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On Certain Arithmetic Functions $\tilde{M}(s; z_1, z_2)$ Associated with Global Fields: Analytic Properties

Dedicated to Professor Mikio Sato

by

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Abstract

The arithmetic functions in the title arose from value-distribution theories related to L-functions of global fields. They are "complexifications" of the Fourier duals of the corresponding density functions. We shall study their complex analytic properties including analytic continuations and the limit behaviors at the critical point $s = 1/2$.

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Introduction

§0.1

In [2], we started our study of the complex analytic function $\tilde{M}(s; z_1, z_2)$ (denoted there as $\tilde{M}_s(z_1, z_2)$) in three variables s, z_1, z_2 ($\Re(s) > 1/2$), in connection with the value-distribution of $\{d \log L(s, \chi)/ds\}_\chi$. Here, χ runs over a suitable family of abelian characters of a global field K and $L(s, \chi)$ denotes the associated L-function. The connection is that for each fixed s with $\Re(s) = \sigma > 1/2$, the inverse Fourier transform $M_\sigma(w)$ of $\tilde{M}_\sigma(z) = \tilde{M}(\sigma; z, \bar{z})$ is the density function for the distribution of $\{d \log L(s, \chi)/ds\}_\chi$ on the complex w-plane (generally conjectural, proved in various cases [2, 4, 6]). In the joint work with K. Matsumoto [5, 6] (cf.

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also a survey [7]), we continued this study treating also the corresponding M- and $\tilde{M}$-functions related to the value-distribution of $\{\log L(s,\chi)\}_\chi$. We use the same symbols M, $\tilde{M}$ etc., and distinguish the former $d\log$-case as Case 1, the latter log-case as Case 2. They are different systems of functions having various properties in common. Each also depends on the pair (K, P_∞), where K is a global field (either an algebraic number field or an algebraic function field of one variable over a finite field) and P_∞ is a given finite set of prime divisors of K including all archimedean primes in the number field case. When $K = \mathbb{Q}$ (the rational number field) and $|P_\infty| = 1$, $\tilde{M}$ is given by

$$\tilde{M}(s; z_1, z_2) = \sum_{n=1}^{\infty} \lambda_{z_1}(n)\lambda_{z_2}(n)n^{-2s} \qquad (\Re(s) > 1/2), \tag{0.1.1}$$

where each $\lambda_z(n)$ $(n = 1, 2, \dots)$ is a polynomial of z determined by

$$\sum_{n=1}^{\infty} \lambda_z(n)n^{-s} := \begin{cases} \exp\left(\dfrac{iz}{2}\dfrac{d}{ds}\log\zeta_{\mathbb{Q}}(s)\right) & \text{(Case 1)} \\ \exp\left(\dfrac{iz}{2}\log\zeta_{\mathbb{Q}}(s)\right) & \text{(Case 2)} \end{cases}$$

$(\Re(s) > 1, \ i = \sqrt{-1})$, $\zeta_{\mathbb{Q}}(s)$ being the Riemann zeta function. Note that $\tilde{M}(s, -2i, -2ix) = \zeta_{\mathbb{Q}}(2s)^x$ $(x \in \mathbb{C})$ in Case 2. It seems to the author that these functions are interesting in themselves.

§0.2

We shall pursue the investigation of analytic properties of the functions $\tilde{M}(s; z_1, z_2)$ and $M_\sigma(w)$. In the present article, we first study the variance μ_σ of the density measure $M_\sigma(w)|dw|$ ($|dw| = dudv/(2\pi)$ for $w = u + vi$) on $\mathbb{C}$ and the "Plancherel volume"

$$\nu_\sigma = \int_{\mathbb{C}} M_\sigma(w)^2\,|dw| = \int_{\mathbb{C}} |\tilde{M}_\sigma(z)|^2\,|dz|; \tag{0.2.1}$$

especially the limits $\lim_{\sigma\to 1/2}$ and $\lim_{\sigma\to+\infty}$ of the natural invariant $\mu_\sigma\nu_\sigma$, the variance-normalized measure $\mu_\sigma M_\sigma(\mu_\sigma^{1/2}w)$ and its Fourier transform $\tilde{M}_\sigma(\mu_\sigma^{-1/2}z)$ (§1, §2). The first limit is of course more difficult. A key point here is the limit behavior at $s = 1/2$ of the complex analytic version

$$\tilde{M}(s; \mu(s)^{-1/2}z_1, \mu(s)^{-1/2}z_2) \tag{0.2.2}$$

of $\tilde{M}_\sigma(\mu_\sigma^{-1/2}z)$, which is partly related to the second main subject of this article, namely analytic continuation. We shall prove (§3) that $\tilde{M}(s; z_1, z_2)$ extends to an analytic function of $(s;\, z_1, z_2)$ on the product space $\mathcal{D} \times \mathbb{C}^2$, where

$$\mathcal{D} = \{\Re(s) > 0\} \setminus \{\rho/(2n);\, n \in \mathbb{N},\, \zeta(\rho) = 0 \text{ or } \infty\}, \tag{0.2.3}$$

$\zeta(s) = \zeta_{K,P_\infty}(s)$ being the zeta function of K without P_∞ factors. In fact, $\tilde{M}(s; z_1, z_2)$ is univalent on $\mathcal{D} \times \mathbb{C}^2$ in Case 1, but multivalent in Case 2 (univalent on $\mathcal{D}^{\mathrm{urab}} \times \mathbb{C}^2$, $\mathcal{D}^{\mathrm{urab}}$ being the maximal unramified abelian cover of $\mathcal{D}$). This property is closely related to the infinite product expansion which, in Case 2, looks like

$$\tilde{M}(s; z_1, z_2) = \prod_{n=1}^{\infty} \zeta(2ns)^{R_n(z_1, z_2)}, \tag{0.2.4}$$

where each $R_n(z_1, z_2)$ is a polynomial of degree $\leq n$ in each variable z_1, z_2. This means that for any $N \in \mathbb{N}$, (i) the quotient of $\tilde{M}(s; z_1, z_2)$ by the partial product over $n \leq N$ on the right hand side extends to a holomorphic function on $\Re(s) > 1/(2N+2)$, and (ii) *on some subdomain* of $\{\Re(s) > 1/2\} \times \mathbb{C}^2$, the remaining product converges absolutely to a non-vanishing holomorphic function which gives that quotient. This result for $N = 1$ will be used to show that (0.2.2) converges to $\exp(-z_1 z_2/4)$ as $s \to 1/2$. Together with a result on the upper bound for $|\tilde{M}_\sigma(z)|^2$ near $\sigma = 1/2$, valid for all $z \in \mathbb{C}$ established in §4, this leads to our limit formulas for $\mu_\sigma \nu_\sigma$ and $\mu_\sigma M_\sigma(\mu_\sigma^{1/2} w)$.

§0.3

In §1.1, we first discuss general density functions $M(x)|dx|$ on $\mathbb{R}^d$ $(d = 1, 2, \dots)$ with center 0, in particular, the best possible lower bound for the quantity $\mu^{d/2}\nu$ (Theorem 1), where μ is the variance and ν is the Plancherel volume. For $d = 2$, this gives $\mu\nu \geq 8/9$. Then in §1.2, we briefly review (from [6, §4]) the definitions and basic properties of our functions $\tilde{M}(s; z_1, z_2)$ and $M_\sigma(w)$.

In §2, we study the limits as $\sigma \to 1/2, +\infty$, of $\mu_\sigma \nu_\sigma$ and $\mu_\sigma M_\sigma(\mu_\sigma^{1/2} w)$ (Theorems 2, 3). Some of the relevant key lemmas will be proved later (§3, §4). This logically inverted ordering of sections is due to the introductory nature of §2 and the "heaviness" of §3 and §4.

In §3, we shall prove the analytic continuation of $\tilde{M}(s; z_1, z_2)$ (Theorem 5).

In §4, we shall study the rapid decay property of $|\tilde{M}_\sigma(z)|^2$, with special care when σ is arbitrarily close to 1/2 and $|z|$ is not bounded (Theorem 7C).

§0.4

Now we mention something about the zero divisor of $\tilde{M}(s; z_1, z_2)$ on which no information appears in the product formula (0.2.4). First, as is already shown in the previous articles (reviewed in §1.2), $\tilde{M}(s;z_1,z_2)$ has an Euler product decomposition

$$\tilde{M} = \tilde{M}(s;z_1,z_2) = \prod_{\mathfrak{p}\notin P_\infty} \tilde{M}_{\mathfrak{p}}(s;z_1,z_2) \quad (\Re(s) > 1/2), \tag{0.4.1}$$

where each local factor $\tilde{M}_{\mathfrak{p}} = \tilde{M}_{\mathfrak{p}}(s;z_1,z_2)$ is holomorphic on $\{\Re(s) > 0\} \times \mathbb{C}^2$. In Case 2, $\tilde{M}_{\mathfrak{p}}$ can be expressed by the Gauss hypergeometric function $F(a,b;c;x)$, as

$$\tilde{M}_{\mathfrak{p}}(s;z_1,z_2) = F(iz_1/2, iz_2/2; 1;\ N(\mathfrak{p})^{-2s}). \tag{0.4.2}$$

Each $\tilde{M}_{\mathfrak{p}}$ has a non-trivial zero divisor $\mathcal{Z}_{\mathfrak{p}}$; $\{\mathcal{Z}_{\mathfrak{p}}\}_{\mathfrak{p}}$ is locally finite, and the intersection with $\mathcal{D} \times \mathbb{C}^2$ of $\sum_{\mathfrak{p}} \mathcal{Z}_{\mathfrak{p}}$ gives the zero divisor of $\tilde{M}$.

The local zero divisor $\mathcal{Z}_{\mathfrak{p}}$ seems worth studying fully.[1] But let us touch here the main property of its restriction to the hyperplane $z_1 + z_2 = 0$, say, in Case 2. Put $t = N(\mathfrak{p})^{-s}$, $x = iz_1$, and consider the "locally normalized" function

$$f_t(x) = F(x/(2\arcsin(t)), -x/(2\arcsin(t)); 1;\ t^2). \tag{0.4.3}$$

Then $f_0(x) = J_0(x)$, the Bessel function of order 0. If $\pm\{\gamma_\nu\}_{\nu=1}^{\infty}$ with $0 < \gamma_1 < \gamma_2 < \cdots$ denote all the zeros of $J_0(x)$, then there exists $0 < t_0 < 1$ such that each γ_ν extends uniquely and holomorphically to a zero $\gamma_\nu(t)$ of $f_t(x)$ for all $|t| < t_0$ (*real* if t is so). Moreover, $f_t(x)$ has no other zeros than $\{\pm\gamma_\nu(t)\}$, and we have the Weierstrass decomposition

$$f_t(x) = \prod_{\nu=1}^{\infty}\left(1 - \frac{x^2}{\gamma_\nu(t)^2}\right). \tag{0.4.4}$$

This gives rise to another infinite product decomposition

$$\tilde{M}(s;z,-z) = \prod_{\mathfrak{p}\notin P_\infty}\prod_{\nu=1}^{\infty}\left(1 + \left(\frac{\arcsin(N(\mathfrak{p})^{-s})}{\gamma_\nu(N(\mathfrak{p})^{-s})}\right)^2 z^2\right) = \prod_{\mu=1}^{\infty}(1+\theta_\mu^2 z^2) \tag{0.4.5}$$

for $\Re(s) > 1/2$, where $\{\theta_\mu\}_\mu$ is a reordering of $\{\arcsin(N(\mathfrak{p})^{-s})/\gamma_\nu(N(\mathfrak{p})^{-s})\}_{\mathfrak{p},\nu}$ according to the absolute values. For $s = \sigma \in \mathbb{R}$, θ_μ^2 are all positive real, as long as $N(\mathfrak{p})^\sigma$ is sufficiently large. Comparing the two decompositions (0.2.4) and (0.4.5) will be a subject of future study.

[1] Left to future articles; cf. [3] for some partial results for Case 1.

§1. Preliminaries

§1.1. The Plancherel volume

Let $\mathbb{R}^d = \{x = (x_1, \dots, x_d);\ x_i \in \mathbb{R}\,(1 \le i \le d)\}$ be the d-dimensional Euclidean space ($d = 1, 2, \dots$), and $|dx| = (dx_1 \dots dx_d)/(2\pi)^{d/2}$ be the self-dual Haar measure with respect to the self-dual pairing $e^{i\langle x, x'\rangle}$ of $\mathbb{R}^d$, where $\langle x, x'\rangle = \sum_{i=1}^d x_i x_i'$. Write, as usual, $|x| = \langle x, x\rangle^{1/2}$. Consider any density measure $M(x)|dx|$ ($M(x)$ a measurable real-valued function) on $\mathbb{R}^d$ with center 0, for which the standard formulas in Fourier analysis hold; namely (the integrals denoting those over $\mathbb{R}^d$),

$$M(x) \ge 0, \qquad \int M(x)\,|dx| = 1; \tag{1.1.1}$$

$$\int M(x) x_i\,|dx| = 0 \qquad (1 \le i \le d); \tag{1.1.2}$$

$$\tilde{M}(y) := \int M(x) e^{i\langle x, y\rangle}\,|dx|, \qquad M(x) = \int \tilde{M}(y) e^{-i\langle x, y\rangle}\,|dy|; \tag{1.1.3}$$

$$\nu := \nu_M = \int M(x)^2\,|dx| = \int |\tilde{M}(y)|^2\,|dy| \qquad \text{(Plancherel formula)}. \tag{1.1.4}$$

We shall compare the two invariants

$$\mu := \mu_M = \int M(x)|x|^2\,|dx| \qquad \text{(the variance)} \tag{1.1.5}$$

and the above ν_M which will be called the *Plancherel volume* of $M(x)$ (or of $M(x)\,|dx|$). Note that ν_M can also be expressed as

$$\nu_M = M(x) * M(-x)\,|_{x=0} \tag{1.1.6}$$

($*$ being the convolution product with respect to $|dx|$). Thus, ν_M may be regarded as the density at the origin of the distribution of the differences of two points in the measure space $(\mathbb{R}^d, M(x)\,|dx|)$.

In general, the two invariants, the average μ of the square of the distance from the center and the density ν at the origin of $x - x'$ ($x, x' \in \mathbb{R}^d$), both with respect to the given density measure $M(x)\,|dx|$, are unrelated invariants. But the product

$$\mu^{d/2}\nu \tag{1.1.7}$$

seems to be an interesting basic invariant. Note that this is invariant under the scalar transform

$$M(x) \mapsto c^d M(cx) \tag{1.1.8}$$

for any $c > 0$; in fact, μ (resp. ν) is multiplied by c^{-2} (resp. c^d).

If we denote by $M^{\star}(x) = \mu^{d/2}M(\mu^{1/2}x)$ the scalar transform (1.1.8) for $c = \mu^{1/2}$, then $M^{\star}(x)$ has Fourier dual $\tilde{M}(\mu^{-1/2}y)$, variance 1, and Plancherel volume $\mu^{d/2}\nu$. This scalar transform $M(x) \mapsto M^{\star}(x)$ will be called the *variance normalization.*

Let us pay attention to the following three special cases and the theorem to come thereafter.

Example 1. If $M(x)\,|dx|$ is Gaussian, i.e., $M(x) = ce^{-a|x|^2}$ $(a,\, c > 0)$, then

$$\mu^{d/2}\nu = (d/2)^{d/2}. \tag{1.1.9}$$

In particular, the two-dimensional Gaussian distribution satisfies $\mu\nu = 1$.

Indeed, we have $c = (2a)^{d/2}$ by (1.1.1), and $\mu = d/(2a)$, $\nu = a^{d/2}$.

Example 2. If $M(x) = c$ $(|x| \le R)$ and $= 0$ $(|x| > R)$, where $c, R > 0$, then

$$\mu^{d/2}\nu = \left(\frac{2d}{d+2}\right)^{d/2}\Gamma\left(\frac{d}{2}+1\right). \tag{1.1.10}$$

In particular, when $d = 2$, we again have $\mu\nu = 1$.

Indeed, $c = 2^{d/2}\Gamma(d/2+1)R^{-d}$, $\mu = \frac{d}{d+2}R^2$, $\nu = 2^{d/2}\Gamma(d/2+1)R^{-d}$.

Thus, when $d = 2$, $\mu\nu = 1$ holds in these two special cases.

Example 3. Define the function $f_d^*(r)$ of $r \ge 0$ by

$$f_d^*(r) = \begin{cases} \dfrac{d(d+2)}{2}\gamma_d \cdot (1-r^2), & 0 \le r \le 1, \\ 0, & r \ge 1, \end{cases} \tag{1.1.11}$$

where

$$\gamma_d = (2\pi)^{d/2}/\mathrm{Vol}(\mathrm{S}_{d-1}) = 2^{d/2-1}\Gamma(d/2), \tag{1.1.12}$$

$\mathrm{Vol}(\mathrm{S}_{d-1})$ being the Euclidean volume of the $(d-1)$-dimensional unit sphere. For any fixed $c > 0$, consider the function $M(x) = c^d \cdot f_d^*(c|x|)$ on $\mathbb{R}^d$. Then $M(x)$ also satisfies (1.1.1) and (1.1.2), and we have

$$\mu^{d/2}\nu = \left(\frac{2d}{d+4}\right)^{d/2}\frac{4\Gamma(\frac{d+4}{2})}{d+4}. \tag{1.1.13}$$

Indeed, $\mu = c^{-2}\mu_d^*$ and $\nu = c^d\nu_d^*$, where

$$\mu_d^* = \frac{d}{d+4}, \qquad \nu_d^* = \frac{2d(d+2)}{d+4}\gamma_d = 2^{d/2}\frac{4\Gamma(\frac{d+4}{2})}{d+4}. \tag{1.1.14}$$

Now, intuitively, μ and ν cannot be too small at the same time and hence there must be some inequality showing this. The following elementary but seemingly basic inequality was obtained in passing. Since I could not find it in the literature (including e.g. [1]), I take this opportunity to present it with a full proof (a sketch was given in [3]).

Theorem 1. *For each $d \geq 1$ and each measurable function $M(x)$ on $\mathbb{R}^d$ satisfying* (1.1.1)–(1.1.2), *we have, for $\mu = \mu_M$ and $\nu = \nu_M$,*[1]

$$\mu^{d/2}\nu \geq \left(\frac{2d}{d+4}\right)^{d/2} \frac{4\,\Gamma(\frac{d+4}{2})}{d+4}. \tag{1.1.15}$$

Moreover, equality holds if and only if $M(x)$ coincides almost everywhere with the function given in Example 3.

The minimum-giving Example 3 was found by using small deformations, which led to a simple differential equation of order 1. And once found, the proof is simple (and somewhat miraculous).

