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Baker–Akhiezer Modules on the Intersections of Shifted Theta Divisors

by

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Abstract

The restriction, on the spectral variables, of the Baker–Akhiezer (BA) module of a g-dimensional principally polarized abelian variety with the non-singular theta divisor to an intersection of shifted theta divisors is studied. It is shown that the restriction to a k-dimensional variety becomes a free module over the ring of differential operators in k variables. The remaining $g-k$ derivations define evolution equations for generators of the BA-module. As a corollary new examples of commutative rings of partial differential operators with matrix coefficients and their non-trivial evolution equations are obtained.

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§1. Introduction

The Baker–Akhiezer (BA) module was introduced in [N1, N2] in order to extend the theory of the BA function due to Krichever [K] to higher dimensions. It is a geometric counterpart of the $\mathcal{D}$-module generated by the wave operator in Sato's theory of KP-hierarchy and universal Grassmann manifold.

A fundamental example of the BA function is a function on an elliptic curve of the form

$$\varphi(z;x) = \frac{\sigma(z+x)}{\sigma(z)\sigma(x)} \exp(-x\zeta(z)),$$

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where $\sigma(z)$, $\zeta(z)$ are Weierstrass' sigma and zeta functions. The corresponding BA-module is the $\mathcal{D}$-module generated by $\varphi(z;x)$:

$$M = \mathcal{D}\varphi(z;x) = \sum_{n=0}^{\infty} \mathcal{O}\partial_x^n \varphi(z;x),$$

where $\mathcal{O}$ is a suitable ring of functions such as the convergent power series ring, its quotient field etc. and $\mathcal{D} = \mathcal{O}[\partial_x]$ is the ring of differential operators in x with coefficients in $\mathcal{O}$. It is a rank one free module over $\mathcal{D}$.

Let $A = \mathbb{C}[\wp(z), \wp'(z)]$ be the affine ring of the elliptic curve. An important property of the BA-module is that it is not only a $\mathcal{D}$-module but also an A-module. As a consequence A is embedded in $\mathcal{D}$ as a commutative subring.

Similarly, in the case of genus g algebraic curves, the BA-module becomes a $(\mathcal{D}_g, A)$-bimodule, where $\mathcal{D}_g = \mathcal{O}[\partial_1, \ldots, \partial_g]$ is the ring of differential operators in g variables and A is the affine ring of the curve. It becomes a rank one free module over the subring $\mathcal{D}_1 = \mathcal{O}[\partial_1]$ of operators in one variable and the affine ring A is embedded in $\mathcal{D}_1$. The action of the commuting derivations $\partial_2, \ldots, \partial_g$ specifies evolution equations of the BA-module, or the deformation of the image of A in $\mathcal{D}$. In this way solutions of integrable nonlinear equations such as KP equation, KdV equation are constructed [K].

In [N1] the BA-module of a g-dimensional principally polarized Abelian variety (X, Θ) with a non-singular Θ is studied. It has been proved that the BA-module becomes a free $\mathcal{D}$-module of rank $g!$, where $\mathcal{D}$ is the ring of differential operators in g variables. Consequently, the affine ring A of $X \backslash \Theta$ is embedded in the ring $\mathrm{Mat}(g!, \mathcal{D})$ of differential operators with coefficients in $g! \times g!$ matrices. However in this case there is no non-trivial deformation of A in $\mathrm{Mat}(g!, \mathcal{D})$. To have a non-trivial deformation it is necessary to consider the BA-module of polarized subvarieties of (X, Θ).

We consider an intersection Y^k of shifted theta divisors as a subvariety of X and the intersection Q^k of it with the theta divisor as a divisor, where k denotes the codimension of Y^k in X. We show that the restriction of the BA-module of (X, Θ) to Y^k is a free $\mathcal{D}_{g-k}$-module of rank $g!$, where $\mathcal{D}_i$ is the ring of differential operators in i variables. As a by-product we have the embedding of the affine ring of $Y^k \backslash Q^k$ in $\mathrm{Mat}(g!, \mathcal{D}_{g-k})$ and k commuting derivations which specify the deformation of the image of the affine ring.

The simplest case of $g = 3$ and $k = 1$ is studied in [N2]. The case of intersections of more general divisors is studied in [Mir]. Unfortunately the proof of the freeness is incomplete in that paper. Other examples of BA-modules which have non-trivial deformations are studied in [R].

The plan of the paper is as follows. In Section 2 the definition of BA-modules and the main result are given. The combinatorial properties of the restriction of the BA-module is studied in Section 3. It is shown that the character of the associated graded module of the BA-module coincides with that of the free module. This means that as far as the dimension is concerned the restriction of the BA-module becomes a free module. In Section 4 the proof of the main theorem is given based on the result of Section 3.

§2. Results

Let Ω be a point of the Siegel upper half space of degree g, X the corresponding principally polarized Abelian variety

$$X = \mathbb{C}^g/\Gamma, \quad \Gamma = \mathbb{Z}^g + \Omega\mathbb{Z}^g,$$

$\theta_{a,b}(z)$ Riemann's theta function with characteristic (a, b), $a, b \in \mathbb{R}^g$,

$$\theta_{a,b}(z) = \theta_{a,b}(z, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i {}^t(n + a)\Omega(n + a) + 2\pi i {}^t(n + a)(z + b)),$$

and Θ the theta divisor on X defined by the zero set of $\theta(z) = \theta_{0,0}(z)$. For $c \in \mathbb{C}^g$, $\mathcal{L}_c$ denotes the holomorphic flat line bundle on X which has $\theta(z + c)/\theta(z)$ as a meromorphic section. A meromorphic section of $\mathcal{L}_c$ can be considered as a meromorphic function $f(z)$ on $\mathbb{C}^g$ which satisfies

$$f(z + m + \Omega n) = \exp(-2\pi i {}^t nc)f(z), \quad m, n \in \mathbb{Z}^g. \tag{2.1}$$

We denote by $L_c(n)$ the space of meromorphic sections of $\mathcal{L}_c$ whose poles are only on Θ of order at most n. It is known [M2] that $\dim L_c(n) = n^g$ and a linear basis is given by the functions $f_{n,a}(z + c/n)/\theta(z)^n$, $a \in \mathbb{Z}^g/n\mathbb{Z}^g$, where

$$f_{n,a}(z) = \theta_{a/n,0}(nz, n\Omega).$$

We define

$$L_c = \bigcup_{n=0}^{\infty} L_c(n).$$

The subspaces $L_c(n)$ define an increasing filtration of L_c.

