Publ. RIMS Kyoto Univ. **47** (2011), 535–551
DOI 10.2977/PRIMS/42

Gelfand–Zetlin Basis, Whittaker Vectors and a Bosonic Formula for the $\mathfrak{sl}_{n+1}$ Principal Subspace

by

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Abstract

We derive a bosonic formula for the character of the principal space in the level k vacuum module for $\widehat{\mathfrak{sl}}_{n+1}$, starting from a known fermionic formula for it. In our previous work, the latter was written as a sum consisting of Shapovalov scalar products of Whittaker vectors for $U_{v^{\pm 1}}(\mathfrak{gl}_{n+1})$. In this paper we compute these scalar products in bosonic form, using the decomposition of Whittaker vectors in the Gelfand–Zetlin basis. We show further that the bosonic formula obtained in this way is the quasi-classical decomposition of the fermionic formula.

2010 Mathematics Subject Classification: Primary 16T30.
Keywords: difference Toda Hamiltonian, quantum groups, fermionic formulas, bosonic formulas.

§1. Introduction

One of the central results in the theory of Kac–Moody algebras is the Weyl formula for the characters of irreducible representations. This formula can be interpreted "quasi-classically". This means the following. Let L_χ be an integrable representation with highest weight χ. In L_χ there are some special vectors called extremal vectors. They are labeled by the Weyl group and have the form $w \cdot v_\chi$, where v_χ is

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This is a contribution to the special issue "The golden jubilee of algebraic analysis".

Communicated by M. Kashiwara. Received December 11, 2009.

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a highest weight vector, w is an element of the Weyl group and $w \cdot v$ denotes the projective action. The Weyl formula reads

$$\mathrm{ch}\, L_\chi = \sum_{w \in W} C_w(\chi),$$

where $C_w(\chi)$ is interpreted as the character of L_χ in the vicinity of the extremal vector $w \cdot v_\chi$. In this interpretation we suppose χ is "big", so that the extremal vector $w \cdot v_\chi$ is well-separated from other extremal vectors. To be more precise, this means that, when $\chi \to \infty$, generically the character L_χ in the vicinity of $w \cdot v_\chi$ stabilizes and gives $C_w(\chi)$. The Weyl formula states that the quasi-classical decomposition is exact for finite χ. One important point in the decomposition is that each term $C_w(\chi)$ is, up to a simple monomial, the inverse of a (possibly infinite) product of simple factors.

Now suppose that $\hat{\mathfrak{g}}$ is an affine Kac–Moody algebra, and let L_k be the vacuum representation of level k with highest weight vector v_k. Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be the Cartan decomposition, and let $\hat{\mathfrak{n}}_+ = \mathfrak{n}_+ \otimes \mathbb{C}[t, t^{-1}] \subset \hat{\mathfrak{g}}$ be the nilpotent subalgebra. Set

$$V^k = U(\hat{\mathfrak{n}}_+) \cdot v_k \subset L_k$$

and call it the *principal subspace* in L_k. The quasi-classical formula for the character of V^k can also be written. For example, if $\mathfrak{g} = \mathfrak{sl}_2$, we have [FL]

$$\mathrm{ch}\, V^k \overset{\mathrm{def}}{=} \mathrm{Tr}_{V^k}\, q^{-d} z^{H/2} = \sum_{m=0}^{\infty} \frac{q^{km^2} z^{km}}{(q^{2m+1}z)_\infty (q)_m (q^{-2m+1}z^{-1})_m},$$

where $d = t\frac{d}{dt}$ is the scaling operator, H is the generator of the Cartan subalgebra, and $(z)_m = \prod_{i=1}^{m}(1 - q^{i-1}z)$. In this formula the right hand side is understood as a power series in z. The $m = 0, 1, 2, \ldots$ terms are the contributions from the extremal vectors v_k, ${e_{-1}}^k v_k$, ${e_{-3}}^k {e_{-1}}^k v_k, \ldots$.

In general

$$\mathrm{ch}\, V^k = \sum_{\gamma \in Q^+} C_\gamma(k)$$

where Q is the root lattice, and the subset Q^+ consists of linear combinations of simple roots with nonnegative integer coefficients.

In [FFJMM1], it was proved that for $\mathfrak{g} = \mathfrak{sl}_3$,

$$\mathrm{ch}\, V^k = \sum_{d_1, d_2 = 0}^{\infty} q^{k(d_1^2 + d_2^2 - d_1 d_2)} z_1^{kd_1} z_2^{kd_2} \tilde{J}_{d_1, d_2}(q, q^{2d_1 - d_2} z_1, q^{2d_2 - d_1} z_2), \tag{1.1}$$

where

$$\tilde{J}_{d_1,d_2}(q,z_1,z_2) = \frac{1}{(qz_1)_\infty (qz_2)_\infty (qz_1z_2)_\infty} J_{d_1,d_2}(q,z_1^{-1},z_2^{-1}),$$
$$J_{d_1,d_2}(q,z_1,z_2) = \frac{(qz_1z_2)_{d_1+d_2}}{(q)_{d_1}(q)_{d_2}(qz_1)_{d_1}(qz_2)_{d_2}(qz_1z_2)_{d_1}(qz_1z_2)_{d_2}}.$$

(The $J_{d_1,d_2}(q,z_1,z_2)$ in the present paper is different from $J_{d_1,d_2}(z_1,z_2)$ used in [FFJMM1].)

The terms in this formula are still factorized but they have nontrivial factors in the numerators. On the other hand, in [FFJMM1], we have also derived another expression for the same character, in which $J_{d_1,d_2}(q,q^{2d_1-d_2}z_1,q^{2d_2-d_1}z_2)$ is split into 12 terms, each of which is a simple power in q,z_1,z_2 with a factorized denominator. We call such a formula a "desingularization". In general, $C_\gamma(k)$ is complicated and cannot be factorized. However, in this paper, we show that at least a desingularization can be found for the character of the principal subspace for the vacuum module where $\mathfrak{g}=\mathfrak{sl}_{n+1}$ (see Theorem 3.1 and Proposition 2.2). For $\mathfrak{g}=\mathfrak{sl}_3$ we have

$$(1.2)\qquad J_{d_1,d_2}(q,z_1,z_2) = \sum_{m=0}^{\min(d_1,d_2)} (-z_1)^m q^{-m(d_2-m)+m(m-1)/2} \frac{1}{(q)_m(q)_{d_1-m}(q)_{d_2-m}} \times \frac{1}{(qz_1)_m(qz_1z_2)_m(qz_2)_{d_2-m}(q^{-d_2+m}z_1)_m(q^{-d_2+2m+1}z_1)_{d_1-m}}.$$

We call such a formula a *bosonic formula*. We note that in [FFJMM1] bosonic formulas for more general modules over $\hat{\mathfrak{n}}_+$ are obtained in the case where $\mathfrak{g}=\mathfrak{sl}_3$, in which we used more terms than in the case of the vacuum module in this paper.

