Publ. RIMS Kyoto Univ. **47** (2011), 897–909
DOI 10.2977/PRIMS/55

Periodicity for Mumford–Morita–Miller Classes of Surface Symmetries

by

Toshiyuki Akita

Abstract

We prove periodicity for mod p Mumford–Morita–Miller classes of surface symmetries and thereby for finite subgroups of mapping class groups. As an application, we obtain a couple of vanishing results for mod p Mumford–Morita–Miller classes for surface symmetries.

2010 Mathematics Subject Classification: Primary 55R40; Secondary 20J06, 57N05, 57S17.
Keywords: surface symmetry, Mumford–Morita–Miller class, group cohomology.

§1. Introduction

By a surface symmetry, we mean a pair (G, C) consisting of a closed oriented surface C and a finite group G acting on C effectively and preserving orientation. Associated with a surface symmetry (G, C), there is an oriented surface bundle

$$\pi : EG \times_G C \to BG$$

where $EG \to BG$ is the universal principal G-bundle. Mumford [18], Morita [17] and Miller [15] introduced a series of characteristic classes of oriented surface bundles which are called Mumford–Morita–Miller classes (MMM classes for short). The k-th MMM class $e_k(G, C) \in H^{2k}(G, \mathbb{Z})$ of a surface symmetry (G, C) is then defined to be the k-th MMM class of the associated surface bundle π. See Section 3 for precise definitions.

MMM classes of surface symmetries were studied in [1, 3, 4, 13, 20]. In case G is a finite cyclic group, Uemura proved, among other things, a certain kind of periodicity for MMM classes of surface symmetries [20, Theorem 2.1]:

Communicated by T. Ohtsuki. Received September 12, 2010.

T. Akita: Department of Mathematics, Hokkaido University, Sapporo, 060-0810 Japan;
e-mail: `akita@math.sci.hokudai.ac.jp`

Theorem 1.1 (Uemura). *Let G be a cyclic group of order m. There exists a cohomology class $v_G \in H^{2\phi(m)}(G,\mathbb{Z})$ such that, for any surface symmetry (G,C),*

$$e_{k+\phi(m)}(G,C) = e_k(G,C) \cup v_G \in H^{2(k+\phi(m))}(G,\mathbb{Z}) \quad \text{for all } k \geq 1,$$

where $\phi(m)$ is the number of positive integers less than m that are coprime to m.

See Remark 3.1 for the definition of v_G. The primary purpose of this paper is to generalize "mod p reduction" of Uemura's result to arbitrary finite groups:

Theorem 1.2. *Let G be an arbitrary finite group, p a prime number which divides the order of G, and $\nu_p(G) := p^m - p^{m-1}$ where p^m is the highest power of p dividing the order of G. There exists a cohomology class $u_G \in H^{2\nu_p(G)}(G,\mathbb{F}_p)$ such that, for any surface symmetry (G,C),*

$$e_{k+\nu_p(G)}(G,C) = e_k(G,C) \cup u_G \in H^{2(k+\nu_p(G))}(G,\mathbb{F}_p) \quad \text{for all } k \geq 1.$$

The cohomology class u_G is a certain multiple of the $\nu_p(G)$-th Chern class of the regular representation of G. See the proof of Theorem 1.2 for the precise definition of u_G. Two remarks are in order. Firstly, if G is a cyclic group of order m, then $\nu_p(G)$ divides $\phi(m)$, and Theorem 1.2 implies "mod p reduction" of Theorem 1.1. Secondly, if $p = 2$ then $e_k(G,C) \in H^{2k}(G,\mathbb{F}_2)$ vanishes for all $k \geq 1$ (see Section 5). Hence Theorem 1.2 is trivial for $p = 2$; however, it is highly nontrivial for odd primes.

Many of the results in this paper, including Theorem 1.2, can be restated in purely algebraic terms. We try to separate topological and algebraic ingredients of our results as much as possible, for we believe that our results have their own meanings in group cohomology as well as topology. To do so, based on a result of Kawazumi and Uemura [13], we will introduce the notion of algebraic MMM classes which is an algebraic counterpart of MMM classes of surface symmetries.

