t(U) \rightarrow 0.$$

Again, it follows from a theorem of Łojasiewicz that  $U \mapsto \mathcal{D}b^t(U)$  is a sheaf on  $X_{\text{sa}}$ . We denote it by  $\mathcal{D}b_{X_{\text{sa}}}^t$  or simply  $\mathcal{D}b_X^t$  if there is no risk of confusion. The sheaf  $\mathcal{D}b_X^t$  is quasi-injective, that is, the functor  $\mathcal{H}om_{\mathbb{C}_{X_{\text{sa}}}}(\bullet, \mathcal{D}b_X^t)$  is exact in the category  $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$ . Moreover, for  $U \in \text{Op}_{X_{\text{sa}}}$ ,  $\mathcal{H}om_{\mathbb{C}_{X_{\text{sa}}}}(\mathbb{C}_U, \mathcal{D}b_X^t)$  is also quasi-injective and  $\text{R}\mathcal{H}om_{\mathbb{C}_{X_{\text{sa}}}}(\mathbb{C}_U, \mathcal{D}b_X^t)$  is concentrated in degree 0. Note that the sheaf

$$\Gamma_{[U]} \mathcal{D}b_X := \rho^{-1} \mathcal{H}om_{\mathbb{C}_{X_{\text{sa}}}}(\mathbb{C}_U, \mathcal{D}b_X^t)$$

is a  $\mathcal{C}_X^\infty$ -module, so that in particular  $\text{R}\Gamma(V; \Gamma_{[U]} \mathcal{D}b_X)$  is concentrated in degree 0 for  $V \subset X$  an open subset.

**Formal extensions.** By Proposition 2.5 the sheaves  $\mathcal{C}_X^{\infty,t}$ ,  $\mathcal{D}b_X^t$  and  $\Gamma_{[U]} \mathcal{D}b_X$  are acyclic for the functor  $(\bullet)^{\hbar}$ . We set

$$\mathcal{C}_X^{\infty,t,\hbar} := (\mathcal{C}_X^{\infty,t})^{\hbar}, \quad \mathcal{D}b_X^{t,\hbar} := (\mathcal{D}b_X^t)^{\hbar}, \quad \Gamma_{[U]} \mathcal{D}b_X^{\hbar} := (\Gamma_{[U]} \mathcal{D}b_X)^{\hbar}.$$

Note that, by Lemmas 2.3 and 2.9,

$$\Gamma_{[U]} \mathcal{D}b_X^{\hbar} \simeq \rho^{-1} \mathcal{H}om_{\mathbb{C}_{X_{\text{sa}}}}(\mathbb{C}_U, \mathcal{D}b_X^{t,\hbar}).$$

By Proposition 2.2 we get:

**Proposition 4.2.** *The sheaves  $\mathcal{C}_X^{\infty,t,\hbar}$ ,  $\mathcal{D}_X^{t,\hbar}$  and  $\Gamma_{[U]}\mathcal{D}_X^{\hbar}$  are cohomologically  $\hbar$ -complete.*

Now assume  $X$  is a complex manifold. Denote by  $\overline{X}$  the complex conjugate manifold and by  $X^{\mathbb{R}}$  the underlying real analytic manifold, identified with the diagonal of  $X \times \overline{X}$ . One defines the sheaf (in fact, an object of the derived category) of tempered holomorphic functions by

$$\mathcal{O}_X^t := R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}}}(\rho_!\mathcal{O}_{\overline{X}}, \mathcal{C}_X^{\infty,t}) \xrightarrow{\sim} R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}}}(\rho_!\mathcal{O}_{\overline{X}}, \mathcal{D}_X^t).$$

Here and below, we write  $\mathcal{C}_X^{\infty,t}$  and  $\mathcal{D}_X^t$  instead of  $\mathcal{C}_{X^{\mathbb{R}}}^{\infty,t}$  and  $\mathcal{D}_{X^{\mathbb{R}}}^t$ , respectively. We set

$$\mathcal{O}_X^{t,\hbar} := (\mathcal{O}_X^t)^{\text{R}\hbar},$$

a cohomologically  $\hbar$ -complete object of  $D^b(\mathbb{C}_{X_{\text{sa}}}^{\hbar})$ . By Lemma 2.3,

$$\mathcal{O}_X^{t,\hbar} \simeq R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}}}(\rho_!\mathcal{O}_{\overline{X}}, \mathcal{C}_X^{\infty,t,\hbar}) \xrightarrow{\sim} R\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}}}(\rho_!\mathcal{O}_{\overline{X}}, \mathcal{D}_X^{t,\hbar}).$$

Note that  $\text{gr}_{\hbar}(\mathcal{O}_X^{t,\hbar}) \simeq \mathcal{O}_X^t$  in  $D^b(\mathbb{C}_{X_{\text{sa}}})$ .

### §5. Riemann–Hilbert correspondence

Let  $X$  be a complex analytic manifold. Consider the functors

$$\text{TH}: D_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X) \rightarrow D_{\text{rh}}^b(\mathcal{D}_X)^{\text{op}}, \quad F \mapsto \rho^{-1}R\mathcal{H}om_{\mathbb{C}_{X_{\text{sa}}}}(\rho_*F, \mathcal{O}_X^t),$$

$$\text{TH}_{\hbar}: D_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^{\hbar}) \rightarrow D^b(\mathcal{D}_X^{\hbar})^{\text{op}}, \quad F \mapsto \rho^{-1}R\mathcal{H}om_{\mathbb{C}_{X_{\text{sa}}}^{\hbar}}(\rho_*F, \mathcal{O}_X^{t,\hbar}).$$

The classical Riemann–Hilbert correspondence of Kashiwara [6] states that the functors  $\text{Sol}$  and  $\text{TH}$  are equivalences of categories between  $D_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X)$  and  $D_{\text{rh}}^b(\mathcal{D}_X)^{\text{op}}$  quasi-inverse to each other. In order to obtain a similar statement for  $\mathbb{C}_X$  and  $\mathcal{D}_X$  replaced with  $\mathbb{C}_X^{\hbar}$  and  $\mathcal{D}_X^{\hbar}$ , respectively, we start by establishing some lemmas.

**Lemma 5.1.** *For  $\mathcal{M}, \mathcal{N} \in D_{\text{hol}}^b(\mathcal{D}_X^{\hbar})$  one has a natural isomorphism in  $D_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^{\hbar})$*

$$R\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{M}, \mathcal{N}) \xrightarrow{\sim} R\mathcal{H}om_{\mathbb{C}_X^{\hbar}}(\text{Sol}_{\hbar}(\mathcal{N}), \text{Sol}_{\hbar}(\mathcal{M})).$$

*Proof.* Applying the functor  $\text{gr}_{\hbar}$  to this morphism, we get an isomorphism by the classical Riemann–Hilbert correspondence. Then the result follows from Corollary 1.17 and Theorem 3.13.  $\square$

Note that there is an isomorphism in  $D^b(\mathcal{D}_X)$

$$(5.1) \quad \text{gr}_{\hbar}(\text{TH}_{\hbar}(F)) \simeq \text{TH}(\text{gr}_{\hbar}(F)).$$

**Lemma 5.2.** *The functor  $\mathrm{TH}_{\hbar}$  induces a functor*

$$(5.2) \quad \mathrm{TH}_{\hbar}: \mathrm{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^{\hbar}) \rightarrow \mathrm{D}_{\mathrm{rh}}^b(\mathcal{D}_X^{\hbar})^{\mathrm{op}}.$$

*Proof.* Let  $F \in \mathrm{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^{\hbar})$ . By (5.1) and the classical Riemann–Hilbert correspondence we know that  $\mathrm{gr}_{\hbar}(\mathrm{TH}_{\hbar}(F))$  is regular holonomic, and in particular coherent. It is thus left to prove that  $\mathrm{TH}_{\hbar}(F)$  is coherent. Note that our problem is of local nature.

We use the Dolbeault resolution of  $\mathcal{O}_X^{t,\hbar}$  with coefficients in  $\mathcal{D}_X^{t,\hbar}$  and we choose a resolution of  $F$  as given in Proposition A.2(i). We find that  $\mathrm{TH}_{\hbar}(F)$  is isomorphic to a bounded complex  $\mathcal{M}^{\bullet}$ , where the  $\mathcal{M}^i$  are locally finite sums of sheaves of the type  $\Gamma_{[U]} \mathcal{D}^{t,\hbar}$  with  $U \in \mathrm{Op}_{X_{\mathrm{sa}}}$ . It follows from Proposition 4.2 that  $\mathrm{TH}_{\hbar}(F)$  is cohomologically  $\hbar$ -complete, and we conclude by Theorem 1.11 with  $\mathcal{A} = \mathcal{D}_X^{\hbar}$ .  $\square$

**Lemma 5.3.** *We have  $\mathrm{R}\mathcal{H}om_{\rho_! \mathcal{D}_X^{\hbar}}(\rho_! \mathcal{O}_X^{\hbar}, \mathcal{O}_X^{t,\hbar}) \simeq \mathbb{C}_{X_{\mathrm{sa}}}^{\hbar}$ .*

*Proof.* This isomorphism is given by the sequence

$$\begin{aligned} \mathrm{R}\mathcal{H}om_{\rho_! \mathcal{D}_X^{\hbar}}(\rho_! \mathcal{O}_X^{\hbar}, \mathcal{O}_X^{t,\hbar}) &\simeq \mathrm{R}\mathcal{H}om_{\rho_! \mathcal{D}_X}(\rho_! \mathcal{O}_X, \mathcal{O}_X^{t,\hbar}) \simeq \mathrm{R}\mathcal{H}om_{\rho_! \mathcal{D}_X}(\rho_! \mathcal{O}_X, \mathcal{O}_X^t)^{\mathrm{Rh}} \\ &\simeq (\rho_* \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X))^{\mathrm{Rh}} \simeq (\mathbb{C}_{X_{\mathrm{sa}}})^{\mathrm{Rh}} \simeq \mathbb{C}_{X_{\mathrm{sa}}}^{\hbar}, \end{aligned}$$

where the first isomorphism is an extension of scalars, the second follows from Lemma 2.3 and the third is given by the adjunction between  $\rho_!$  and  $\rho^{-1}$ .  $\square$

**Theorem 5.4.** *The functors  $\mathrm{Sol}_{\hbar}$  and  $\mathrm{TH}_{\hbar}$  are equivalences of categories between  $\mathrm{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^{\hbar})$  and  $\mathrm{D}_{\mathrm{rh}}^b(\mathcal{D}_X^{\hbar})^{\mathrm{op}}$  quasi-inverse to each other.*