Following some geometrical ideas from [BrFi], one naturally expects that in the desingularization of the $\mathfrak{sl}_{n+1}$ formula the terms are labeled by some basis in the Verma modules of $U_v(\mathfrak{gl}_{n+1})$ where $q=v^2$, actually by the Gelfand–Zetlin basis. Our proof goes as follows. In [FFJMM2], we managed to rewrite the fermionic formula [FS] for $\mathrm{ch}\, V^k$ in terms of the eigenfunctions of the quantum difference Toda Hamiltonian. Such eigenfunctions were written by using the Whittaker vectors in the Verma modules for $U_v(\mathfrak{gl}_{n+1})$. In this paper, we decompose the Whittaker vectors in the Gelfand–Zetlin basis. This decomposition produces the decomposition of the coefficients of the eigenfunctions. Moreover, each term of this decomposition has a factorized form. As a by-product we get some interesting fermionic formulas and their quasi-classical decompositions.

Fermionic formulas are statistical sums over configurations of particles with color and weight. A configuration of particles is determined by a set of nonnegative integers $\mathbf{m} = (m_{i,t})_{(i,t)\in\mathcal{S}}$ which represents the number of particles with color i and weight t. Given a function $B(\mathbf{m})$, the fermionic sum is of the form

$$F(\mathcal{S}, B) = \sum_{\mathbf{m}} \frac{q^{B(\mathbf{m})}}{\prod_{(i,t)\in\mathcal{S}} (q)_{m_{i,t}}}.$$

See (4.2) for the case we study in this paper. Let us discuss the fermionic formula for the character $\operatorname{ch} V^k$ for $\mathfrak{g} = \mathfrak{sl}_3$. In this case we take $\mathcal{S}_k = \{1,2\} \times [1,k]$ for $\mathcal{S}$. In [FFJMM2] we have shown that the quasi-classical decomposition is valid in the following sense. Fix $(m_1, m_2, n_1, n_2) \in \mathbb{Z}^4_{\geq 0}$, and consider the above sum with the restriction that

$$\sum_{1\leq t\ll k} m_{i,t} = m_i, \qquad \sum_{1\ll t\leq k} m_{i,t} = n_i.$$

In the limit $k \to \infty$ this sum approaches some rational function $F_{1,k}(m_1, m_2, n_1, n_2)$. In [FFJMM2] we have shown that for finite $k \geq 0$, we have the equality

$$F(\mathcal{S}_k, B) = \sum_{m_1,m_2,n_1,n_2} F_{1,k}(m_1, m_2, n_1, n_2).$$

We call this equality the *quasi-classical decomposition.* In this paper we consider the case where we take

$$\mathcal{S}_{k',k} = \{(i,t) \mid 1 \leq t \leq k\delta_{i,1} + k'\delta_{i,2}\}.$$

In the limit $1 \ll k' \ll k$, we have a similar decomposition:

$$F(\mathcal{S}_{k',k}, B) = \sum_{m_1,m_2,n_1,n_2,l_1} F_{1,k',k}(m_1, m_2, n_1, n_2, l_1).$$

The restriction for the sum for $F_{1,k',k}(m_1, m_2, n_1, n_2, l_1)$ is such that

$$\sum_{1\leq t\ll k'} m_{i,t} = m_i, \qquad \sum_{1\ll t\leq k'} m_{1,t} + \sum_{k'<t\ll k} m_{1,t} = n_1,$$
$$\sum_{1\ll t\leq k'} m_{2,t} = n_2, \qquad \sum_{k'\ll t\leq k} m_{1,t} = l_1.$$

There are two remarkable features. First, the decomposition is exact for finite $k \geq k' \geq 1$. Therefore, if $k = k'$, it gives another formula for $\operatorname{ch} V^k$. Second, each summand in this decomposition is factorized. In fact, summing up over m_1, m_2 we obtain (1.1), (1.2). We will derive such a decomposition for general $\mathfrak{g} = \mathfrak{sl}_{n+1}$ by using Drinfeld Casimir elements of smaller rank.

Finally, we note that our paper is inspired by [BrFi]. Actually we study the structure of the singular points on some moduli spaces by using the equivalent language from the representation theory of affine Lie algebras.

§2. Whittaker vectors for $\mathfrak{gl}_{n+1}$

In this section we recall some known facts about Whittaker vectors for $\mathfrak{gl}_{n+1}$ and their Shapovalov scalar product, including the Toda recursion and fermionic formulas. We give explicit formulas for them using the Gelfand–Zetlin basis of Verma modules.

§2.1. Gelfand–Zetlin basis

Throughout, we consider the complex Lie algebra $\mathfrak{gl}_{n+1}$. Let $\epsilon_0, \dots, \epsilon_n$ be a basis of the Cartan subalgebra orthonormal with respect to the invariant scalar product $(\ , \ )$. The simple roots and fundamental weights are expressed as $\alpha_i = \epsilon_{i-1} - \epsilon_i$, $\omega_i = \epsilon_0 + \cdots + \epsilon_{i-1}$, $1 \le i \le n$. We set $Q = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$, $P = \bigoplus_{i=0}^n \mathbb{Z}\epsilon_i$, and $\rho = \sum_{i=1}^n \omega_i$.

Let $U_v(\mathfrak{gl}_{n+1})$ be the corresponding quantum group over $\mathbb{K} = \mathbb{C}(v)$, with generators $\{E_i, F_i\}_{1\le i\le n}$, $\{v^{\pm\epsilon_i}\}_{0\le i\le n}$ and standard defining relations. We set $K_i = v^{\epsilon_{i-1}-\epsilon_i}$. For $\lambda = \sum_{i=0}^n \lambda_i \epsilon_i \in P$, let $\mathcal{V}^\lambda$ be the Verma module over $U_v(\mathfrak{gl}_{n+1})$ generated by the highest weight vector $\mathbf{1}^\lambda$ with defining relations

$$E_i \mathbf{1}^\lambda = 0 \quad (1 \le i \le n), \qquad v^{\epsilon_i}\mathbf{1}^\lambda = v^{\lambda_i}\mathbf{1}^\lambda \quad (0 \le i \le n).$$

Recall that $\mathcal{V}^\lambda$ has a distinguished basis (known as the *Gelfand–Zetlin basis*) relative to the tower of subalgebras