Now let us describe the content of this paper very briefly. In Section 2 we recall relevant definitions and facts concerning surface symmetries. In Section 3 we recall the definitions of MMM classes, together with Chern classes of linear representations. Algebraic MMM classes will also be defined there. Section 4 is devoted to the proof of Theorem 1.2. There are two key ingredients of the proof. One is a result of Kawazumi and Uemura mentioned above. The other is a result of Kahn [11] concerning the total Chern classes of the regular representations of finite groups. The final section is devoted to applications of the main result. We will prove a couple of vanishing results for mod p MMM classes of surface symmetries.

For other results concerning mod p reduction of MMM classes, we refer the reader to [1, 2, 4, 6, 7, 12] and Section 5.

Notation. For a finite group G, the order of G is denoted by $|G|$, the commutator subgroup of G is denoted by $[G, G]$, and the abelianization $G/[G, G]$ of G is denoted by G_{ab}. For a subgroup H of G, the corestriction (or transfer) is denoted by $\mathrm{Cor}_H^G : H^*(H, \mathbb{Z}) \to H^*(G, \mathbb{Z})$, while the restriction is denoted by $\mathrm{Res}_H^G : H^*(G, \mathbb{Z}) \to H^*(H, \mathbb{Z})$. For a prime number p, the field of p elements is denoted by $\mathbb{F}_p$.

§2. Surface symmetries

§2.1. Ramification data

By a *surface symmetry* we mean a pair (G, C), where C is a closed oriented surface, not necessarily connected, and G is a finite group acting on C effectively and preserving the orientation of C. Throughout this paper, we assume that surfaces and G-actions on them are smooth.

For each $x \in C$, let G_x be the isotropy subgroup at x. Note that G_x is necessarily cyclic. Set $\mathcal{S} = \{x \in C \mid G_x \neq 1\}$, and let $\mathcal{S}/G = \{x_1, \ldots, x_q\}$ be a set of representatives of G-orbits of elements of $\mathcal{S}$. For each $x_i \in \mathcal{S}/G$, choose a generator γ_i of G_{x_i} such that γ_i acts on the tangent space $T_{x_i}C$ by rotation through the angle $2\pi/|G_{x_i}|$ with respect to a G-invariant metric on C. Let $\hat{\gamma}_i$ be the conjugacy class of γ_i $(1 \leq i \leq q)$. The *ramification data* of (G, C), abbreviated by $\delta(G, C)$, is the unordered q-tuple $\langle \hat{\gamma}_1, \ldots, \hat{\gamma}_q \rangle$. Note that $\delta(G, C)$ is independent of various choices made and hence well-defined for (G, C).

Proposition 2.1. *If $\langle \hat{\gamma}_1, \ldots, \hat{\gamma}_q \rangle$ is the ramification data of a surface symmetry (G, C), then $\gamma_1 \cdots \gamma_q \in [G, G]$.*

Proof. If the quotient surface $D := C/G$ is connected, the conclusion is proved as follows. Let $\pi : C \to D$ be the natural projection and put $y_i := \pi(x_i)$ $(x_i \in \mathcal{S}/G)$. Then $\pi : C \setminus \mathcal{S} \to D \setminus \{y_1, \ldots, y_q\}$ is a regular covering. Such a covering yields a homomorphism $\rho : \pi_1(D \setminus \{y_1, \ldots, y_q\}, y_0) \to G$ where $y_0 \in D \setminus \{y_1, \ldots, y_q\}$ is a base point. Note that ρ is surjective if and only if C is connected. Now one has a presentation

$$\pi_1(D \setminus \{y_1, \ldots, y_q\}, y_0) \cong \left\langle a_i,\, b_i\ (1 \leq i \leq h),\, c_j\ (1 \leq j \leq q) \;\middle|\; \prod_{i=1}^{h} [a_i, b_i] \prod_{j=1}^{q} c_j \right\rangle,$$

where h is the genus of the quotient surface D and c_j is a small loop around y_j $(1 \leq j \leq q)$. If c_j's are oriented in a standard way, then $\gamma_j' := \rho(c_j) \in G$ is conjugate to γ_j $(1 \leq j \leq q)$ (γ_j' depends on the choices of y_0 and c_j). The proposition follows from the equality

$$\prod_{j=1}^{q} \gamma'_j = \rho\Big(\prod_{j=1}^{q} c_j\Big) = \rho\Big(\prod_{i=1}^{h}[a_i, b_i]\Big)^{-1} \in [G, G].$$