*Proof.* In view of Lemma 5.1, the functor  $\mathrm{Sol}_{\hbar}$  is fully faithful. It is then enough to show that  $\mathrm{Sol}_{\hbar}(\mathrm{TH}_{\hbar}(F)) \simeq F$  for  $F \in \mathrm{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^{\hbar})$ . By Theorem 3.15, this is equivalent to  $\mathrm{DR}_{\hbar}(\mathrm{TH}_{\hbar}(F)) \simeq D'_{\hbar}F$ . Since we already know by Lemma 5.2 that  $\mathrm{TH}_{\hbar}(F)$  is holonomic, we may use (3.9). We have the sequence of isomorphisms

$$\begin{aligned} \rho_* \mathrm{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{O}_X^{\hbar}, \mathrm{TH}_{\hbar}(F)) &= \rho_* \mathrm{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar}}(\mathcal{O}_X^{\hbar}, \rho^{-1} \mathrm{R}\mathcal{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}^{\hbar}}(\rho_* F, \mathcal{O}_X^{t,\hbar})) \\ &\simeq \mathrm{R}\mathcal{H}om_{\rho_! \mathcal{D}_X^{\hbar}}(\rho_! \mathcal{O}_X^{\hbar}, \mathrm{R}\mathcal{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}^{\hbar}}(\rho_* F, \mathcal{O}_X^{t,\hbar})) \\ &\simeq \mathrm{R}\mathcal{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}^{\hbar}}(\rho_* F, \mathrm{R}\mathcal{H}om_{\rho_! \mathcal{D}_X^{\hbar}}(\rho_! \mathcal{O}_X^{\hbar}, \mathcal{O}_X^{t,\hbar})) \\ &\simeq \mathrm{R}\mathcal{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}^{\hbar}}(\rho_* F, \mathbb{C}_{X_{\mathrm{sa}}}^{\hbar}) \simeq \mathrm{R}\mathcal{H}om_{\mathbb{C}_{X_{\mathrm{sa}}}^{\hbar}}(\rho_* F, \rho_* \mathbb{C}_X^{\hbar}) \simeq \rho_* D'_{\hbar}F, \end{aligned}$$

where we have used the adjunction between  $\rho_!$  and  $\rho^{-1}$ , the isomorphism of Lemma 5.3 and the commutation of  $\rho_*$  with  $\mathrm{R}\mathcal{H}om$ . One concludes by recalling the isomorphism of functors  $\rho^{-1} \rho_* \simeq \mathrm{id}$ .  $\square$

**$t$ -structure.** Recall the definition of the middle perversity  $t$ -structure for complex constructible sheaves. Let  $\mathbb{K}$  denote either the field  $\mathbb{C}$  or the ring  $\mathbb{C}^{\hbar}$ . For  $F \in \mathbf{D}_{\mathbb{C}\text{-c}}^b(\mathbb{K}_X)$ , we have  $F \in {}^p\mathbf{D}_{\mathbb{C}\text{-c}}^{\leq 0}(\mathbb{K}_X)$  if and only if

$$(5.3) \quad \forall i \in \mathbb{Z} \quad \dim \operatorname{supp} H^i(F) \leq d_X - i,$$

and  $F \in {}^p\mathbf{D}_{\mathbb{C}\text{-c}}^{\geq 0}(\mathbb{K}_X)$  if and only if, for any locally closed complex analytic subset  $S \subset X$ ,

$$(5.4) \quad H_S^i(F) = 0 \quad \text{for all } i < d_X - \dim(S).$$

One denotes by  $\operatorname{Perv}(\mathbb{K}_X)$  the heart of this  $t$ -structure.

With the above convention, the de Rham functor

$$\operatorname{DR}: \mathbf{D}_{\operatorname{hol}}^b(\mathcal{D}_X) \rightarrow {}^p\mathbf{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X)$$

is  $t$ -exact, when  $\mathbf{D}_{\operatorname{hol}}^b(\mathcal{D}_X)$  is equipped with the natural  $t$ -structure.

**Theorem 5.5.** *The de Rham functor  $\operatorname{DR}_{\hbar}: \mathbf{D}_{\operatorname{hol}}^b(\mathcal{D}_X^{\hbar}) \rightarrow {}^p\mathbf{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^{\hbar})$  is  $t$ -exact, and induces a  $t$ -exact equivalence between  $\mathbf{D}_{\operatorname{rh}}^b(\mathcal{D}_X^{\hbar})$  and  ${}^p\mathbf{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^{\hbar})$ . In particular, it induces an equivalence between  $\operatorname{Mod}_{\operatorname{rh}}(\mathcal{D}_X^{\hbar})$  and  $\operatorname{Perv}(\mathbb{C}_X^{\hbar})$ .*

*Proof.* (i) Let  $\mathcal{M} \in \mathbf{D}_{\operatorname{hol}}^{\leq 0}(\mathcal{D}_X^{\hbar})$ . Let us prove that  $\operatorname{DR}_{\hbar}\mathcal{M} \in {}^p\mathbf{D}_{\mathbb{C}\text{-c}}^{\leq 0}(\mathbb{C}_X^{\hbar})$ . Since  $\operatorname{DR}_{\hbar}\mathcal{M}$  is constructible, by Proposition 1.19 it is enough to check (5.3) for  $\operatorname{gr}_{\hbar}(\operatorname{DR}_{\hbar}\mathcal{M}) \simeq \operatorname{DR}(\operatorname{gr}_{\hbar}\mathcal{M})$ . In other words, it is enough to check that  $\operatorname{DR}(\operatorname{gr}_{\hbar}\mathcal{M}) \in {}^p\mathbf{D}_{\mathbb{C}\text{-c}}^{\leq 0}(\mathbb{C}_X)$ . Since  $\operatorname{gr}_{\hbar}\mathcal{M} \in \mathbf{D}_{\operatorname{hol}}^{\leq 0}(\mathcal{D}_X)$ , this result follows from the  $t$ -exactness of the functor  $\operatorname{DR}$ .

(ii) Let  $\mathcal{M} \in \mathbf{D}_{\operatorname{hol}}^{\geq 0}(\mathcal{D}_X^{\hbar})$ . Let us prove that  $\operatorname{DR}_{\hbar}\mathcal{M} \in {}^p\mathbf{D}_{\mathbb{C}\text{-c}}^{\geq 0}(\mathbb{C}_X^{\hbar})$ . We set  $\mathcal{N} = (H^0\mathcal{M})_{\hbar\text{-tor}}$ . We have a morphism  $u: \mathcal{N} \rightarrow \mathcal{M}$  induced by  $H^0\mathcal{M} \rightarrow \mathcal{M}$  and we let  $\mathcal{M}'$  be the mapping cone of  $u$ . We have a distinguished triangle

$$\operatorname{DR}_{\hbar}\mathcal{N} \rightarrow \operatorname{DR}_{\hbar}\mathcal{M} \rightarrow \operatorname{DR}_{\hbar}\mathcal{M}' \xrightarrow{+1}$$

so that it is enough to show that  $\operatorname{DR}_{\hbar}\mathcal{N}$  and  $\operatorname{DR}_{\hbar}\mathcal{M}'$  belong to  ${}^p\mathbf{D}_{\mathbb{C}\text{-c}}^{\geq 0}(\mathbb{C}_X^{\hbar})$ .

(ii-a) By Propositions 3.6(ii) and 3.8,  $\mathcal{N}$  is holonomic as a  $\mathcal{D}_X$ -module. Hence  $\operatorname{DR}_{\hbar}\mathcal{N} \simeq \operatorname{DR}\mathcal{N}$  is a perverse sheaf (over  $\mathbb{C}$ ) and satisfies (5.4). Since (5.4) does not depend on the coefficient ring,  $\operatorname{DR}_{\hbar}\mathcal{N} \in {}^p\mathbf{D}_{\mathbb{C}\text{-c}}^{\geq 0}(\mathbb{C}_X^{\hbar})$ .

(ii-b) We note that  $H^0\mathcal{M}' \simeq (H^0\mathcal{M})_{\hbar\text{-tf}}$ . Hence by Proposition 1.14,  $\operatorname{gr}_{\hbar}\mathcal{M}' \in \mathbf{D}_{\operatorname{hol}}^{\geq 0}(\mathcal{D}_X)$  and  $\operatorname{DR}(\operatorname{gr}_{\hbar}\mathcal{M}') \in {}^p\mathbf{D}_{\mathbb{C}\text{-c}}^{\geq 0}(\mathbb{C}_X)$ , that is,  $\operatorname{DR}(\operatorname{gr}_{\hbar}\mathcal{M}')$  satisfies (5.4). Let  $S \subset X$  be a locally closed complex subanalytic subset. We have

$$\operatorname{R}\Gamma_S(\operatorname{DR}(\operatorname{gr}_{\hbar}\mathcal{M}')) \simeq \operatorname{gr}_{\hbar}(\operatorname{R}\Gamma_S(\operatorname{DR}_{\hbar}\mathcal{M}'))$$

and it follows from Proposition 1.19 that  $\operatorname{DR}_{\hbar}\mathcal{M}'$  also satisfies (5.4) and thus belongs to  ${}^p\mathbf{D}_{\mathbb{C}\text{-c}}^{\geq 0}(\mathbb{C}_X^{\hbar})$ .

(iii) Consider the restriction  $DR_{\hbar} : D_{\text{rh}}^b(\mathcal{D}_X^{\hbar}) \rightarrow {}^pD_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^{\hbar})$  to regular holonomic complexes. In view of Lemma A.1, it follows from Theorems 5.4 and 3.15 that the functor  $\text{TH}_{\hbar} \circ D'_{\hbar}$  is a quasi-inverse to  $DR_{\hbar}$ . As quasi-inverse to a  $t$ -exact functor,  $\text{TH}_{\hbar} \circ D'_{\hbar}$  is also  $t$ -exact. Thus  $DR_{\hbar}$  is a  $t$ -exact equivalence, and it induces an equivalence between the respective hearts, i.e. between  $\text{Mod}_{\text{rh}}(\mathcal{D}_X^{\hbar})$  and  $\text{Perv}(\mathbb{C}_X^{\hbar})$ .  $\square$

**§6. Duality and  $\hbar$ -torsion**

The duality functors  $\mathbb{D}$  on  $D_{\text{rh}}(\mathcal{D}_X)$  and  $D'$  on  ${}^pD_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X)$  are  $t$ -exact. We will discuss here the finer  $t$ -structures needed in order to obtain a similar result when replacing  $\mathbb{C}_X$  and  $\mathcal{D}_X$  by their formal extensions  $\mathbb{C}_X^{\hbar}$  and  $\mathcal{D}_X^{\hbar}$ .