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_n, \tag{2.1}$$

where $\mathcal{A}_k \simeq U_v(\mathfrak{gl}_{k+1})$ $(k = 0, \dots, n)$ denotes the subalgebra of $U_v(\mathfrak{gl}_{n+1})$ generated by $\{E_i, F_i\}_{1\le i\le k}$ and $\{v^{\pm\epsilon_i}\}_{0\le i\le k}$. Each subspace of $\mathcal{V}^\lambda$ which is jointly invariant under $\mathcal{A}_k$'s is one-dimensional. Such subspaces are labeled by arrays of numbers

$$\boldsymbol{\lambda} = \begin{matrix} \lambda_{0,n} & & \lambda_{1,n} & & \cdots & \lambda_{n-1,n} & & \lambda_{n,n} \\ & \lambda_{0,n-1} & & \lambda_{1,n-1} & & \cdots & \lambda_{n-1,n-1} & \\ & & \ddots & & \ddots & & & \\ & & & \lambda_{0,1} & & \lambda_{1,1} & & \\ & & & & \lambda_{0,0} & & & \end{matrix} \tag{2.2}$$

called *Gelfand patterns*. Here we set

$$\lambda_{k,i} = \lambda_k - m_{k,i}, \tag{2.3}$$

and $m_{k,i}$ are nonnegative integers satisfying

$$0 = m_{k,n} \le m_{k,n-1} \le m_{k,n-2} \le \cdots \le m_{k,k} \quad (0 \le k \le n). \tag{2.4}$$

In particular, we have

$$\lambda_{k,n} = \lambda_k.$$

For economy of space we shall also write $\boldsymbol{\lambda}$ as

$$\boldsymbol{\lambda} = (\lambda^{(n)}, \dots, \lambda^{(0)}), \qquad \lambda^{(i)} = \lambda_{0,i}\epsilon_0 + \cdots + \lambda_{i,i}\epsilon_i.$$

By choosing an appropriate generator $|\boldsymbol{\lambda}\rangle = |\lambda^{(n)}, \dots, \lambda^{(0)}\rangle$ of each subspace corresponding to (2.2), the action of Chevalley generators can be described explicitly. For this purpose it is convenient to extend the base field from $\mathbb{K}$ to $\mathcal{R}$ obtained by adjoining all elements of the form $\sqrt{f}$ $(f \in \mathbb{K})$. We use the same symbols $U_v(\mathfrak{gl}_{n+1})$ (resp. $\mathcal{V}^\lambda$) to denote $\mathcal{R} \otimes_{\mathbb{K}} U_v(\mathfrak{gl}_{n+1})$ (resp. $\mathcal{R} \otimes_{\mathbb{K}} \mathcal{V}^\lambda$). Then the Chevalley generators act by the formula [J]

$$v^{\epsilon_i}|\boldsymbol{\lambda}\rangle = v^{h_i(\boldsymbol{\lambda}) - h_{i-1}(\boldsymbol{\lambda})}|\boldsymbol{\lambda}\rangle, \tag{2.5}$$

$$E_i|\boldsymbol{\lambda}\rangle = \sum_{k=0}^{i-1} c_{k,i-1}(\boldsymbol{\lambda})|\boldsymbol{\lambda}_+^{(k,i-1)}\rangle, \tag{2.6}$$

$$F_i|\boldsymbol{\lambda}\rangle = \sum_{k=0}^{i-1} c_{k,i-1}(\boldsymbol{\lambda}_-^{(k,i-1)})|\boldsymbol{\lambda}_-^{(k,i-1)}\rangle. \tag{2.7}$$

Here $h_i(\boldsymbol{\lambda}) = \sum_{k=0}^{i} \lambda_{k,i}$, and $\boldsymbol{\lambda}_\pm^{(k,i-1)}$ signifies the Gelfand pattern wherein $\lambda_{k,i-1}$ is replaced by $\lambda_{k,i-1} \pm 1$ while keeping all other $\lambda_{l,j}$'s unchanged. The coefficients $c_{k,i-1}(\boldsymbol{\lambda})$ have the factorized form

$$c_{k,i-1}(\boldsymbol{\lambda})^2 = -\frac{\prod_{0 \le l \le i-2}[\lambda_{l,i-2} - \lambda_{k,i-1} - l + k - 1]\prod_{0 \le l \le i}[\lambda_{l,i} - \lambda_{k,i-1} - l + k]}{\prod_{\substack{0 \le l \le i-1 \\ l \ne k}}[\lambda_{l,i-1} - \lambda_{k,i-1} - l + k - 1][\lambda_{l,i-1} - \lambda_{k,i-1} - l + k]}.$$

Hereafter, we use the symbols $[m] = (v^m - v^{-m})/(v - v^{-1})$, $[m]! = [m][m-1]\cdots[1]$, and $[m]_k = [m][m+1]\cdots[m+k-1]$.

§2.2. Whittaker vectors

The Verma module carries an obvious grading $\mathcal{V}^\lambda = \bigoplus_{\beta \in Q^+}(\mathcal{V}^\lambda)_\beta$ where

$$(\mathcal{V}^\lambda)_\beta = \{w \in \mathcal{V}^\lambda \mid K_i w = v^{(\alpha_i, \lambda - \beta)} w \ (1 \le i \le n)\}. \tag{2.8}$$

A *Whittaker vector* $\theta^\lambda = \sum_{\beta\in Q^+} \theta^\lambda_\beta$ is an element of a completion $\prod_{\beta\in Q^+}(\mathcal{V}^\lambda)_\beta$ of the Verma module. It is uniquely defined by the conditions that $\theta^\lambda_0 = \mathbf{1}^\lambda$ and

$$E_i K_i^{i-1}\,\theta^\lambda = \frac{1}{1-v^2}\,\theta^\lambda \quad (1\le i\le n). \tag{2.9}$$

Let us give an explicit formula for θ^λ in terms of the Gelfand–Zetlin basis. For $i=1,\dots,n$ and parameters $\mu=(\mu_0,\dots,\mu_i)$, $\nu=(\nu_0,\dots,\nu_{i-1})$ satisfying $\mu_k-\nu_k \in \mathbb{Z}_{\ge 0}$, define

$$\begin{aligned} A_i(\mu,\nu)^2 &= \frac{1}{\prod_{k=0}^{i-1}[\mu_k-\nu_k]!} \\ &\cdot \frac{1}{\prod_{0\le k<l\le i-1}[\nu_k-\nu_l-k+l+1]_{\mu_k-\nu_k}} \cdot \frac{1}{\prod_{0\le k<l\le i}[\nu_k-\mu_l-k+l]_{\mu_k-\nu_k}}. \end{aligned} \tag{2.10}$$