Set

$$\zeta_i(z) = \frac{\partial}{\partial z_i} \log \theta(z).$$

It satisfies, for $m, n \in \mathbb{Z}^g$,

$$\zeta_j(z + m + \Omega n) = \zeta_j(z) - 2\pi i n_j.$$

Let $\mathcal{O}$ be the convergent power series ring in $x = (x_1, \ldots, x_g)$, $\mathcal{K}$ its quotient field, $\partial_{x_i} = \partial/\partial x_i$, $\mathcal{D} = \mathcal{K}[\partial_{x_1}, \ldots, \partial_{x_g}]$ the ring of differential operators with coefficients in $\mathcal{K}$, $\mathcal{D}(n) = \{\sum_{|\alpha|\le n} a_\alpha \partial_x^\alpha \in \mathcal{D}\}$ the differential operators of order at most n and $\operatorname{gr}\mathcal{D} = \bigoplus \mathcal{D}(n)/\mathcal{D}(n-1)$ the commutative ring of principal symbols.

In general, for a module with an increasing filtration $M = \bigcup_n M(n)$, the associated graded module is defined by

$$\operatorname{gr} M = \bigoplus_n \operatorname{gr}_n M, \qquad \operatorname{gr}_n M = M(n)/M(n-1).$$

Let us consider the spaces of functions of the form:

$$M_c(n) = \sum_{a\in\mathbb{Z}^g/n\mathbb{Z}^g} \mathcal{K}\frac{f_{n,a}(z+\frac{c+x}{n})}{\theta(z)^n}\exp\Big(-\sum_{i=1}^g x_i\zeta_i(z)\Big),$$
$$M_c = \bigcup_{n=0}^{\infty} M_c(n).$$

The space $M_c(n)$ is an n^g-dimensional vector space over $\mathcal{K}$ and the subspaces $\{M_c(n)\}$ specify an increasing filtration of M_c. Then, for $c \notin \Gamma$,

$$\dim_{\mathcal{K}} \operatorname{gr}_n M_c = \dim_{\mathcal{K}} \operatorname{gr}_n L_c = n^g - (n-1)^g, \quad n \ge 1. \tag{2.2}$$

As functions of z variables, elements of M_c satisfy the equation (2.1). The differentiation in x_i preserves this equation and satisfies

$$\partial_{x_i} M_c(n) \subset M_c(n+1).$$

Thus M_c becomes a $\mathcal{D}$-module and $\operatorname{gr} M_c$ becomes a $\operatorname{gr}\mathcal{D}$-module. The $\mathcal{D}$-module M_c is introduced in [N1] and called the *Baker–Akhiezer* (BA) module of (X, Θ).

Let A be the affine ring of $X\backslash\Theta$. Analytically it is described as

$$A = \left\{ \frac{F(z)}{\theta(z)^n} \;\middle|\; F(z) \text{ is holomorphic on } \mathbb{C}^g,\ \frac{F}{\theta^n}(z+\gamma) = \frac{F}{\theta^n}(z) \text{ for any } \gamma\in\Gamma \right\}.$$

If $f(z)$ satisfies (2.1) and $a(z) \in A$ then $f(z)a(z)$ satisfies (2.1). Therefore L_c is an A-module. Consequently, M_c becomes an A-module whose action obviously commutes with that of $\mathcal{D}$.

Remark. In [N1] the BA-module is defined globally on the dual abelian variety $\widehat{X}$ of X as a sheaf of $\mathcal{D}_{\widehat{X}}$-modules by the Fourier–Mukai transform, M_c being a scalar extension of the stalk at the point specified by c.

Let r_i, $i \geq 1$, be integers defined by

$$r_i = i^g - (i-1)^g - \sum_{j=1}^{i-1} r_j \binom{g+i-j-1}{g-1}.$$

They satisfy $r_1 = 1$, $r_i = r_{g+1-i}$, $r_i = 0$ for $i > g$ and $\sum_{i=1}^{g} r_i = g!$.

The following theorem is proved in [N1].

Theorem 2.1 ([N1]). *Suppose that* Θ *is non-singular and* $c \notin \Gamma$. *Then* $\operatorname{gr} M_c$ *is a free* $\operatorname{gr}\mathcal{D}$*-module of rank* $g!$ *and* M_c *is a free* $\mathcal{D}$*-module of rank* $g!$. *More precisely*

$$\operatorname{gr} M_c = \bigoplus_{i=1}^{g} \bigoplus_{j=1}^{r_i} (\operatorname{gr}\mathcal{D})\psi_{ij}, \quad \psi_{ij} \in \operatorname{gr}_i M_c,$$
$$M_c = \bigoplus_{i=1}^{g} \bigoplus_{j=1}^{r_i} \mathcal{D}\phi_{ij}, \quad \phi_{ij} \in M_c(i),$$

where ψ_{ij} *is the projection of* ϕ_{ij} *in* $\operatorname{gr}_i M_c$.