Following [2, Chapter I.2], let us start by recalling some facts related to torsion pairs and  $t$ -structures. We need in particular Proposition 6.2 below, which can also be found in [3].

**Definition 6.1.** Let  $\mathcal{C}$  be an abelian category. A *torsion pair* on  $\mathcal{C}$  is a pair  $(\mathcal{C}_{\text{tor}}, \mathcal{C}_{\text{tf}})$  of full subcategories such that

- (i) for all objects  $T$  in  $\mathcal{C}_{\text{tor}}$  and  $F$  in  $\mathcal{C}_{\text{tf}}$ , we have  $\text{Hom}_{\mathcal{C}}(T, F) = 0$ ,
- (ii) for any object  $M$  in  $\mathcal{C}$ , there are objects  $M_{\text{tor}}$  in  $\mathcal{C}_{\text{tor}}$  and  $M_{\text{tf}}$  in  $\mathcal{C}_{\text{tf}}$  and a short exact sequence  $0 \rightarrow M_{\text{tor}} \rightarrow M \rightarrow M_{\text{tf}} \rightarrow 0$ .

**Proposition 6.2.** Let  $D$  be a triangulated category endowed with a  $t$ -structure  $({}^pD^{\leq 0}, {}^pD^{\geq 0})$ . Let us denote its heart by  $\mathcal{C}$  and its cohomology functors by  ${}^pH^i : D \rightarrow \mathcal{C}$ . Suppose that  $\mathcal{C}$  is endowed with a torsion pair  $(\mathcal{C}_{\text{tor}}, \mathcal{C}_{\text{tf}})$ . Then we can define a new  $t$ -structure  $(\pi D^{\leq 0}, \pi D^{\geq 0})$  on  $D$  by setting

$$\pi D^{\leq 0} = \{M \in {}^pD^{\leq 1} : {}^pH^1(M) \in \mathcal{C}_{\text{tor}}\}, \quad \pi D^{\geq 0} = \{M \in {}^pD^{\geq 0} : {}^pH^0(M) \in \mathcal{C}_{\text{tf}}\}.$$

With the notation of Definition 3.2, there is a natural torsion pair attached to  $\text{Mod}(\mathcal{D}_X^{\hbar})$  given by the full subcategories

$$\text{Mod}(\mathcal{D}_X^{\hbar})_{\hbar\text{-tor}} = \{\mathcal{M} : \mathcal{M}_{\hbar\text{-tor}} \xrightarrow{\sim} \mathcal{M}\}, \quad \text{Mod}(\mathcal{D}_X^{\hbar})_{\hbar\text{-tf}} = \{\mathcal{M} : \mathcal{M} \xrightarrow{\sim} \mathcal{M}_{\hbar\text{-tf}}\}.$$

**Definition 6.3.** (a) We call the torsion pair on  $\text{Mod}(\mathcal{D}_X^{\hbar})$  defined above, the  $\hbar$ -torsion pair.

- (b) We denote by  $(D^{\leq 0}(\mathcal{D}_X^{\hbar}), D^{\geq 0}(\mathcal{D}_X^{\hbar}))$  the natural  $t$ -structure on  $D(\mathcal{D}_X^{\hbar})$ .
- (c) We denote by  $({}^tD^{\leq 0}(\mathcal{D}_X^{\hbar}), {}^tD^{\geq 0}(\mathcal{D}_X^{\hbar}))$  the  $t$ -structure on  $D^b(\mathcal{D}_X^{\hbar})$  associated via Proposition 6.2 with the  $\hbar$ -torsion pair on  $\text{Mod}(\mathcal{D}_X^{\hbar})$ .

Proposition 1.14 implies the following equivalences for  $\mathcal{M} \in \mathbf{D}_{\text{coh}}^{\text{b}}(\mathcal{D}_X^{\hbar})$ :

$$(6.1) \quad \mathcal{M} \in {}^t\mathbf{D}^{\geq 0}(\mathcal{D}_X^{\hbar}) \Leftrightarrow \text{gr}_{\hbar}\mathcal{M} \in \mathbf{D}^{\geq 0}(\mathcal{D}_X),$$

$$(6.2) \quad \mathcal{M} \in \mathbf{D}^{\leq 0}(\mathcal{D}_X^{\hbar}) \Leftrightarrow \text{gr}_{\hbar}\mathcal{M} \in \mathbf{D}^{\leq 0}(\mathcal{D}_X).$$

**Proposition 6.4.** *Let  $\mathcal{M}$  be a holonomic  $\mathcal{D}_X^{\hbar}$ -module.*

- (i) *If  $\mathcal{M}$  has no  $\hbar$ -torsion, then  $\mathbb{D}_{\hbar}\mathcal{M}$  is concentrated in degree 0 and has no  $\hbar$ -torsion.*
- (ii) *If  $\mathcal{M}$  is an  $\hbar$ -torsion module, then  $\mathbb{D}_{\hbar}\mathcal{M}$  is concentrated in degree 1 and is an  $\hbar$ -torsion module.*

*Proof.* By (1.2) we have  $\text{gr}_{\hbar}(\mathbb{D}_{\hbar}\mathcal{M}) \simeq \mathbb{D}(\text{gr}_{\hbar}\mathcal{M})$ . Since  $\text{gr}_{\hbar}\mathcal{M}$  is concentrated in degrees 0 and  $-1$ , with holonomic cohomology,  $\mathbb{D}(\text{gr}_{\hbar}\mathcal{M})$  is concentrated in degrees 0 and 1. By Proposition 1.14,  $\mathbb{D}_{\hbar}\mathcal{M}$  itself is concentrated in degrees 0 and 1 and  $H^0(\mathbb{D}_{\hbar}\mathcal{M})$  has no  $\hbar$ -torsion.

(i) The short exact sequence

$$0 \rightarrow \mathcal{M} \xrightarrow{\hbar} \mathcal{M} \rightarrow \mathcal{M}/\hbar\mathcal{M} \rightarrow 0$$

induces the long exact sequence

$$\dots \rightarrow H^1(\mathbb{D}_{\hbar}(\mathcal{M}/\hbar\mathcal{M})) \rightarrow H^1(\mathbb{D}_{\hbar}\mathcal{M}) \xrightarrow{\hbar} H^1(\mathbb{D}_{\hbar}\mathcal{M}) \rightarrow 0.$$

By Nakayama's lemma  $H^1(\mathbb{D}_{\hbar}\mathcal{M}) = 0$  as required.

(ii) Since  $\mathcal{M}$  is locally annihilated by some power of  $\hbar$ , the cohomology groups  $H^i(\mathbb{D}_{\hbar}\mathcal{M})$  also are  $\hbar$ -torsion modules. As  $H^0(\mathbb{D}_{\hbar}\mathcal{M})$  has no  $\hbar$ -torsion, we get  $H^0(\mathbb{D}_{\hbar}\mathcal{M}) = 0$ . □

**Theorem 6.5.** *The duality functor  $\mathbb{D}_{\hbar}: \mathbf{D}_{\text{hol}}^{\text{b}}(\mathcal{D}_X^{\hbar})^{\text{op}} \rightarrow {}^t\mathbf{D}_{\text{hol}}^{\text{b}}(\mathcal{D}_X^{\hbar})$  is  $t$ -exact. In other words,  $\mathbb{D}_{\hbar}$  interchanges  $\mathbf{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_X^{\hbar})$  with  ${}^t\mathbf{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_X^{\hbar})$  and  $\mathbf{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_X^{\hbar})$  with  ${}^t\mathbf{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_X^{\hbar})$ .*

*Proof.* (i) Let us first prove, for  $\mathcal{M} \in \mathbf{D}_{\text{hol}}^{\text{b}}(\mathcal{D}_X^{\hbar})$ ,

$$(6.3) \quad \mathcal{M} \in \mathbf{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_X^{\hbar}) \Leftrightarrow \mathbb{D}_{\hbar}(\mathcal{M}) \in {}^t\mathbf{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_X^{\hbar}).$$

By (1.2) we have  $\text{gr}_{\hbar}(\mathbb{D}_{\hbar}\mathcal{M}) \simeq \mathbb{D}(\text{gr}_{\hbar}\mathcal{M})$  and we know that the analog of (6.3) holds true for  $\mathcal{D}_X$ -modules:

$$\mathcal{N} \in \mathbf{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_X) \Leftrightarrow \mathbb{D}(\mathcal{N}) \in \mathbf{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_X).$$

Hence (6.3) follows easily from (6.1) and (6.2).

(ii) We recall the general fact for a  $t$ -structure  $(\mathcal{D}, \mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$  and  $A \in \mathcal{D}$ :

$$\begin{aligned} A \in \mathcal{D}^{\leq 0} &\Leftrightarrow \text{Hom}(A, B) = 0 \text{ for any } B \in \mathcal{D}^{\geq 1}, \\ A \in \mathcal{D}^{\geq 0} &\Leftrightarrow \text{Hom}(B, A) = 0 \text{ for any } B \in \mathcal{D}^{\leq -1}. \end{aligned}$$

Since  $\mathbb{D}_{\hbar}$  is an involutive equivalence of categories we deduce from (6.3) the dual statement:

$$\mathcal{M} \in \mathcal{D}_{\text{hol}}^{\geq 0}(\mathcal{D}_X^{\hbar}) \Leftrightarrow \mathbb{D}_{\hbar}(\mathcal{M}) \in {}^t\mathcal{D}_{\text{hol}}^{\leq 0}(\mathcal{D}_X^{\hbar}). \quad \square$$

**Remark 6.6.** The above result can be stated as follows in the language of quasi-abelian categories of [19]. We will follow the notation of [8, Chapter 2]. The category  $\mathcal{C} = \text{Mod}(\mathcal{D}_X^{\hbar})_{\hbar\text{-tf}}$  is quasi-abelian. Hence its derived category has a natural generalized  $t$ -structure  $(\mathcal{D}^{\leq s}(\mathcal{C}), \mathcal{D}^{> s-1}(\mathcal{C}))_{s \in \frac{1}{2}\mathbb{Z}}$ . Note that  $\mathcal{D}^{[-1/2, 0]}(\mathcal{C})$  is equivalent to  $\text{Mod}(\mathcal{D}_X^{\hbar})$ , and  $\mathcal{D}^{[0, 1/2]}(\mathcal{C})$  is equivalent to the heart of  ${}^t\mathcal{D}^b(\mathcal{D}_X^{\hbar})$ . Then Theorem 6.5 states that the duality functor  $\mathbb{D}_{\hbar}$  is  $t$ -exact on  $\mathcal{D}_{\text{hol}}^b(\mathcal{C})$ .