Proof. Let $M(x)$ be as at the beginning of this subsection, with invariants μ, ν. We shall prove

$$\mu^{d/2}\nu \geq (\mu_d^*)^{d/2}\nu_d^*, \tag{1.1.16}$$

where μ_d^*, ν_d^* are as defined by (1.1.14). We may assume that $M(x)$ is rotation-invariant, because averaging over $|x| = r$ does not change μ, while ν either decreases or remains the same. Therefore, $M(x) = f(|x|)$ with some non-negative real-valued function $f(r)$ of $r \geq 0$, and

(1.1.17)
$$\frac{1}{\gamma_d}\int_0^\infty f(r)r^{d-1}\,dr = 1,\ \frac{1}{\gamma_d}\int_0^\infty f(r)r^{d+1}\,dr = \mu,\ \frac{1}{\gamma_d}\int_0^\infty f(r)^2 r^{d-1}\,dr = \nu.$$

By a suitable scalar transform (1.1.8) we may assume that μ is any given positive real number, and so we assume $\mu = \mu_d^*$. We then have

(1.1.18)
$$\frac{1}{\gamma_d}\int_0^1 f(r)(1-r^2)r^{d-1}\,dr \geq \frac{1}{\gamma_d}\int_0^\infty f(r)(1-r^2)r^{d-1}\,dr = 1 - \mu_d^* = \frac{4}{d+4},$$

because the corresponding integral over $(1, \infty)$ is obviously non-positive. Now the Schwarz inequality gives

$$\left(\int_0^1 f_d^*(r)^2 r^{d-1}\,dr\right)\left(\int_0^1 f(r)^2 r^{d-1}\,dr\right) \geq \left(\int_0^1 f_d^*(r)f(r)r^{d-1}\,dr\right)^2. \tag{1.1.19}$$

[1]Here we just need the first definition of ν in (1.1.4) involving only $M(x)$.

Here, the first integral on the left hand side is nothing but $\gamma_d \nu_d^*$, while the right hand side is

$$(1.1.20)\quad \left(\frac{d(d+2)}{2}\right)^2 \gamma_d^2 \left(\int_0^1 f(r)(1-r^2)r^{d-1}\,dr\right)^2 \geq \left(\frac{d(d+2)}{2}\right)^2 \gamma_d^4 \left(\frac{4}{d+4}\right)^2,$$

by (1.1.11) and (1.1.18). Therefore, (1.1.19) gives

$$(1.1.21)\qquad \frac{1}{\gamma_d}\int_0^1 f(r)^2 r^{d-1}\,dr \geq \gamma_d^2 (\nu_d^*)^{-1} \left(\frac{2d(d+2)}{d+4}\right)^2 = \nu_d^*$$

by (1.1.14), and hence the desired inequality $\nu \geq \nu_d^*$. The last statement of Theorem 1 is clear from the above proof. □

In particular, for $d = 1, 2$, we obtain

Corollary 1.1.22. *We have*

$$(1.1.23)\qquad \mu^{1/2}\nu \geq (18\pi/125)^{1/2} \qquad (d = 1),$$

$$(1.1.24)\qquad \mu\nu \geq 8/9 \qquad (d = 2).$$

On the other hand, there is no upper bound for $\mu^{d/2}\nu$; indeed, if the support of $M(x)$ is concentrated on the sphere with center 0 and radius r, then μ is close to r^2 while ν can be arbitrarily large.

§1.2. The function $\tilde{M}(s; z_1, z_2)$

We shall review, mainly from [6, §4], the definition and some main properties of the function $\tilde{M}(s; z_1, z_2)$ and its local factors $\tilde{M}_{\mathfrak{p}}(s; z_1, z_2)$. Let K be any global field, i.e., either an algebraic number field of finite degree, or an algebraic function field of one variable over a finite field. Let $\mathfrak{p}$ be any non-archimedean prime of K. Define $\lambda_z(\mathfrak{p}^n)$ ($z \in \mathbb{C}$, $n \geq 0$) to be the coefficient of the power series

$$(1.2.1)\quad \sum_{n=0}^{\infty} \lambda_z(\mathfrak{p}^n) N(\mathfrak{p})^{-ns} = \begin{cases} \exp\left(\frac{iz}{2}\frac{d}{ds}\log((1-N(\mathfrak{p})^{-s})^{-1})\right) & \text{(Case 1)}, \\ \exp\left(\frac{iz}{2}\log((1-N(\mathfrak{p})^{-s})^{-1})\right) & \text{(Case 2)}, \end{cases}$$

of $N(\mathfrak{p})^{-s}$. It is a polynomial of z given by

$$(1.2.2)\qquad \lambda_z(\mathfrak{p}^n) = \begin{cases} F_n\left(-\frac{iz}{2}\log N(\mathfrak{p})\right) & \text{(Case 1)}, \\ F_n\left(\frac{iz}{2}\right) & \text{(Case 2)}, \end{cases}$$

with

$$F_n(x) = \begin{cases} \displaystyle\sum_{k=1}^{n} \frac{1}{k!}\binom{n-1}{k-1} x^k & \text{(Case 1)}, \\ \displaystyle\sum_{k=1}^{n} \frac{1}{k!}\delta_k(n) x^k = \frac{1}{n!} x(x+1)\dots(x+n-1) & \text{(Case 2)}, \end{cases} \tag{1.2.3}$$

for $n \geq 1$, and $F_0(x) = 1$, where

$$\delta_k(n) = \sum_{\substack{n=n_1+\cdots+n_k \\ n_1,\dots,n_k \geq 1}} \frac{1}{n_1 \dots n_k} \leq \sum_{\substack{n=n_1+\cdots+n_k \\ n_1,\dots,n_k \geq 1}} 1 = \binom{n-1}{k-1}. \tag{1.2.4}$$

Now the local $\mathfrak{p}$-factor $\tilde{M}_{\mathfrak{p}}(s; z_1, z_2)$ of $\tilde{M}(s; z_1, z_2)$ is a holomorphic function of (s, z_1, z_2) on $\{\Re(s) > 0\} \times \mathbb{C}^2$ defined by the following power series of $N(\mathfrak{p})^{-2s}$:

$$\tilde{M}_{\mathfrak{p}}(s; z_1, z_2) = \sum_{n=0}^{\infty} \lambda_{z_1}(\mathfrak{p}^n)\lambda_{z_2}(\mathfrak{p}^n) N(\mathfrak{p})^{-2ns}. \tag{1.2.5}$$

For a given finite set P_∞ of prime divisors of K including all the archimedean primes in the number field case, the global function $\tilde{M}(s; z_1, z_2)$, which is a holomorphic function of (s, z_1, z_2) on $\{\Re(s) > 1/2\} \times \mathbb{C}^2$, is defined by the Euler product

$$\tilde{M}(s; z_1, z_2) = \prod_{\mathfrak{p} \notin P_\infty} \tilde{M}_{\mathfrak{p}}(s; z_1, z_2), \tag{1.2.6}$$

which is absolutely convergent on $\Re(s) > 1/2$ in the following sense. For any given $\sigma_0 > 1/2$, $R > 0$, let $|z_1|, |z_2| \leq R$ and $\Re(s) \geq \sigma_0$. Then for all but finitely many primes $\mathfrak{p}$, we have $|\tilde{M}_{\mathfrak{p}}(s; z_1, z_2) - 1| < 1$, and the sum of $\log \tilde{M}_{\mathfrak{p}}(s; z_1, z_2)$ (the principal branch) over these $\mathfrak{p}$ converges absolutely and uniformly. It has a Dirichlet series expansion

$$\tilde{M}(s; z_1, z_2) = \sum_{D \text{ integral}} \lambda_{z_1}(D)\lambda_{z_2}(D) N(D)^{-2s} \qquad (\Re(s) > 1/2), \tag{1.2.7}$$

where D runs over the *integral divisors*, i.e., divisors of K of the form $D = \prod_{\mathfrak{p} \notin P_\infty} \mathfrak{p}^{n_{\mathfrak{p}}}$ ($n_{\mathfrak{p}} \geq 0$, $n_{\mathfrak{p}} = 0$ for almost all $\mathfrak{p}$), and $\lambda_z(D) = \prod_{\mathfrak{p} \notin P_\infty} \lambda_z(\mathfrak{p}^{n_{\mathfrak{p}}})$.

Other expressions. The local function $\tilde{M}_{\mathfrak{p}}(s; z_1, z_2)$ has an integral expression

$$\tilde{M}_{\mathfrak{p}}(s; z_1, z_2) = \int_{\mathbb{C}^1} \exp\left(\frac{i}{2}(z_1 g_{s,\mathfrak{p}}(t^{-1}) + z_2 g_{s,\mathfrak{p}}(t))\right) d^\times t, \tag{1.2.8}$$

where $g_{s,\mathfrak{p}}(t)$ is a continuous function on $\mathbb{C}^1 = \{t \in \mathbb{C};\ |t| = 1\}$ defined by

$$g_{s,\mathfrak{p}}(t) = \begin{cases} \dfrac{-(\log N(\mathfrak{p}))N(\mathfrak{p})^{-s}t}{1 - N(\mathfrak{p})^{-s}t} & \text{(Case 1)}, \\ -\log(1 - N(\mathfrak{p})^{-s}t) & \text{(Case 2)}, \end{cases} \tag{1.2.9}$$

(the principal branch of the logarithm), and $d^\times t$ is the normalized Haar measure on $\mathbb{C}^1$ (so that the total measure of $\mathbb{C}^1$ is 1). It also has the following power series expansion in z_1, z_2:

$$\tilde{M}_{\mathfrak{p}}(s; z_1, z_2) = 1 + \sum_{a,b \geq 1} (\pm i/2)^{a+b} \mu_{\mathfrak{p}}^{(a,b)}(s) \frac{z_1^a z_2^b}{a!b!}, \tag{1.2.10}$$

where the sign is *minus* (resp. *plus*) for Case 1 (resp. Case 2), and

$$\mu_{\mathfrak{p}}^{(a,b)}(s) = \begin{cases} (\log N(\mathfrak{p}))^{a+b} \displaystyle\sum_{n \geq \max(a,b)} \binom{n-1}{a-1}\binom{n-1}{b-1} N(\mathfrak{p})^{-2ns} & \text{(Case 1)}, \\ \displaystyle\sum_{n \geq \max(a,b)} \delta_a(n)\delta_b(n) N(\mathfrak{p})^{-2ns} & \text{(Case 2)}. \end{cases} \tag{1.2.11}$$

In particular,

$$\mu_{\mathfrak{p}}^{(1,1)}(s) = \begin{cases} (\log N(\mathfrak{p}))^2/(N(\mathfrak{p})^{2s} - 1) & \text{(Case 1)}, \\ \displaystyle\sum_{n \geq 1} n^{-2} N(\mathfrak{p})^{-2ns} & \text{(Case 2)}. \end{cases} \tag{1.2.12}$$

The global function $\tilde{M}(s; z_1, z_2)$, for each s with $\Re(s) > 1/2$, has an everywhere absolutely convergent power series expansion in z_1, z_2:

$$\tilde{M}(s; z_1, z_2) = 1 + \sum_{a,b \geq 1} (\pm i/2)^{a+b} \mu^{(a,b)}(s) \frac{z_1^a z_2^b}{a!b!}, \tag{1.2.13}$$

with the same choice of the sign as above. Here, each $\mu^{(a,b)}(s)$ denotes the following Dirichlet series which is absolutely convergent on $\Re(s) > 1/2$:

$$\mu^{(a,b)}(s) = \sum_{D \text{ integral}} \Lambda_a(D)\Lambda_b(D) N(D)^{-2s}, \tag{1.2.14}$$

where $\Lambda_k(D)$ (≥ 0) for each integral divisor D is defined by

$$\Lambda_k(D) = \sum_{D = D_1 \dots D_k} \Lambda_1(D_1) \dots \Lambda_1(D_k), \tag{1.2.15}$$

where

$$\Lambda_1(D) = \begin{cases} \log N(\mathfrak{p}) & \text{(Case 1)}, \\ 1/n & \text{(Case 2)}, \end{cases} \tag{1.2.16}$$

if $D = \mathfrak{p}^n$ with some $\mathfrak{p} \notin P_\infty$ and $n \geq 1$, and $\Lambda_1(D) = 0$ otherwise. By comparing the coefficients of $z_1 z_2$ for $\tilde{M}_\mathfrak{p}(s; z_1, z_2)$ and $\tilde{M}(s; z_1, z_2)$ in the formula (1.2.6), we obtain the Euler *sum* expansion (only for $(a, b) = (1, 1)$):

$$\mu(s) := \mu^{(1,1)}(s) = \sum_{\mathfrak{p} \notin P_\infty} \mu_\mathfrak{p}^{(1,1)}(s) \qquad (\Re(s) > 1/2). \tag{1.2.17}$$

Finally, let $M_\sigma(w)$ ($\sigma > 1/2$, $w \in \mathbb{C}$) denote the "M-function" defined and studied in [2] (Case 1) and [5] (Case 2). (In the latter, it is denoted by $\mathcal{M}_\sigma(w)$.) Then its Fourier dual is $\tilde{M}_\sigma(z) := \tilde{M}(\sigma; z, \bar{z})$. In fact, if ψ_{z_1,z_2} ($z_1, z_2 \in \mathbb{C}$) denotes the quasi-character $\mathbb{C} \to \mathbb{C}^\times$ defined by

$$\psi_{z_1,z_2}(w) = \exp\left(\frac{i}{2}(z_1 \overline{w} + z_2 w)\right), \tag{1.2.18}$$

and if we put $\psi_z = \psi_{z,\bar{z}}$ (which is a character $\mathbb{C} \to \mathbb{C}^1$), then we have

$$\tilde{M}(\sigma; z_1, z_2) = \int M_\sigma(w) \psi_{z_1,z_2}(w)\, |dw|, \tag{1.2.19}$$

$$M_\sigma(w) = \int \tilde{M}_\sigma(z) \psi_{-w}(z)\, |dz|, \tag{1.2.20}$$

where $|dw| = dudv/(2\pi)$ for $w = u + vi$, and the integrals are over the whole complex plane. Both $M_\sigma(w)$ and $\tilde{M}_\sigma(z)$ are continuous functions on $\mathbb{C}$ belonging to L^1; hence the Plancherel formula holds. Recall also ([6, §4.2]) that the center of gravity of $M_\sigma(w)\,|dw|$ is 0, and that $\mu(\sigma) = \mu^{(1,1)}(\sigma)$ ($\sigma > 1/2$) is equal to the variance

$$\mu_\sigma := \mu(\sigma) = \int M_\sigma(w) |w|^2\, |dw|. \tag{1.2.21}$$

It is easy to see (cf. §3 below) that $\lim_{\sigma \to 1/2} \mu_\sigma = +\infty$ and $\lim_{\sigma \to +\infty} \mu_\sigma = 0$ (Cases 1, 2).

Now let ν_σ denote the Plancherel volume of $M_\sigma(w)$. In connection with Examples 1, 2, 3 (§1.1), where $\mu\nu = 1$, 1, 8/9 (the minimal possible value) respectively for $d = 2$, we are interested in studying the product $\mu_\sigma \nu_\sigma$. First, some numerical evidences suggest that $\mu_\sigma \nu_\sigma$ is often quite close to 1. For example in Case 1, when $K = \mathbb{Q}$ (resp. $\mathbb{Q}(\sqrt{-1})$) and P_∞ consists of the unique archimedean prime of K, then $1 - \mu_1 \nu_1 = 0.017\ldots$ (resp. $0.018\ldots$). In §2, we shall study the limit behaviors of the variance-normalized function $\mu_\sigma M_\sigma(\mu_\sigma^{1/2} w)$ and that of

$\mu_\sigma\nu_\sigma$ as $\sigma \to 1/2$ and $\sigma \to \infty$ for general cases of (K, P_∞). Here, we just add, without proof (cf. [3] for a sketch of proof) the following

Example 4. Let $K = \mathbb{F}_q(x)$ be the rational function field over a finite field $\mathbb{F}_q$ and $P_\infty = \{\mathfrak{p}_\infty\}$, the unique prime at which $x = \infty$. Write $\mu_\sigma^{(q)}$ (resp. $\nu_\sigma^{(q)}$) for the variance (resp. the Plancherel volume) of $M_\sigma(w)\,|dw|$. Then for *any fixed* $\sigma > 1/2$, at least in Case 1, we have

$$\lim_{q\to\infty} \mu_\sigma^{(q)}\nu_\sigma^{(q)} = 1. \tag{1.2.22}$$

Is $\mu_\sigma\nu_\sigma$ related to some invariant with a different origin? Is there a complex analytic version of ν_σ?

§2. Limits at $\sigma = 1/2$ and $\sigma = +\infty$

Let $\tilde{M}(s; z_1, z_2)$, $M_\sigma(w)$, etc. be the functions defined in §1.2 associated with a given pair (K, P_∞) of a global field K and a finite set P_∞ of prime divisors of K including all archimedean primes in the number field case. Let μ_σ (resp. ν_σ) denote the variance (resp. the Plancherel volume; cf. §1.1) of the measure $M_\sigma(w)\,|dw|$ $(\sigma > 1/2)$ on $\mathbb{C}$ (considered as the two-dimensional Euclidean space in the obvious way). We shall study the limits, first at $\sigma = 1/2$, then briefly those at $\sigma = +\infty$, of the invariant $\mu_\sigma\nu_\sigma$ and of the variance-normalized function $\mu_\sigma M_\sigma(\mu_\sigma^{1/2} w)$. In this section, we shall state the main results, Theorem 2 for $\sigma \to 1/2$ and Theorem 3 for $\sigma \to +\infty$, and reduce the proof of Theorem 2 to Lemmas A, B and that of Theorem 3 to Lemmas A′, B′. Lemmas A, A′ concern the limits of

$$\tilde{M}\left(s; \frac{z_1}{\mu(s)^{1/2}}, \frac{z_2}{\mu(s)^{1/2}}\right)$$

as $s \to 1/2, +\infty$ respectively, where $\mu(s) = \mu^{(1,1)}(s)$ is the complex analytic version of μ_σ. Lemmas B, B′ are on the rapid decay property as $|z| \to \infty$ of the normalized Fourier dual $\tilde{M}_\sigma(z/\mu_\sigma^{1/2})$, when σ belongs to a small neighborhood of $1/2$, $+\infty$, respectively. The proofs of these lemmas will be postponed to later sections, except for Lemma A′. Because of its introductory nature, we have put this section right after §1, in spite of its logical dependence on later sections.

§2.1. The main results for $\sigma \to 1/2$

Theorem 2. (i) *As* $\sigma \to 1/2$,

$$\mu_\sigma \sim \begin{cases} (2\sigma - 1)^{-2} & \text{(Case 1)}, \\ \log \dfrac{1}{2\sigma - 1} & \text{(Case 2)}, \end{cases} \tag{2.1.1}$$

where $\sim$ *means that the ratio of two sides tends to* 1.

(ii) *We have*

$$\lim_{\sigma\to 1/2} \mu_\sigma \nu_\sigma = 1 \qquad \text{(Cases 1, 2)}. \tag{2.1.2}$$

(iii)[1] *We have*

$$\lim_{\sigma\to 1/2} \mu_\sigma M_\sigma(\mu_\sigma^{1/2} w) = 2e^{-|w|^2} \qquad (w \in \mathbb{C}) \ \text{(Cases 1, 2)}. \tag{2.1.3}$$

These answer "the $\lim_{\sigma\to 1/2}$-version" of the questions raised in [2, Remark 3.11.17].