Let Φ be the column vector of dimension $g!$ whose components consist of the functions ϕ_{ij} and $\operatorname{Mat}(m, \mathcal{D})$ the ring of $m \times m$ matrices with components in $\mathcal{D}$. Since M_c is an A-module, for $a \in A$, there exists a unique element $\ell(a) \in \operatorname{Mat}(g!, \mathcal{D})$ such that

$$a\Phi = \ell(a)\Phi.$$

This defines an embedding of A into $\operatorname{Mat}(g!, \mathcal{D})$ as a ring. Thus A is realized as a commutative subring of the ring of differential operators in g variables with matrix coefficients.

In this paper we shall extend Theorem 2.1 to the BA-modules on intersections of shifted theta divisors.

For $a \in \mathbb{C}^g$ we set

$$\Theta_a = \{\theta(z-a) = 0\} \subset X.$$

Take $a_1, \ldots, a_{g-1} \in \mathbb{C}^g$. Beginning from $(Y^0, Q^0) = (X, \Theta)$ we define (Y^k, Q^k) for $k \geq 1$ by

$$Y^k = \Theta_{a_1} \cap \cdots \cap \Theta_{a_k},$$
$$Q^k = Y^k \cap \Theta.$$

We assume that, for any $k < g$, $\Theta_{a_1}, \ldots, \Theta_{a_k}$ intersect transversally and so do $\Theta, \Theta_{a_1}, \ldots, \Theta_{a_k}$. This means in particular that Y^k and Q^k are non-singular subvarieties of X of dimensions $g-k$ and $g-k-1$ respectively. It can be shown that for generic $a_1, \ldots, a_{g-1}$ the assumption is satisfied.

We denote the restriction of $\mathcal{L}_c$ to Y^k by the same symbol for simplicity. Let $\mathcal{L}_c(nQ^k)$ be the sheaf of germs of meromorphic sections of $\mathcal{L}_c$ on Y^k with poles only on Q^k of order at most n. We set

$$L_c^k(n) = H^0(Y^k, \mathcal{L}_c(nQ^k)).$$

We define the space $\tilde{M}_c^k(n)$ as the set of functions of the form

$$f(z;x)\exp\Big(-\sum_{i=1}^{g} x_i\zeta_i(z)|_{Y^k}\Big),$$

where $f(z;x)$ satisfies the following conditions.

There is an open neighborhood U of $0 \in \mathbb{C}^g$, which can depend on f, with the following properties.

(i) For each $x \in U$, $f(z;x)$ belongs to $L_{c+x}^k(n)$ as a function of z.
(ii) As a function of x, $f(z;x)$ is analytic on U.

It is obvious that $\tilde{M}_c^k(n)$ is an $\mathcal{O}$-module.

Lemma 2.2. *Let k satisfy $0 \le k \le g-1$. Suppose that $c + \sum_{i\in I} a_i \notin \Gamma$ for any subset I of $\{1,\dots,k\}$. Then*

(i) *$H^i(Y^k, \mathcal{L}_c(nQ^k)) = 0$, $i \neq 0, g-k$, $n \in \mathbb{Z}$.*
(ii) *The restriction map $L_c^{k-1}(n) \to L_c^k(n)$ is surjective for any $n \in \mathbb{Z}$.*

Proof. We have the following exact sequence and isomorphism:

$$0 \to \mathcal{L}_c(nQ^{k-1} - Y^k) \to \mathcal{L}_c(nQ^{k-1}) \to \mathcal{L}_c(nQ^k) \to 0, \tag{2.3}$$

$$\mathcal{L}_c(nQ^{k-1} - Y^k) \simeq \mathcal{L}_{c+a_k}((n-1)Q^{k-1}). \tag{2.4}$$

The isomorphism (2.4) follows from $\mathcal{O}_X(-\Theta_a) \simeq \mathcal{O}_X(-\Theta) \otimes \mathcal{L}_a$. Assertion (i) can be proved by induction on k using the cohomology sequence of (2.3) and the vanishing [M1]

$$H^i(X, \mathcal{L}_c(n\Theta)) = 0$$

for $i \ge 1$, $n \ge 1$ or $i \ge 0$, $n = 0$ or $i \neq g$, $n < 0$. Assertion (ii) follows from the cohomology sequence of (2.3) and (i). □

Lemma 2.3. *Assume the same conditions as in Lemma 2.2. Then $\dim L_c^k(n)$ does not depend on c and satisfies*

$$\dim L_c^k(n) = \dim L_c^{k-1}(n) - \dim L_c^{k-1}(n-1). \tag{2.5}$$

Proof. The lemma can easily be proved by induction on k using the cohomology sequence of (2.3), the isomorphism (2.4), Lemma 2.2(i) and $\dim L_c^0(n) = n^g$ for $n \geq 0$, $c \notin \Gamma$. □

Example 2.4.

$$\dim L_c^1(n) = n^g - (n-1)^g \quad \text{for } n \geq 1,$$
$$\dim L_c^2(n) = n^g - 2(n-1)^g + (n-2)^g \quad \text{for } n \geq 2 \quad \text{and} \quad \dim L_c^2(1) = 1.$$

Lemma 2.5 ([M1]). *Assume the same conditions as in Lemma* 2.2. *Then $\tilde{M}_c^k$ is a free $\mathcal{O}$-module of rank* $\dim L_c^k(n)$.