Recall that  $\text{Perv}(\mathbb{C}_X^{\hbar})$  denotes the heart of the middle perversity  $t$ -structure on  $\mathcal{D}_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^{\hbar})$ . Consider the full subcategories of  $\text{Perv}(\mathbb{C}_X^{\hbar})$

$$\begin{aligned} \text{Perv}(\mathbb{C}_X^{\hbar})_{\hbar\text{-tor}} &= \{F : \text{locally } \hbar^N F = 0 \text{ for some } N \in \mathbb{N}\}, \\ \text{Perv}(\mathbb{C}_X^{\hbar})_{\hbar\text{-tf}} &= \{F : F \text{ has no non-zero subobjects in } \text{Perv}(\mathbb{C}_X^{\hbar})_{\hbar\text{-tor}}\}. \end{aligned}$$

**Lemma 6.7.** (i) *Let  $F \in \text{Perv}(\mathbb{C}_X^{\hbar})$ . Then the inductive system of sub-perverse sheaves  $\text{Ker}(\hbar^n : F \rightarrow F)$  is locally stationary.*

(ii) *The pair  $(\text{Perv}(\mathbb{C}_X^{\hbar})_{\hbar\text{-tor}}, \text{Perv}(\mathbb{C}_X^{\hbar})_{\hbar\text{-tf}})$  is a torsion pair.*

*Proof.* (i) Set  $\mathcal{M} = \mathbb{D}_{\hbar}\text{TH}_{\hbar}(F)$ . By the Riemann–Hilbert correspondence, one has  $\text{Ker}(\hbar^n : F \rightarrow F) \simeq \text{DR}_{\hbar}(\text{Ker}(\hbar^n : \mathcal{M} \rightarrow \mathcal{M}))$ . Since  $\mathcal{M}$  is coherent, the inductive system  $\text{Ker}(\hbar^n : \mathcal{M} \rightarrow \mathcal{M})$  is locally stationary. Hence so is the system  $\text{Ker}(\hbar^n : F \rightarrow F)$ .

(ii) By (i) it makes sense to define, for  $F \in \text{Perv}(\mathbb{C}_X^{\hbar})$ ,

$$F_{\hbar\text{-tor}} = \bigcup_n \text{Ker}(\hbar^n : F \rightarrow F), \quad F_{\hbar\text{-tf}} = F/F_{\hbar\text{-tor}}.$$

It is easy to check that  $F_{\hbar\text{-tor}} \in \text{Perv}(\mathbb{C}_X^{\hbar})_{\hbar\text{-tor}}$  and  $F_{\hbar\text{-tf}} \in \text{Perv}(\mathbb{C}_X^{\hbar})_{\hbar\text{-tf}}$ . Then property (ii) in Definition 6.1 is clear. For property (i) let  $u : F \rightarrow G$  be a morphism in  $\text{Perv}(\mathbb{C}_X^{\hbar})$  with  $F \in \text{Perv}(\mathbb{C}_X^{\hbar})_{\hbar\text{-tor}}$  and  $G \in \text{Perv}(\mathbb{C}_X^{\hbar})_{\hbar\text{-tf}}$ . Then  $\text{Im } u$  is also in  $\text{Perv}(\mathbb{C}_X^{\hbar})_{\hbar\text{-tor}}$  and so it is zero by the definition of  $\text{Perv}(\mathbb{C}_X^{\hbar})_{\hbar\text{-tf}}$ .  $\square$

Denote by  $(\pi\mathcal{D}_{\mathbb{C}\text{-c}}^{\leq 0}(\mathbb{C}_X^{\hbar}), \pi\mathcal{D}_{\mathbb{C}\text{-c}}^{\geq 0}(\mathbb{C}_X^{\hbar}))$  the  $t$ -structure on  $\mathcal{D}_{\mathbb{C}\text{-c}}(\mathbb{C}_X^{\hbar})$  induced by the perversity  $t$ -structure and this torsion pair as in Proposition 6.2. We also set  $\pi\text{Perv}(\mathbb{C}_X^{\hbar}) = \pi\mathcal{D}_{\mathbb{C}\text{-c}}^{\leq 0}(\mathbb{C}_X^{\hbar}) \cap \pi\mathcal{D}_{\mathbb{C}\text{-c}}^{\geq 0}(\mathbb{C}_X^{\hbar})$ .

**Theorem 6.8.** *There is a quasi-commutative diagram of  $t$ -exact functors*

$$\begin{array}{ccc} D_{\text{hol}}^b(\mathcal{D}_X^{\hbar})^{\text{op}} & \xrightarrow{\text{DR}_{\hbar}} & {}^pD_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^{\hbar})^{\text{op}} \\ \mathbb{D}_{\hbar} \downarrow & & \downarrow \mathbb{D}'_{\hbar} \\ {}^tD_{\text{hol}}^b(\mathcal{D}_X^{\hbar}) & \xrightarrow{\text{DR}_{\hbar}} & \pi D_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^{\hbar}) \end{array}$$

where the duality functors are equivalences of categories and the de Rham functors become equivalences when restricted to the subcategories of regular objects.

**Example 6.9.** Let  $X = \mathbb{C}$ ,  $U = X \setminus \{0\}$  and denote by  $j: U \hookrightarrow X$  the embedding. Let  $L$  be the local system on  $U$  with stalk  $\mathbb{C}^{\hbar}$  and monodromy  $1 + \hbar$ . The sheaf  $Rj_*L \simeq D'_h(j_!(D'_h L))$  is perverse for both  $t$ -structures, as is the sheaf  $H^0(Rj_*L) = j_*L \simeq j_!L$ . The sheaf  $H^1(Rj_*L) \simeq \mathbb{C}_{\{0\}}$  has  $\hbar$ -torsion. From the distinguished triangle  $j_*L \rightarrow Rj_*L \rightarrow \mathbb{C}_{\{0\}}[-1] \xrightarrow{+1}$ , one gets the short exact sequences

$$\begin{aligned} 0 \rightarrow j_*L \rightarrow Rj_*L \rightarrow \mathbb{C}_{\{0\}}[-1] \rightarrow 0 & \quad \text{in } \text{Perv}(\mathbb{C}_X^{\hbar}), \\ 0 \rightarrow \mathbb{C}_{\{0\}}[-2] \rightarrow j_*L \rightarrow Rj_*L \rightarrow 0 & \quad \text{in } \pi\text{Perv}(\mathbb{C}_X^{\hbar}). \end{aligned}$$

**§7.  $\mathcal{D}((\hbar))$ -modules**

Denote by

$$\mathbb{C}^{\hbar,\text{loc}} := \mathbb{C}((\hbar)) = \mathbb{C}[[\hbar^{-1}, \hbar]]$$

the field of Laurent series in  $\hbar$ , that is, the fraction field of  $\mathbb{C}^{\hbar}$ . Recall the exact functor

$$(7.1) \quad (\bullet)^{\text{loc}}: \text{Mod}(\mathbb{C}_X^{\hbar}) \rightarrow \text{Mod}(\mathbb{C}_X^{\hbar,\text{loc}}), \quad F \mapsto \mathbb{C}^{\hbar,\text{loc}} \otimes_{\mathbb{C}^{\hbar}} F,$$

and note that by [9, Proposition 5.4.14] one has the inclusion

$$(7.2) \quad \text{SS}(F^{\text{loc}}) \subset \text{SS}(F).$$

For  $G \in D^b(\mathbb{C}_X)$ , we write  $G^{\hbar,\text{loc}}$  instead of  $(G^{\hbar})^{\text{loc}}$ . We will consider in particular

$$\mathcal{O}_X^{\hbar,\text{loc}} = \mathcal{O}_X((\hbar)), \quad \mathcal{D}_X^{\hbar,\text{loc}} = \mathcal{D}_X((\hbar)).$$

**Lemma 7.1.** *Let  $\mathcal{M}$  be a coherent  $\mathcal{D}_X^{\hbar,\text{loc}}$ -module. Then  $\mathcal{M}$  is pseudo-coherent over  $\mathcal{D}_X^{\hbar}$ . In other words, if  $\mathcal{L} \subset \mathcal{M}$  is a finitely generated  $\mathcal{D}_X^{\hbar}$ -module, then  $\mathcal{L}$  is  $\mathcal{D}_X^{\hbar}$ -coherent.*

*Proof.* The proof follows from [7, Appendix, A1]. □

**Definition 7.2.** A lattice  $\mathcal{L}$  of a coherent  $\mathcal{D}_X^{\hbar,\text{loc}}$ -module  $\mathcal{M}$  is a coherent  $\mathcal{D}_X^{\hbar}$ -submodule of  $\mathcal{M}$  which generates it.

Since  $\mathcal{M}$  has no  $\hbar$ -torsion, none of its lattices has  $\hbar$ -torsion. In particular, one has  $\mathcal{M} \simeq \mathcal{L}^{\text{loc}}$  and  $\text{gr}_{\hbar} \mathcal{L} \simeq \mathcal{L}_0 = \mathcal{L}/\hbar\mathcal{L}$ .

It follows from Lemma 7.1 that lattices locally exist: for a local system of generators  $(m_1, \dots, m_N)$  of  $\mathcal{M}$ , define  $\mathcal{L}$  as the  $\mathcal{D}_X^{\hbar}$ -submodule with the same generators.

**Lemma 7.3.** *Let  $0 \rightarrow \mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}'' \rightarrow 0$  be an exact sequence of coherent  $\mathcal{D}_X^{\hbar, \text{loc}}$ -modules. Locally there exist lattices  $\mathcal{L}', \mathcal{L}, \mathcal{L}''$  of  $\mathcal{M}', \mathcal{M}, \mathcal{M}''$ , respectively, inducing an exact sequence of  $\mathcal{D}_X^{\hbar}$ -modules*

$$0 \rightarrow \mathcal{L}' \rightarrow \mathcal{L} \rightarrow \mathcal{L}'' \rightarrow 0.$$

*Proof.* Let  $\mathcal{L}$  be a lattice of  $\mathcal{M}$  and let  $\mathcal{L}''$  be its image in  $\mathcal{M}''$ . We set  $\mathcal{L}' := \mathcal{L} \cap \mathcal{M}'$ . These sub- $\mathcal{D}_X^{\hbar}$ -modules give rise to an exact sequence.