Proposition 2.1. *In the Gelfand–Zetlin basis* (2.2), *the Whittaker vector* θ^λ *has the following representation:*

$$\theta^\lambda = \sum_{\boldsymbol{\lambda}} \left(\frac{1}{1-v^2}\right)^{\mathrm{ht}(\boldsymbol{\lambda})} \prod_{i=1}^n C_i(\lambda^{(i)},\lambda^{(i-1)})|\boldsymbol{\lambda}\rangle. \tag{2.11}$$

Here we have set

$$\begin{aligned} \mathrm{ht}(\boldsymbol{\lambda}) &= \sum_{0\le k\le i\le n-1}(\lambda_{k,n}-\lambda_{k,i}),\\ C_i(\lambda^{(i)},\lambda^{(i-1)}) &= v^{p_i(\lambda^{(i)},\lambda^{(i-1)})}A_i(\lambda^{(i)},\lambda^{(i-1)}),\\ p_i(\lambda^{(i)},\lambda^{(i-1)}) &= (i-1)\Big(\sum_{k=0}^{i-1}\lambda_{k,i-1}\Big(\sum_{k=0}^{i-1}\lambda_{k,i-1}-\sum_{k=0}^{i}\lambda_{k,i}\Big)\\ &\quad -\sum_{0\le k<l\le i-1}\lambda_{k,i-1}\lambda_{l,i-1}+\sum_{0\le k<l\le i}\lambda_{k,i}\lambda_{l,i}\Big)\\ &\quad -\sum_{k=1}^{i-1}k(i-k)(\lambda_{k,i-1}-\lambda_{k,i}). \end{aligned}$$

The sum ranges over all Gelfand patterns (2.2)–(2.4) *with fixed* $\lambda_0,\dots,\lambda_n$.

Proof. The proof is a direct calculation using formulas (2.5), (2.6) for the action of E_i, K_i. The defining relations (2.9) reduce to the identities

$$\sum_{l=0}^{i}\frac{\prod_{k=0}^{i-1}[a_k-b_l]}{\prod_{\substack{k=0\\k\ne l}}^{i}[b_k-b_l]}v^{-\sum_{k=0}^{i-1}a_k+\sum_{\substack{k=0\\k\ne l}}^{i}b_k} = 1. \qquad \square$$

§2.3. Scalar product

The main object of our interest is the scalar product of the Whittaker and the dual Whittaker vectors. To define the latter, we consider the quantum group $U_{v^{-1}}(\mathfrak{gl}_{n+1})$ with parameter v^{-1}. Its generators are denoted by $\{\bar{E}_i, \bar{F}_i\}_{1\le i\le n}$, $\{\bar{v}^{\pm\epsilon_i}\}_{0\le i\le n}$. Let $\overline{\mathcal{V}}^{\lambda}$ be the Verma module over $U_{v^{-1}}(\mathfrak{gl}_{n+1})$ generated by the highest weight vector $\bar{\mathbf{1}}^{\lambda}$ with defining relations

$$\bar{E}_i\bar{\mathbf{1}}^{\lambda} = 0 \quad (1 \le i \le n), \qquad \bar{v}^{\epsilon_i}\bar{\mathbf{1}}^{\lambda} = v^{-\lambda_i}\bar{\mathbf{1}}^{\lambda} \quad (0 \le i \le n).$$

The *dual Whittaker vector* is defined similarly as an element $\bar{\theta}^{\lambda} \in \prod_{\beta\in Q^+}(\bar{\mathcal{V}}^{\lambda})_{\beta}$, imposing $\bar{\theta}^{\lambda}_0 = \bar{\mathbf{1}}^{\lambda}$ and

$$\bar{E}_i\bar{K}_i^{i-1}\bar{\theta}^{\lambda} = \frac{1}{1-v^{-2}}\bar{\theta}^{\lambda} \quad (1 \le i \le n) \tag{2.12}$$

in place of (2.9).

Let σ be the $\mathcal{R}$-linear anti-isomorphism of algebras given by

$$\sigma : U_v(\mathfrak{gl}_{n+1}) \to U_{v^{-1}}(\mathfrak{gl}_{n+1}), \qquad E_i \mapsto \bar{F}_i,\ F_i \mapsto \bar{E}_i,\ K_i \mapsto \bar{K}_i^{-1}. \tag{2.13}$$

There is a unique nondegenerate $\mathcal{R}$-bilinear pairing $(\ ,\ ) : \mathcal{V}^{\lambda} \times \overline{\mathcal{V}}^{\lambda} \to \mathcal{R}$ such that $(\mathbf{1}^{\lambda}, \bar{\mathbf{1}}^{\lambda}) = 1$ and

$$(xw, w') = (w, \sigma(x)w') \tag{2.14}$$

for all $x \in U_v(\mathfrak{gl}_{n+1})$ and $w \in \mathcal{V}^{\lambda}$, $w' \in \overline{\mathcal{V}}^{\lambda}$. We call (2.14) the *Shapovalov pairing.* The Gelfand–Zetlin basis $\{|\boldsymbol{\lambda}\rangle\}$ of $\mathcal{V}^{\lambda}$ and $\{\overline{|\boldsymbol{\lambda}\rangle}\}$ of $\overline{\mathcal{V}}^{\lambda}$ are orthonormal with respect to the Shapovalov pairing: $(|\boldsymbol{\lambda}\rangle, \overline{|\boldsymbol{\lambda}'\rangle}) = \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}'}$.

In [FFJMM2], we considered the scalar product

$$J^{\lambda}_{\beta} = J^{\lambda}_{\beta}[0,\infty) = v^{-(\beta,\beta)/2+(\lambda,\beta)}(\theta^{\lambda}_{\beta}, \bar{\theta}^{\lambda}_{\beta}). \tag{2.15}$$

We set $J^{\lambda}_{\beta} = 0$ unless $\beta \in Q^+$. The notation $J^{\lambda}_{\beta}[0,\infty)$ comes from the fact that the corresponding fermionic formula is related to the interval $[0,\infty)$ (see Theorem 3.2 in [FFJMM2] and Proposition 2.4 below).

In what follows, we choose the variables[1] $z_i = q^{-(\lambda+\rho,\alpha_i)}$ and write

$$J_{d_1,\dots,d_n}(q, z_1, \dots, z_n) = J^{\lambda}_{\beta}[0,\infty) \quad \text{for } \beta = \sum_{i=1}^{n} d_i\alpha_i. \tag{2.16}$$

These are rational functions in $q = v^2$ and $z_1, \dots, z_n$.

[1] The present definition for z_i is different from [FFJMM2] where $z_i = q^{-(\lambda,\alpha_i)}$ was used.