Proof. We take some basis $\{f_i(z)\}$ of $L_c^k(n)$ and lift it to $\{f_i(z;x)\}$ such that the conditions (i), (ii) of $\tilde{M}_c^k(n)$ are satisfied and $f_i(z;0) = f_i(z)$. This gives an $\mathcal{O}$-free basis of $\tilde{M}_c^k(n)$. Such analytic lift can be constructed among the functions $\{f_{n,a}(z + \frac{c+x}{n})\}$ restricted to Y^k since the restriction map $L_{c+x}^0(n) \to L_{c+x}^k(n)$ is surjective for x sufficiently close to $0 \in \mathbb{C}^g$ by Lemma 2.2(ii). □

We set

$$M_c^k(n) = \mathcal{K} \otimes_{\mathcal{O}} \tilde{M}_c^k(n), \qquad M_c^k = \bigcup M_c^k(n).$$

The set of subspaces $\{M_c^k(n)\}$ defines an increasing filtration on M_c^k.

Let $\pi_k : M_c^k \to M_c^{k+1}$ be the restriction map. It is surjective for $k \geq 0$ as shown above. In particular $\pi_{0k} := \pi_{k-1} \cdots \pi_0 : M_c \to M_c^k$ is surjective. Thus M_c^k can be directly described as the restriction of M_c to Y^k with respect to the z variables:

$$M_c^k = M_c|_{Y^k}.$$

It is obvious that the restriction in z variables commutes with the action of ∂_{x_i}. Therefore M_c^k becomes a $\mathcal{D}$-module. Moreover the action of ∂_{x_i} satisfies $\partial_{x_i} M_c^k(n) \subset M_c^k(n+1)$. Thus $\operatorname{gr} M_c^k$ becomes a $\operatorname{gr}\mathcal{D}$-module.

The main result of this paper is

Theorem 2.6. *Suppose that Θ is non-singular and $c + \sum_{i \in I} a_i \notin \Gamma$ for any subset I of $\{1, \ldots, g-1\}$. Then there exists a set of linearly independent vector fields $D_i = \sum_{j=1}^g c_{ij}\partial_{x_j}$, c_{ij} being constants, such that the following properties are valid.*

(i) *Let $\mathcal{D}_i = \mathcal{K}[D_1, \ldots, D_i]$. Then M_c^k is a free $\mathcal{D}_{g-k}$-module of rank $g!$. More precisely it is of the form*

$$M_c^k = \bigoplus_{i=1}^{g} \bigoplus_{j=1}^{r_i} \mathcal{D}_{g-k}\phi_{ij}^k, \qquad \phi_{ij}^k \in M_c^k(i).$$

(ii) *The module* $\mathrm{gr}\, M_c^k$ *is a free* $\mathrm{gr}\, \mathcal{D}_{g-k}$*-module of rank* $g!$. *More precisely it is of the form*

$$\mathrm{gr}\, M_c^k = \bigoplus_{i=1}^{g} \bigoplus_{j=1}^{r_i} (\mathrm{gr}\, \mathcal{D}_{g-k}) \psi_{ij}^k, \qquad \psi_{ij}^k \in \mathrm{gr}_i\, M_c^k,$$

where ψ_{ij}^k *is the image of* ϕ_{ij}^k *in* $\mathrm{gr}_i\, M_c^k$ *and the filtration of* $\mathcal{D}_{g-k}$ *is specified by* $\mathcal{D}_k(n) = \mathcal{D}_k \cap \mathcal{D}(n)$.

Let Φ^k be the column vector whose components consist of the functions ϕ_{ij}^k and A^k be the affine ring of $Y^k \backslash Q^k$. We have $\Phi^0 = \Phi$, $A^0 = A$. The ring A^k acts on M_c^k. Thus for any $a \in A^k$ there exists a unique operator $\ell^k(a) \in \mathrm{Mat}(g!, \mathcal{D}_{g-k})$ such that

$$a\Phi^k = \ell^k(a)\Phi^k.$$

This defines an embedding of A^k in $\mathrm{Mat}(g!, \mathcal{D}_{g-k})$. Since M_c^k is a $\mathcal{D}$-module and $\mathcal{D} = \mathcal{D}_g$, for D_i with $g - k + 1 \le i \le g$, there exists a unique operator $B_i^k \in \mathrm{Mat}(g!, \mathcal{D}_{g-k})$ such that

$$D_i \Phi^k = B_i^k \Phi^k.$$

Those operators satisfy, for any a, b, i, j,

$$[\ell^k(a), \ell^k(b)] = 0, \qquad [D_i - B_i^k, D_j - B_j^k] = 0, \qquad [D_i - B_i^k, \ell^k(a)] = 0.$$

This system of equations is an analogue of the Zakharov–Shabat equations [K].

§3. Combinatorial freeness

For a graded $\mathcal{K}$-vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$ such that each V_n is finite-dimensional we define the character $\mathrm{ch}\, V$ by

$$\mathrm{ch}\, V = \sum (\dim_{\mathcal{K}} V_n) t^n.$$

Obviously

$$\mathrm{ch}\, \mathrm{gr}\, \mathcal{D}_j = \frac{1}{(1-t)^j}.$$

We have

$$\mathrm{ch}\, \mathrm{gr}\, M_c^0 = \sum_{n=1}^{\infty} (n^g - (n-1)^g) t^n = (1-t)\left(t\frac{d}{dt}\right)^g (1-t)^{-1}.$$

Let $\{\psi_{ij}\}$ be a $\mathrm{gr}\, \mathcal{D}$-free basis of $\mathrm{gr}\, M_c$ as in Theorem 2.1 and $F = \bigoplus \mathcal{K}\psi_{ij}$ the subspace of $\mathrm{gr}\, M_c$. By Theorem 2.1, $\mathrm{gr}\, M_c \simeq (\mathrm{gr}\, \mathcal{D}_g) \otimes F$. The module F naturally inherits a grading from $\mathrm{gr}\, M_c$. Then

$$\mathrm{ch}\, \mathrm{gr}\, M_c = (\mathrm{ch}\, \mathrm{gr}\, \mathcal{D}_g) \cdot \mathrm{gr}\, F = (1-t)^{-g} \sum_{i=1}^{g} r_i t^i.$$