§2.2. Proof of Theorem 2(i)

This follows directly from (1.2.12) and (1.2.17). But a more economical way is to rely on a result of §3 below; namely (with the notations of §3.1, §3.3), by the last assertion of Theorem 4, the difference $\mu(s) - \phi^{(2\kappa)}(2s)$ extends to a holomorphic function on $\Re(s) > 1/4$. Hence

$$\lim_{\substack{s\to 1/2 \\ |\mathrm{Arg}(2s-1)|<\pi}} \frac{\mu(s)}{\phi^{(2\kappa)}(2s)} = 1; \tag{2.2.1}$$

hence

$$\lim_{s\to 1/2} (2s-1)^2 \mu(s) = 1 \qquad \text{(Case 1)}, \tag{2.2.2}$$

$$\lim_{\substack{s\to 1/2 \\ |\mathrm{Arg}(2s-1)|<\pi}} \frac{\mu(s)}{\log \frac{1}{2s-1}} = 1 \qquad \text{(Case 2)}, \tag{2.2.3}$$

as desired.

For any s with $|2s-1| \ll 1$ and $|\mathrm{Arg}(2s-1)| < \pi$, we define $\mu(s)^{1/2}$ to be the square root taking positive value when s is real and $> 1/2$.

§2.3. The Key Lemmas A, B

The first key lemma is Corollary 3.4.8 (§3.4) of Theorem 5, to be proved in §3.

Lemma A. *We have*

$$\lim_{\substack{s\to 1/2 \\ |\mathrm{Arg}(2s-1)|<\pi}} \tilde{M}\left(s; \frac{z_1}{\mu(s)^{1/2}}, \frac{z_2}{\mu(s)^{1/2}}\right) = \exp\left(-\frac{z_1 z_2}{4}\right), \tag{2.3.1}$$

and the convergence is uniform on $|z_1|, |z_2| \le R$ for any given $R > 0$.

[1]The author is grateful to S. Takanobu for helpful discussions which led to this generalized form of the result.

The second key lemma states a rapid decay property of the function $\tilde{M}_\sigma(z) := \tilde{M}(\sigma; z, \bar{z})$ of $z \in \mathbb{C}$, to be proved in §4.6.

Lemma B. *Fix any ϵ with $0 < \epsilon < 1$. If $(2\sigma - 1)^{-1} \gg_\epsilon 1$, then*

$$|\tilde{M}_\sigma(z)|^2 \leq \exp\left(-\frac{1-\epsilon}{2}\mu_\sigma |z|^{2(1-\epsilon')}\right) \tag{2.3.2}$$

for all $z \in \mathbb{C}$, where $\epsilon' = \epsilon$ (resp. 0) for $|z| \geq 1$ (resp. $|z| < 1$).

§2.4. Proofs of Theorem 2(ii)&(iii) assuming Lemmas A, B

Proof of (ii). Note first that

$$\mu_\sigma \nu_\sigma = \int |\tilde{M}_\sigma(\mu_\sigma^{-1/2} z)|^2 \, |dz|. \tag{2.4.1}$$

For each fixed z, when $\sigma \to 1/2$, the integrand tends to $\exp(-|z|^2/2)$ by Lemma A. In order to apply Lebesgue's convergence theorem to deduce that the $\lim_{\sigma \to 1/2}$ operation commutes with integration, we only need to show that the integrand is uniformly bounded near $\sigma = 1/2$ (we may thus assume $\mu_\sigma > 1$) by an integrable function of z. But this follows directly from Lemma B. In fact, Lemma B for $\epsilon = 1/2$ gives

$$|\tilde{M}_\sigma(\mu_\sigma^{-1/2} z)|^2 \leq \max(\exp(-|z|/4), \exp(-|z|^2/4)), \tag{2.4.2}$$

which is integrable. Therefore,

$$\begin{aligned} \lim_{\sigma \to 1/2} \mu_\sigma \nu_\sigma &= \int \lim_{\sigma \to 1/2} |\tilde{M}_\sigma(\mu_\sigma^{-1/2} z)|^2 \, |dz| = \int \exp(-|z|^2/2) \, |dz| \\ &= \int_0^\infty e^{-r^2/2} r \, dr = 1, \end{aligned} \tag{2.4.3}$$

as desired.

Proof of (iii). The Fourier inversion formula (1.2.20) gives

$$\mu_\sigma M_\sigma(\mu_\sigma^{1/2} w) = \int \tilde{M}_\sigma(\mu_\sigma^{-1/2} z) \psi_{-w}(z) \, |dz|. \tag{2.4.4}$$

By Lemma A and (2.4.2), we can also apply Lebesgue's convergence theorem and hence obtain

$$\lim_{\sigma \to 1/2} \mu_\sigma M_\sigma(\mu_\sigma^{1/2} w) = \int \exp(-|z|^2/4) \psi_{-w}(z) \, |dz| = 2e^{-|w|^2}, \tag{2.4.5}$$

as desired. □

§2.5. The main results for $\sigma \to +\infty$

The following numerical invariants of the pair (K, P_∞),

$$\alpha := \min_{\mathfrak{p} \notin P_\infty} N(\mathfrak{p}), \qquad m := |\{\mathfrak{p} \notin P_\infty;\, N(\mathfrak{p}) = \alpha\}|$$

($|\ |$ denoting cardinality), and the Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n} \tag{2.5.1}$$

will be involved. Clearly, $\alpha \geq 2$ and $m \geq 1$. The main result corresponding to Theorem 2 is the following:

Theorem 3. (i) *As* $\sigma \to +\infty$,

$$\mu_\sigma \sim \begin{cases} (\log\alpha)^2 m\alpha^{-2\sigma} & \text{(Case 1)}, \\ m\alpha^{-2\sigma} & \text{(Case 2)}. \end{cases} \tag{2.5.2}$$

(ii) *In each of Cases* 1, 2,

$$\lim_{\sigma\to+\infty} \mu_\sigma \nu_\sigma = m \int_0^\infty J_0(x)^{2m} x\, dx \begin{cases} = \infty & (m \leq 2), \\ < \infty & (m \geq 3). \end{cases} \tag{2.5.3}$$

(iii) *In each of Cases* 1, 2, *at least if* $m \geq 5$, *we have*

$$\lim_{\sigma\to+\infty} \mu_\sigma M_\sigma(\mu_\sigma^{1/2} w) = \int_0^\infty J_0(|w|x) J_0(x/\sqrt{m})^m x\, dx. \tag{2.5.4}$$

Moreover, the support of this function is compact, being contained in $\{w \in \mathbb{C};\ |w| \leq \sqrt{m}\}$.

§2.6. Proof of Theorem 3(i)

We shall prove a slightly stronger result;

$$\lim_{\sigma=\Re(s)\to+\infty} \alpha^{2s}\mu(s) = \begin{cases} (\log\alpha)^2 m & \text{(Case 1)}, \\ m & \text{(Case 2)}, \end{cases} \tag{2.6.1}$$

with the uniformity of convergence with respect to $\Im(s)$. First, by (1.2.12) and (1.2.17) we have

$$\alpha^{2s}\mu(s) = \alpha^{2s} \sum_{\mathfrak{p}\notin P_\infty} \mu_{\mathfrak{p}}^{(1,1)}(s) = \sum_{\substack{\mathfrak{p}\notin P_\infty \\ n\geq 1}} a(\mathfrak{p}^n)(\alpha/N(\mathfrak{p})^n)^{2s}, \tag{2.6.2}$$

where $a(\mathfrak{p}^n) = (\log N(\mathfrak{p}))^2$ (resp. $1/n^2$) for Case 1 (resp. Case 2). Now decompose the sum into three parts; the first sum, over those $(\mathfrak{p}, n)$ satisfying $N(\mathfrak{p}) = \alpha$

and $n = 1$, gives the right hand side of (2.6.1); the second, over $N(\mathfrak{p}) > \alpha$, is $\ll (\alpha/\alpha')^{2\sigma-2}$, where α' denotes the second smallest norm of primes outside P_∞; the rest is over $N(\mathfrak{p}) = \alpha$, $n \geq 2$, which is $\ll \alpha^{-2\sigma}$. Since the latter two partial sums tend to 0 uniformly with respect to $\Im(s)$, this proves (2.6.1).

In particular, $\mu(s) \neq 0$ for $\Re(s)$ sufficiently large. We shall denote by $\mu(s)^{1/2}$ its unique square root that takes positive values when $s = \sigma > 1$.

§2.7. The Key Lemmas A′, B′

The counterparts of Lemmas A, B for $\lim_{\sigma\to+\infty}$ are the following.

Lemma A′. *We have*

$$\lim_{\sigma=\Re(s)\to+\infty} \tilde{M}\left(s; \frac{z_1}{\mu(s)^{1/2}}, \frac{z_2}{\mu(s)^{1/2}}\right) = J_0\left(\sqrt{\frac{z_1 z_2}{m}}\right)^m, \tag{2.7.1}$$

and the convergence is uniform on $|z_1|, |z_2| \leq R$ *for any given* $R > 0$ *and with respect to* $\Im(s)$.

The proof will be sketched in §2.9.

Lemma B′. *There exists a constant* $C > 0$ *depending only on* (K, P_∞) *such that*

$$|\tilde{M}_\sigma(\mu_\sigma^{-1/2} z)| \leq C|z|^{-m/2} \tag{2.7.2}$$

for all $\sigma \geq 1$ *and all* $z \in \mathbb{C}$.

The proof will be postponed to §4.1.

§2.8. Proofs of Theorem 3(ii)&(iii) assuming Lemmas A′, B′

Proof of (ii). The limit formula (2.5.3) for $m \geq 3$ can be obtained from Lemmas A′, B′ exactly in the same manner as in the proof of Theorem 2(ii). For $m \leq 2$, the divergences can be checked easily.

Proof of (iii). When $m \geq 5$, there is again no problem. (The term $J_0(|w|x)$ appears as the average of $\psi_{-w}(z)$ over the circle $|z| = x$.) It is likely that the same equality holds also for smaller m. But it should be noted that the limit function of w need not be continuous. Especially when $m = 1$, the limit is not even a function, but a hyperfunction with support on the unit circle $|w| = 1$. This is because in the limit $\sigma \to +\infty$, only the contribution of the unique prime $\mathfrak{p}$ with $N(\mathfrak{p}) = \alpha$ remains.

As for the statement on the support, we can see this in two ways. Firstly, by construction [2, 5], the support of $M_\sigma(w)$ for $\sigma > 1$ is contained in $|w| \leq \rho_\sigma$, where

$$\rho_\sigma = \begin{cases} -\dfrac{d}{d\sigma}\log\zeta_{K,P_\infty}(\sigma) \sim m\alpha^{-\sigma}\log\alpha & \text{(Case 1)}, \\ \log\zeta_{K,P_\infty}(\sigma) \sim m\alpha^{-\sigma} & \text{(Case 2)}; \end{cases} \tag{2.8.1}$$

hence $\lim_{\sigma\to+\infty} \rho_\sigma/\mu_\sigma^{1/2} = \sqrt{m}$. Secondly, from the right hand side of (2.5.4), one can also see this by a result of Nicholson (cf. [9, §13.46]), which asserts that if $\Re(\nu) > -1$, $a_1, \ldots, a_m > 0$, $b > a_1 + \cdots + a_m$, then

$$\int_0^\infty x^{\nu(1-m)+1} J_\nu(bx) \prod_{i=1}^m J_\nu(a_i x)\, dx = 0 \tag{2.8.2}$$

(our m corresponds to $m-1$ in [9]). Apply this for $\nu = 0$, $a_1 = \cdots = a_m = 1/\sqrt{m}$, $b = |w|$, to see that the right hand side of (2.5.4) vanishes for $|w| \geq \sqrt{m}$.

Remark 2.8.3. As for the value of the right hand side of (2.5.3), i.e.,

$$a(m) := m \int_0^\infty J_0(x)^{2m} x\, dx, \tag{2.8.4}$$

we have $a(3) = 1.01\ldots$, $a(4) = 0.951\ldots$, $a(5) = 0.953\ldots$, etc., and one can prove that $\lim_{m\to\infty} a(m) = 1$. Numerical evidence suggests that $\lim_{m\to\infty} m(1 - a(m)) = 1/4$.

§2.9. Sketch of the proof of Lemma A′

Let $\Re(s)$ be sufficiently large and choose $\mu(s)^{1/2}$ as at the end of §2.6. The power series expansion (1.2.13) of $\tilde{M}(s; z_1, z_2)$ gives

$$\tilde{M}\left(s; \frac{z_1}{\mu(s)^{1/2}}, \frac{z_2}{\mu(s)^{1/2}}\right) = 1 + \sum_{a,b\geq 1} (\pm i/2)^{a+b} \frac{\mu^{(a,b)}(s)}{\mu(s)^{(a+b)/2}} \frac{z_1^a z_2^b}{a!b!}. \tag{2.9.1}$$

Here, as in (1.2.13), the sign of $i/2$ is *minus* (Case 1), or *plus* (Case 2). On the other hand, the expansion (2.5.1) for $J_0(x)$ gives

$$J_0\left(\left(\frac{z_1 z_2}{m}\right)^{1/2}\right)^m = 1 + \sum_{a,b\geq 1} (-i/2)^{a+b} \tilde{\mu}^{(a,b)} \frac{z_1^a z_2^b}{a!b!}, \tag{2.9.2}$$

where

$$\tilde{\mu}^{(a,b)} = \begin{cases} 0 & (a \neq b), \\ m^{-k} \displaystyle\sum_{k=k_1+\cdots+k_m} \binom{k}{k_1, \ldots, k_m}^2 & (a = b = k \geq 1). \end{cases} \tag{2.9.3}$$

So, it is enough to prove the existence of constants $\sigma_0 > 1$ and $C > 0$, each depending only on (K, P_∞), such that

$$\frac{\mu^{(a,b)}(s)}{\mu(s)^{(a+b)/2}} - \tilde{\mu}^{(a,b)} \ll \frac{C^{a+b}}{\sigma - \sigma_0 - 1} \qquad (\sigma = \Re(s) > \sigma_0 + 1). \tag{2.9.4}$$

Note that the left hand side of (2.9.4) is 0 when $a = b = 1$.

To prove (2.9.4), note first that (1.2.14) gives

$$\alpha^{(a+b)s}\mu^{(a,b)}(s) = \sum_{D\,\text{integral}} \Lambda_a(D)\Lambda_b(D)(\alpha^{a+b}/N(D)^2)^s = \sum_{D\,\text{integral}}{}' \Lambda_a(D)\Lambda_b(D)(\alpha^{a+b}/N(D)^2)^s, \tag{2.9.5}$$

where $\sum'$ denotes the sum over the non-vanishing terms.

Proposition 2.9.6. *Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ be all the distinct primes $\notin P_\infty$ with norm α. Let $k \geq 1$ and D be any integral divisor. If $\Lambda_k(D) \neq 0$, then $N(D) \geq \alpha^k$, and equality holds if and only if D has the form $D = \prod_{i=1}^m \mathfrak{p}_i^{k_i}$ with $\sum k_i = k$. Moreover, in this case,*

$$\Lambda_k(D) = \binom{k}{k_1, \ldots, k_m}(\log\alpha)^{\kappa k}, \tag{2.9.7}$$

where $\kappa = 1$ (Case 1) *or $\kappa = 0$* (Case 2).

This is almost obvious. By this proposition, we may rewrite (2.9.5) as $I^{(a,b)} + II^{(a,b)}(s)$, where

$$I^{(a,b)} = \sum_{N(D)^2=\alpha^{a+b}}{}' \Lambda_a(D)\Lambda_b(D) = \begin{cases} 0 & (a \neq b), \\ \left(\sum_{k=k_1+\cdots+k_m} \binom{k}{k_1, \ldots, k_m}^2\right)(\log\alpha)^{\kappa(a+b)} & (a = b = k), \end{cases}$$

$$II^{(a,b)}(s) = \sum_{N(D)^2>\alpha^{a+b}}{}' \Lambda_a(D)\Lambda_b(D)(\alpha^{a+b}/N(D)^2)^s.$$

In particular, $I^{(1,1)} = m(\log\alpha)^{2\kappa}$; hence

$$\tilde{\mu}^{(a,b)} = \frac{I^{(a,b)}}{(I^{(1,1)})^{(a+b)/2}}. \tag{2.9.8}$$

Therefore,

$$\frac{\mu^{(a,b)}(s)}{\mu(s)^{(a+b)/2}} - \tilde{\mu}^{(a,b)} = \frac{I^{(a,b)} + II^{(a,b)}(s)}{(I^{(1,1)} + II^{(1,1)}(s))^{(a+b)/2}} - \frac{I^{(a,b)}}{(I^{(1,1)})^{(a+b)/2}}. \tag{2.9.9}$$

In order to estimate the quantity $II^{(a,b)}(s)$, we need the following

Proposition 2.9.10. *There exists $\sigma_0 > 1$ depending only on (K, P_∞) such that*

$$\Lambda_k(D) < N(D)^{\sigma_0} \tag{2.9.11}$$

for any D and any $k \geq 1$.

The point is that the present bound is independent of k.

Proof. Since $\lim_{\sigma\to+\infty}(\zeta'_{K,P_\infty}/\zeta_{K,P_\infty})(\sigma) = \lim_{\sigma\to+\infty}\log\zeta_{K,P_\infty}(\sigma) = 0$, we have

$$0 < \sum_{D \text{ integral}} \frac{\Lambda_1(D)}{N(D)^{\sigma_0}} < 1$$

for sufficiently large $\sigma_0 > 1$. But then its k-th power is also < 1; hence

$$\sum_{D \text{ integral}} \frac{\Lambda_k(D)}{N(D)^{\sigma_0}} < 1. \tag{2.9.12}$$

Since each summand is non-negative, this implies $\Lambda_k(D) < N(D)^{\sigma_0}$ for each D, as desired. □

By using Proposition 2.9.10, we can easily derive

$$|II^{(a,b)}(s)| \ll \frac{(\alpha^{\sigma_0+1})^{a+b}}{\sigma - \sigma_0 - 1} \qquad (\sigma = \Re(s) > \sigma_0 + 1), \tag{2.9.13}$$

and by combining these we obtain (2.9.4) directly.

§3. Analytic continuations

§3.1. Local formal power series

In connection with the local factors of $\tilde{M}(s; z_1, z_2)$, we consider, in each of Cases 1, 2, the following formal power series $F = F(x_1, x_2; t)$ in three variables:

$$F(x_1, x_2; t) = \sum_{n=0}^{\infty} F_n(x_1)F_n(x_2)t^n = 1 + \sum_{n=1}^{\infty} F_n(x_1)F_n(x_2)t^n, \tag{3.1.1}$$

where each $F_n(x)$ is a polynomial of x of degree n defined by (1.2.3), or equivalently, by the generating functions

$$\exp\left(\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} F_n(x)t^n \qquad \text{(Case 1)}, \tag{3.1.2}$$

$$\exp(-x\log(1-t)) = (1-t)^{-x} = \sum_{n=0}^{\infty} F_n(x)t^n \qquad \text{(Case 2)}. \tag{3.1.3}$$

Note that each monomial $x_1^a x_2^b t^n$ appearing in $F - 1$ satisfies $1 \le a, b \le n$, and has a positive rational coefficient. Define also the formal power series $\log F$ by

$\sum_{k=1}^{\infty}(-1)^{k-1}(F-1)^k/k$, and express it as a power series of x_1, x_2, t, as

$$\log F(x_1,x_2;t) = \sum_{a,b,n\geq 1} \beta_n^{(a,b)} \frac{x_1^a x_2^b}{a!b!} t^n \qquad (\beta_n^{(a,b)} \in \mathbb{Q}) \tag{3.1.4}$$

$$= \sum_{n\geq 1} B_n(x_1,x_2)t^n = \sum_{a,b\geq 1} B^{(a,b)}(t)\frac{x_1^a x_2^b}{a!b!}.$$

Note that $\beta_n^{(a,b)} = 0$ if $n < \max(a,b)$; hence

$$B_n(x_1,x_2) = \sum_{1\leq a,b\leq n} \beta_n^{(a,b)} \frac{x_1^a x_2^b}{a!b!}, \tag{3.1.5}$$

$$B^{(a,b)}(t) = \sum_{n\geq \max(a,b)} \beta_n^{(a,b)} t^n. \tag{3.1.6}$$

For example,

$$B^{(1,1)}(t) = \begin{cases} t(1-t)^{-1} & \text{(Case 1)}, \\ \displaystyle\sum_{n=1}^{\infty} n^{-2}t^n & \text{(Case 2)}; \end{cases} \tag{3.1.7}$$

hence $\beta_n^{(1,1)} = 1$ (Case 1) and $\beta_n^{(1,1)} = n^{-2}$ (Case 2).