The explicit formula (2.11) (and for its dual) yields the following expression for (2.16). Set

$$z_{k,l}=\prod_{j=k+1}^{l} z_j.$$

Proposition 2.2. *We have*

$$\begin{aligned}(2.17)\qquad J_{d_1,\dots,d_n}(q,z_1,\dots,z_n)=&\sum_{\substack{m_{0,i-1}+\dots+m_{i-1,i-1}=d_i\\ 1\le i\le n}}(-1)^{\sum_{i=1}^n d_i-\sum_{i=0}^{n-1}m_{i,i}}\\ &\times q^{p(m)}\prod_{j=1}^{n} z_j^{\sum_{k=0}^{j-1}\sum_{i=j+1}^{n} m_{k,i-1}}\\ &\times\prod_{0\le k<i\le n}\frac{1}{(q)_{m_{k,i-1}-m_{k,i}}}\times\prod_{0\le k<l<i\le n}\frac{1}{(q^{m_{k,i}-m_{l,i-1}}z_{k,l})_{m_{k,i-1}-m_{k,i}}}\\ &\times\prod_{0\le k<l\le i\le n}\frac{1}{(q^{m_{k,i}-m_{l,i}+1}z_{k,l})_{m_{k,i-1}-m_{k,i}}},\end{aligned}$$

where

$$\begin{aligned}p(m)=&-\sum_{0\le k<l\le i\le n-1} m_{k,i}m_{l,i}+\sum_{0\le k<l<i\le n-1} m_{k,i}m_{l,i-1}\\ &+\frac{1}{2}\sum_{0\le k<i\le n-1} m_{k,i}(m_{k,i}-1).\end{aligned}$$

The sum is taken over all nonnegative integers $m_{k,i}$ satisfying (2.4) *and* $\sum_{k=0}^{i-1} m_{k,i-1}=d_i$.

Example. We have, for $n=1$,

$$J_{d_1}(q,z_1)=\frac{1}{(q)_{d_1}(qz_1)_{d_1}},$$

and for $n=2$,

$$\begin{aligned}J_{d_1,d_2}(q,z_1,z_2)=&\sum_{m=0}^{\min(d_1,d_2)}(-z_1)^m q^{-m(d_2-m)+m(m-1)/2}\frac{1}{(q)_m(q)_{d_1-m}(q)_{d_2-m}}\\ &\times\frac{1}{(qz_1)_m(qz_1z_2)_m(qz_2)_{d_2-m}(q^{-d_2+m}z_1)_m(q^{-d_2+2m+1}z_1)_{d_1-m}}.\end{aligned}$$

The second formula can be further simplified to

$$J_{d_1,d_2}(q,z_1,z_2)=\frac{(qz_1z_2)_{d_1+d_2}}{(q)_{d_1}(q)_{d_2}(qz_1)_{d_1}(qz_2)_{d_2}(qz_1z_2)_{d_1}(qz_1z_2)_{d_2}}.$$

The existence of a factorized form is a specific (and rather accidental) feature of $n=1,2$. It does not hold for $n\geq 3$.

§2.4. Toda Hamiltonian and fermionic formula

The quantity $J_{d_1,\ldots,d_n}(q,z_1,\ldots,z_n)$ admits, besides the explicit formula (2.17), other ways of characterization. For completeness, we quote these facts from the literature adapting them to the present notation.

The first is through the quantum difference Toda Hamiltonian of type A. It is a q-difference operator which acts on functions $f(y_1,\ldots,y_n)$:

$$Hf=\sum_{i=0}^{n} D_i^{-1}D_{i+1}(z_{i,n}(1-y_i)f). \tag{2.18}$$

Here D_i stands for the q-shift operator $(D_if)(y_1,\ldots,y_i,\ldots,y_n)=f(y_1,\ldots,qy_i,\ldots,y_n)$, and we set $y_0=0$, $D_0=D_{n+1}=1$.

Proposition 2.3 ([Sev, Et]). *The generating series*

$$F(q,y_1,\ldots,y_n;z_1,\ldots,z_n)=\sum_{d_1,\ldots,d_n\geq 0} J_{d_1,\ldots,d_n}(q,z_1,\ldots,z_n)y_1^{d_1}\cdots y_n^{d_n} \tag{2.19}$$

is an eigenfunction of the Toda Hamiltonian

$$HF=\Big(\sum_{i=0}^{n} z_{i,n}\Big)F.$$

The second way is the fermionic formula. Here we restrict the general consideration in [FFJMM2] to the Cartan matrix of type A. For a (possibly infinite) interval $[r,s]$, consider the sum[2]

$$\begin{aligned} &I_{d_1,\ldots,d_n}(q,z_1,\ldots,z_n|r,s)\\ &=\sum_{\substack{l_{r,i}+\cdots+l_{s,i}=d_i\\ 1\leq i\leq n}} \frac{q^{\sum_{t,t'=r}^{s}\min(t,t')(\sum_{i=1}^{n} l_{t,i}l_{t',i}-\sum_{i=1}^{n-1} l_{t,i}l_{t',i+1})}\prod_{i=1}^{n} z_i^{\sum_{t=r}^{s} tl_{t,i}}}{\prod_{i=1}^{n}\prod_{t=r}^{s}(q)_{l_{t,i}}}. \end{aligned} \tag{2.20}$$

Then we have

Proposition 2.4 ([FFJMM2]). *The following formula holds:*

$$J_{d_1,\ldots,d_n}(q,z_1,\ldots,z_n)=I_{d_1,\ldots,d_n}(q,z_1,\ldots,z_n|0,\infty).$$

[2] The definition is modified from that of [FFJMM2, (2.3)], in order to match the change of the definition of z_i.

§3. Character of the principal subspace

Consider the affine Lie algebra $\widehat{\mathfrak{sl}}_{n+1} = \mathfrak{sl}_{n+1}[t,t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$. Let M^k be the integrable highest weight vacuum module of level $k \in \mathbb{Z}_{\geq 0}$. Namely M^k is the irreducible highest weight $\widehat{\mathfrak{sl}}_{n+1}$-module generated by the highest weight vector w such that

$$(x \otimes t^j)w = 0 \qquad (x \in \mathfrak{sl}_{n+1},\ j \geq 0),$$

and the canonical central element c acts as the scalar k. Let $\widehat{\mathfrak{n}}_+ = \mathfrak{n}_+ \otimes \mathbb{C}[t,t^{-1}]$ be the current algebra over the nilpotent subalgebra $\mathfrak{n}_+$ of $\mathfrak{sl}_{n+1}$. The $\widehat{\mathfrak{n}}_+$-submodule generated by w,

$$V^k = U(\widehat{\mathfrak{n}}_+)w \ \subset M^k,$$

is called the *principal subspace* of M^k.