Example 3.1. For $g = 1, 2, 3, 4$, $\operatorname{ch}\operatorname{gr} M_c$ is given by

$$\frac{t}{1-t}, \quad \frac{t+t^2}{(1-t)^2}, \quad \frac{t+4t^2+t^3}{(1-t)^3}, \quad \frac{t+11t^2+11t^3+t^4}{(1-t)^4}.$$

Lemma 3.2. *Assume the same conditions for $c, a_1, \ldots, a_{g-1}$ as in Theorem 2.6. Then*

(i) $\dim_{\mathcal{K}} \operatorname{gr}_n M_c^{j+1} = \dim_{\mathcal{K}} \operatorname{gr}_n M_c^j - \dim_{\mathcal{K}} \operatorname{gr}_{n-1} M_c^j$.

(ii) $\operatorname{ch}\operatorname{gr} M_c^{j+1} = (1-t)\operatorname{ch}\operatorname{gr} M_c^j$.

Proof. Asertion (ii) follows from (i), and (i) follows from Lemmas 2.3 and 2.5. $\square$

By Lemma 3.2 we have

$$\operatorname{ch}\operatorname{gr} M_c^j = (1-t)^j \operatorname{ch}\operatorname{gr} M_c = (\operatorname{ch}\mathcal{D}_{g-j}) \cdot \operatorname{ch} F. \tag{3.1}$$

§4. Proof of Theorem 2.6

Notice that (i) of Theorem 2.6 follows from (ii).

We shall prove

Proposition 4.1. *Assume the same conditions as in Theorem 2.6. Set $y^{(0)} = (y_1^{(0)}, \ldots, y_g^{(0)}) = (x_1, \ldots, x_g)$. Then, for each $k \geq 1$, there exist a coordinate $y^{(k)} = (y_1^{(k)}, \ldots, y_{g-k+1}^{(k)})$, a linear change of coordinates from $(y_1^{(k-1)}, \ldots, y_{g-k+1}^{(k-1)})$ to $y^{(k)}$ and $\psi_{ij}^k \in \operatorname{gr}_i M_c^k$, $1 \leq i \leq g$, $1 \leq j \leq r_i$ such that the following properties hold. Let $\mathcal{D}_{g-k} = \mathcal{K}[\partial_{y_1^{(k)}}, \ldots, \partial_{y_{g-k}^{(k)}}]$, $\mathcal{D}_{g-k}(n) = \mathcal{D}_{g-k} \cap \mathcal{D}(n)$ and $\xi_i^{(k)}$ the image of $\partial_{y_i^{(k)}}$ in $\operatorname{gr}_1 \mathcal{D}$. Then*

$$\xi_{g-k+1}^{(k)} \psi_{ij}^k \in \sum_{i'j'} (\operatorname{gr}\mathcal{D}_{g-k}) \psi_{i'j'}^k,$$

$$\operatorname{gr} M_c^k = \bigoplus_{i=1}^{g} \bigoplus_{j=1}^{r_i} (\operatorname{gr}\mathcal{D}_{g-k}) \psi_{ij}^k.$$

If we define $D_k = \partial_{y_k^{(g+1-k)}}$ for $2 \leq k \leq g$ and $D_1 = \partial_{y_1^{(g-1)}}$, then Theorem 2.6(ii) follows from this proposition.

We prove Proposition 4.1 by induction on k, where the case $k = 0$ is established by Theorem 2.1. We assume that the assertion is valid for k if $c + \sum_{i\in I} a_i \notin \Gamma$ for any subset I of $\{1, \ldots, k\}$. We shall prove that it is true for $k+1$ if $c + \sum_{i\in I} a_i \notin \Gamma$ for any subset I of $\{1, \ldots, k+1\}$.

Let us set

$$\tilde{\psi}_{ij}^{k+1} = \psi_{ij}^k|_{Y^{k+1}}.$$

Lemma 4.2. *For each (i,j) there exist a non-zero element $P_{ij} \in \mathrm{gr}_{N^{ij}} \mathcal{D}_{g-k}$ for some $N^{ij} \geq 0$ and a linear change of coordinates from $(y_1^{(k)}, \ldots, y_{g-k}^{(k)})$ to $(y_1^{(k+1)}, \ldots, y_{g-k}^{(k+1)})$ such that the following properties hold:*

(i) $P_{ij}\tilde{\psi}_{ij}^{k+1} = 0$ *in* $\mathrm{gr}\, M_c^{k+1}$.

(ii) *Let* $\xi_i = \xi_i^{(k+1)}$. *Then* P_{ij} *is of the form*

$$P_{ij} = \xi_{g-k}^{N^{ij}} + \sum_{|\alpha|=N^{ij},\, \alpha_{g-k}<N^{ij}} a_{ij;\alpha} \xi_1^{\alpha_1} \cdots \xi_{g-k}^{\alpha_{g-k}},$$

where $\alpha = (\alpha_1, \ldots, \alpha_{g-k})$, $|\alpha| = \sum_{i=1}^{g-k} \alpha_i$.

Proof. (i) By (3.1) the dimension of $\mathrm{gr}_n M_c^{k+1}$ is a polynomial in n of degree $g-k-2$ for sufficiently large n. If there are no non-trivial linear relations among $\xi_1^{\alpha_1} \cdots \xi_{g-k}^{\alpha_{g-k}} \tilde{\psi}_{ij}^{k+1}$, $\sum \alpha_l = n-i$ for any n, then $\dim_{\mathcal{K}}((\mathrm{gr}_{n-i} \mathcal{D}_{g-k})\tilde{\psi}_{ij}^{k+1}) = \dim_{\mathcal{K}} \mathrm{gr}_{n-i} \mathcal{D}_{g-k}$ and it is a polynomial in n of degree $g-k-1$ for sufficiently large n. Thus there should be a relation as in the assertion.