If the quotient surface D is not connected, the conclusion can be proved by applying the above argument to each connected component of D. The details are left to the reader. □

§2.2. The Grieder monoid of a finite group

Let G be a finite group and $\hat{\gamma}$ the conjugacy class of an element $\gamma \in G$. We denote by $\langle \hat{\gamma}_1, \ldots, \hat{\gamma}_q \rangle$ (or $\langle \hat{\gamma}_1, \ldots, \hat{\gamma}_q \rangle_G$) an unordered q-tuple ($q \geq 0$) of the conjugacy classes of nontrivial elements of G satisfying $\gamma_1 \cdots \gamma_q \in [G, G]$, and by $\mathcal{M}_G$ the set of all such tuples. We can define a commutative monoid structure on $\mathcal{M}_G$ by

$$\langle \hat{\gamma}_1, \ldots, \hat{\gamma}_q \rangle + \langle \hat{\gamma}_{q+1}, \ldots, \hat{\gamma}_r \rangle = \langle \hat{\gamma}_1, \ldots, \hat{\gamma}_q, \hat{\gamma}_{q+1}, \ldots, \hat{\gamma}_r \rangle.$$

The identity element is the empty tuple $\langle\rangle$. We call $\mathcal{M}_G$ the *Grieder monoid* of G, since $\mathcal{M}_G$ and its variants were introduced and investigated by Grieder [8, 9] to study surface symmetries.

According to Proposition 2.1, the ramification data $\delta(G, C)$ of any surface symmetry (G, C) is an element of the Grieder monoid $\mathcal{M}_G$. Conversely, any element of $\mathcal{M}_G$ can be realized as the ramification data for some surface symmetry. Hence $\mathcal{M}_G$ can be identified with the set of ramification data of (G, C)'s:

Proposition 2.2. *For any $\langle \hat{\gamma}_1, \ldots, \hat{\gamma}_q \rangle \in \mathcal{M}_G$, there exists a surface symmetry (G, C) whose ramification data $\delta(G, C)$ coincides with $\langle \hat{\gamma}_1, \ldots, \hat{\gamma}_q \rangle$. Moreover, C can be chosen to be connected.*

Proof. We continue to use the notation of the proof of Proposition 2.1. To prove the proposition, it suffices to construct a surjective homomorphism

$$\rho : \pi_1(D \setminus \{y_1, \ldots, y_q\}, y_0) \to G$$

with $\rho(c_j) = \gamma_j$ ($1 \leq j \leq q$) for some genus $h = g(D)$. Since $\gamma_1 \cdots \gamma_q \in [G, G]$, one has

$$(\gamma_1 \cdots \gamma_q)^{-1} = \prod_{i=1}^{r} [\alpha_i, \alpha'_i]$$

for some $\alpha_i, \alpha'_i \in G$ ($1 \leq i \leq r$). Choose a set of generators $\{\beta_1, \ldots, \beta_s\}$ of G and set $h := r + s$. Then a homomorphism $\rho : \pi_1(D \setminus \{y_1, \dots, y_q\}, y_0) \to G$ defined by

$$\begin{cases} \rho(a_i) = \alpha_i, \quad \rho(b_i) = \alpha_i' \quad (1 \le i \le r), \\ \rho(a_{i+r}) = \rho(b_{i+r}) = \beta_i \quad (1 \le i \le s), \\ \rho(c_j) = \gamma_j \quad (1 \le j \le q) \end{cases}$$

has the desired properties. □

For a subgroup H of G, define a homomorphism $\mathrm{Cor}_H^G : \mathcal{M}_H \to \mathcal{M}_G$ of monoids by

$$\mathrm{Cor}_H^G(\langle \hat{\gamma}_1, \dots, \hat{\gamma}_q \rangle_H) := \langle \hat{\gamma}_1, \dots, \hat{\gamma}_q \rangle_G.$$

One can also define a homomorphism $\mathrm{Res}_H^G : \mathcal{M}_G \to \mathcal{M}_H$ as follows. Given $\rho \in \mathcal{M}_G$, choose a surface symmetry (G, C) with $\rho = \delta(G, C)$, and define $\mathrm{Res}_H^G(\rho)$ by $\mathrm{Res}_H^G(\rho) := \delta(H, C) \in \mathcal{M}_H$. The definition of $\mathrm{Res}_H^G(\rho)$ is independent of the surface symmetry chosen. We will call Cor_H^G and Res_H^G the *corestriction* and the *restriction*, respectively.