In connection with the local factors of higher logarithmic derivatives of the zeta function, we also consider the power series

$$\ell(t) = \ell_0(t) = -\log(1-t), \tag{3.1.8}$$

and for each $k \geq 0$,

$$\ell_k(t) = \left(t\frac{d}{dt}\right)^k \ell(t) = \sum_{n=1}^{\infty} n^{k-1}t^n = t + \cdots. \tag{3.1.9}$$

They have the generating function

$$\ell(te^u) = \sum_{k=0}^{\infty} \frac{\ell_k(t)}{k!} u^k. \tag{3.1.10}$$

Put

$$\kappa = \begin{cases} 1 & \text{(Case 1)}, \\ 0 & \text{(Case 2)}. \end{cases}$$

For each fixed $a, b \geq 1$, $\{\ell_{\kappa(a+b)}(t^n)\}_{n=1,2,\ldots}$ forms a $\mathbb{Q}$-linear topological basis of the power series algebra $\mathbb{Q}[[t]]$ equipped with the t-adic topology. Hence there exists a unique system $\{\gamma_n^{(a,b)}\}_{n,a,b\geq 1}$ of rational numbers such that

$$B^{(a,b)}(t) = \sum_{n \geq 1} \gamma_n^{(a,b)} \ell_{\kappa(a+b)}(t^n) \tag{3.1.11}$$

for any $a, b \geq 1$. It is clear from the definition that $\gamma_n^{(a,b)} = 0$ if $n < \max(a, b)$, and that

$$\beta_m^{(a,b)} = \sum_{n|m} \gamma_n^{(a,b)} (m/n)^{\kappa(a+b)-1} \tag{3.1.12}$$

$(m = 1, 2, \dots)$; hence the Möbius inversion formula gives

$$\gamma_n^{(a,b)} = \sum_{d|n} \mu(n/d)(n/d)^{\kappa(a+b)-1} \beta_d^{(a,b)}. \tag{3.1.13}$$

For example, $\gamma_1^{(1,1)} = 1$, and for $n > 1$, $\gamma_n^{(1,1)} = \prod_{\ell|n}(1 - \ell)$ in Case 1, and n^{-2} times this quantity in Case 2, where ℓ runs over all prime factors of n. By (3.1.4) and (3.1.11), we have the formal equality

$$\log F(x_1, x_2; t) = \sum_{\substack{n,a,b \geq 1 \\ n \geq \max(a,b)}} \gamma_n^{(a,b)} \ell_{\kappa(a+b)}(t^n) \frac{x_1^a x_2^b}{a!b!}. \tag{3.1.14}$$

§3.2. Local analytic functions

We start with the following

Proposition 3.2.1. (i) *$F(x_1, x_2; t)$ defines a holomorphic function of $x_1, x_2, t \in \mathbb{C}$ on $|t| < 1$.*

(ii) *Let $R > 0$, $0 < r < 1$ and $|x_1|, |x_2| \leq R$, $|t| \leq r$, and suppose that one of r, R is fixed and the other is sufficiently small. Then $|F(x_1, x_2; t) - 1| < 1$; hence the (principal branch) logarithm $\log F(x_1, x_2; t)$ is holomorphic on this domain.*

Proof. Note first that the equalities (3.1.2) for Case 1 and (3.1.3) for Case 2 are valid also as formulas for analytic functions of x, t on $|t| < 1$. Note also that the coefficients of $F_n(x)$ are non-negative. Thus, for any $N \geq 1$ and $|x_1|, |x_2| \leq R$, $|t| \leq r$, we have

$$\begin{aligned} \sum_{n=1}^{N} |F_n(x_1)F_n(x_2)t^n| &\leq \sum_{n=1}^{N} F_n(R)^2 r^n \leq \Big(\sum_{n=1}^{N} F_n(R) r^{n/2}\Big)^2 \\ &< \Big(\sum_{n=1}^{\infty} F_n(R) r^{n/2}\Big)^2 = \begin{cases} \left(\exp\left(\dfrac{Rr^{1/2}}{1 - r^{1/2}}\right) - 1\right)^2 & \text{(Case 1)}, \\ ((1 - r^{1/2})^{-R} - 1)^2 & \text{(Case 2)}. \end{cases} \end{aligned} \tag{3.2.2}$$

The rest is obvious.

Corollary 3.2.3. (i) *For each* $a, b \geq 1$*, the series* (3.1.6) *converges absolutely on* $|t| < 1$ *and hence defines a holomorphic function* $B^{(a,b)}(t)$ *on* $|t| < 1$*.*

(ii) *Under the assumptions of Proposition* 3.2.1(ii)*, the three series in* (3.1.4) *are absolutely convergent, and the three equalities there are valid as equalities for analytic functions.*

Now let $\mathfrak{p}$ be any non-archimedean prime divisor of the base field K, and put

$$\lambda_{\mathfrak{p}} = (-\log N(\mathfrak{p}))^{\kappa} = \begin{cases} -\log N(\mathfrak{p}) & \text{(Case 1)}, \\ 1 & \text{(Case 2)}. \end{cases} \tag{3.2.4}$$

Then it follows directly from the definitions (§1.2) that

$$\tilde{M}_{\mathfrak{p}}(s; z_1, z_2) = F((i\lambda_{\mathfrak{p}}/2)z_1, (i\lambda_{\mathfrak{p}}/2)z_2; N(\mathfrak{p})^{-2s}) \tag{3.2.5}$$

($s, z_1, z_2 \in \mathbb{C}$, $\Re(s) > 0$). For each pair (a, b) of $a, b \geq 1$, define the holomorphic function $\mathbf{B}_{\mathfrak{p}}^{(a,b)}(s)$ on $\Re(s) > 0$ by

$$\mathbf{B}_{\mathfrak{p}}^{(a,b)}(s) = \lambda_{\mathfrak{p}}^{a+b} B^{(a,b)}(N(\mathfrak{p})^{-2s}). \tag{3.2.6}$$

In the special case $a = b = 1$, we have, by (1.2.12) and (3.1.7),

$$\mathbf{B}_{\mathfrak{p}}^{(1,1)}(s) = \mu_{\mathfrak{p}}^{(1,1)}(s) = \begin{cases} (\log N(\mathfrak{p}))^2 (N(\mathfrak{p})^{2s} - 1)^{-1} & \text{(Case 1)}, \\ \sum_{n=1}^{\infty} n^{-2} N(\mathfrak{p})^{-2ns} & \text{(Case 2)}. \end{cases} \tag{3.2.7}$$

Corollary 3.2.8. *Let* $R > 0$*,* $\alpha \geq 2$*,* $\sigma_0 > 0$*, and* $|z_1|, |z_2| \leq R$*,* $N(\mathfrak{p}) \geq \alpha$*,* $\Re(s) \geq \sigma_0$*. Suppose that two of* R, α, σ_0 *are fixed and the remaining one, if* R*, is sufficiently small while if* α *or* σ_0*, is sufficiently large. Then* $|\tilde{M}_{\mathfrak{p}}(s; z_1, z_2) - 1| < 1$*, and*

$$\log \tilde{M}_{\mathfrak{p}}(s; z_1, z_2) = \sum_{a,b \geq 1} \mathbf{B}_{\mathfrak{p}}^{(a,b)}(s) (i/2)^{a+b} \frac{z_1^a z_2^b}{a!b!}. \tag{3.2.9}$$

Proof. In Case 2, this is obvious by (3.2.5) and Corollary 3.2.3(ii). In Case 1, the difference between $|z_\nu|$ and $|x_\nu|$ ($\nu = 1, 2$) involves $\log N(\mathfrak{p})$. But since $(\log N(\mathfrak{p})) N(\mathfrak{p})^{-\sigma_0}$ is bounded and it tends to 0 when one of α, σ_0 tends to ∞, the same proof works. □

Now put

$$\phi_{\mathfrak{p}}(s) = \ell(N(\mathfrak{p})^{-s}) = -\log(1 - N(\mathfrak{p})^{-s}) \quad (\Re(s) > 0), \tag{3.2.10}$$

and for $k = 0, 1, 2, \ldots,$

$$\phi_{\mathfrak{p}}^{(k)}(s) = \frac{d^k \phi_{\mathfrak{p}}}{ds^k}(s) = (-\log N(\mathfrak{p}))^k \ell_k(N(\mathfrak{p})^{-s}) \qquad (\Re(s) > 0). \tag{3.2.11}$$

In particular, for $k = \kappa(a+b)$ and $n = 1, 2, \ldots,$

$$\phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns) = \lambda_{\mathfrak{p}}^{a+b} \ell_{\kappa(a+b)}(N(\mathfrak{p})^{-2ns}). \tag{3.2.12}$$

The formal equalities (3.1.11), (3.1.14) suggest that the corresponding analytic equalities

$$\mathbf{B}_{\mathfrak{p}}^{(a,b)}(s) = \sum_{n \geq \max(a,b)} \gamma_n^{(a,b)} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns), \tag{3.2.13}$$

$$\log \tilde{M}_{\mathfrak{p}}(s; z_1, z_2) = \sum_{a,b,n \geq 1} \gamma_n^{(a,b)} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns)(i/2)^{a+b} \frac{z_1^a z_2^b}{a!b!} \tag{3.2.14}$$

would hold on some suitable domain where $\tilde{M}_{\mathfrak{p}}(s; z_1, z_2)$ does not vanish. Note that the coefficients $\gamma_n^{(a,b)}$ are independent of $\mathfrak{p}$, so that under some further conditions the globalization would be possible. Our aim is to establish these results (Theorems 4, 5).

§3.3. The global analytic functions of s

First, we define the functions $\mathbf{B}^{(a,b)}(s)$ $(a, b \geq 1)$ of s.

Proposition 3.3.1. *Let $a, b \geq 1$. Then the sum*

$$\mathbf{B}^{(a,b)}(s) := \sum_{\mathfrak{p} \notin P_\infty} \mathbf{B}_{\mathfrak{p}}^{(a,b)}(s) \tag{3.3.2}$$

converges absolutely and uniformly on $\sigma = \Re(s) \geq (1+\epsilon)/(2\max(a,b))$ for any $\epsilon > 0$, thereby defining a holomorphic function on $\sigma > 1/(2\max(a,b))$.

Proof. Since $N(\mathfrak{p})^{-2\sigma} \leq 2^{-1/\max(a,b)}$, and since $B^{(a,b)}(t)/t^{\max(a,b)}$ is holomorphic on $|t| < 1$ and hence bounded on $|t| \leq 2^{-1/\max(a,b)}$, we have, by (3.2.6),

$$|\mathbf{B}_{\mathfrak{p}}^{(a,b)}(s)| \ll_{a,b} (\log N(\mathfrak{p}))^{a+b} N(\mathfrak{p})^{-2\sigma \max(a,b)} \leq (\log N(\mathfrak{p}))^{a+b} N(\mathfrak{p})^{-1-\epsilon},$$

whose sum over $\mathfrak{p} \notin P_\infty$ converges. ☐

In the special case $a = b = 1$, we have, by (1.2.17) and (3.2.7),

$$\mathbf{B}^{(1,1)}(s) = \mu^{(1,1)}(s). \tag{3.3.3}$$

Now we shall define the functions $\phi^{(k)}(s)$. Let $\zeta(s) = \zeta_{K,P_\infty}(s)$ be the zeta function of K without P_∞ factors, defined by the Euler product expansion

$$\prod_{\mathfrak{p} \notin P_\infty} (1 - N(\mathfrak{p})^{-s})^{-1} \qquad (\Re(s) > 1) \tag{3.3.4}$$

and by analytic continuation on the whole complex plane. Let

$$\phi(s) = \log \zeta(s), \tag{3.3.5}$$

where the branch of the logarithm is the one that tends to 0 as $\Re(s)$ tends to $+\infty$. It is holomorphic on $\Re(s) > 1$ and is a multivalued analytic function on $\mathbb{C}$ where $\zeta(s) \neq \infty, 0$. For each $k \geq 0$, $\phi^{(k)}(s)$ will denote the kth derivative of $\phi(s)$ with respect to s. Thus, $\phi^{(0)}(s) = \log \zeta(s)$, and for each $k \geq 1$,

$$\phi^{(k)}(s) = \frac{d^{k-1}}{ds^{k-1}}(\zeta'(s)/\zeta(s)) \tag{3.3.6}$$

is a meromorphic function on $\mathbb{C}$. By these definitions we have, for each $k \geq 0$,

$$\phi^{(k)}(s) = \sum_{\mathfrak{p} \notin P_\infty} \phi_{\mathfrak{p}}^{(k)}(s) \qquad (\Re(s) > 1); \tag{3.3.7}$$

hence for each $n \geq 1$,

$$\phi^{(k)}(2ns) = \sum_{\mathfrak{p} \notin P_\infty} \phi_{\mathfrak{p}}^{(k)}(2ns) \qquad (\Re(s) > 1/(2n)). \tag{3.3.8}$$

In particular, if $n \geq \max(a, b)$, then $\phi^{(\kappa(a+b))}(2ns)$ is holomorphic on $\Re(s) > 1/(2\max(a, b))$.

Theorem 4. *Let $a, b \geq 1$. Then the equality*

$$\mathbf{B}^{(a,b)}(s) = \sum_{n \geq \max(a,b)} \gamma_n^{(a,b)} \phi^{(\kappa(a+b))}(2ns) \tag{3.3.9}$$

holds in the following sense. (i) *For any $N \geq \max(a, b) - 1$ and $\epsilon > 0$, the sum over $n \geq N + 1$ on the right hand side converges absolutely and uniformly on $\sigma = \Re(s) \geq (1+\epsilon)/(2(N+1))$, and* (ii) *the equality* (3.3.9) *holds on $\sigma > 1/(2\max(a, b))$.*

In other words, the holomorphic function

$$\mathbf{B}^{(a,b)}(s) - \sum_{n \leq N} \gamma_n^{(a,b)} \phi^{(\kappa(a+b))}(2ns) \tag{3.3.10}$$

on $\sigma > 1/(2\max(a, b))$ extends to a holomorphic function

$$\sum_{n \geq N+1} \gamma_n^{(a,b)} \phi^{(\kappa(a+b))}(2ns) \tag{3.3.11}$$

on $\sigma > 1/(2(N + 1))$. In particular, $\mu^{(1,1)}(s) - \phi^{(2\kappa)}(2s)$ extends to a holomorphic function on $\sigma > 1/4$.

The proof will be given in §3.7 after some preliminaries.

§3.4. The analytic continuation of $\tilde{M}(s; z_1, z_2)$

Theorem 5. (i) *For any $N \geq 0$, the holomorphic function*

$$\tilde{M}(s; z_1, z_2) \exp\Bigg(- \sum_{1 \leq a,b \leq n \leq N} \gamma_n^{(a,b)} \phi^{(\kappa(a+b))}(2ns)(i/2)^{a+b} \frac{z_1^a z_2^b}{a!b!}\Bigg) \tag{3.4.1}$$

of (s, z_1, z_2) on $\Re(s) > 1/2$ extends to $\Re(s) > 1/(2(N+1))$. In particular ($N = 1$),

$$\tilde{M}(s; z_1, z_2) \exp\bigg(\frac{1}{4}\phi^{(2\kappa)}(2s) z_1 z_2\bigg) \tag{3.4.2}$$

extends to a holomorphic function of s, z_1, z_2 on the domain defined by $\sigma > 1/4$.
(ii) *Let $\sigma_0 > 1/2$, $R > 0$, and $\Re(s) \geq \sigma_0$, $|z_1|, |z_2| \leq R$. Suppose that either σ_0 is fixed and R is sufficiently small, or R is fixed and σ_0 is sufficiently large. Then the two series*

$$\sum_{a,b \geq 1} \mathbf{B}^{(a,b)}(s)(i/2)^{a+b} \frac{z_1^a z_2^b}{a!b!}, \tag{3.4.3}$$

$$\sum_{\substack{a,b,n \\ n \geq \max(a,b)}} \gamma_n^{(a,b)} \phi^{(\kappa(a+b))}(2ns)(i/2)^{a+b} \frac{z_1^a z_2^b}{a!b!} \tag{3.4.4}$$

both converge absolutely and uniformly to $\log \tilde{M}(s; z_1, z_2)$. In Case 2, this means that $\tilde{M}(s; z_1, z_2)$ has an absolutely convergent infinite product expansion

$$\tilde{M}(s; z_1, z_2) = \prod_{n=1}^{\infty} \zeta(2ns)^{R_n(z_1, z_2)} \qquad (\Re(s) \geq \sigma_0,\ |z_1|, |z_2| \leq R) \tag{3.4.5}$$

(σ_0, R as above), where

$$R_n(z_1, z_2) = \sum_{a,b=1}^{n} \gamma_n^{(a,b)} (i/2)^{a+b} \frac{z_1^a z_2^b}{a!b!}. \tag{3.4.6}$$

The proof will be given in §3.8 after the preliminary subsections.

For example, let $K = \mathbb{F}_q(x)$ and $P_\infty = \{\mathfrak{p}_\infty\}$ be as in Example 4 (§1.2). Then $\zeta_{K,P_\infty}(s)$ is simply $= (1 - q^{1-s})^{-1}$; hence

$$\tilde{M}(s; z_1, z_2) = \prod_{n=1}^{\infty} (1 - q^{1-2ns})^{-R_n(z_1, z_2)}. \tag{3.4.7}$$

Corollary 3.4.8 (Lemma A, §2.3). *We have*

$$\lim_{\substack{s \to 1/2 \\ |\mathrm{Arg}(2s-1)| < \pi}} \tilde{M}\bigg(s;\ \frac{z_1}{\mu(s)^{1/2}}, \frac{z_2}{\mu(s)^{1/2}}\bigg) = \exp\bigg(-\frac{z_1 z_2}{4}\bigg), \tag{3.4.9}$$

and the convergence is uniform on $|z_1|, |z_2| \leq R$ for any given $R > 0$.