(ii) Let us write

$$P_{ij} = \sum_{\sum \alpha_l = N^{ij}} q_{\alpha_1,\ldots,\alpha_{g-k}} \xi_1^{\alpha_1} \cdots \xi_{g-k}^{\alpha_{g-k}},$$

Renaming the variables if necessary one can assume that $q_{\alpha_1,\ldots,\alpha_{g-k}} \neq 0$ for some $(\alpha_1, \ldots, \alpha_{g-k})$ with α_{g-k}. If $q_{0,\ldots,0,N^{ij}} \neq 0$ then we get the desired element by dividing P_{ij} by $q_{0,\ldots,0,N^{ij}} \neq 0$. If this is not the case, we consider the change of variables of the form

$$\xi_i^{(k)} = \sum_{l=i}^{g-k} c_{i,l} \xi_l^{(k+1)}.$$

Let $c_i = c_{i,g-k}$. Then in the resulting expression of P_{ij} the coefficient of $(\xi_{g-k}^{(k+1)})^{N^{ij}}$ is

$$\sum_{|\alpha|=N^{ij}} c_1^{\alpha_1} \cdots c_{g-k}^{\alpha_{g-k}} q_{\alpha_1,\ldots,\alpha_{g-k}}. \tag{4.1}$$

This is a non-zero homogeneous polynomial in $c_1, \ldots, c_{g-k}$. Thus it is non-zero on a non-empty open subset of $\mathbb{C}^{g-k}$. Take a point in this set, make a change of coordinates and dividing P_{ij} by (4.1) we get the desired result. □

Let

$$\mathrm{gr}\, \pi_k : \mathrm{gr}\, M_c^k \to \mathrm{gr}\, M_c^{k+1}$$

be the restriction map induced by π_k and $K^k = \bigoplus K_n^k$ the kernel of $\mathrm{gr}\, \pi_k$. Since $\mathrm{gr}\, \pi_k$ is a homomorphism of $\mathrm{gr}\, \mathcal{D}$-modules, K^k is a $\mathrm{gr}\, \mathcal{D}$-submodule of $\mathrm{gr}\, M_c^k$. We denote by $\tilde{K}^k = \bigoplus \tilde{K}^k_n$ the $\operatorname{gr}\mathcal{D}$-module obtained from K^k by shifting the grading by -1, that is, $\tilde{K}^k_n = K^k_{n+1}$.

Lemma 4.3. *The map*

$$\operatorname{gr}_n M^k_{c+a_{k+1}} \to K^k_{n+1}, \qquad \phi(z) \mapsto \frac{\theta(z-a_{k+1})}{\theta(z)}\phi(z), \tag{4.2}$$

gives an isomorphism of $\operatorname{gr} M^k_{c+a_{k+1}}$ *and* $\tilde{K}^k$ *as* $\operatorname{gr}\mathcal{D}_{g-k}$*-modules.*

Proof. We can assume $g-k \geq 2$. Using Lemma 2.2(i), (2.4) and the cohomology sequence of (2.3) we have

$$\operatorname{Ker}(\pi_k|_{M^k_c(n)}) = \frac{\theta(z-a_{k+1})}{\theta(z)}\bigg|_{Y^k} M^k_{c+a_{k+1}}(n-1). \tag{4.3}$$

Let $\operatorname{gr}_n(\mathcal{L}_c(-mY^{k+1})|_{Y^k})$ be the sheaf on Y^k defined by the exact sequence

$$\begin{aligned} 0 \to \mathcal{L}_c((n-1)Q^k - mY^{k+1}) \to \mathcal{L}_c(nQ^k - mY^{k+1}) \\ \to \operatorname{gr}_n(\mathcal{L}_c(-mY^{k+1})|_{Y^k}) \to 0. \end{aligned} \tag{4.4}$$

Then one can easily verify that the following sequence is exact:

$$0 \to \operatorname{gr}_n(\mathcal{L}_c(-Y^{k+1})|_{Y^k}) \to \operatorname{gr}_n(\mathcal{L}_c|_{Y^k}) \to \operatorname{gr}_n(\mathcal{L}_c|_{Y^{k+1}}) \to 0. \tag{4.5}$$

By the cohomology sequence of (4.4) with $m=0$ and Lemma 2.2(i), the natural map

$$\operatorname{gr}_n L^k_c := L^k_c(n)/L^k_c(n-1) \to H^0(Y^k, \operatorname{gr}_n(\mathcal{L}_c|_{Y^k})) \tag{4.6}$$

is always injective and becomes isomorphic if $g-k \geq 2$. Then we have

$$\begin{aligned} \operatorname{Ker}(\operatorname{gr}_n L^k_c \to \operatorname{gr}_n L^{k+1}_c) &\simeq H^0(Y^k, \operatorname{gr}_n(\mathcal{L}_c(-Y^{k+1})|_{Y^k})) \\ &\simeq \frac{H^0(Y^k, \mathcal{L}_c(nQ^k - Y^{k+1}))}{H^0(Y^k, \mathcal{L}_c((n-1)Q^k - Y^{k+1}))} \\ &\simeq \frac{\theta(z-a_{k+1})}{\theta(z)}\bigg|_{Y^k} \operatorname{gr}_{n-1} L^k_{c+a_{k+1}}, \end{aligned}$$

where we use the cohomology sequences of (4.5), (4.4), Lemma 2.2(i), (2.4), (4.3). □