The following fermionic formula is known [FS] (see also [FJMMT]):

$$\text{(3.1)} \quad \operatorname{ch} V^k = \sum_{l_{t,i}\geq 0} \frac{q^{\sum_{t,t'=1}^{k} \min(t,t')(\sum_{i=1}^{n} l_{t,i}l_{t',i} - \sum_{i=1}^{n-1} l_{t,i}l_{t',i+1})} \prod_{i=1}^{n} z_i^{\sum_{t=1}^{k} t l_{t,i}}}{\prod_{i=1}^{n}\prod_{t=1}^{k}(q)_{l_{t,i}}}.$$

In the notation of (2.20), we have

$$\operatorname{ch} V^k = \sum_{d_1,\dots,d_n\geq 0} I_{d_1,\dots,d_n}(z_1,\dots,z_n|1,k).$$

The main result of the present note is the following bosonic formula, which generalizes a result of [FFJMM1] for $n=2$.

Theorem 3.1. *The character of the principal subspace of the level k vacuum module over $\widehat{\mathfrak{sl}}_{n+1}$ is given by*

$$\begin{aligned}\operatorname{ch} V^k = \sum_{d_1,\dots,d_n\geq 0} & q^{k(\sum_{i=1}^{n} d_i^2 - \sum_{i=1}^{n-1} d_i d_{i+1})} z_1^{kd_1}\cdots z_n^{kd_n} \\ & \times \tilde{J}_{d_1,\dots,d_n}(q, q^{2d_1-d_2}z_1, q^{-d_1+2d_2-d_3}z_2, \dots, q^{-d_{n-1}+2d_n}z_n),\end{aligned}$$

where

$$\tilde{J}_{d_1,\dots,d_n}(q,z_1,\dots,z_n) = \frac{1}{\prod_{0\leq i<j\leq n}(qz_{i,j})_\infty} \cdot J_{d_1,\dots,d_n}(q, z_1^{-1},\dots,z_n^{-1}),$$

and $J_{d_1,\dots,d_n}(q,z_1,\dots,z_n)$ is given by (2.17).

Proof. Writing $(q)_\beta = (q)_{d_1}\cdots(q)_{d_n}$ for $\beta = d_1\alpha_1 + \cdots + d_n\alpha_n$, let us introduce the notation

$$\text{(3.2)} \qquad J_\beta^\lambda[r,s] = \sum_{\sum_{t=r}^{s}\gamma_t = \beta} \frac{q^{(1/2)\sum_{r\leq t,t'\leq s}\min(t,t')(\gamma_t,\gamma_{t'}) - (\lambda+\rho, \sum_{t=r}^{s} t\gamma_t)}}{\prod_{t=r}^{s}(q)_{\gamma_t}}.$$

Then (2.20) can be written as

$$I_{d_1,\dots,d_n}(q,z_1,\dots,z_n|r,s)=J^{\lambda_0\epsilon_0+\cdots+\lambda_n\epsilon_n}_{d_1\alpha_1+\cdots+d_n\alpha_n}[r,s],$$

and hence

$$\operatorname{ch}V^k=\sum_{\beta\in Q^+}J^\lambda_\beta[1,k].$$

The following formula was proved in [FFJMM2, (4.26), Theorem 4.13]:

$$J^\lambda_\beta[0,k]=\sum_{\alpha\in Q^+}J^{\alpha-\lambda-2\rho}_\alpha[0,\infty)J^{\lambda-\alpha}_{\beta-\alpha}[0,\infty)\times q^{k((\alpha,\alpha)/2-(\lambda+\rho,\alpha))}.$$

Using the relation

$$J^\lambda_\beta[r+1,s+1]=q^{(\beta,\beta)/2-(\lambda+\rho,\beta)}J^\lambda_\beta[r,s],$$

we deduce that

$$J^\lambda_\beta[1,k]=\sum_{\alpha\in Q^+}J^{\alpha-\lambda-2\rho}_\alpha[0,\infty)J^{\lambda-\alpha}_{\beta-\alpha}[1,\infty)\times q^{k((\alpha,\alpha)/2-(\lambda+\rho,\alpha))}. \tag{3.3}$$

On the other hand, in the limit $k\to\infty$ the formula (3.1) reduces to

$$\sum_{\gamma\in Q^+}J^\lambda_\gamma[1,\infty)=\frac{1}{\prod_{0\le i<j\le n}(qz_{i,j})_\infty}.$$

Summing (3.3) over β, setting $\alpha=\sum_{i=1}^n d_i\alpha_i$ and noting that $q^{-(\lambda-\alpha+\rho,\alpha_i)}=q^{(\alpha,\alpha_i)}z_i$, we obtain the desired formula. □

Remark. The formula for the character of the principal subspace is a quasi-classical decomposition in the sense that it is written as a sum over the positive root lattice. Each term represents an asymptotic behavior when k is large. In this sense the formula is quasi-classical. However, a remarkable fact is that the formula is exact for finite k. Moreover, each summand is desingularized in the sense that it is a sum of factorized terms (2.17).

§4. Quasi-classical expansion

In this section we extend the fermionic formula (2.20) to the setting corresponding to the tower of subalgebras (2.1), and discuss its 'quasi-classical' decomposition. In the following, we indicate by subscript k the quantities associated with the subalgebra $\mathcal{A}_k\simeq U_v(\mathfrak{gl}_{k+1})$: for instance, $P_k=\bigoplus_{i=0}^k\mathbb{Z}\epsilon_i$ and $Q_k^+=\bigoplus_{i=1}^k\mathbb{Z}_{\ge0}\alpha_i$.

Let $-\infty\le r_1\le\cdots\le r_n\le\infty$ be a non-decreasing sequence of integers (possibly including $\pm\infty$), and set $I=[r_1,\infty)$. Generalizing (3.2), we define, for

$\lambda \in P_n$ and $\beta \in Q_n^+$,

$$J\left(\begin{matrix} r_1, r_2 \\ Q_1^+ \end{matrix}\middle|\cdots\middle|\begin{matrix} r_n, \infty \\ Q_n^+ \end{matrix}\middle|\lambda, \beta\right) = \sum_{\{\gamma_t\}_{t\in I}} \frac{1}{\prod_{t\in I}(q)_{\gamma_t}} q^{B(\{\gamma_t\}|\lambda)}, \tag{4.1}$$

$$B(\{\gamma_t\}|\lambda) = \frac{1}{2}\sum_{t,t'\in I} \min(t,t')(\gamma_t, \gamma_{t'}) - \Big(\lambda+\rho, \sum_{t\in I} t\gamma_t\Big). \tag{4.2}$$

The sum in (4.1) is taken over $\gamma_t \in Q_n^+$ $(t \in I)$ such that

$$\sum_{t\in I}\gamma_t = \beta, \qquad \gamma_t \in Q_i^+ \quad \text{for } r_i \le t < r_{i+1}\ (i = 1, \ldots, n).$$

In the new notation we have $J^\lambda_\beta[r,s] = J\left(\begin{smallmatrix} r,s \\ Q_n^+ \end{smallmatrix}\middle|\lambda,\beta\right)$.