Proof. The above theorem shows in particular that

$$f(s;z_1,z_2) := \tilde{M}(s;z_1,z_2)\exp\left(\frac{\phi^{(2\kappa)}(2s)}{4}z_1z_2\right) \tag{3.4.10}$$

extends to a holomorphic function of (s,z_1,z_2) on $\Re(s)>1/4$. Clearly, $f(s,0,0) = 1$, $f(s,z_1,z_2)$ is continuous at $(1/2,0,0)$, and $\lim_{s\to 1/2}\mu(s)^{-1/2}=0$. Therefore,

$$\begin{aligned} &f(s;z_1/\mu(s)^{1/2},z_2/\mu(s)^{1/2}) \\ &\qquad = \tilde{M}(s;\, z_1/\mu(s)^{1/2},z_2/\mu(s)^{1/2})\exp\left(\frac{\phi^{(2\kappa)}(2s)}{4\mu(s)}z_1z_2\right) \end{aligned} \tag{3.4.11}$$

tends uniformly to 1 as $s\to 1/2$ (on $|z_1|,|z_2|\le R$). But by (2.2.1), the exponential factor tends uniformly to $\exp(z_1z_2/4)$. These together prove the corollary. □

Now let

$$\mathcal{D} = \{s\in\mathbb{C};\, \Re(s)>0,\ \zeta(2ns)\neq 0,\infty\ (n=1,2,\dots)\}, \tag{3.4.12}$$

where $\zeta(s)=\zeta_{K,P_\infty}(s)$. (In the number field case, the condition $\zeta(2ns)\neq\infty$ is of course equivalent to $s\neq 1/(2n)$.) Then Theorem 5 gives directly:

Corollary 3.4.13. *$\tilde{M}(s;z_1,z_2)$ extends to a single-valued* (Case 1) *or multi-valued* (Case 2) *analytic function of* (s,z_1,z_2) *on* $\mathcal{D}\times\mathbb{C}^2$.

As regards Case 2, if s_0 is a point with $\Re(s_0)>0$, $s_0\notin\mathcal{D}$, and s encircles s_0 in its small neighborhood in the positive direction (z_1,z_2 being fixed), then $\tilde{M}(s;z_1,z_2)$ is multiplied by

$$\exp\left(2\pi i\sum_{\nu=1}^{r}k_\nu R_{n_\nu}(z_1,z_2)\right). \tag{3.4.14}$$

Here, $(n_\nu)_{\nu=1}^r$ are the distinct positive integers such that $\zeta(2n_\nu s_0)=0$ or ∞, and k_ν is the order of $\zeta(s)$ at $s=2n_\nu s_0$. Thus, $\tilde{M}(s;z_1,z_2)$ can be regarded as a univalent analytic function on $\mathcal{D}^{\mathrm{urab}}\times\mathbb{C}^2$, where $\mathcal{D}^{\mathrm{urab}}$ denotes the maximal unramified *abelian* covering of $\mathcal{D}$. Moreover, although $\tilde{M}(s;z_1,z_2)$ is multi-valued, its *divisor* on $\mathcal{D}\times\mathbb{C}^2$ is well-defined. Note also that for $y_1,y_2\in\mathbb{R}$, $|\tilde{M}(s,iy_1,iy_2)|$ is a univalent function on $\mathcal{D}\times\mathbb{R}^2$ (because $R_n(iy_1,iy_2)\in\mathbb{R}$).

Now each local factor $\tilde{M}_{\mathfrak{p}}(s;z_1,z_2)$ is a holomorphic function on $\{\Re(s)>0\}\times\mathbb{C}^2$, having a non-trivial zero divisor. It is clear from the Euler product expansion

$$\tilde{M}(s;z_1,z_2) = \prod_{\mathfrak{p}\notin P_\infty}\tilde{M}_{\mathfrak{p}}(s;z_1,z_2) \qquad (\Re(s)>1/2) \tag{3.4.15}$$

(cf. §1.2) that the zero divisor of $\tilde{M}(s;z_1,z_2)$ on $\{\Re(s)>1/2\}\times\mathbb{C}^2$ is simply the sum of the zero divisors of the local factors. But moreover, we have

Corollary 3.4.16. *The zero divisor of $\tilde{M}(s;z_1,z_2)$ on $\mathcal{D}\times\mathbb{C}^2$ is the sum of the zero divisors of $\tilde{M}_{\mathfrak{p}}(s;z_1,z_2)$ (restricted to $\mathcal{D}\times\mathbb{C}^2$) for all $\mathfrak{p}\notin P_\infty$.*

This will be proved in the course of the proof of Theorem 5(i) (in §3.8).

§3.5. Preliminaries for the proofs of Theorems 4, 5; Some estimates

A main point in the proofs is the exchangeability of the order of (various) summations, over $\mathfrak{p}$, n, (a,b). To justify this, we need the absolute convergence of various sums over all $\mathfrak{p}$, n, (a,b), and for this, some estimates of each summand will be needed. In this subsection, we shall estimate $|B^{(a,b)}(t)|$, $|\beta_n^{(a,b)}|$ and $|\gamma_n^{(a,b)}|$.

Proposition 3.5.1. *Let $|t|<1$. Then*

$$|B^{(a,b)}(t)|\le(2\min(a,b))^{a+b}|t|^{\max(a,b)}(1-|t|^{1/2})^{-2(a+b)}. \tag{3.5.2}$$

Proposition 3.5.3.[1] *We have*

$$\text{(i)}\quad |\beta_n^{(a,b)}|<(4en)^{a+b-1}\qquad\text{(Cases 1, 2)},$$

$$\text{(ii)}\quad \sum_{a,b,n}\left(\frac{\beta_n^{(a,b)}}{3^{a+b}a!b!}\right)^2<\frac{1}{2}\qquad\text{(Case 2)}.$$

Proposition 3.5.4. *We have*

$$\text{(i)}\quad |\gamma_n^{(a,b)}|<(4en)^{a+b}\qquad\text{(Case 1)},$$

$$\text{(ii)}\quad |\gamma_n^{(a,b)}|<3^{a+b}a!b!\qquad\text{(Case 2)}.$$

Before proving these propositions, we need several basic remarks. First, by (1.2.3), the coefficient of $x_1^a x_2^b t^n$ in $F(x_1,x_2;t)=\sum_{n\ge0}F_n(x_1)F_n(x_2)t^n$ for $n\ge\max(a,b)$ is given by $\binom{n-1}{a-1}\binom{n-1}{b-1}/(a!b!)$ in Case 1 and by $\delta_a(n)\delta_b(n)/a!b!$ in Case 2, and is 0 for $n<\max(a,b)$; hence $F(x_1,x_2;t)$ may be rewritten as

$$F(x_1,x_2;t)=1+\sum_{a,b\ge1}f^{(a,b)}(t)\frac{x_1^a x_2^b}{a!b!}, \tag{3.5.5}$$

[1] Since so many positive absolute constants appear, instead of denoting them C_1, C_2, etc., we shall simply give an explicit choice for each (e.g., $4e$ in (i) below). Later arguments will not depend on these specific choices.

where

$$f^{(a,b)}(t)=\begin{cases}\displaystyle\sum_{n\geq\max(a,b)}\binom{n-1}{a-1}\binom{n-1}{b-1}t^n & \text{(Case 1)},\\ \displaystyle\sum_{n\geq\max(a,b)}\delta_n(a)\delta_n(b)t^n & \text{(Case 2)}.\end{cases}\tag{3.5.6}$$

Therefore,

$$\log F(x_1,x_2;t)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\Big(\sum_{a,b\geq 1}f^{(a,b)}(t)\frac{x_1^a x_2^b}{a!b!}\Big)^k;\tag{3.5.7}$$

hence the coefficient $B^{(a,b)}(t)$ of $\frac{x_1^a x_2^b}{a!b!}$ in (3.5.7) is given by

$$\begin{aligned}&B^{(a,b)}(t)\\ &=\sum_{k=1}^{\min(a,b)}\frac{(-1)^{k-1}}{k}\sum_{\substack{a=a_1+\cdots+a_k\\ a_1,\ldots,a_k\geq 1}}\sum_{\substack{b=b_1+\cdots+b_k\\ b_1,\ldots,b_k\geq 1}}\binom{a}{a_1,\ldots,a_k}\binom{b}{b_1,\ldots,b_k}\prod_{\nu=1}^{k}f^{(a_\nu,b_\nu)}(t).\end{aligned}\tag{3.5.8}$$

(A priori, the outer sum is over all $k\geq 1$, but the inner sum is 0 unless $k\leq\min(a,b)$.)

For any formal power series f and g with *non-negative* real coefficients, $f\leq_{\text{cf}} g$ will denote the *coefficientwise inequality* $\leq$. Note that this inequality is preserved by additions and multiplications. By (3.1.6) and (3.5.8), we have

$$\begin{aligned}&\sum_{n\geq\max(a,b)}|\beta_n^{(a,b)}|t^n\\ &\leq_{\text{cf}}\sum_{k=1}^{\min(a,b)}\frac{1}{k}\sum_{\substack{a=a_1+\cdots+a_k\\ a_1,\ldots,a_k\geq 1}}\sum_{\substack{b=b_1+\cdots+b_k\\ b_1,\ldots,b_k\geq 1}}\binom{a}{a_1,\ldots,a_k}\binom{b}{b_1,\ldots,b_k}\prod_{\nu=1}^{k}f^{(a_\nu,b_\nu)}(t).\end{aligned}\tag{3.5.9}$$

We shall need the following two $\leq_{\text{cf}}$ inequalities:

$$f^{(a,b)}(u^2)\leq_{\text{cf}}\begin{cases}(u(1-u)^{-1})^{a+b},\\ \displaystyle\binom{\max(a,b)-1}{\min(a,b)-1}(u(1-u)^{-1})^{2\max(a,b)}.\end{cases}\tag{3.5.10}$$

To verify these we may assume $a \geq b$. By (1.2.4) and (3.5.6),

$$f^{(a,b)}(u^2) \leq_{\text{cf}} \sum_{n\geq a} \binom{n-1}{a-1}\binom{n-1}{b-1} u^{2n} \tag{3.5.11}$$
$$\leq_{\text{cf}} \left(\sum_{n\geq a} \binom{n-1}{a-1} u^n\right)\left(\sum_{n\geq a} \binom{n-1}{b-1} u^n\right).$$

But

$$\sum_{n\geq a} \binom{n-1}{a-1} u^n = (u(1-u)^{-1})^a, \tag{3.5.12}$$

and hence

$$\sum_{n\geq a} \binom{n-1}{b-1} u^n \leq_{\text{cf}} \begin{cases} (u(1-u)^{-1})^b \\ \binom{a-1}{b-1}(u(1-u)^{-1})^a. \end{cases} \tag{3.5.13}$$

(The first is obvious by (3.5.12) and $b \leq a$; the second is by (3.5.12) and $\binom{n-1}{b-1} \leq \binom{n-b}{a-b}\binom{n-1}{b-1} = \binom{a-1}{b-1}\binom{n-1}{a-1}$). Therefore, (3.5.10) follows directly from (3.5.11)–(3.5.13).

Proof of Proposition 3.5.3(i). By (3.5.9) and the first inequality of (3.5.10), we obtain

$$\sum_{n\geq \max(a,b)} |\beta_n^{(a,b)}| u^{2n} \leq_{\text{cf}} \sum_{k=1}^{\min(a,b)} k^{a+b-1}(u(1-u)^{-1})^{a+b} \tag{3.5.14}$$
$$\leq_{\text{cf}} \min(a,b)^{a+b}(u(1-u)^{-1})^{a+b}.$$

Therefore, by (3.5.12),

$$|\beta_n^{(a,b)}| \leq \min(a,b)^{a+b}\binom{2n-1}{a+b-1} \leq (a+b-1)^{a+b}\frac{(2n-1)^{a+b-1}}{(a+b-1)!}.$$

By using $n! > e^{-n}n^n$ and $a+b-1 \leq 2^{a+b-1}$, we obtain

$$|\beta_n^{(a,b)}| < (a+b-1)e^{a+b-1}(2n-1)^{a+b-1} < (4en)^{a+b-1},$$

as desired. □

Proof of Proposition 3.5.1. We use the second inequality of (3.5.10), and proceed similarly. The only difference is that we finally turn to "real inequalities" by using $|t| < 1$ and $a + b \geq \sum_\nu \max(a_\nu, b_\nu) \geq \max(a, b)$. □

Proof of Proposition 3.5.3(ii). This is more delicate. In Case 2, by (3.1.1) and (1.2.3), our $F(x_1, x_2; t)$ is nothing but the Gauss hypergeometric series

$$F(a, b; c; t) = 1 + \frac{a \cdot b}{1 \cdot c} t + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} t^2 + \cdots , \tag{3.5.15}$$

for $a = x_1, b = x_2, c = 1$;

$$F(x_1, x_2; t) = F(x_1, x_2; 1; t). \tag{3.5.16}$$

When $\Re(c) > 0$ and $\Re(c - a - b) > 0$, the series (3.5.15) converges also for $t = 1$, and the Gauss formula

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \tag{3.5.17}$$

holds. In particular, $F(1/3, 1/3; 1) = \Gamma(1/3)\Gamma(2/3)^{-2} = 1.461\ldots < 3/2$. Therefore, when $|x_1|, |x_2| \leq 1/3$, $|t| < 1$, we have

$$|F(x_1, x_2; t) - 1| \leq \sum_{n \geq 1} F_n(1/3)^2 = F(1/3, 1/3; 1) - 1 < 1/2. \tag{3.5.18}$$

Note now the following. For any $\alpha_1, \alpha_2, \ldots \in \mathbb{C}$, if we define the formal power series

$$\sum_{n \geq 1} \beta_n t^n = \log\Big(1 + \sum_{n \geq 1} \alpha_n t^n\Big), \tag{3.5.19}$$

then for any sequence $\{a_n\}_{n \geq 1}$ of non-negative real numbers with $|\alpha_n| \leq a_n$, the coefficientwise inequality

$$\sum_{n \geq 1} |\beta_n| t^n \leq_{\mathrm{cf}} -\log\Big(1 - \sum_{n \geq 1} a_n t^n\Big) \tag{3.5.20}$$

holds. Apply this for $\alpha_n = F_n(x_1)F_n(x_2)$ (for $|x_1|, |x_2| \leq 1/3$), $a_n = F_n(1/3)^2$ and $\beta_n = B_n(x_1, x_2)$, to obtain

$$\sum_{n \geq 1} |B_n(x_1, x_2)| t^n \leq_{\mathrm{cf}} -\log\Big(1 - \sum_{n \geq 1} F_n(1/3)^2 t^n\Big). \tag{3.5.21}$$

This of course carries over to an actual inequality for any t with $0 \leq t < 1$. Therefore, by letting $t \to 1$ and by using (3.5.18) and Abel's theorem, we obtain

$$\sum_{n \geq 1} |B_n(x_1, x_2)| \leq \log 2 \quad (|x_1|, |x_2| \leq 1/3); \tag{3.5.22}$$

hence

$$\sum_{n \geq 1} |B_n(x_1, x_2)|^2 \leq (\log 2)^2 < 1/2 \quad (|x_1|, |x_2| \leq 1/3). \tag{3.5.23}$$

Now by (3.1.5) and the orthogonality relation we obtain

$$\int_{|x_1|=|x_2|=1/3} |B_n(x_1, x_2)|^2 \, d^\times x_1 \, d^\times x_2 = \sum_{a,b=1}^{n} \left(\frac{\beta_n^{(a,b)}}{a!b!} \left(\frac{1}{3}\right)^{a+b} \right)^2, \tag{3.5.24}$$

where $d^\times x_\nu$ ($\nu = 1, 2$) denotes the normalized Haar measure on the circle $|x_\nu| = 1/3$ (note that $\beta_n^{(a,b)}$ are rational and hence real). Summing over all n, we obtain

(3.5.25)

$$\sum_{n,a,b} \left(\frac{\beta_n^{(a,b)}}{a!b!} \left(\frac{1}{3}\right)^{a+b} \right)^2 = \int_{|x_1|=|x_2|=1/3} \sum_{n=1}^{\infty} |B_n(x_1, x_2)|^2 \, d^\times x_1 \, d^\times x_2 < 1/2,$$

as desired. □

Proof of Proposition 3.5.4. (Case 1) By (3.1.13), we have

$$\gamma_n^{(a,b)} = n^{a+b-1} \sum_{d|n} \mu(n/d) d^{1-a-b} \beta_d^{(a,b)},$$

and by Proposition 3.5.3(i), we have $d^{1-a-b}|\beta_d^{(a,b)}| < (4e)^{a+b-1}$; hence

$$|\gamma_n^{(a,b)}| < (4en)^{a+b-1} \sum_{d|n} 1 \le (4e)^{a+b-1} n^{a+b}.$$

(Case 2) In this case,

$$\gamma_n^{(a,b)} = \frac{1}{n} \sum_{d|n} \mu(n/d) d \beta_d^{(a,b)};$$

hence

$$|\gamma_n^{(a,b)}| \le \frac{1}{n} \sum_{d|n} d|\beta_d^{(a,b)}| \le \frac{1}{n} \Big(\Big(\sum_{d|n} d^2\Big) \Big(\sum_{d|n} |\beta_d^{(a,b)}|^2\Big) \Big)^{1/2}. \tag{3.5.26}$$

By Proposition 3.5.3(ii) we have

$$\sum_{d|n} |\beta_d^{(a,b)}|^2 < (3^{a+b} a!b!)^2/2 \tag{3.5.27}$$

for each $a, b \ge 1$, and on the other hand, $n^{-2} \sum_{d|n} d^2 < \sum_{m \ge 1} m^{-2} = \pi^2/6$; hence

$$|\gamma_n^{(a,b)}| < \frac{\pi}{\sqrt{12}} 3^{a+b} a!b! < 3^{a+b} a!b!, \tag{3.5.28}$$

as desired. □

Remark 3.5.29. From (3.5.16) and (3.5.17) and the power series expansion

$$\log\Gamma(1-x) = \gamma x + \sum_{n=2}^{\infty} \frac{\zeta_{\mathbb{Q}}(n)}{n} x^n \qquad (|x|<1) \tag{3.5.30}$$

(γ the Euler constant, $\zeta_{\mathbb{Q}}(s)$ the Riemann zeta function), we obtain, in Case 2 for $|x_1|, |x_2| < 1/2$,

$$\begin{aligned}\log F(x_1,x_2;1) &= \log\Gamma(1-x_1-x_2) - \log\Gamma(1-x_1) - \log\Gamma(1-x_2) \\ &= \sum_{n=2}^{\infty} \frac{\zeta_{\mathbb{Q}}(n)}{n}((x_1+x_2)^n - x_1^n - x_2^n) = \sum_{a,b\geq 1} (a+b-1)!\zeta_{\mathbb{Q}}(a+b)\frac{x_1^a x_2^b}{a!b!},\end{aligned} \tag{3.5.31}$$

and hence

$$B^{(a,b)}(1) = (a+b-1)!\zeta_{\mathbb{Q}}(a+b) \qquad (a,b\geq 1) \text{ (Case 2)}. \tag{3.5.32}$$

§3.6. Further estimates and convergences

Proposition 3.6.1. *Fix $a,b \geq 1$, $0<r<1$ and let $|t| \leq r$. Then the series*

$$\sum_{n\geq \max(a,b)} \gamma_n^{(a,b)} \ell_{\kappa(a+b)}(t^n) \tag{3.6.2}$$

is absolutely and uniformly convergent, and has $B^{(a,b)}(t)$ as its limit. Moreover,

$$\Big|B^{(a,b)}(t) - \sum_{n=\max(a,b)}^{N} \gamma_n^{(a,b)} \ell_{\kappa(a+b)}(t^n)\Big| \ll_{a,b,r} (N+1)^{\kappa(a+b)+1}|t|^{N+1} \tag{3.6.3}$$

for any $N \geq \max(a,b)-1$.