If $c+a_{k+1}+\sum_{i\in I} a_i \notin \Gamma$ for any subset I of $\{1,\ldots,k\}$ the induction hypothesis can be applied to $M^k_{c+a_{k+1}}$. By the lemma and the inductive assumption, K^k is described as

$$K^k = \bigoplus_{i=1}^{g} \bigoplus_{j=1}^{r_i} (\operatorname{gr} \mathcal{D}_{g-k}) \varphi_{ij},$$

$$\varphi_{ij} = \frac{\theta(z - a_{k+1})}{\theta(z)} \psi_{ij}^k |_{c \to c + a_{k+1}} \in K_{i+1}^k.$$

Since $\varphi_{ij} \in \operatorname{gr}_{i+1} M_c^k$, it can be written as a linear combination of $\{\psi_{ij}^k\}$ with coefficients in $\operatorname{gr} \mathcal{D}_{g-k}$ as

$$\varphi_{ij} = \sum Q_{ij:i'j'} \psi_{i'j'}^k, \qquad Q_{ij:i'j'} = \sum_{|\alpha| + i' = i+1} q_{ij;i'j'}^{\alpha} \xi^{\alpha}. \tag{4.7}$$

By Lemma 4.2(i) we have $P_{ij} \psi_{ij}^k \in K_{i+N^{ij}}^k$. Thus it can be written as a linear combination of $\{\varphi_{ij}\}$ with coefficients in $\operatorname{gr} \mathcal{D}_{g-k}$ as

$$P_{ij} \psi_{ij}^k = \sum R_{ij;i'j'} \varphi_{i'j'}, \qquad R_{ij;i'j'} = \sum_{|\alpha| + i' + 1 = i + N^{ij}} r_{ij;i'j'}^{\alpha} \xi^{\alpha}. \tag{4.8}$$

Composing these relations we get

$$P_{ij} \psi_{ij}^k = \sum R_{ij;i'j'} Q_{i'j':i''j''} \psi_{i''j''}^k. \tag{4.9}$$

In matrix form it is written as

$$P = RQ, \tag{4.10}$$

where P is the diagonal matrix whose diagonal entries consist of the P_{ij} and $R = (R_{ij;i'j'})$, $Q = (Q_{ij;i'j'})$ are $g! \times g!$ matrices.

We shall construct a basis $\{\widehat{\psi}_{ij}^k\}$ of $\operatorname{gr} M_c^k$ as a $\operatorname{gr} \mathcal{D}_{g-k}$-module modifying $\{\psi_{ij}^k\}$ appropriately so that they satisfy

$$\xi_{g-k} \widehat{\psi}_{ij}^k \in \sum_{i' \le i+1} \mathcal{K}[\xi_1, \ldots, \xi_{g-k-1}] \widehat{\psi}_{i'j'}^k + K^k. \tag{4.11}$$

To this end we use the relation (4.7). Let us write it more explicitly as

$$\begin{aligned} \varphi_{ij} = & \sum q_{ij;i+1j'} \psi_{i+1,j'}^k + \sum_{l=1}^{g-k} \sum_{j'} q_{ij;ij'}^l \xi_l \psi_{ij'}^k \\ & + \sum_{i' < i,\, |\alpha| + i' = i+1} q_{ij;i'j'}^{\alpha} \xi^{\alpha} \psi_{i'j'}^k. \end{aligned} \tag{4.12}$$

For the sake of simplicity we identify $q_{ij;ij'}^l$ with $q_{ij;ij'}^{\alpha}$, $\alpha = (0, \ldots, 1, \ldots, 0)$, where 1 is in the l-th place.

We construct $\{\widehat{\psi}_{ij}^k\}$ satisfying (4.11) by induction on i.

Let us consider the case $i = 1$. Then (4.12) becomes

$$\varphi_{11} = \sum q_{11;2j}\psi^k_{2j} + \sum_{l=1}^{g-k} q^l_{11;11}\xi_l\psi^k_{11}.$$

We proceed by considering several cases

(i) Assume $q^{g-k}_{11;11} \neq 0$. In this case (4.11) holds for $(i,j) = (1,1)$ by defining $\widehat{\psi}^k_{11} = \psi^k_{11}$ and $\widehat{\psi}^k_{2j} = \psi^k_{2j}$.

(ii) Assume $q^{g-k}_{11;11} = 0$ and $q_{11;2j} \neq 0$ for some j. In this case we modify ψ^k_{2j} to $\widehat{\psi}^k_{2j} = \psi^k_{2j} - \xi_{g-k}\psi^k_{11}$. The operators P_{ij} are changed correspondingly. We can take the product $P_{2j}P_{11}$ as an operator which annihilates $\widehat{\psi}^k_{2j}|_{Y^{k+1}}$, since $P_{2j}P_{11}\widehat{\psi}^k_{2j} \in K^k$. We set $\widehat{\psi}^k_{2j'} = \psi^k_{2j'}$ for $j' \neq j$ and $\widehat{\psi}^k_{11} = \psi^k_{11}$. Then (4.11) holds for $(i,j) = (1,1)$.

(iii) Assume $q^{g-k}_{11;11} = 0$ and $q_{11;2j} = 0$ for all j. This is impossible. In fact, suppose that this is the case. We consider the equation (4.10) modulo the ideal

$$I = \sum_{l=1}^{g-k-1} (\operatorname{gr}\mathcal{D}_{g-k})\xi_l.$$

Then the right hand side is degenerate while the left hand side is non-degenerate.

On the whole we have constructed $\widehat{\psi}^k_{ij}$, $i \leq 2$, such that (4.11) holds for $(i,j) = (1,1)$.