Since  $\mathcal{L}''$  is of finite type over  $\mathcal{D}_X^{\hbar}$ , it is a lattice of  $\mathcal{M}''$ . Let us show that  $\mathcal{L}'$  is a lattice of  $\mathcal{M}'$ . Being the kernel of a morphism  $\mathcal{L} \rightarrow \mathcal{L}''$  between coherent  $\mathcal{D}_X^{\hbar}$ -modules,  $\mathcal{L}'$  is coherent. To show that  $\mathcal{L}'$  generates  $\mathcal{M}'$ , note that any  $m' \in \mathcal{M}' \subset \mathcal{M}$  may be written as  $m' = \hbar^{-N}m$  for some  $N \geq 0$  and  $m \in \mathcal{L}$ . Hence  $m = \hbar^N m' \in \mathcal{M}' \cap \mathcal{L} = \mathcal{L}'$ .  $\square$

For an abelian category  $\mathcal{C}$ , we denote by  $K(\mathcal{C})$  its Grothendieck group. For an object  $M$  of  $\mathcal{C}$ , we denote by  $[M]$  its class in  $K(\mathcal{C})$ . We let  $\mathcal{K}(\mathcal{D}_X)$  be the sheaf on  $X$  associated to the presheaf

$$U \mapsto K(\text{Mod}_{\text{coh}}(\mathcal{D}_X|_U)).$$

We define  $\mathcal{K}(\mathcal{D}_X^{\hbar, \text{loc}})$  in the same way.

**Lemma 7.4.** *Let  $\mathcal{L}$  be a coherent  $\mathcal{D}_X^{\hbar}$ -module without  $\hbar$ -torsion. Then, for any  $i > 0$ , the  $\mathcal{D}_X$ -module  $\mathcal{L}/\hbar^i\mathcal{L}$  is coherent, and we have the equality  $[\mathcal{L}/\hbar^i\mathcal{L}] = i \cdot [\text{gr}_{\hbar}(\mathcal{L})]$  in  $K(\text{Mod}_{\text{coh}}(\mathcal{D}_X))$ .*

*Proof.* Since the functor  $(\bullet) \otimes_{\mathbb{C}^{\hbar}} \mathbb{C}^{\hbar}/\hbar^i\mathbb{C}^{\hbar}$  is right exact,  $\mathcal{L}/\hbar^i\mathcal{L}$  is a coherent  $\mathcal{D}_X$ -module. Since  $\mathcal{L}$  has no  $\hbar$ -torsion, multiplication by  $\hbar^i$  induces an isomorphism  $\mathcal{L}/\hbar\mathcal{L} \xrightarrow{\sim} \hbar^i\mathcal{L}/\hbar^{i+1}\mathcal{L}$ . We conclude by induction on  $i$  with the exact sequence

$$0 \rightarrow \hbar^i\mathcal{L}/\hbar^{i+1}\mathcal{L} \rightarrow \mathcal{L}/\hbar^{i+1}\mathcal{L} \rightarrow \mathcal{L}/\hbar^i\mathcal{L} \rightarrow 0. \quad \square$$

**Lemma 7.5.** *For  $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_X^{\hbar, \text{loc}})$ ,  $U \subset X$  an open set and  $\mathcal{L} \subset \mathcal{M}|_U$  a lattice of  $\mathcal{M}|_U$ , the class  $[\text{gr}_{\hbar}(\mathcal{L})] \in K(\text{Mod}_{\text{coh}}(\mathcal{D}_X|_U))$  only depends on  $\mathcal{M}$ . This defines a morphism of abelian sheaves  $\mathcal{K}(\mathcal{D}_X^{\hbar, \text{loc}}) \rightarrow \mathcal{K}(\mathcal{D}_X)$ .*

*Proof.* (i) We first prove that  $[\text{gr}_{\hbar}(\mathcal{L})]$  only depends on  $\mathcal{M}$ . We consider another lattice  $\mathcal{L}'$  of  $\mathcal{M}|_U$ . Since  $\mathcal{L}$  is a  $\mathcal{D}_X^{\hbar}$ -module of finite type, and  $\mathcal{L}'$  generates  $\mathcal{M}$ , there exists  $n > 1$  such that  $\mathcal{L} \subset \hbar^{-n}\mathcal{L}'$ . Similarly, there exists  $m > 1$  with

$\mathcal{L}' \subset \hbar^{-m} \mathcal{L}$ , so that we have the inclusions

$$\hbar^{m+n+2} \mathcal{L} \subset \hbar^{m+n+1} \mathcal{L} \subset \hbar^{m+1} \mathcal{L}' \subset \hbar^m \mathcal{L}' \subset \mathcal{L}.$$

Any inclusion  $A \subset B \subset C$  yields an identity  $[C/A] = [C/B] + [B/A]$  in the Grothendieck group, and we obtain in particular

$$\begin{aligned} [\hbar^m \mathcal{L}' / \hbar^{m+n+1} \mathcal{L}] &= [\hbar^m \mathcal{L}' / \hbar^{m+1} \mathcal{L}'] + [\hbar^{m+1} \mathcal{L}' / \hbar^{m+n+1} \mathcal{L}], \\ [\mathcal{L} / \hbar^{m+n+1} \mathcal{L}] &= [\mathcal{L} / \hbar^{m+1} \mathcal{L}'] + [\hbar^{m+1} \mathcal{L}' / \hbar^{m+n+1} \mathcal{L}], \\ [\mathcal{L} / \hbar^{m+n+2} \mathcal{L}] &= [\mathcal{L} / \hbar^{m+1} \mathcal{L}'] + [\hbar^{m+1} \mathcal{L}' / \hbar^{m+n+2} \mathcal{L}]. \end{aligned}$$

Note that we have isomorphisms of the type  $\hbar^k \mathcal{M}_1 / \hbar^k \mathcal{M}_2 \simeq \mathcal{M}_1 / \mathcal{M}_2$  for modules without  $\hbar$ -torsion. Then Lemma 7.4 and the above equalities give:

$$\begin{aligned} [\mathcal{L}' / \hbar^{n+1} \mathcal{L}] &= [\mathrm{gr}_{\hbar}(\mathcal{L}')] + [\mathcal{L}' / \hbar^n \mathcal{L}], \\ (m+n+1)[\mathrm{gr}_{\hbar}(\mathcal{L})] &= [\mathcal{L} / \hbar^{m+1} \mathcal{L}'] + [\mathcal{L}' / \hbar^n \mathcal{L}], \\ (m+n+2)[\mathrm{gr}_{\hbar}(\mathcal{L})] &= [\mathcal{L} / \hbar^{m+1} \mathcal{L}'] + [\mathcal{L}' / \hbar^{n+1} \mathcal{L}]. \end{aligned}$$

A suitable combination of these lines gives  $[\mathrm{gr}_{\hbar}(\mathcal{L})] = [\mathrm{gr}_{\hbar}(\mathcal{L}')]$ , as desired.

(ii) Now we consider an open subset  $V \subset X$  and  $\mathcal{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X^{\hbar, \mathrm{loc}}|_V)$ . We choose an open covering  $\{U_i\}_{i \in I}$  of  $V$  such that for each  $i \in I$ ,  $\mathcal{M}|_{U_i}$  admits a lattice, say  $\mathcal{L}^i$ . We have seen that  $[\mathrm{gr}_{\hbar}(\mathcal{L}^i)] \in \mathrm{K}(\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X|_{U_i}))$  only depends on  $\mathcal{M}$ . This implies that

$$[\mathrm{gr}_{\hbar}(\mathcal{L}^i)]|_{U_{i,j}} = [\mathrm{gr}_{\hbar}(\mathcal{L}^j)]|_{U_{i,j}} \quad \text{in } \mathrm{K}(\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X|_{U_{i,j}})).$$

Hence the  $[\mathrm{gr}_{\hbar}(\mathcal{L}^i)]$ 's define a section, say  $c(\mathcal{M})$ , of  $\mathcal{K}(\mathcal{D}_X)$  over  $V$ . By Lemma 7.3,  $c(\mathcal{M})$  only depends on the class  $[\mathcal{M}]$  in  $\mathrm{K}(\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X^{\hbar, \mathrm{loc}}|_V))$ , and  $\mathcal{M} \mapsto c(\mathcal{M})$  induces the morphism  $\mathcal{K}(\mathcal{D}_X^{\hbar, \mathrm{loc}}) \rightarrow \mathcal{K}(\mathcal{D}_X)$ .  $\square$

By Lemma 7.5, the following definition is correct.

**Definition 7.6.** The *characteristic variety* of a coherent  $\mathcal{D}_X^{\hbar, \mathrm{loc}}$ -module  $\mathcal{M}$  is defined by

$$\mathrm{char}_{\hbar, \mathrm{loc}}(\mathcal{M}) = \mathrm{char}_{\hbar}(\mathcal{L}),$$

for  $\mathcal{L} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X^{\hbar})$  a (local) lattice. For  $\mathcal{M} \in \mathrm{D}_{\mathrm{coh}}^{\mathrm{b}}(\mathcal{D}_X^{\hbar, \mathrm{loc}})$ , one sets  $\mathrm{char}_{\hbar, \mathrm{loc}}(\mathcal{M}) = \bigcup_j \mathrm{char}_{\hbar, \mathrm{loc}}(H^j(\mathcal{M}))$ .

**Proposition 7.7.** *The characteristic variety  $\mathrm{char}_{\hbar, \mathrm{loc}}$  is additive both on  $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X^{\hbar, \mathrm{loc}})$  and on  $\mathrm{D}_{\mathrm{coh}}^{\mathrm{b}}(\mathcal{D}_X^{\hbar, \mathrm{loc}})$ .*

*Proof.* This follows from Proposition 3.6(ii) and Lemma 7.3.  $\square$

Consider the functor

$$\mathrm{Sol}_{\hbar,\mathrm{loc}} : \mathrm{D}^b(\mathcal{D}_X^{\hbar,\mathrm{loc}})^{\mathrm{op}} \rightarrow \mathrm{D}^b(\mathbb{C}_X^{\hbar,\mathrm{loc}}), \quad \mathcal{M} \mapsto \mathrm{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar,\mathrm{loc}}}(\mathcal{M}, \mathcal{O}_X^{\hbar,\mathrm{loc}}).$$

**Proposition 7.8.** *Let  $\mathcal{M} \in \mathrm{D}_{\mathrm{coh}}^b(\mathcal{D}_X^{\hbar,\mathrm{loc}})$ . Then*

$$\mathrm{SS}(\mathrm{Sol}_{\hbar,\mathrm{loc}}(\mathcal{M})) \subset \mathrm{char}_{\hbar,\mathrm{loc}}(\mathcal{M}).$$

*Proof.* By dévissage, we can assume that  $\mathcal{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X^{\hbar,\mathrm{loc}})$ . Moreover, since the problem is local, we may assume that  $\mathcal{M}$  admits a lattice  $\mathcal{L}$ .