The above inequality will be needed for globalization.

Proof. Since (a,b) is fixed, $|\gamma_n^{(a,b)}| \ll n^{\kappa(a+b)}$ by Proposition 3.5.4; and clearly, $|\ell_k(t)| \ll_{k,r} |t|$ for $|t|\leq r$. Thus,

$$|\gamma_n^{(a,b)} \ell_{\kappa(a+b)}(t^n)| \ll_{a,b,r} n^{\kappa(a+b)} r^n. \tag{3.6.4}$$

Therefore, (3.6.2) is absolutely and uniformly convergent. Now, by definitions,

$$\begin{aligned}\mathrm{Coeff}\Big(B^{(a,b)}(t) - \sum_{n=\max(a,b)}^{N} \gamma_n^{(a,b)} \ell_{\kappa(a+b)}(t^n), t^m\Big) \\ = \beta_m^{(a,b)} - \sum_{\substack{n\leq N \\ n|m}} \gamma_n^{(a,b)} (m/n)^{\kappa(a+b)-1},\end{aligned} \tag{3.6.5}$$

where Coeff$(\cdot, t^m)$ stands for the coefficient of t^m. This is $= 0$ when $m \le N$, and is $\ll_{a,b} m^{\kappa(a+b)+1}$ for $m \ge N+1$, by Propositions 3.5.3 and 3.5.4. Therefore, the left hand side of (3.6.3) is

$$\ll_{a,b} \sum_{m \ge N+1} m^{\kappa(a+b)+1} |t|^m \ll_{a,b,r} (N+1)^{\kappa(a+b)+1} |t|^{N+1},$$

as desired. □

By (3.2.6) and (3.2.12), this gives:

Corollary 3.6.6. *The holomorphic function* $\mathbf{B}_{\mathfrak{p}}^{(a,b)}(s)$ *on* $\Re(s) > 0$ *can be expressed as an absolutely* (*and uniformly on* $\Re(s) \ge \epsilon > 0$) *convergent series*

$$\mathbf{B}_{\mathfrak{p}}^{(a,b)}(s) = \sum_{n \ge \max(a,b)} \gamma_n^{(a,b)} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns). \tag{3.6.7}$$

Proposition 3.6.8. *Let* $|x_1|, |x_2| \le R$, $|t| \le r < 1$, *and consider the triple series*

$$\sum_{n,a,b \ge 1} \gamma_n^{(a,b)} \frac{x_1^a x_2^b}{a!b!} \ell_{\kappa(a+b)}(t^n). \tag{3.6.9}$$

If one of R, r *is fixed and the other is sufficiently small, then* (3.6.9) *converges absolutely and uniformly to* $\log F(x_1, x_2; t)$. *Moreover, for any given* $0 < c < 1$, *if* R, r *are such that* $re^{8eR} \le c$, *then*

$$\sum_{\substack{a,b,n \\ n \ge N+1}} \left| \gamma_n^{(a,b)} \frac{x_1^a x_2^b}{a!b!} \ell_{\kappa(a+b)}(t^n) \right| \ll_c (N+1)^2 (re^{8eR})^{N+1} \qquad (\textit{for all } N \ge 0). \tag{3.6.10}$$

Proof. We shall first prove (3.6.10).

(Case 1) By Proposition 3.5.4 and (3.1.10) (which holds analytically for $|t|, |te^u| < 1$), the left hand side of (3.6.10) is

$$\begin{aligned} &\le \sum_{n \ge N+1} \sum_{k \ge 2} \sum_{\substack{a+b=k \\ 1 \le a,b \le n}} \frac{(4enR)^k}{a!b!} \ell_k(r^n) \le \sum_{n \ge N+1} \left(\sum_{k \ge 0} \frac{1}{k!} (8enR)^k \ell_k(r^n) \right) \\ &= \sum_{n \ge N+1} \ell((re^{8eR})^n) \ll_c \sum_{n \ge N+1} (re^{8eR})^n \ll_c (re^{8eR})^{N+1}, \end{aligned}$$

provided that $re^{8eR} \le c < 1$.

(Case 2) Since $\ell(r^n) = -\log(1-r^n) \leq r^n(1-r^n)^{-1} \leq r^n(1-r)^{-1}$, Proposition 3.5.4(ii) gives

$$\sum_{a,b=1}^{n} |\gamma_n^{(a,b)} \frac{x_1^a x_2^b}{a!b!} \ell(t^n)| \leq \sum_{a,b=1}^{n} (3R)^{a+b} r^n (1-r)^{-1} \ll_r \begin{cases} (3R)^2 n^2 r^n & (3R < 1), \\ (3R)^{2n} n^2 r^n & (3R \geq 1). \end{cases} \tag{3.6.11}$$

But since $(aR)^2/2 < e^{aR}$ $(a > 0)$ and hence $(3R)^2 \leq e^{3\sqrt{2}R} < e^{8eR}$, this gives $(3R)^2 < e^{8eRn}$ and also $(3R)^{2n} < e^{8eRn}$ $(n \geq 1)$; hence the left hand side of (3.6.10) in this case is $\ll_c (N+1)^2(re^{8eR})^{N+1}$ if $re^{8eR} \leq c$, as desired. This settles the proof of (3.6.10) for both cases.

By (3.6.10), the series (3.6.9) converges absolutely and uniformly, as long as $re^{8eR} \leq c$. Therefore, we may change the order of summation. Since we already know by Proposition 3.6.1 that

$$\sum_{n \geq 1} \gamma_n^{(a,b)} \ell_{\kappa(a+b)}(t^n) = B^{(a,b)}(t) \qquad (|t| < 1), \tag{3.6.12}$$

and by Corollary 3.2.3 that

$$\sum_{a,b \geq 1} B^{(a,b)}(t) \frac{x_1^a x_2^b}{a!b!} = \log F(x_1, x_2; t) \tag{3.6.13}$$

holds when one of r, R is fixed and the other is sufficiently small, we conclude that (3.6.9) tends to $\log F(x_1, x_2; t)$ for such r, R. □

Corollary 3.6.14. *Let $\sigma_0, R > 0$, and $\Re(s) \geq \sigma_0$, $|z_1|, |z_2| \leq R$. Suppose that either σ_0 is fixed and R is sufficiently small, or R is fixed and σ_0 is sufficiently large. Then for any non-archimedean prime $\mathfrak{p}$, $\log \tilde{M}_{\mathfrak{p}}(s; z_1, z_2)$ (cf. Corollary 3.2.8) can be expressed as an absolutely convergent series*

$$\log \tilde{M}_{\mathfrak{p}}(s; z_1, z_2) = \sum_{\substack{n,a,b \geq 1 \\ n \geq \max(a,b)}} \gamma_n^{(a,b)} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns)(i/2)^{a+b} \frac{z_1^a z_2^b}{a!b!}. \tag{3.6.15}$$

Proof. Again, in Case 2, this follows immediately from the above proposition. In Case 1, we may take $r = N(\mathfrak{p})^{-2\sigma_0}$, but R is replaced by $(\log N(\mathfrak{p}))R/2$; hence re^{8eR} will be replaced by $N(\mathfrak{p})^{4eR-2\sigma_0}$. The exponent $4eR - 2\sigma_0$ is $\leq -\sigma_0$ if and only if $4eR \leq \sigma_0$; which is satisfied under our assumptions on σ_0 and R. Hence this case is also settled. □

§3.7. Proof of Theorem 4

Write $\sigma = \Re(s)$. Fix $N \geq \max(a,b) - 1$ and $\epsilon > 0$. We shall prove first that the double sum

$$\sum_{\substack{n \geq N+1 \\ \mathfrak{p} \notin P_\infty}} |\gamma_n^{(a,b)} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns)| \tag{3.7.1}$$

is finite and bounded on $\sigma \geq (1+\epsilon)/(2(N+1))$. First, $\phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns) = \lambda_{\mathfrak{p}}^{a+b} \ell_{\kappa(a+b)}(N(\mathfrak{p})^{-2ns})$ by (3.2.12). But $|\gamma_n^{(a,b)}| \ll_{a,b} n^{\kappa(a+b)} \leq n^{a+b}$ (by Proposition 3.5.4), $|\lambda_{\mathfrak{p}}| \leq \log N(\mathfrak{p})$, $\ell_{\kappa(a+b)}(t) \ll_{a,b,r} |t|$ for $|t| \leq r < 1$, and $N(\mathfrak{p})^{-2\sigma n} \leq 2^{-2\sigma(N+1)} \leq 2^{-1-\epsilon} < 1/2$; hence (3.7.1) is

$$\ll_{a,b} \sum_{n \geq N+1} n^{a+b} \sum_{\mathfrak{p} \notin P_\infty} (\log N(\mathfrak{p}))^{a+b} N(\mathfrak{p})^{-2n\sigma}. \tag{3.7.2}$$

Put $\alpha = \min_{\mathfrak{p} \notin P_\infty} N(\mathfrak{p})$. Then since $2n\sigma \geq 1+\epsilon$ and $\alpha N(\mathfrak{p})^{-1} \leq 1$, we have $(\alpha N(\mathfrak{p})^{-1})^{2n\sigma} \leq (\alpha N(\mathfrak{p})^{-1})^{1+\epsilon}$; hence

$$\begin{aligned} \alpha^{2n\sigma} &\sum_{\mathfrak{p} \notin P_\infty} (\log N(\mathfrak{p}))^{a+b} N(\mathfrak{p})^{-2n\sigma} \\ &\leq \alpha^{1+\epsilon} \sum_{\mathfrak{p} \notin P_\infty} (\log N(\mathfrak{p}))^{a+b} N(\mathfrak{p})^{-1-\epsilon} \ll_{a,b,\epsilon} 1; \end{aligned} \tag{3.7.3}$$

hence (3.7.2) is

$$\ll_{a,b,\epsilon} \sum_{n \geq N+1} n^{a+b} \alpha^{-2n\sigma} \leq \sum_{n \geq N+1} n^{a+b} (\alpha^{-(1+\epsilon)/(N+1)})^n \ll_{a,b,\epsilon,N} 1, \tag{3.7.4}$$

as desired.

Since the sum

$$\phi^{(k)}(2ns) = \sum_{\mathfrak{p} \notin P_\infty} \phi_{\mathfrak{p}}^{(k)}(2ns) \tag{3.7.5}$$

is absolutely convergent, because $2n\sigma \geq 2(N+1)\sigma \geq 1+\epsilon$, the convergence of (3.7.1) implies that the global sum

$$\sum_{n \geq N+1} \gamma_n^{(a,b)} \phi^{(\kappa(a+b))}(2ns) \tag{3.7.6}$$

is also absolutely and uniformly convergent on $\sigma \geq (1+\epsilon)/(2(N+1))$; whence (i).

To prove (ii), let $\sigma > 1/(2\max(a,b))$. By Proposition 3.3.1 and Corollary 3.6.6,

$$\mathbf{B}^{(a,b)}(s) = \sum_{\mathfrak{p} \notin P_\infty} \mathbf{B}_{\mathfrak{p}}^{(a,b)}(s) = \sum_{\mathfrak{p} \notin P_\infty} \sum_{n \geq \max(a,b)} \gamma_n^{(a,b)} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns), \tag{3.7.7}$$

which, by the absolute convergence of (3.7.1) (for $N=0$) can be reordered as

$$\sum_{n\ge\max(a,b)} \gamma_n^{(a,b)} \sum_{\mathfrak{p}\notin P_\infty} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns) = \sum_{n\ge\max(a,b)} \gamma_n^{(a,b)} \phi^{(\kappa(a+b))}(2ns) \tag{3.7.8}$$

(cf. (3.3.8)), as desired. This completes the proof of Theorem 4. □

§3.8. Proof of Theorem 5

Proof of (i). Fix $N\ge 0$ and $R\ge 1$, and assume $|z_1|,|z_2|\le R$. We may remove any finite set of prime components $\mathfrak{p}$ from both $\tilde{M}(s;z_1,z_2)$ and $\phi^{(\kappa(a+b))}(2ns)$ in proving Theorem 5. So we may assume $N(\mathfrak{p})$ is so large that the following conditions (a) and (b) are satisfied;

(a) $N(\mathfrak{p})^{-1/(2(N+1))} \le 1/2$, *and more strongly,* $4eR(\log N(\mathfrak{p}))N(\mathfrak{p})^{-1/(2(N+1))} \le 1/2$;

(b) $\alpha = \min(N(\mathfrak{p}))$ *is so large that the assumption of Corollary* 3.2.8 *is satisfied for* $\sigma_0 = 1/(2(N+1))$ (*and for the above* R).

Thus, $|\tilde{M}_{\mathfrak{p}}(s;z_1,z_2)-1|<1$ and

$$\log \tilde{M}_{\mathfrak{p}}(s;z_1,z_2) = \sum_{a,b\ge 1} \mathbf{B}_{\mathfrak{p}}^{(a,b)}(s)(i/2)^{a+b}\frac{z_1^a z_2^b}{a!b!} \qquad (\Re(s)\ge\sigma_0,\ |z_1|,|z_2|\le R) \tag{3.8.1}$$

(absolutely convergent). Write

$$\log \tilde{M}_{\mathfrak{p}}(s;z_1,z_2) - \sum_{1\le a,b\le n\le N} \gamma_n^{(a,b)} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns)(i/2)^{a+b}\frac{z_1^a z_2^b}{a!b!} = I_{\mathfrak{p}} + II_{\mathfrak{p}}, \tag{3.8.2}$$

with

$$I_{\mathfrak{p}} = \sum_{a,b=1}^{N}\Big(\mathbf{B}_{\mathfrak{p}}^{(a,b)}(s) - \sum_{n=\max(a,b)}^{N} \gamma_n^{(a,b)} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns)\Big)(i/2)^{a+b}\frac{z_1^a z_2^b}{a!b!}, \tag{3.8.3}$$

$$II_{\mathfrak{p}} = \sum_{\max(a,b)\ge N+1} \mathbf{B}_{\mathfrak{p}}^{(a,b)}(s)(i/2)^{a+b}\frac{z_1^a z_2^b}{a!b!}. \tag{3.8.4}$$

Note that $I_{\mathfrak{p}}$ is a finite sum. Let $I_{\mathfrak{p}}^{\star}$ (resp. $II_{\mathfrak{p}}^{\star}$) denote the modifications of the sums (3.8.3) (resp. (3.8.4)) where each outer summand is replaced by its absolute value.

First, when $\Re(s)>1/2$, we have

$$\sum_{\mathfrak{p}} \log \tilde{M}_{\mathfrak{p}}(s;z_1,z_2) = \log \tilde{M}(s;z_1,z_2) \quad \text{(absolute convergence)}, \tag{3.8.5}$$

by the argument of [6, §4] applied to the present situation, and also

$$\sum_{\mathfrak{p}} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns) = \phi^{(\kappa(a+b))}(2ns) \quad \text{(absolute convergence)}. \tag{3.8.6}$$

Hence the sum over $\mathfrak{p}$ of the left hand side of (3.8.2) for $\Re(s) > 1/2$ converges to

$$\log \tilde{M}(s; z_1, z_2) - \sum_{\substack{a,b,n \\ 1 \le a,b \le n \le N}} \gamma_n^{(a,b)} \phi^{(\kappa(a+b))}(2ns)(i/2)^{a+b} \frac{z_1^a z_2^b}{a!b!}. \tag{3.8.7}$$

In order to prove Theorem 5(i) and Corollary 3.4.16, it suffices to show that (3.8.7) extends to a holomorphic function on $\sigma > 1/(2(N+1))$, and for this it remains to prove that $\sum_{\mathfrak{p}} I_{\mathfrak{p}}^{\star}$ and $\sum_{\mathfrak{p}} II_{\mathfrak{p}}^{\star}$ are finite and uniformly bounded on $\sigma \ge (1+\epsilon)/(2(N+1))$.

As for $I_{\mathfrak{p}}^{\star}$, by Proposition 3.6.1,

$$\begin{aligned} I_{\mathfrak{p}}^{\star} &\le \sum_{a,b=1}^{N} \left| \mathbf{B}_{\mathfrak{p}}^{(a,b)}(s) - \sum_{\max(a,b) \le n \le N} \gamma_n^{(a,b)} \phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns) \right| \frac{(R/2)^{a+b}}{a!b!} \\ &\ll_{N,\epsilon} \sum_{a,b=1}^{N} (N+1)^{\kappa(a+b)+1} (\log N(\mathfrak{p}))^{a+b} N(\mathfrak{p})^{-2\sigma(N+1)} \frac{(R/2)^{a+b}}{a!b!} \\ &\ll_{N,\epsilon,R} (\log N(\mathfrak{p}))^{2N} N(\mathfrak{p})^{-1-\epsilon}; \end{aligned}$$

hence $\sum_{\mathfrak{p}} I_{\mathfrak{p}}^{\star} \ll \sum_{\mathfrak{p}} (\log N(\mathfrak{p}))^{2N} N(\mathfrak{p})^{-1-\epsilon} \ll 1$.

As for $II_{\mathfrak{p}}^{\star}$, we first estimate this by using Proposition 3.5.1, which together with (3.2.6) gives

$$\begin{aligned} &|\mathbf{B}_{\mathfrak{p}}^{(a,b)}(s)| \\ &\quad \le (\log N(\mathfrak{p}))^{a+b} (2\min(a,b))^{a+b} N(\mathfrak{p})^{-2\sigma \max(a,b)} (1 - N(\mathfrak{p})^{-\sigma})^{-2(a+b)}. \end{aligned} \tag{3.8.8}$$

But since $N(\mathfrak{p})^{-\sigma} < N(\mathfrak{p})^{-1/(2(N+1))} \le 1/2$ (by the assumption (a) above) and $\min(a,b)^{a+b} \le a^a b^b \le e^{a+b} a!b!$, we obtain

$$\frac{1}{a!b!} |\mathbf{B}_{\mathfrak{p}}^{(a,b)}(s)| \le (8e \log N(\mathfrak{p}))^{a+b} N(\mathfrak{p})^{-2\sigma \max(a,b)}. \tag{3.8.9}$$

Since $a + b \le 2\max(a,b)$ and $R \ge 1$, by reordering the sum using $\nu = \max(a,b)$ we obtain

$$\begin{aligned} II_{\mathfrak{p}}^{\star} &\le \sum_{\max(a,b) \ge N+1} (4eR \log N(\mathfrak{p}))^{2\max(a,b)} N(\mathfrak{p})^{-2\sigma \max(a,b)} \\ &\le 2 \sum_{\nu \ge N+1} \nu \left(\frac{4eR \log N(\mathfrak{p})}{N(\mathfrak{p})^{\sigma}} \right)^{2\nu}. \end{aligned} \tag{3.8.10}$$

By the assumption (a) for $N(\mathfrak{p})$, we have $4eR(\log N(\mathfrak{p}))N(\mathfrak{p})^{-\sigma} \le 1/2$; hence

$$II_{\mathfrak{p}}^{\star} \ll_N \left(\frac{4eR\log N(\mathfrak{p})}{N(\mathfrak{p})^{\sigma}}\right)^{2(N+1)} \le \frac{(4eR\log N(\mathfrak{p}))^{2(N+1)}}{N(\mathfrak{p})^{1+\epsilon}}, \tag{3.8.11}$$

because $2(N+1)\sigma \ge 1+\epsilon$. Therefore, $\sum_{\mathfrak{p}} II_{\mathfrak{p}}^{\star} \ll_{N,R,\epsilon} 1$.