Assume that $\widehat{\psi}^k_{i'j'}$, $i' \leq i$, $1 \leq j' \leq r_{i'}$, are constructed in such a way that (4.11) holds for (i',j'), $i' < i$, by solving (4.12). As a consequence of the change from $\{\psi^k_{i'j'} \mid i' \leq i\}$ to $\{\widehat{\psi}^k_{i'j'} \mid i' \leq i\}$ the matrices P, Q, R may change. However the equation (4.10) still holds and P remains non-degenerate. Therefore we use the same symbols P, Q, R and their components in the argument below for the sake of simplicity.

Using the relation (4.11) for $\widehat{\psi}^k_{i'j'}$ for $i' < i$, the formula (4.7) for φ_{ij} can be written as

$$\tilde{\varphi}_{ij} = \sum q_{ij;i+1j'}\psi^k_{i+1,j'} + \sum_{l=1}^{g-k}\sum_{j'=1}^{r_i} \tilde{q}^l_{ij;ij'}\xi_l\widehat{\psi}^k_{ij'} + \sum_{i'<i,\, |\alpha|+i'=i+1,\, \alpha_{g-k}=0} \tilde{q}^\alpha_{ij;i'j'}\xi^\alpha\widehat{\psi}^k_{i'j'}$$

for some $\tilde{\varphi}_{ij} \in K^k_{i+1}$ and some $\tilde{q}^\alpha_{ij;i'j'}$, $i' \leq i$. Here $q^\alpha_{ij;i'j'}$ changes to $\tilde{q}^\alpha_{ij;i'j'}$ as a consequence of the use of (4.11). Notice that the effect on the matrix Q of applying the relation (4.11) is equivalent to making fundamental transformations on rows of Q, since (4.11) is obtained by solving (4.12).

We have

$$\tilde{\varphi}_{ij} = \sum q_{ij;i+1,j'}\psi^k_{i+1,j'} + \sum_{j'=1}^{r_i} \tilde{q}^{g-k}_{ij;ij'}\xi_{g-k}\widehat{\psi}^k_{ij'} \quad \mod I \operatorname{gr} M^k_c.$$

The rank of the $r_i \times (r_i + r_{i+1})$ matrix

$$\big((q_{ij;i+1,j'})_{1\le j'\le r_{i+1}}, (\tilde{q}^{g-k}_{ij;ij'})_{1\le j'\le r_i}\big)_{1\le j\le r_i}$$

is maximal, for otherwise P is not non-degenerate. Thus, as in the case of $i = 1$, modifying $\psi^k_{i+1,j'}$ to $\widehat{\psi}^k_{i+1,j'} = \psi^k_{i+1,j'} - \xi_{g-k}\psi^k_{ij''}$ for some j' and j'' if necessary, we get $\{\widehat{\psi}^k_{i',j'} \mid i' \le i+1\}$ such that (4.11) holds for (i', j'), $i' \le i$. Notice that, in the last step $i = g$, $\tilde{q}^{g-k}_{g+1,1;g1} \ne 0$ is automatic.

Thus a $\operatorname{gr}\mathcal{D}_{g-k}$-free basis $\{\widehat{\psi}^k_{ij}\}$ of $\operatorname{gr} M^k_c$ satisfying the condition (4.11) is constructed. Set

$$\psi^{k+1}_{ij} = \widehat{\psi}^k_{ij}|_{Y^{k+1}}.$$

Then (4.11) implies that

$$\xi_{g-k}\psi^{k+1}_{ij} \in \sum(\operatorname{gr}\mathcal{D}_{g-k-1})\psi^{k+1}_{i'j'}. \tag{4.13}$$

Lemma 4.4. *If $c + \sum_{i\in I} a_i \notin \Gamma$ for any subset I of $\{1, \dots, k\}$, the restriction map $\operatorname{gr}\pi_k$ is surjective.*

Proof. Use a similar argument to the proof of Lemma 4.3. □

By the lemma and the inductive assumption on $\operatorname{gr} M^k_c$, $\operatorname{gr} M^{k+1}_c$ is generated by $\{\psi^{k+1}_{ij}\}$ over $\operatorname{gr}\mathcal{D}_{g-k}$. Then (4.13) implies that

$$\operatorname{gr} M^{k+1}_c = \sum(\operatorname{gr}\mathcal{D}_{g-k-1})\psi^{k+1}_{ij}.$$

It follows from (3.1) that the sum on the right hand side is a direct sum. This completes the proof of Proposition 4.1.

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References

[K] I. M. Krichever, Methods of algebraic geometry in the theory of non-linear equations, Russian Math. Surveys **32** (1977), 185–213. Zbl 0386.35002 MR 0516323

[Mir] A. E. Mironov, Commutative rings of differential operators corresponding to multidimensional algebraic varieties, Siberian Math. J. **43** (2002), 888–898. Zbl 1006.14016 MR 1946267

[M1] D. Mumford, *Abelian varieties*, Oxford Univ. Press, 1970. Zbl 0223.14022 MR 0282985

[M2] D. Mumford, *Tata lectures on theta I*, Birkhäuser, 1983. Zbl 1112.14002 MR 2352717

[N1] A. Nakayashiki, Structure of Baker–Akhiezer modules of principally polarized abelian varieties, Duke Math. J. **62** (1991), 315–358. Zbl 0732.14008 MR 1104527

[N2] ———, Commuting partial differential operators and vector bundles over abelian varieties, Amer. J. Math. **116** (1994), 65–100. Zbl 0809.14016 MR 1262427

[R] M. Rothstein, Dynamics of the Krichever construction in several variables, J. Reine Angew. Math. **572** (2004), 111–138. Zbl 1142.37362 MR 2076122