One has the isomorphism  $\mathrm{Sol}_{\hbar,\mathrm{loc}}(\mathcal{M}) \simeq \mathrm{R}\mathcal{H}om_{\mathcal{D}_X^{\hbar,\mathrm{loc}}}(\mathcal{L}, \mathcal{O}_X^{\hbar,\mathrm{loc}})$  by extension of scalars. Taking a local resolution of  $\mathcal{L}$  by free  $\mathcal{D}_X^{\hbar}$ -modules of finite type, we deduce that  $\mathrm{Sol}_{\hbar,\mathrm{loc}}(\mathcal{M}) \simeq F^{\mathrm{loc}}$  for  $F = \mathrm{Sol}_{\hbar}(\mathcal{L})$ . The statement follows by (7.2) and Corollary 3.14.  $\square$

One says that  $\mathcal{M}$  is *holonomic* if its characteristic variety is isotropic.

**Proposition 7.9.** *The functor  $\mathrm{Sol}_{\hbar,\mathrm{loc}}$  induces a functor*

$$\mathrm{Sol}_{\hbar,\mathrm{loc}} : \mathrm{D}_{\mathrm{hol}}^b(\mathcal{D}_X^{\hbar,\mathrm{loc}})^{\mathrm{op}} \rightarrow \mathrm{D}_{\mathrm{C-c}}^b(\mathbb{C}_X^{\hbar,\mathrm{loc}}).$$

*Proof.* By the same arguments and with the same notation as in the proof of Proposition 7.8, we reduce to the case  $\mathrm{Sol}_{\hbar,\mathrm{loc}}(\mathcal{M}) \simeq F^{\mathrm{loc}}$ , for  $F = \mathrm{Sol}_{\hbar}(\mathcal{L})$  and  $\mathcal{L}$  a lattice of  $\mathcal{M} \in \mathrm{Mod}_{\mathrm{hol}}(\mathcal{D}_X^{\hbar,\mathrm{loc}})$ . Hence  $\mathcal{L}$  is a holonomic  $\mathcal{D}_X^{\hbar}$ -module, and  $F \in \mathrm{D}_{\mathrm{C-c}}^b(\mathbb{C}_X^{\hbar})$ .  $\square$

**Remark 7.10.** In general the functor

$$\mathrm{Sol}_{\hbar,\mathrm{loc}} : \mathrm{D}_{\mathrm{hol}}^b(\mathcal{D}_X^{\hbar,\mathrm{loc}})^{\mathrm{op}} \rightarrow \mathrm{D}_{\mathrm{C-c}}^b(\mathbb{C}_X^{\hbar,\mathrm{loc}})$$

is not locally essentially surjective. In fact, consider the quasi-commutative diagram of categories

$$\begin{array}{ccc} \mathrm{D}_{\mathrm{hol}}^b(\mathcal{D}_X^{\hbar})^{\mathrm{op}} & \xrightarrow{\mathrm{Sol}_{\hbar}} & \mathrm{D}_{\mathrm{C-c}}^b(\mathbb{C}_X^{\hbar}) \\ (\cdot)^{\mathrm{loc}} \downarrow & & \downarrow (\cdot)^{\mathrm{loc}} \\ \mathrm{D}_{\mathrm{hol}}^b(\mathcal{D}_X^{\hbar,\mathrm{loc}})^{\mathrm{op}} & \xrightarrow{\mathrm{Sol}_{\hbar,\mathrm{loc}}} & \mathrm{D}_{\mathrm{C-c}}^b(\mathbb{C}_X^{\hbar,\mathrm{loc}}) \end{array}$$

By the local existence of lattices the left vertical arrow is locally essentially surjective. If  $\mathrm{Sol}_{\hbar,\mathrm{loc}}$  were also locally essentially surjective, so should be the right vertical arrow. The following example shows that it is not the case.

One can interpret this phenomenon by remarking that  $\mathrm{D}_{\mathrm{hol}}^b(\mathcal{D}_X^{\hbar,\mathrm{loc}})$  is equivalent to the localization of the category  $\mathrm{D}_{\mathrm{hol}}^b(\mathcal{D}_X^{\hbar})$  with respect to the morphism  $\hbar$ , in contrast to the category  $\mathrm{D}_{\mathrm{C-c}}^b(\mathbb{C}_X^{\hbar,\mathrm{loc}})$ .

**Example 7.11.** Let  $X = \mathbb{C}$ ,  $U = X \setminus \{0\}$  and denote by  $j: U \hookrightarrow X$  the embedding. Set  $F = Rj_!L$ , where  $L$  is the local system on  $U$  with stalk  $\mathbb{C}^{\hbar, \text{loc}}$  and monodromy  $\hbar$  around the origin. Since  $\hbar$  is not invertible in  $\mathbb{C}^{\hbar}$ , there is no  $F_0 \in D_{\mathbb{C}\text{-c}}^b(\mathbb{C}_X^{\hbar})$  such that  $F \simeq (F_0)^{\text{loc}}$ .

**§8. Links with deformation quantization**

In this last section, we shall briefly explain how the study of deformation quantization algebras on complex symplectic manifolds is related to  $\mathcal{D}_X^{\hbar}$ . We follow the terminology of [13].

The cotangent bundle  $\mathfrak{X} = T^*X$  to the complex manifold  $X$  has the structure of a complex symplectic manifold and is endowed with the  $\mathbb{C}^{\hbar}$ -algebra  $\widehat{\mathcal{W}}_{\mathfrak{X}}$ , a non-homogeneous version of the algebra of microdifferential operators. Its subalgebra  $\widehat{\mathcal{W}}_{\mathfrak{X}}(0)$  of operators of order at most zero is a deformation quantization algebra. In a system  $(x, u)$  of local symplectic coordinates,  $\widehat{\mathcal{W}}_{\mathfrak{X}}(0)$  is identified with the star algebra  $(\mathcal{O}_{\mathfrak{X}}^{\hbar}, \star)$  in which the star product is given by the Leibniz product

$$(8.1) \quad f \star g = \sum_{\alpha \in \mathbb{N}^n} \frac{\hbar^{|\alpha|}}{\alpha!} (\partial_u^{\alpha} f)(\partial_x^{\alpha} g) \quad \text{for } f, g \in \mathcal{O}_{\mathfrak{X}}.$$

In this section we will set for short  $\mathcal{A} := \widehat{\mathcal{W}}_{\mathfrak{X}}(0)$ , so that  $\mathcal{A}^{\text{loc}} \simeq \widehat{\mathcal{W}}_{\mathfrak{X}}$ . Note that  $\mathcal{A}$  satisfies Assumption 1.8.

Let us identify  $X$  with the zero section of the cotangent bundle  $\mathfrak{X}$ . Recall that  $X$  is a local model for any smooth Lagrangian submanifold of  $\mathfrak{X}$ , and that  $\mathcal{O}_X^{\hbar}$  is a local model of any simple  $\mathcal{A}$ -module along  $X$ . As  $\mathcal{O}_X^{\hbar}$  has both a  $\mathcal{D}_X^{\hbar}$ -module and an  $\mathcal{A}|_X$ -module structure, there are morphisms of  $\mathbb{C}^{\hbar}$ -algebras

$$(8.2) \quad \mathcal{D}_X^{\hbar} \rightarrow \text{End}_{\mathbb{C}^{\hbar}}(\mathcal{O}_X^{\hbar}) \leftarrow \mathcal{A}|_X.$$

**Lemma 8.1.** *The morphisms in (8.2) are injective and induce an embedding  $\mathcal{A}|_X \hookrightarrow \mathcal{D}_X^{\hbar}$ .*

*Proof.* Since the problem is local, we may choose a local symplectic coordinate system  $(x, u)$  on  $\mathfrak{X}$  such that  $X = \{u = 0\}$ . Then  $\mathcal{A}|_X$  is identified with  $\mathcal{O}_{\mathfrak{X}}^{\hbar}|_X$ . As the action of  $u_i$  on  $\mathcal{O}_X^{\hbar}$  is given by  $\hbar \partial_{x_i}$ , the morphism  $\mathcal{A}|_X \rightarrow \text{End}_{\mathbb{C}^{\hbar}}(\mathcal{O}_X^{\hbar})$  factors through  $\mathcal{D}_X^{\hbar}$ , and the induced morphism  $\mathcal{A}|_X \rightarrow \mathcal{D}_X^{\hbar}$  is described by

$$(8.3) \quad \sum_{i \in \mathbb{N}} f_i(x, u) \hbar^i \mapsto \sum_{j \in \mathbb{N}} \left( \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq j} \partial_u^{\alpha} f_{j-|\alpha|}(x, 0) \partial_x^{\alpha} \right) \hbar^j,$$

which is clearly injective. □

Consider the following subsheaves of  $\mathcal{D}_X^{\hbar}$ :

$$\mathcal{D}_X^{\hbar,m} = \prod_{i \geq 0} (F_{i+m} \mathcal{D}_X) \hbar^i, \quad \mathcal{D}_X^{\hbar,f} = \bigcup_{m \geq 0} \mathcal{D}_X^{\hbar,m}.$$

Note that  $\mathcal{D}_X^{\hbar,0}$  and  $\mathcal{D}_X^{\hbar,f}$  are subalgebras of  $\mathcal{D}_X^{\hbar}$ , that  $\mathcal{D}_X^{\hbar,0}$  is  $\hbar$ -complete while  $\mathcal{D}_X^{\hbar,f}$  is not, and that  $\mathcal{D}_X^{\hbar,0,\text{loc}} \simeq \mathcal{D}_X^{\hbar,f,\text{loc}}$ . By (8.3), the image of  $\mathcal{A}|_X$  in  $\mathcal{D}_X^{\hbar}$  is contained in  $\mathcal{D}_X^{\hbar,0}$ . (The ring  $\mathcal{D}_X^{\hbar,0}$  should be compared with the ring  $\mathcal{B}_{X \times \mathbb{C}}$  of [16].)