This settles the proof of (i) and Corollary 3.4.16.

Proof of (ii). First, we shall prove the statement relating to (3.4.3). By Corollary 3.2.8, we have $|\tilde{M}_{\mathfrak{p}}(s;z_1,z_2)-1|<1$ and

$$\log \tilde{M}_{\mathfrak{p}}(s;z_1,z_2) = \sum_{a,b\ge 1} \mathbf{B}_{\mathfrak{p}}^{(a,b)}(s)(i/2)^{a+b}\frac{z_1^a z_2^b}{a!b!}. \tag{3.8.12}$$

Moreover, by the finiteness of $\sum_{\mathfrak{p}} II_{\mathfrak{p}}^{\star}$ for $N=0$ shown above, the double sum

$$\sum_{\mathfrak{p}}\sum_{a,b} \mathbf{B}_{\mathfrak{p}}^{(a,b)}(s)(i/2)^{a+b}\frac{z_1^a z_2^b}{a!b!} \tag{3.8.13}$$

is absolutely convergent. Therefore, we may interchange the summation order, and since $\sigma_0 > 1/2$, we have

$$\begin{aligned}\log \tilde{M}(s;z_1,z_2) &= \sum_{\mathfrak{p}} \log \tilde{M}_{\mathfrak{p}}(s;z_1,z_2) = \sum_{\mathfrak{p}}\sum_{a,b} \mathbf{B}_{\mathfrak{p}}^{(a,b)}(s)(i/2)^{a+b}\frac{z_1^a z_2^b}{a!b!} \\ &= \sum_{a,b}\Big(\sum_{\mathfrak{p}} \mathbf{B}_{\mathfrak{p}}^{(a,b)}(s)\Big)(i/2)^{a+b}\frac{z_1^a z_2^b}{a!b!} \\ &= \sum_{a,b} \mathbf{B}^{(a,b)}(s)(i/2)^{a+b}\frac{z_1^a z_2^b}{a!b!},\end{aligned} \tag{3.8.14}$$

as desired.

As regards (3.4.4), by Corollary 3.6.14,

$$\log \tilde{M}_{\mathfrak{p}}(s;z_1,z_2) = \sum_{n,a,b} \gamma_n^{(a,b)}\phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns)(i/2)^{a+b}\frac{z_1^a z_2^b}{a!b!} \tag{3.8.15}$$

for all $\mathfrak{p}$. Put

$$III_{\mathfrak{p}} = \sum_{n,a,b}\left|\gamma_n^{(a,b)}\phi_{\mathfrak{p}}^{(\kappa(a+b))}(2ns)(i/2)^{a+b}\frac{z_1^a z_2^b}{a!b!}\right|. \tag{3.8.16}$$

We shall show, by using Proposition 3.6.8, that $\sum_{\mathfrak{p}} III_{\mathfrak{p}} < \infty$. In Case 1, $r = N(\mathfrak{p})^{-2\sigma_0}$, and R should be replaced by $(R/2)\log N(\mathfrak{p})$. Hence re^{8eR} in Proposition 3.6.8 is $N(\mathfrak{p})^{-2\sigma_0+4eR}$. Hence if either $\sigma_0 \gg_R 1$ or $R \ll_{\sigma_0} 1$, then $-2\sigma_0+4eR <$

$-1-\epsilon$; hence by (3.6.10) for $N = 0$, $III_{\mathfrak{p}} \ll N(\mathfrak{p})^{-1-\epsilon}$; hence $\sum_{\mathfrak{p}} III_{\mathfrak{p}} < \infty$. In Case 2, the same conclusion follows more directly. Therefore, we may interchange $\sum_{\mathfrak{p}}$ with $\sum_{n,a,b}$ in (3.8.15) and use (3.3.8) to conclude the convergence of (3.4.4) to $\log \tilde{M}(s; z_1, z_2)$. This settles the proof of (ii), and hence completes that of Theorem 5. □

§4. Rapid decay of $|\tilde{M}_\sigma(z)|$

The main purpose of §4 is to give some reasonably strong estimates of $|\tilde{M}_\sigma(z)|^2$, for $\tilde{M}_\sigma(z) = \tilde{M}(\sigma; z, \bar{z})$ ($\sigma > 1/2$, $z \in \mathbb{C}$). The main results are Theorem 6 (§4.3) and Theorem 7C (§4.6). The proofs of Lemmas B resp. B′ of §2 will also be supplied (cf. Theorem 7C, resp. Corollary 4.1.6).

§4.1. Local estimates; large $|z|$

For any non-archimedean prime $\mathfrak{p}$ of K and a positive real number σ, write as before

$$\mu_{\sigma,\mathfrak{p}} := \mu_{\mathfrak{p}}^{(1,1)}(\sigma) = \begin{cases} (\log N(\mathfrak{p}))^2/(N(\mathfrak{p})^{2\sigma}-1) & \text{(Case 1)}, \\ \displaystyle\sum_{n\geq 1} n^{-2} N(\mathfrak{p})^{-2n\sigma} & \text{(Case 2)}, \end{cases} \tag{4.1.1}$$

(cf. (1.2.12)), and put

$$\tilde{M}_{\sigma,\mathfrak{p}}(z) = \tilde{M}_{\mathfrak{p}}(\sigma; z, \bar{z}) = \int_{\mathbb{C}^1} \exp(i\Re(z g_{\sigma,\mathfrak{p}}(t^{-1})))\, d^\times t \tag{4.1.2}$$

(cf. (1.2.8)). Note that

$$|\tilde{M}_{\sigma,\mathfrak{p}}(z)| \leq 1. \tag{4.1.3}$$

A basic universal estimate of $|\tilde{M}_{\sigma,\mathfrak{p}}(z)|$ is the following:

Lemma C. *Fix any $\sigma_0 > 0$. Then*

$$|\tilde{M}_{\sigma,\mathfrak{p}}(z)|^2 \ll_{\sigma_0} (\mu_{\sigma,\mathfrak{p}}^{1/2}|z|)^{-1} \qquad (\sigma \geq \sigma_0), \tag{4.1.4}$$

where $\ll_{\sigma_0}$ depends only on σ_0.

Proof. Roughly speaking, this follows from the integral expression (4.1.2) and classical analysis: if $f(\theta)$ ($\theta \in \mathbb{R}/(2\pi)$) is a real-valued C^2-function such that $f'(\theta)$, $f''(\theta)$ are "sufficiently close" to trigonometric functions $\sin\theta$, $\cos\theta$ respectively, then

$$\int_0^{2\pi} e^{i|z|f(\theta)}\, d\theta \ll |z|^{-1/2}.$$

But to save space, we shall just reduce its proof for each Case to an established result.

In Case 1, this is proved in [2, §3.3]. (In fact, by (3.3.12), $|\tilde{M}_{\sigma,\mathfrak{p}}(z)|^2 = |H_{\sigma,\mathfrak{p}}(z)|^2 \ll_{\sigma_0} (r_{\sigma,\mathfrak{p}}|z|)^{-1}$, but $r_{\sigma,\mathfrak{p}}^{-1} = (N(\mathfrak{p})^{\sigma} - N(\mathfrak{p})^{-\sigma})/\log N(\mathfrak{p}) < \mu_{\sigma,\mathfrak{p}}^{-1/2}$.) In Case 2, this follows directly from [8, §7 Theorem 13], applied to $F(z) = -\log(1-z)$ in which case we can take $\rho_0 = 1$ (cf. the first paragraph of [8, §10]). This asserts that for any $\rho_1 < 1$,

$$\frac{1}{2\pi}\int_0^{2\pi} \exp\{-i\Re(\bar{z}\log(1-re^{i\theta}))\}\, d\theta \ll_{\rho_1} r^{-1/2}|z|^{-1/2} \quad (0 < r \le \rho_1). \tag{4.1.5}$$

Since the left hand side of (4.1.5) for $r = N(\mathfrak{p})^{-\sigma}$ gives $\tilde{M}_{\sigma,\mathfrak{p}}(z)$, and since $\mu_{\sigma,\mathfrak{p}} N(\mathfrak{p})^{2\sigma} \ll_{\sigma_0} 1$ $(\sigma \ge \sigma_0)$, (4.1.5) gives $\tilde{M}_{\sigma,\mathfrak{p}}(z) \ll_{\sigma_0} \mu_{\sigma,\mathfrak{p}}^{-1/4}|z|^{-1/2}$, and hence the desired result. □

Corollary 4.1.6 (Lemma B′ of §2.7). *As in* §2.5, *put* $\alpha = \min_{\mathfrak{p}\notin P_\infty} N(\mathfrak{p})$ *and* $m = |\{\mathfrak{p} \notin P_\infty; N(\mathfrak{p}) = \alpha\}|$. *Then there exists a constant* $C > 0$ *depending only on* (K, P_∞) *such that*

$$|\tilde{M}_\sigma(\mu_\sigma^{-1/2} z)| \le C|z|^{-m/2} \tag{4.1.7}$$

for all $\sigma \ge 1$ *and all* $z \in \mathbb{C}$.

Proof. By (4.1.1) we have $\mu_{\sigma,\mathfrak{p}} \gg N(\mathfrak{p})^{-2\sigma}$; hence

$$\mu_{\sigma,\mathfrak{p}}^{-1/2} \ll N(\mathfrak{p})^{\sigma}, \tag{4.1.8}$$

and Theorem 3(i) (§2) gives $\alpha^{2\sigma}\mu_\sigma \ll_K 1$ for $\sigma \ge 1$; hence by (4.1.3) and Lemma C,

$$\begin{aligned} |\tilde{M}_\sigma(\mu_\sigma^{-1/2} z)|^2 &\le \prod_{\substack{\mathfrak{p}\notin P_\infty \\ N(\mathfrak{p})=\alpha}} |\tilde{M}_{\sigma,\mathfrak{p}}(\mu_\sigma^{-1/2} z)|^2 \ll \prod_{\substack{\mathfrak{p}\notin P_\infty \\ N(\mathfrak{p})=\alpha}} (\mu_{\sigma,\mathfrak{p}}^{1/2}|\mu_\sigma^{-1/2} z|)^{-1} \\ &\ll (\alpha^{\sigma}|\mu_\sigma^{-1/2} z|^{-1})^m = (\alpha^{2\sigma}\mu_\sigma)^{m/2}|z|^{-m} \ll |z|^{-m}, \end{aligned} \tag{4.1.9}$$

as desired. □

§4.2. Local estimates; relatively small $|z|$

Since we always have (4.1.3), the bound (4.1.4) is effective only when $\mu_{\sigma,\mathfrak{p}}^{1/2}|z| \gg 1$. If we fix both $z \in \mathbb{C}$ and $\sigma > 0$, then $\mu_{\sigma,\mathfrak{p}}^{1/2} z$ tends to 0 as $N(\mathfrak{p}) \to \infty$. For small $\mu_{\sigma,\mathfrak{p}}^{1/2}|z|$, the following estimate will be useful.

Lemma D. *There exists an absolute constant* $q_0 > 1$ *such that*

$$|\tilde{M}_{\sigma,\mathfrak{p}}(z)|^2 \le \exp\left(-\frac{\mu_{\sigma,\mathfrak{p}}}{2}|z|^2\right) \tag{4.2.1}$$

holds whenever $N(\mathfrak{p})^{\sigma} \ge q_0$ *and* $\mu_{\sigma,\mathfrak{p}}^{1/2}|z| \le 2$.

Proof. Put $t = N(\mathfrak{p})^{-\sigma}$. Then (4.1.1) gives a power series expansion of $\mu_{\sigma,\mathfrak{p}}$ in t^2 starting with $\lambda_{\mathfrak{p}}^2 t^2$, where $\lambda_{\mathfrak{p}} = \log N(\mathfrak{p})$ (resp. $= 1$) for Case 1 (resp. Case 2). Note that this power series is nowhere vanishing on $|t| < 1$ (in Case 1 this is obvious; in Case 2 note only that $\sum_{n\geq 2} n^{-2} = \pi^2/6 - 1 < 1$). Thus, $\mu_{\sigma,\mathfrak{p}}^{1/2} = \lambda_{\mathfrak{p}} t + \cdots$ extends to a holomorphic and nowhere vanishing function of t on $|t| < 1$. We shall first show that for any $\vartheta \in \mathbb{R}/(2\pi)$,

$$\left| \tilde{M}_{\sigma,\mathfrak{p}}\left(\frac{re^{i\vartheta}}{\mu_{\sigma,\mathfrak{p}}^{1/2}} \right) \right|^2 - J_0(r)^2 \tag{4.2.1}$$

extends to a holomorphic function of (r, t) on $|t| < 1$ *whose Taylor series at* $(0, 0)$ *is divisible by* $t^2 r^4$ (in the ring of power series of r, t). By (1.2.10),

$$\tilde{M}_{\sigma,\mathfrak{p}}\left(\frac{re^{i\vartheta}}{\mu_{\sigma,\mathfrak{p}}^{1/2}} \right) = 1 + \sum_{a,b\geq 1} (\pm i/2)^{a+b} \frac{\mu_{\sigma,\mathfrak{p}}^{(a,b)}}{(\mu_{\sigma,\mathfrak{p}}^{1/2})^{a+b}} \frac{r^{a+b}}{a!b!} \cos((a-b)\vartheta), \tag{4.2.2}$$

where $\mu_{\sigma,\mathfrak{p}}^{(a,b)} = \mu_{\mathfrak{p}}^{(a,b)}(\sigma)$. On the other hand, by (1.2.11) we see easily that

$$\frac{\mu_{\sigma,\mathfrak{p}}^{(a,b)}}{(\mu_{\sigma,\mathfrak{p}}^{1/2})^{a+b}} \equiv \begin{cases} 0 \bmod t^{|b-a|} & (a \neq b), \\ 1 \bmod t^2 & (a = b). \end{cases} \tag{4.2.3}$$

Note also that this quotient is a power series of t depending only on Cases and (a, b). Therefore, the real (resp. imaginary) part $f_1(r, t)$ (resp. $f_2(r, t)$) of (4.2.2) (for $r > 0$, $\vartheta \in \mathbb{R}/(2\pi)$) are:

$$\begin{aligned} f_1(r,t) &= 1 + \sum_{a\geq 1} (-1)^a \frac{\mu_{\sigma,\mathfrak{p}}^{(a,a)}}{(\mu_{\sigma,\mathfrak{p}})^a} \frac{(r/2)^{2a}}{a!^2} \\ &\quad + 2 \sum_{\substack{b>a\geq 1 \\ b\equiv a \bmod 2}} (-1/4)^{(a+b)/2} \frac{\mu_{\sigma,\mathfrak{p}}^{(a,b)}}{(\mu_{\sigma,\mathfrak{p}}^{1/2})^{a+b}} \frac{r^{a+b}}{a!b!} \cos((a-b)\vartheta) \\ &\equiv J_0(r) \bmod t^2 r^4, \end{aligned} \tag{4.2.4}$$

$$\begin{aligned} f_2(r,t) &= \pm \sum_{\substack{b>a\geq 1 \\ b\equiv a+1 \bmod 2}} (-1/4)^{(a+b-1)/2} \frac{\mu_{\sigma,\mathfrak{p}}^{(a,b)}}{(\mu_{\sigma,\mathfrak{p}}^{1/2})^{a+b}} \frac{r^{a+b}}{a!b!} \cos((a-b)\vartheta) \\ &\equiv 0 \bmod tr^3. \end{aligned} \tag{4.2.5}$$

Hence $f_1^2 + f_2^2 - J_0(r)^2 \equiv 0 \bmod t^2 r^4$, as desired. Therefore, the quotient

$$(f_1^2 + f_2^2 - J_0(r)^2)/(t^2 r^4) \tag{4.2.6}$$

is bounded on $|t| \le 1/\sqrt{2}$ and $|r| \le 2$ (say), independent of the continuous parameter $\vartheta \in \mathbb{R}/(2\pi)$. Call an upper bound c_1, so that

$$(4.2.7)\quad |\tilde{M}_{\sigma,\mathfrak{p}}(\mu_{\sigma,\mathfrak{p}}^{-1/2}w)|^2 - J_0(|w|)^2 \le c_1 N(\mathfrak{p})^{-2\sigma}|w|^4 \quad (N(\mathfrak{p})^\sigma \ge \sqrt{2},\ |w| \le 2).$$

Now we shall verify another inequality

$$(4.2.8)\qquad \exp(-|w|^2/2) - J_0(|w|)^2 \ge c_2|w|^4 \quad (|w| \le 2),$$

where c_2 is another positive absolute constant. These two combined will give Lemma D; indeed, if $q_0 \ge \sqrt{2}$ and $q_0^2 > c_1/c_2$, then $c_1 N(\mathfrak{p})^{-2\sigma} \le c_1 q_0^{-2} < c_2$; hence by (4.2.7) and (4.2.8),

$$(4.2.9)\qquad |\tilde{M}_{\sigma,\mathfrak{p}}(\mu_{\sigma,\mathfrak{p}}^{-1/2}w)|^2 \le J_0(|w|)^2 + c_2|w|^4 \le \exp(-|w|^2/2).$$

Verification of (4.2.8). First, the power series expansion at $r = 0$ gives

$$(4.2.10)\qquad \exp(-r^2/2) - J_0(r)^2 \equiv \left(1 - \frac{r^2}{2} + \frac{r^4}{8}\right) - \left(1 - \frac{r^2}{4} + \frac{r^4}{64}\right)^2 \equiv \frac{r^4}{32} \bmod r^6;$$

hence

$$(4.2.11)\qquad \frac{1}{r^4}(\exp(-r^2/2) - J_0(r)^2) > 0 \quad (0 \le r \le r_0)$$

with some $r_0 > 0$. That we may take $r_0 = 2.72$ can be checked by computer. That we may take $r_0 = 2$ (which is what we need here) can also be shown as follows. Put $f(r) = \exp(r^2/4)J_0(r)$. Then $f(0) = 1$, and

$$f'(r) = \left(\frac{r}{2}J_0(r) - J_1(r)\right)\exp(r^2/4) = -\frac{r}{2}J_2(r)\exp(r^2/4).$$

But $J_0(r) > 0$ for $r < 2.4$, and $J_2(r) > 0$ for $r < 5.1$; hence for $0 < r < 2.4$, we have $f(r) > 0$ and $f'(r) < 0$; hence $f(r) < f(0) = 1$; hence $f(r)^2 < 1$, i.e., $J_0(r)^2 < \exp(-r^2/2)$ on this region. Therefore, (4.2.11) takes a positive minimal value c_2 on $0 \le r \le 2$. This settles the proof of (4.2.8) and hence that of Lemma D.