Remark. If $-\infty < r \le s < \infty$, the fermionic sum $J^\lambda_\beta(r,s)$ is a finite sum and it is a rational function in $z_1, \ldots, z_n$ and q. If $[r,s] = [r,\infty]$ with $r > -\infty$, or $[r,s] = [-\infty, s]$ with $s < \infty$, the fermionic sum is a Laurent series with coefficients which are rational functions in q. However, as discussed in [FFJMM2, Section 2.2], the Laurent series is well-defined as a rational function in $z_1, \ldots, z_n$ and q. If $[r,s] = [-\infty,\infty]$, we split the interval $[-\infty,\infty]$ into $[-\infty,-1] \sqcup [0,\infty]$ and define the fermionic sum as a sum of products where each summand is the product of two rational functions, one corresponding to $[-\infty,-1]$ and the other to $[0,\infty]$.

Recall that in the completion of $U_v(\mathfrak{gl}_{n+1})$ there is an element u which satisfies

$$K_i u = uK_i, \quad E_i u = uK_i^2 E_i, \quad F_i u = uF_i K_i^{-2} \quad \text{for all } i = 1, \ldots, n.$$

Up to multiplication by a simple factor, u is the Drinfeld Casimir element. On each weight component $\mathcal{V}^\lambda_\beta$ of the Verma module, u acts as the scalar $q^{-(\beta,\beta)/2+(\lambda+\rho,\beta)}$. In [FFJMM2], the fermionic formula (2.20) was derived by inserting u in the scalar product (2.15) which defines the Whittaker vectors and calculating it in two different ways. The fermionic formula thus obtained is equivalent to the following recursive formula for $J^\lambda_\beta = J\left(\begin{smallmatrix} 0,\infty \\ Q_n^+ \end{smallmatrix}\middle|\lambda,\beta\right)$:

$$J^\lambda_\beta = \sum_\gamma \frac{1}{(q)_{\beta-\gamma}} q^{(\gamma,\gamma)/2-(\lambda+\rho,\gamma)} J^\lambda_\gamma.$$

The same calculation can be repeated using a 'partial' Drinfeld Casimir element. Namely let u_k denote the counterpart of u corresponding to the subalgebra $\mathcal{A}_k$, $k = 1, \ldots, n$.

Proposition 4.1. *Let $r_1 \le \cdots \le r_n \le 0$, and set $r_{n+1} = 0$. Then*

$$v^{-(\beta,\beta)/2+(\lambda,\beta)}\Big(\prod_{k=1}^n u_k^{-r_k+r_{k+1}} \cdot \theta^\lambda_\beta, \bar\theta^\lambda_\beta\Big) = J\left(\begin{matrix} r_1, r_2 \\ Q_1^+ \end{matrix}\middle|\cdots\middle|\begin{matrix} r_n, \infty \\ Q_n^+ \end{matrix}\middle|\lambda, \beta\right). \tag{4.3}$$

Proof. The calculation is the same as in [FFJMM2, Theorem 3.1], and the proof following it. □

For each $k = 1, \ldots, n-1$, the Whittaker vector (2.15) admits a decomposition in terms of those for the lower rank subalgebra $\mathcal{A}_k$:

$$\theta^\lambda_\beta = \sum_{\lambda^{(n-1)},\ldots,\lambda^{(k)}} (1-q)^{-\sum_{i=k+1}^n \sum_{l=0}^{i-1}(\lambda_{l,n}-\lambda_{l,i-1}) - \sum_{l=0}^k (k-l)(\lambda_{l,n}-\lambda_{l,k})} \times \prod_{i=k+1}^n C_i(\lambda^{(i)}, \lambda^{(i-1)}) \cdot \theta(\lambda^{(n)}, \ldots, \lambda^{(k)} | \beta^{(k)}), \tag{4.4}$$

$$\theta(\lambda^{(n)}, \ldots, \lambda^{(k)} | \beta^{(k)}) = \sum_{\lambda^{(k-1)},\ldots,\lambda^{(0)}} (1-q)^{-\sum_{i=1}^k \sum_{l=0}^{i-1}(\lambda_{l,k}-\lambda_{l,i-1})} \times \prod_{i=1}^k C_i(\lambda^{(i)}, \lambda^{(i-1)}) \cdot |\lambda^{(n)}, \ldots, \lambda^{(0)}\rangle. \tag{4.5}$$

Here $\beta^{(i)}$ are defined by

$$\beta^{(n)} = \beta, \tag{4.6}$$

$$(\lambda^{(i+1)} - \beta^{(i+1)})|_{P_i} = \lambda^{(i)} - \beta^{(i)} \qquad (\beta^{(i)} \in Q_i^+,\ i = 1, \ldots, n-1), \tag{4.7}$$

where $\epsilon_k|_{P_i} = \sum_{j=0}^i \delta_{j,k}\epsilon_k$ is the projection to P_i. Note that from (4.7) we see that

$$\begin{aligned}\beta^{(i)} - \beta^{(i-1)} &= (\lambda_{0,i} - \lambda_{0,i-1})(\alpha_1 + \cdots + \alpha_i) + (\lambda_{1,i} - \lambda_{1,i-1})(\alpha_2 + \cdots + \alpha_i) \\ &\quad + \cdots + (\lambda_{i-1,i} - \lambda_{i-1,i-1})\alpha_i.\end{aligned}$$

Therefore, the sum $\sum_{\lambda^{(n-1)},\ldots,\lambda^{(1)}}$ is equivalent to the sum over a partition of β,

$$\beta = \gamma^{(1)} + \cdots + \gamma^{(n)}$$

where

$$\begin{aligned}\gamma^{(i)} &= \beta^{(i)} - \beta^{(i-1)} \\ &\in R_i^+ \stackrel{\text{def}}{=} \mathbb{Z}_{\geq 0}(\alpha_1 + \cdots + \alpha_i) \oplus \mathbb{Z}_{\geq 0}(\alpha_2 + \cdots + \alpha_i) \oplus \cdots \oplus \mathbb{Z}_{\geq 0}\alpha_i.\end{aligned}$$

Note that $\lambda^{(n)} = \lambda$ and other $\lambda^{(i)}$ are determined by

$$\lambda^{(i)} = (\lambda^{(i+1)} - \gamma^{(i+1)})|_{P_i}. \tag{4.8}$$

Here $\theta(\lambda^{(n)}, \ldots, \lambda^{(k)}|\beta^{(k)})$ is a weight component of a Whittaker vector with respect to the subalgebra $\mathcal{A}_k \simeq U_v(\mathfrak{gl}_{k+1})$ and its Verma module with highest weight $\lambda^{(k)}$. It is so normalized that the coefficient of the vector $|\lambda^{(n)}, \ldots, \lambda^{(0)}\rangle$ satisfying $\lambda_{l,k} = \lambda_{l,k-1} = \cdots = \lambda_{l,l}$ for all $0 \leq l \leq k-1$ is 1.