**Remark 8.2.** More precisely, denote by  $\mathcal{O}_{\mathfrak{X}}^{\hbar}|_X \simeq (\mathcal{O}_{\mathfrak{X}}^{\hbar})^{\hbar}|_X$  the formal completion of  $\mathcal{O}_{\mathfrak{X}}^{\hbar}$  along the submanifold  $X$ . Then the star product in (8.1) extends to this sheaf, and (8.3) induces an isomorphism  $(\mathcal{O}_{\mathfrak{X}}^{\hbar}|_X, \star) \simeq \mathcal{D}_X^{\hbar,0}$ .

Summarizing, one has the compatible embeddings of algebras

$$\begin{array}{ccccccc} \mathcal{A}^{\text{loc}}|_X & \hookrightarrow & \mathcal{D}_X^{\hbar,0,\text{loc}} & \xrightarrow{\sim} & \mathcal{D}_X^{\hbar,f,\text{loc}} & \hookrightarrow & \mathcal{D}_X^{\hbar,\text{loc}} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathcal{A}|_X & \hookrightarrow & \mathcal{D}_X^{\hbar,0} & \hookrightarrow & \mathcal{D}_X^{\hbar,f} & \hookrightarrow & \mathcal{D}_X^{\hbar} \end{array}$$

One has

$$\text{gr}_{\hbar} \mathcal{A}|_X \simeq \mathcal{O}_{\mathfrak{X}}|_X, \quad \text{gr}_{\hbar} \mathcal{D}_X^{\hbar,0} \simeq \mathcal{O}_{\mathfrak{X}}^{\hbar}|_X, \quad \text{gr}_{\hbar} \mathcal{D}_X^{\hbar,f} \simeq \text{gr}_{\hbar} \mathcal{D}_X^{\hbar} \simeq \mathcal{D}_X.$$

**Proposition 8.3.** (i) The algebra  $\mathcal{D}_X^{\hbar,0}$  is faithfully flat over  $\mathcal{A}|_X$ .

(ii) The algebra  $\mathcal{D}_X^{\hbar,\text{loc}}$  is flat over  $\mathcal{A}^{\text{loc}}|_X$ .

*Proof.* (i) follows from Theorem 1.12.

(ii) follows from (i) and the isomorphism  $(\mathcal{D}_X^{\hbar,0})^{\text{loc}} \simeq \mathcal{D}_X^{\hbar,\text{loc}}$ . □

The next examples show that the scalar extension functor

$$\text{Mod}_{\text{coh}}(\mathcal{D}_X^{\hbar,0}) \rightarrow \text{Mod}_{\text{coh}}(\mathcal{D}_X^{\hbar})$$

is neither exact nor full.

**Example 8.4.** Let  $X = \mathbb{C}^2$  with coordinates  $(x, y)$ . Then  $\hbar \partial_y$  is injective as an endomorphism of  $\mathcal{D}_X^{\hbar,0} / \langle \hbar \partial_x \rangle$  but it is not injective as an endomorphism of  $\mathcal{D}_X^{\hbar} / \langle \hbar \partial_x \rangle$ , since  $\partial_x$  belongs to its kernel. This shows that  $\mathcal{D}_X^{\hbar}$  is not flat over  $\mathcal{D}_X^{\hbar,0}$ .

**Example 8.5.** This example was communicated to us by Masaki Kashiwara. Let  $X = \mathbb{C}$  with coordinate  $x$ , and denote by  $(x, u)$  the symplectic coordinates on  $\mathfrak{X} = T^*\mathbb{C}$ . Consider the cyclic  $\mathcal{A}$ -modules

$$\mathcal{M} = \mathcal{A} / \langle x - u \rangle, \quad \mathcal{N} = \mathcal{A} / \langle x \rangle,$$

and their images in  $\text{Mod}(\mathcal{D}_X^{\hbar})$

$$\mathcal{M}' = \mathcal{D}_X^{\hbar}/\langle x - \hbar\partial_x \rangle, \quad \mathcal{N}' = \mathcal{D}_X^{\hbar}/\langle x \rangle.$$

As their supports in  $\mathfrak{X}$  differ,  $\mathcal{M}$  and  $\mathcal{N}$  are not isomorphic as  $\mathcal{A}$ -modules. On the other hand, in  $\mathcal{D}_X^{\hbar}$  one has the relation

$$(8.4) \quad x \cdot e^{\hbar\partial_x^2/2} = e^{\hbar\partial_x^2/2} \cdot (x - \hbar\partial_x),$$

and hence an isomorphism  $\mathcal{M}' \xrightarrow{\sim} \mathcal{N}'$  given by  $[P] \mapsto [P \cdot e^{-\hbar\partial_x^2/2}]$ . In fact, one checks that

$$\text{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})|_X = 0, \quad \text{Hom}_{\mathcal{D}_X^{\hbar}}(\mathcal{M}', \mathcal{N}') \simeq \mathbb{C}_X^{\hbar}.$$

**§A. Complements on constructible sheaves**

Let us review some results, well-known to specialists (see, e.g., [18, Proposition 3.10]), but which are usually stated over a field, and we need to work here over the ring  $\mathbb{C}^{\hbar}$ .

Let  $\mathbb{K}$  be a commutative unital Noetherian ring of finite global dimension. Assume that  $\mathbb{K}$  is syzygic, i.e. any finitely generated  $\mathbb{K}$ -module admits a finite projective resolution by finite free modules. (For our purposes we will either have  $\mathbb{K} = \mathbb{C}$  or  $\mathbb{K} = \mathbb{C}^{\hbar}$ ).

Let  $X$  be a real analytic manifold. Denote by  $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X)$  the abelian category of  $\mathbb{R}$ -constructible sheaves on  $X$  and by  $\text{D}_{\mathbb{R}\text{-c}}^b(\mathbb{K}_X)$  the bounded derived category of sheaves of  $\mathbb{K}$ -modules with  $\mathbb{R}$ -constructible cohomology. Under the above assumptions on the base ring, by [9, Propositions 3.4.3, 8.4.9] one has

**Lemma A.1.** *The duality functor  $\text{D}'_{\mathbb{K}}(\bullet) = \text{R}\mathcal{H}om_{\mathbb{K}_X}(\bullet, \mathbb{K}_X)$  induces an involution of  $\text{D}_{\mathbb{R}\text{-c}}^b(\mathbb{K}_X)$ .*

For the next proposition we recall some notation and results of [6, 9]. We consider a simplicial complex  $\mathbf{S} = (S, \Delta)$ , with set of vertices  $S$  and set of simplices  $\Delta$ . We let  $|\mathbf{S}|$  be the realization of  $\mathbf{S}$ . Thus  $|\mathbf{S}|$  is the disjoint union of the realizations  $|\sigma|$  of the simplices. For a simplex  $\sigma \in \Delta$ , the open set  $U(\sigma)$  is defined in [9, (8.1.3)]. A sheaf  $F$  of  $\mathbb{K}$ -modules on  $|\mathbf{S}|$  is said to be *weakly  $\mathbf{S}$ -constructible* if  $F|_{|\sigma|}$  is constant for any  $\sigma \in \Delta$ . An object  $F \in \text{D}^b(\mathbb{K}_{|\mathbf{S}|})$  is said to be *weakly  $\mathbf{S}$ -constructible* if its cohomology sheaves are so. If moreover, all stalks  $F_x$  are perfect complexes,  $F$  is called  *$\mathbf{S}$ -constructible*. By [9, Proposition 8.1.4] we have isomorphisms, for a weakly  $\mathbf{S}$ -constructible sheaf  $F$  and for any  $\sigma \in \Delta$  and  $x \in |\sigma|$ ,

$$(A.1) \quad \Gamma(U(\sigma); F) \xrightarrow{\sim} \Gamma(|\sigma|; F) \xrightarrow{\sim} F_x,$$

$$(A.2) \quad H^j(U(\sigma); F) = H^j(|\sigma|; F) = 0 \quad \text{for } j \neq 0.$$

It follows that, for a weakly  $\mathbf{S}$ -constructible  $F \in D^b(\mathbb{K}_{|\mathbf{S}|})$ , the natural morphisms of complexes of  $\mathbb{K}$ -modules

$$(A.3) \quad \Gamma(U(\sigma); F) \rightarrow \Gamma(|\sigma|; F) \rightarrow F_x$$

are quasi-isomorphisms.

For  $U \subset X$  an open subset, we denote by  $\mathbb{K}_U := (\mathbb{K}_X)_U$  the extension by 0 of the constant sheaf on  $U$ .

**Proposition A.2.** *Let  $F \in D_{\mathbb{R}\text{-c}}^b(\mathbb{K}_X)$ . Then*

(i)  *$F$  is isomorphic to a complex*

$$0 \rightarrow \bigoplus_{i_a \in I_a} \mathbb{K}_{U_{a,i_a}} \rightarrow \cdots \rightarrow \bigoplus_{i_b \in I_b} \mathbb{K}_{U_{b,i_b}} \rightarrow 0,$$

where the  $\{U_{k,i_k}\}_{k,i_k}$ 's are locally finite families of relatively compact subanalytic open subsets of  $X$ .

(ii)  *$F$  is isomorphic to a complex*

$$0 \rightarrow \bigoplus_{i_a \in I_a} \Gamma_{V_{a,i_a}} \mathbb{K}_X \rightarrow \cdots \rightarrow \bigoplus_{i_b \in I_b} \Gamma_{V_{b,i_b}} \mathbb{K}_X \rightarrow 0,$$

where the  $\{V_{k,i_k}\}_{k,i_k}$ 's are locally finite families of relatively compact subanalytic open subsets of  $X$ .

*Proof.* (i) By the triangulation theorem for subanalytic sets (see for example [9, Proposition 8.2.5]) we may assume that  $F$  is an  $\mathbf{S}$ -constructible object in  $D^b(\mathbb{K}_{|\mathbf{S}|})$  for some simplicial complex  $\mathbf{S} = (S, \Delta)$ . For  $i$  an integer, let  $\Delta_i \subset \Delta$  be the subset of simplices of dimension  $\leq i$  and set  $\mathbf{S}_i = (S, \Delta_i)$ . We denote by  $\mathcal{K}^b(\mathbb{K})$  (resp.  $\mathcal{K}^b(\mathbb{K}_{|\mathbf{S}|})$ ) the category of bounded complexes of  $\mathbb{K}$ -modules (resp. sheaves of  $\mathbb{K}$ -modules on  $|\mathbf{S}|$ ) with morphisms up to homotopy. We shall prove by induction on  $i$  that there exists a morphism  $u_i: G_i \rightarrow F$  in  $\mathcal{K}^b(\mathbb{K}_{|\mathbf{S}|})$  such that:

- (a) the  $G_i^k$  are finite direct sums of  $\mathbb{K}_{U(\sigma_\alpha)}$ 's for some  $\sigma_\alpha \in \Delta_i$ ,
- (b)  $u_i|_{|\mathbf{S}_i|}: G_i|_{|\mathbf{S}_i|} \rightarrow F|_{|\mathbf{S}_i|}$  is a quasi-isomorphism.