§4.3. Global estimates; large $|z|$

Here and in what follows, all primes $\mathfrak{p}$ considered are those *outside* P_∞; in particular, $\sum_{\mathfrak{p}\notin P_\infty}$ will be abbreviated as $\sum_{\mathfrak{p}}$. An easy consequence of Lemma C and the prime number theorems (on "$\pi(x)$" and "$\psi(x)$") is:

Theorem 6. *For any fixed $\sigma_1 > 1/2$, $\delta > 0$, $a > 0$, there exists $R = R_{\sigma_1,\delta,a} > 0$ such that*

$$(4.3.1)\qquad |\tilde{M}_\sigma(z)|^2 < \exp(-a|z|^{1/(\sigma+\delta)}) \quad (1/2 < \sigma \le \sigma_1,\ |z| \ge R).$$

Remark 4.3.2. The exponent $1/(\sigma+\delta)$ (< 2) of $|z|$ cannot be replaced by 2. This is because for each fixed $\sigma > 1/2$ the Fourier dual $M_\sigma(w)$ satisfies $M_\sigma(w) \ll e^{-\lambda|w|^2}$ for any given $\lambda > 0$ (cf. [6, §5.2]). By Hardy's theorem,[1] this implies that there cannot exist any $c > 0$ such that $\tilde{M}_\sigma(z) \ll e^{-c|z|^2}$.

Proof of Theorem 6. We may assume $|z| > 1$. For each $y > 1$, write

$$P_y = \{\mathfrak{p};\, N(\mathfrak{p}) \le y\}, \qquad \tilde{M}_{\sigma,P_y}(z) = \prod_{\mathfrak{p}\in P_y} \tilde{M}_{\sigma,\mathfrak{p}}(z), \tag{4.3.3}$$

so that $|\tilde{M}_\sigma(z)| \le |\tilde{M}_{\sigma,P_y}(z)|$. By Lemma C (for, say, $\sigma_0 = 1/2$) and (4.1.8), $|\tilde{M}_{\sigma,\mathfrak{p}}(z)|^2 \le CN(\mathfrak{p})^\sigma |z|^{-1}$ holds with some $C > 1$; hence

$$|\tilde{M}_{\sigma,P_y}(z)|^2 \le C^{|P_y|}\Big(\prod_{\mathfrak{p}\in P_y} N(\mathfrak{p})\Big)^\sigma |z|^{-|P_y|}. \tag{4.3.4}$$

Choose

$$y = |z|^{1/(\sigma+\delta/2)}. \tag{4.3.5}$$

Since $\sigma \le \sigma_1$, $|z| \gg 1$ implies $y \gg 1$. We shall give a proof in the number field case; the function field case can be treated with minor modifications. For any $\epsilon > 0$, we have

$$(1-\epsilon)y/\log y \le |P_y| \le (1+\epsilon)y/\log y, \tag{4.3.6}$$

$$\sum_{\mathfrak{p}\in P_y} \log N(\mathfrak{p}) \le (1+\epsilon)y \tag{4.3.7}$$

for $y \gg_\epsilon 1$. Hence by (4.3.4) (including $\log 0 = -\infty$ in the inequality)

$$\begin{aligned}\log(|\tilde{M}_{\sigma,P_y}(z)|^2) &\le y((1+\epsilon)\log C/\log y + (1+\epsilon)\sigma - (1-\epsilon)\log|z|/\log y)\\ &= |z|^{1/(\sigma+\delta/2)}(I + II),\end{aligned} \tag{4.3.8}$$

with

$$I \le (1+\epsilon)(\sigma_1+\delta/2)(\log C)/\log|z|, \tag{4.3.9}$$

$$II = (1+\epsilon)\sigma - (1-\epsilon)(\sigma+\delta/2) \le -\delta/2 + \epsilon(2\sigma_1+\delta/2). \tag{4.3.10}$$

But $I < \delta/8$ for $|z| \gg 1$, and if we take ϵ that satisfies $\epsilon(2\sigma_1+\delta/2) = \delta/8$, then $I + II < -\delta/4$. Therefore,

$$\log(|\tilde{M}_{\sigma,P_y}(z)|^2) < -\frac{\delta}{4}|z|^{1/(\sigma+\delta/2)} \le -a|z|^{1/(\sigma+\delta)} \tag{4.3.11}$$

for $|z| \gg_{a,\delta,\sigma_1} 1$, as desired. □

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[1] Recall that in the one-dimensional case, it asserts that $f(x) \ll e^{-a|x|^2/2}$, $f^\wedge(\xi) \ll e^{-b|\xi|^2/2}$ ($a, b > 0$) with $ab > 1$ implies $f \equiv 0$. Apply this to $f(x) = (1/\sqrt{2\pi})\int_{-\infty}^{\infty} M_\sigma(x+yi)\,dy$, $f^\wedge(\xi) = \tilde{M}_\sigma(\xi)$.

§4.4. Small $|z|$, large $(2\sigma-1)^{-1}$

An easy consequence of Lemma D is:

Theorem 7A. *Fix any* ϵ, R *with* $0<\epsilon<1$, $R>0$. *If* $|z|\le R$ *and* $(2\sigma-1)^{-1} \gg_{\epsilon,R} 1$, *then*

$$|\tilde{M}_\sigma(z)|^2 \le \exp\Big(-\frac{1-\epsilon}{2}\mu_\sigma|z|^2\Big). \tag{4.4.1}$$

Proof. Take a finite set P of primes such that if $\mathfrak{p}\notin P$ then $N(\mathfrak{p})$ is so large that both $\mu_{1/2,\mathfrak{p}}^{1/2}R\le 2$ and $N(\mathfrak{p})^{1/2}\ge q_0$ (the constant in Lemma D) hold. Then $\mu_{\sigma,\mathfrak{p}}^{1/2}|z|\le 2$ whenever $\mathfrak{p}\notin P$, $|z|\le R$, $\sigma>1/2$. Thus, Lemma D gives

$$\begin{aligned}|\tilde{M}_\sigma(z)|^2 &\le \exp\Big(-\frac{|z|^2}{2}\sum_{\mathfrak{p}\notin P}\mu_{\sigma,\mathfrak{p}}\Big)\\ &\le \exp\Big(\frac{|z|^2}{2}\Big(\sum_{\mathfrak{p}\in P}\mu_{1/2,\mathfrak{p}}-\mu_\sigma\Big)\Big) \qquad (|z|\le R).\end{aligned} \tag{4.4.2}$$

Since $\lim_{\sigma\to 1/2}\mu_\sigma=\infty$, this gives

$$|\tilde{M}_\sigma(z)|^2 \le \exp\Big(-\frac{1-\epsilon}{2}\mu_\sigma|z|^2\Big) \tag{4.4.3}$$

for $|z|\le R$ and $(2\sigma-1)^{-1}\gg_{\epsilon,R} 1$. □

§4.5. Large $|z|$, large $(2\sigma-1)^{-1}$

Theorem 7B. *Fix any* ϵ *with* $0<\epsilon<1$. *If* $|z|\gg_\epsilon 1$ *and* $(2\sigma-1)^{-1}\gg_\epsilon 1$, *then*

$$|\tilde{M}_\sigma(z)|^2 \le \exp\Big(-\frac{\mu_\sigma}{2}|z|^{2(1-\epsilon)}\Big). \tag{4.5.1}$$

The proof requires some global estimate, Lemma E below in §4.7.

§4.6. Large $(2\sigma-1)^{-1}$, all $|z|$

Now, Theorems 7A, 7B combined give immediately:

Theorem 7C (Lemma B, §2.3). *Fix any* ϵ *with* $0<\epsilon<1$. *If* $(2\sigma-1)^{-1}\gg_\epsilon 1$, *then*

$$|\tilde{M}_\sigma(z)|^2 \le \exp\Big(-\frac{1-\epsilon}{2}\mu_\sigma|z|^{2(1-\epsilon')}\Big) \tag{4.6.1}$$

for all $z\in\mathbb{C}$, *where* $\epsilon'=\epsilon$ (*resp.* 0) *for* $|z|\ge 1$ (*resp.* $|z|<1$).

In fact, Theorem 7B shows that (4.5.1) and hence also (4.6.1) holds for $|z| \geq R_\epsilon$ with some $R_\epsilon \geq 1$. Now take $R = R_\epsilon$ in Theorem 7A and let $(2\sigma-1)^{-1} \gg_{\epsilon, R_\epsilon} 1$. Then (4.4.1) and hence also (4.6.1) holds for $|z| \leq R_\epsilon$ too. Thus, Theorem 7C is reduced to Theorem 7B.

§4.7. Key Lemma E

The key points for the proof of Theorem 7B are Lemma D and the following global estimate *from below* of the error term for sums over primes.

Lemma E. *Fix any* ϵ *with* $0 < \epsilon < 1/2$. *If* $(2\sigma - 1)^{-1} \gg_\epsilon 1$ *and* $T \gg_\epsilon 1$, *then*

$$\sum_{N(\mathfrak{p}) \geq T} \mu_{\sigma,\mathfrak{p}} > \begin{cases} (1-\epsilon)\mu_\sigma T^{1-2\sigma} & \text{(Case 1)}, \\ (1-\epsilon)\mu_\sigma T^{1-2\sigma}/\log T & \text{(Case 2)}. \end{cases}$$

Proof. We shall give a proof for the number field case. The function field case can be treated with minor modifications.

(Case 1) By (4.1.1), we have $\mu_{\sigma,\mathfrak{p}} > (\log N(\mathfrak{p}))^2 / N(\mathfrak{p})^{2\sigma}$. As usual, set

$$\pi(T) = \sum_{N(\mathfrak{p}) \leq T} 1 \sim T/\log T, \qquad \psi(T) = \sum_{N(\mathfrak{p}) \leq T} \log N(\mathfrak{p}) \sim T, \tag{4.7.1}$$

and also set

$$\psi_2(T) = \sum_{N(\mathfrak{p}) \leq T} (\log N(\mathfrak{p}))^2 \sim T \log T. \tag{4.7.2}$$

The last estimate follows from the first two by using only the trivial inequalities $\psi_2(T) \leq (\log T)\psi(T)$ and $\psi_2(T)/\pi(T) \geq (\psi(T)/\pi(T))^2$ (the Schwarz inequality). By partial summation and by (4.7.2), we easily obtain, for $T \gg_\epsilon 1$,

$$\sum_{N(\mathfrak{p}) \geq T} \mu_{\sigma,\mathfrak{p}} > -(1+\epsilon) T^{1-2\sigma} \log T + (1-\epsilon) \int_T^\infty \frac{\log t}{t^{2\sigma}}\, dt \tag{4.7.3}$$

for any $\sigma > 1/2$. But since the last integral can be explicitly given by

$$\left(\frac{1}{(2\sigma-1)^2} + \frac{\log T}{2\sigma - 1} \right) T^{1-2\sigma}, \tag{4.7.4}$$

we obtain

$$\begin{aligned} \sum_{N(\mathfrak{p}) \geq T} \mu_{\sigma,\mathfrak{p}} &> (1-\epsilon)\left(\frac{T^{1-2\sigma}}{(2\sigma-1)^2} + \left(\frac{1}{2\sigma-1} - \frac{1+\epsilon}{1-\epsilon} \right) T^{1-2\sigma} \log T \right) \\ &> (1-\epsilon) \frac{T^{1-2\sigma}}{(2\sigma-1)^2} > (1-2\epsilon)\mu_\sigma T^{1-2\sigma} \end{aligned} \tag{4.7.5}$$

for σ sufficiently close to $1/2$, by Theorem 2(i) (§2.1). This settles Case 1.

(Case 2) In this case, where $\mu_{\sigma,\mathfrak{p}} > N(\mathfrak{p})^{-2\sigma}$, we first obtain easily

$$\sum_{N(\mathfrak{p})\geq T} \mu_{\sigma,\mathfrak{p}} > -(1+\epsilon)\frac{T^{1-2\sigma}}{\log T} + (1-\epsilon)\int_T^\infty \frac{dt}{t^{2\sigma}\log t}. \tag{4.7.6}$$

But a more delicate treatment of the integral

$$\int_T^\infty \frac{dt}{t^{2\sigma}\log t} = \int_{(2\sigma-1)\log T}^\infty e^{-u}u^{-1}\,du \tag{4.7.7}$$

is required.

Sublemma 4.7.8. *For any $b > 0$,*

$$\int_b^\infty e^{-u}u^{-1}du = -\gamma + \log(1/b) + \int_0^b \frac{1-e^{-t}}{t}\,dt \tag{4.7.9}$$

$$\geq \begin{cases} c_0(\log(1/b)+1) & (0 < b \leq 2), \\ (b+1)^{-1}e^{-b-1} & (\textit{all } b > 0), \end{cases} \tag{4.7.10}$$

where γ is the Euler constant $\gamma = 0.5772\ldots$, and c_0 is an absolute positive constant.

Proof. As for the first equality, the derivatives d/db of the two sides are equal, and the formula for $b = 1$ can be found, e.g., in [10, §12.2 Ex. 4]. When $0 < b \leq 2$, so that $\log(1/b) + 1 > 1/4$, the quotient

$$\left(\int_b^\infty e^{-u}u^{-1}\,du\right)/(\log(1/b)+1) \tag{4.7.11}$$

is a continuous positive-valued function, which, by the equality (4.7.9) tends to 1 as $b \to 0$. Therefore, (4.7.11) attains a positive minimal value $c_0 > 0$ on $0 < b \leq 2$. The second inequality is obvious, because

$$\int_b^\infty e^{-u}u^{-1}\,du > \int_b^{b+1} e^{-u}u^{-1}\,du > e^{-b-1}(b+1)^{-1}. \qquad \square$$

Corollary 4.7.12.

$$e^{y/x}\int_{y/x}^\infty e^{-u}u^{-1}du > (1+\log x)/y \qquad (x, y \gg 1). \tag{4.7.13}$$

Proof. Put $b = y/x$. Let *LHS* (resp. *RHS*) be the left (resp. right) hand side of (4.7.13). First, let $0 < b \leq 2$. Then $1 + \log(1/b) > 1/4$, and by Sublemma 4.7.8,

$$LHS > e^b c_0(\log(1/b)+1) > c_0(\log(1/b)+1).$$

If y is so large that $1/y < c_0/2$ and $(\log y)/y < c_0/8$, then

$$RHS = (1 + \log(1/b) + \log y)/y < \frac{c_0}{2}(1 + \log(1/b)) + \frac{c_0}{8}.$$

But since $1/4 < 1 + \log(1/b)$, this is $< LHS$.

Now let $b \geq 2$. Then $(b+1)^{-1} \geq (2/3)b^{-1}$; hence by Sublemma 4.7.8,

$$LHS > e^b (b+1)^{-1} e^{-b-1} \geq 2/(3eb).$$

On the other hand, if x is so large that $(1 + \log x)/x < 2/(3e)$, then

$$RHS = (1 + \log x)/(bx) < 2/(3eb) < LHS. \qquad \square$$

Now by (4.7.6), (4.7.7) and Corollary 4.7.12 applied to $x = (2\sigma - 1)^{-1}$ and $y = \log T$ (hence $e^{y/x} = T^{2\sigma - 1}$), we obtain

$$\begin{aligned} \text{(4.7.14)} \qquad \sum_{N(\mathfrak{p}) \geq T} \mu_{\sigma,\mathfrak{p}} &> -(1+\epsilon)\frac{T^{1-2\sigma}}{\log T} + (1-\epsilon)\frac{T^{1-2\sigma}}{\log T}\left(1 + \log\frac{1}{2\sigma - 1}\right) \\ &= \frac{T^{1-2\sigma}}{\log T}\left((1-\epsilon)\log\frac{1}{2\sigma-1} - 2\epsilon\right) > \frac{T^{1-2\sigma}}{\log T}\left((1-2\epsilon)\log\frac{1}{2\sigma-1}\right), \end{aligned}$$

since we may assume $\log(1/(2\sigma - 1)) > 2$. Since $\mu_\sigma - \log(1/(2\sigma - 1))$ is bounded near $\sigma = 1/2$ (say, by Theorem 4 of §3.3), this is

$$> \frac{T^{1-2\sigma}}{\log T}((1 - 3\epsilon)\mu_\sigma).$$

This settles the proof of Lemma E also for Case 2.

§4.8. Proof of Theorem 7B

Let $z \in \mathbb{C}$ with $|z| > 1$ and put

$$T = T_z = \begin{cases} (2|z|\log|z|)^2 & \text{(Case 1)}, \\ |z|^2 & \text{(Case 2)}. \end{cases}$$

It is easy to see that if $|z| \gg 1$ (depending only on (K, P_∞)) and if $N(\mathfrak{p}) \geq T_z$ then the assumptions $N(\mathfrak{p})^\sigma \geq q_0$ and $\mu_{\sigma,\mathfrak{p}}^{1/2}|z| \leq 2$, both for any $\sigma > 1/2$, of Lemma D are satisfied and hence we have

$$\text{(4.8.1)} \qquad |\tilde{M}_{\sigma,\mathfrak{p}}(z)|^2 \leq \exp\left(-\frac{\mu_{\sigma,\mathfrak{p}}}{2}|z|^2\right).$$

(Note only that $x \geq 2y\log y$ and $y \gg 1$ implies $(x-1)/\log x > y$, and that $\mu_{\sigma,\mathfrak{p}} < \mu_{1/2,\mathfrak{p}}$.)

(Case 1) Let $0 < \epsilon < 1$, and $|z| \gg_\epsilon 1$, $(2\sigma - 1)^{-1} \gg_\epsilon 1$. Then by the above claim and Lemmas D, E, we have, for $T = T_z$ as above,

$$\prod_{N(\mathfrak{p}) \geq T} |\tilde{M}_{\sigma,\mathfrak{p}}(z)|^2 \leq \prod_{N(\mathfrak{p}) \geq T} \exp\biggl(-\frac{\mu_{\sigma,\mathfrak{p}}}{2}|z|^2\biggr) \leq \exp\biggl(-\frac{1-\epsilon}{2}\mu_\sigma T^{1-2\sigma}|z|^2\biggr). \tag{4.8.2}$$

But if $2\sigma - 1 < \epsilon/2$ and $|z| \gg_\epsilon 1$, then $T^{1-2\sigma} > T^{-\epsilon/2} = (2|z|\log|z|)^{-\epsilon} > (1-\epsilon)^{-1}|z|^{-2\epsilon}$; hence

$$|\tilde{M}_\sigma(z)|^2 \leq \prod_{N(\mathfrak{p}) \geq T} |\tilde{M}_{\sigma,\mathfrak{p}}(z)|^2 \leq \exp\biggl(-\frac{\mu_\sigma}{2}|z|^{2(1-\epsilon)}\biggr), \tag{4.8.3}$$

as desired.

(Case 2) In this case, $T = |z|^2$, and we obtain, similarly,

$$\prod_{N(\mathfrak{p}) \geq T} |\tilde{M}_{\sigma,\mathfrak{p}}(z)|^2 \leq \exp\biggl(-\frac{1-\epsilon}{2}\mu_\sigma \frac{T^{1-2\sigma}}{\log T}|z|^2\biggr). \tag{4.8.4}$$

But

$$T^{1-2\sigma}/\log T = \frac{1}{2}|z|^{2(1-2\sigma)}/\log|z| > \frac{1}{2}|z|^{-\epsilon}/\log|z| > (1-\epsilon)^{-1}|z|^{-2\epsilon}$$

for $2\sigma - 1 < \epsilon/2$ and $|z| \gg_\epsilon 1$; hence (4.8.4) is $\leq \exp(-(\mu_\sigma/2)|z|^{2(1-\epsilon)})$ also in this case. This completes the proof of Theorem 7B. □

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