Lemma 4.2. *Formula* (4.1) *can be decomposed as*

$$J\left(\begin{matrix} r_1, r_2 \\ Q_1^+ \end{matrix} \middle| \cdots \middle| \begin{matrix} r_n, \infty \\ Q_n^+ \end{matrix} \middle| \lambda, \beta \right) = \sum_{\lambda^{(n-1)}, \dots, \lambda^{(1)}} J\left(\begin{matrix} r_1, \infty \\ Q_1^+ \end{matrix} \middle| \lambda^{(1)}, \gamma^{(1)} \right) \prod_{i=2}^{n} d(\lambda^{(i)}, \gamma^{(i)} | r_i) A_i(\lambda^{(i)}, \lambda^{(i-1)})^2, \tag{4.9}$$

where $A_i(\mu, \nu)$ *is given in* (2.10)*. The coefficients* $d(\mu, \nu | r)$ *have the factorized form*

$$d(\mu, \nu | r) = v^{-(\nu,\nu)/2 + (\mu,\nu)} q^{r((\nu,\nu)/2 - (\mu+\rho,\nu))} ((1-q)(1-q^{-1}))^{-(\rho,\nu)}. \tag{4.10}$$

Proof. The action of $\prod_{k=1}^{n} u_k^{-r_k + r_{k+1}}$ can be calculated by using the decomposition (4.4) and

$$u_k \theta(\lambda^{(n)}, \dots, \lambda^{(k)} | \beta^{(k)}) = q^{-(\beta^{(k)}, \beta^{(k)})/2 + (\lambda^{(k)} + \rho, \beta^{(k)})} \theta(\lambda^{(n)}, \dots, \lambda^{(k)} | \beta^{(k)}).$$

Taking the scalar product with $\bar{\theta}^{\lambda}_{\beta}$ and simplifying the result, we obtain the assertion. □

Lemma 4.3. *We have*

$$J\left(\begin{matrix} -\infty, r \\ Q_{n-1}^+ \end{matrix} \middle| \begin{matrix} r, \infty \\ Q_n^+ \end{matrix} \middle| \lambda^{(n)}, \gamma^{(n)} \right) = d(\lambda^{(n)}, \gamma^{(n)} | r) A_n(\lambda^{(n)}, \lambda^{(n-1)})^2. \tag{4.11}$$

Proof. Consider the decomposition (4.4) with $k = n-1$ and apply $u_{n-1}^{-r_{n-1}+r_n} u_n^{-r_n}$. By the same computation as in the previous lemma, we find

$$J\left(\begin{matrix} r_{n-1}, r_n \\ Q_{n-1}^+ \end{matrix} \middle| \begin{matrix} r_n, \infty \\ Q_n^+ \end{matrix} \middle| \lambda^{(n)}, \beta^{(n)} \right) = \sum_{\gamma^{(n)} \in R_n^+, \, \beta^{(n-1)} = \beta^{(n)} - \gamma^{(n)} \geq 0} J\left(\begin{matrix} r_{n-1}, \infty \\ Q_{n-1}^+ \end{matrix} \middle| \lambda^{(n-1)}, \beta^{(n-1)} \right) \times d(\lambda^{(n)}, \gamma^{(n)} | r_n) A_n(\lambda^{(n)}, \lambda^{(n-1)})^2.$$

Now let $r_{n-1} \to -\infty$. In this limit, only one term $\beta^{(n-1)} = 0$ in the sum contributes. With this choice the factor J on the right hand side is 1 and $\beta^{(n)} = \gamma^{(n)}$, hence we obtain the desired result. □

Substituting (4.11) (with n replaced by $i = 2, \dots, n$) back into (4.9), we arrive at the following result.

Theorem 4.4. *Notation being as in* (4.7), *we have*

$$J\left(\begin{matrix} r_1, r_2 \\ Q_1^+ \end{matrix}\middle| \cdots \middle| \begin{matrix} r_n, \infty \\ Q_n^+ \end{matrix}\middle| \lambda, \beta\right) \tag{4.12}$$

$$= \sum_{\gamma^{(1)}+\cdots+\gamma^{(n)}=\beta, \gamma^{(i)}\in R_i^+} J\left(\begin{matrix} r_1, \infty \\ Q_1^+ \end{matrix}\middle| \lambda^{(1)}, \gamma^{(1)}\right) \times \prod_{i=2}^{n} J\left(\begin{matrix} -\infty, r_i \\ Q_{i-1}^+ \end{matrix}\middle| \begin{matrix} r_i, \infty \\ Q_i^+ \end{matrix}\middle| \lambda^{(i)}, \gamma^{(i)}\right).$$

This theorem has the following interpretation. In formula (4.1), let us consider the limiting situation where $r_1 \ll \cdots \ll r_n$. Imagine that we take the sum separately over the variables γ_t taking t to be 'in the vicinity' I_i of each end point r_i, $i = 1, \ldots, n$. Then the contribution to (4.2) would become

$$\sum_{i=1}^{n} B_i + \sum_{i=1}^{n-1} \Big(\sum_{t\in I_i} t\gamma_t, \sum_{j>i} \sum_{t'\in I_j} \gamma_{t'} \Big)$$

where B_i stands for (4.2) with $t, t' \in I_i$. The corresponding sum, with $\sum_{t\in I_i} \gamma_t = \gamma^{(i)}$ being fixed, gives a summand on the right hand side of (4.12). Theorem 4.4 tells us that this 'quasi-classical decomposition' in fact gives an exact answer.

Acknowledgements

Research of BF is partially supported by SS-3472.2008.2, Programma RAS, "Elementary particles and fundamental nuclear physics", RFBR 08-01-00720, CNRS-RFBR 09-02-93106 and RFBR 09-01-00242. Research of MJ is supported by the Grant-in-Aid for Scientific Research B-20340027 and B-20340011.

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