The desired result is obtained for  $i$  equal to the dimension of  $X$ .

(i)-(1) For  $i = 0$  we consider  $F|_{|\mathbf{S}_0|} \simeq \bigoplus_{\sigma \in \Delta_0} F_\sigma$ . The complexes  $\Gamma(U(\sigma); F)$ ,  $\sigma \in \Delta_0$ , have finite bounded cohomology by the quasi-isomorphisms (A.3). Hence we may choose bounded complexes of finite free  $\mathbb{K}$ -modules,  $R_{0,\sigma}$ , and morphisms  $u_{0,\sigma}: R_{0,\sigma} \rightarrow \Gamma(U(\sigma); F)$  which are quasi-isomorphisms.

We have the natural isomorphism  $\Gamma(U(\sigma); F) \simeq a_* \mathcal{H}om_{\mathcal{K}^b(\mathbb{K}_{|\mathbf{S}|})}(\mathbb{K}_{U(\sigma)}, F)$  in  $\mathcal{K}^b(\mathbb{K})$ , where  $a: |\mathbf{S}| \rightarrow \text{pt}$  is the projection and  $\mathcal{H}om$  is the internal Hom functor.

We deduce the adjunction formula, for  $R \in \mathbb{K}^b(\mathbb{K})$  and  $F \in \mathbb{K}^b(\mathbb{K}_{|\mathbf{S}|})$ ,

$$(A.4) \quad \text{Hom}_{\mathbb{K}^b(\mathbb{K})}(R, \Gamma(U(\sigma); F)) \simeq \text{Hom}_{\mathbb{K}^b(\mathbb{K}_{|\mathbf{S}|})}(R_{U(\sigma)}, F).$$

Hence the  $u_{0,\sigma}$  induce  $u_0: G_0 := \bigoplus_{\sigma \in \Delta_0} (R_{0,\sigma})_{U(\sigma)} \rightarrow F$ . By (A.3),  $(u_0)_x$  is a quasi-isomorphism for all  $x \in |\mathbf{S}_0|$ , so that  $u_0|_{|\mathbf{S}_0|}$  also is a quasi-isomorphism, as required.

(i)-(2) We assume that  $u_i$  is built and let  $H_i = M(u_i)[-1]$  be the mapping cone of  $u_i$ , shifted by  $-1$ . By the distinguished triangle in  $\mathbb{K}^b(\mathbb{K}_{|\mathbf{S}|})$

$$(A.5) \quad H_i \xrightarrow{v_i} G_i \xrightarrow{u_i} F \xrightarrow{+1}$$

$H_i|_{|\mathbf{S}_i|}$  is quasi-isomorphic to 0. Hence  $\bigoplus_{\sigma \in \Delta_{i+1} \setminus \Delta_i} (H_i)|_{\sigma} \rightarrow H_i|_{|\mathbf{S}_{i+1}|}$  is a quasi-isomorphism. As above we choose quasi-isomorphisms  $u_{i+1,\sigma}: R_{i+1,\sigma} \rightarrow \Gamma(U(\sigma); H_i)$ ,  $\sigma \in \Delta_{i+1} \setminus \Delta_i$ , where the  $R_{i+1,\sigma}$  are bounded complexes of finite free  $\mathbb{K}$ -modules. By (A.4) again the  $u_{i+1,\sigma}$  induce a morphism in  $\mathbb{K}^b(\mathbb{K}_{|\mathbf{S}|})$

$$u'_{i+1}: G'_{i+1} := \bigoplus_{\sigma \in \Delta_{i+1} \setminus \Delta_i} (R_{i+1,\sigma})_{U(\sigma)} \rightarrow H_i.$$

For  $x \in |\mathbf{S}_{i+1}| \setminus |\mathbf{S}_i|$ ,  $(u'_{i+1})_x$  is a quasi-isomorphism by (A.3), and, for  $x \in |\mathbf{S}_i|$ , this is trivially true. Hence  $u'_{i+1}|_{|\mathbf{S}_{i+1}|}$  is a quasi-isomorphism.

Now we let  $H_{i+1}$  and  $G_{i+1}$  be the mapping cones of  $u'_{i+1}$  and  $v_i \circ u'_{i+1}$ , respectively. We have distinguished triangles in  $\mathbb{K}^b(\mathbb{K}_{|\mathbf{S}|})$

$$(A.6) \quad G'_{i+1} \xrightarrow{u'_{i+1}} H_i \rightarrow H_{i+1} \xrightarrow{+1}, \quad G'_{i+1} \xrightarrow{v_i \circ u'_{i+1}} G_i \rightarrow G_{i+1} \xrightarrow{+1}.$$

By the construction of the mapping cone, the definition of  $G'_{i+1}$  and the induction hypothesis,  $G_{i+1}$  satisfies property (a) above. The octahedral axiom applied to triangles (A.5) and (A.6) gives a morphism  $u_{i+1}: G_{i+1} \rightarrow F$  and a distinguished triangle  $H_{i+1} \rightarrow G_{i+1} \xrightarrow{u_{i+1}} F \xrightarrow{+1}$ . By construction  $H_{i+1}|_{|\mathbf{S}_{i+1}|}$  is quasi-isomorphic to 0 so that  $u_{i+1}$  satisfies property (b) above.

(ii) Set  $G = D'_{\mathbb{K}}(F)$ , and represent it by a bounded complex as in (i). Since  $U_{k,i_k}$  corresponds to an open subset of the form  $U(\sigma)$  in  $|\mathbf{S}|$ , the sheaves  $\mathbb{K}_{U_{k,i_k}}$  are acyclic for the functor  $D'_{\mathbb{K}}$ . Hence  $F \simeq D'_{\mathbb{K}}(G)$  can be represented as claimed.  $\square$

**Lemma A.3.** *Let  $F \rightarrow G \rightarrow 0$  be an exact sequence in  $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X)$ . Then for any relatively compact subanalytic open subset  $U \subset X$ , there exists a finite covering  $U = \bigcup_{i \in I} U_i$  by subanalytic open subsets such that, for each  $i \in I$ , the morphism  $F(U_i) \rightarrow G(U_i)$  is surjective.*

*Proof.* As in the proof of Proposition A.2 we may assume that  $F, G$  and  $\mathbb{K}_U$  are constructible sheaves on the realization of some finite simplicial complex  $(S, \Delta)$ .

For  $\sigma \in \Delta$  the morphism  $\Gamma(U(\sigma); F) \rightarrow \Gamma(U(\sigma); G)$  is surjective, by (A.1). Since the image of  $U$  in  $|\mathbf{S}|$  is a finite union of  $U(\sigma)$ 's, this proves the lemma.  $\square$

**§B. Complements on subanalytic sheaves**

We review here some well-known results (see [11, Chapter 7] and [15]) but which are usually stated over a field, and we need to work here over the ring  $\mathbb{C}^h$ .

Let  $\mathbb{K}$  be a commutative unital Noetherian ring of finite global dimension (for our purposes we will have either  $\mathbb{K} = \mathbb{C}$  or  $\mathbb{K} = \mathbb{C}^h$ ). Let  $X$  be a real analytic manifold, and consider the natural morphism  $\rho: X \rightarrow X_{\text{sa}}$ .

**Lemma B.1.** *The functor  $\rho_*: \text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X) \rightarrow \text{Mod}(\mathbb{K}_{X_{\text{sa}}})$  is exact and  $\rho^{-1}\rho_*$  is isomorphic to the canonical functor  $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X) \rightarrow \text{Mod}(\mathbb{K}_X)$ .*

*Proof.* Being a direct image functor,  $\rho_*$  is left exact. It is right exact thanks to Lemma A.3. The composition  $\rho^{-1}\rho_*$  is isomorphic to the identity on  $\text{Mod}(\mathbb{K}_X)$  since the open sets of the site  $X_{\text{sa}}$  give a basis of the topology of  $X$ .  $\square$

In the following, we denote by  $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_{X_{\text{sa}}})$  the image under the functor  $\rho_*$  of  $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X)$  in  $\text{Mod}(\mathbb{K}_{X_{\text{sa}}})$ . Hence  $\rho_*$  induces an equivalence of categories  $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X) \simeq \text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_{X_{\text{sa}}})$ . We also denote by  $\text{D}_{\mathbb{R}\text{-c}}^b(\mathbb{K}_{X_{\text{sa}}})$  the full triangulated subcategory of  $\text{D}^b(\mathbb{K}_{X_{\text{sa}}})$  consisting of objects with cohomology in  $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_{X_{\text{sa}}})$ .

**Corollary B.2.** *The subcategory  $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_{X_{\text{sa}}})$  of  $\text{Mod}(\mathbb{K}_{X_{\text{sa}}})$  is thick.*

*Proof.* Since  $\rho_*$  is fully faithful and exact,  $\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_{X_{\text{sa}}})$  is stable under taking kernels and cokernels. It remains to see that, for  $F, G \in \text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X)$ ,

$$\text{Ext}_{\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X)}^1(F, G) \simeq \text{Ext}_{\text{Mod}(\mathbb{K}_{X_{\text{sa}}})}^1(\rho_*F, \rho_*G).$$

By [6] we know that the first  $\text{Ext}^1$  may as well be computed in  $\text{Mod}(\mathbb{K}_X)$ . Note that the functors  $\rho^{-1}$  and  $R\rho_*$  between  $\text{D}^b(\mathbb{K}_X)$  and  $\text{D}^b(\mathbb{K}_{X_{\text{sa}}})$  are adjoint, and moreover  $\rho^{-1}R\rho_* \simeq \text{id}$ . Thus, for  $F', G' \in \text{D}^b(\mathbb{K}_X)$  we have

$$\text{Hom}_{\text{D}^b(\mathbb{K}_{X_{\text{sa}}})}(R\rho_*F', R\rho_*G') \simeq \text{Hom}_{\text{D}^b(\mathbb{K}_X)}(F', G'),$$

and this gives the result.  $\square$

This corollary gives the equivalence  $\text{D}_{\mathbb{R}\text{-c}}^b(\mathbb{K}_X) \simeq \text{D}_{\mathbb{R}\text{-c}}^b(\mathbb{K}_{X_{\text{sa}}})$ , both categories being equivalent to  $\text{D}^b(\text{Mod}_{\mathbb{R}\text{-c}}(\mathbb{K}_X))$ .

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