Remark 2.3. The ramification data of a surface symmetry as well as the monoid $\mathcal{M}_G$ were introduced by Grieder [8]. Propositions 2.1 and 2.2 were stated in [8, Section 3] without details. We wrote down the proof for completeness.

§3. Mumford–Morita–Miller classes

§3.1. The definition of Mumford–Morita–Miller classes

Let C be a closed oriented surface. Let $\pi : E \to B$ be an oriented surface bundle with fiber C, T^vE the tangent bundle along the fibers of π, and $e \in H^2(E; \mathbb{Z})$ the Euler class of T^vE. Define $e_k(\pi) \in H^{2k}(B; \mathbb{Z})$ by $e_k(\pi) := \pi_!(e^{k+1})$ where $\pi_! : H^*(E; \mathbb{Z}) \to H^{*-2}(B; \mathbb{Z})$ is the Gysin homomorphism (or integration along the fiber). The cohomology class $e_k(\pi)$ is called the *k-th Mumford–Morita–Miller class* of π (MMM class for short), as it was introduced and studied by Mumford [18], Morita [17] and Miller [15].

Now let (G, C) be a surface symmetry as in Section 2.1. Associated with (G, C), there is an oriented surface bundle $\pi : EG \times_G C \to BG$ called the *Borel construction* (or the homotopy orbit space), where $EG \to BG$ is the universal principal G-bundle. The k-th MMM class $e_k(G, C) \in H^{2k}(G, \mathbb{Z})$ of (G, C) is defined to be the k-th MMM class of the Borel construction π. For a subgroup H of G, we have

$$\mathrm{Res}_H^G(e_k(G, C)) = e_k(H, C) \in H^{2k}(H, \mathbb{Z}), \tag{3.1}$$

where (H, C) is the surface symmetry obtained by restricting the action of G to the subgroup H.

§3.2. Chern classes of linear representations

Before proceeding further, we introduce Chern classes of linear representations of finite groups, for they are used in the definition of algebraic MMM classes as well as the proof of Theorem 1.2. Let G be a finite group and V an n-dimensional complex linear representation of G. The k-th Chern class of V, denoted by $c_k(V) \in H^{2k}(G, \mathbb{Z})$, is defined to be the k-th Chern class of the n-dimensional complex vector bundle $EG \times_G V \to BG$ associated with V. The total Chern class of V is denoted by $c_\bullet(V) := 1 + \sum_{k=1}^n c_k(V)$.

Now let $\langle\gamma\rangle$ be a cyclic group of order m generated by γ, and L_γ the 1-dimensional complex linear representation of $\langle\gamma\rangle$ defined by $\gamma \mapsto \exp(2\pi\sqrt{-1}/m)$. Define $c(\gamma) \in H^2(\langle\gamma\rangle, \mathbb{Z})$ to be the first Chern class of L_γ. Alternatively, $c(\gamma)$ can be defined in purely algebraic terms, as follows. For any finite group G, there are natural isomorphisms

$$\mathrm{Hom}(G, \mathbb{C}^\times) \xrightarrow{\cong} H^1(G, \mathbb{C}^\times) \xrightarrow{\cong} H^2(G, \mathbb{Z}),$$

where $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$. Here, the latter isomorphism is the connecting homomorphism associated to the short exact sequence $0 \to \mathbb{Z} \hookrightarrow \mathbb{C} \to \mathbb{C}^\times \to 0$, where $\mathbb{C} \to \mathbb{C}^\times$ is the map defined by $z \mapsto \exp(2\pi\sqrt{-1}z)$. Then the first Chern class of a 1-dimensional complex linear representation L of G can be identified with the image of L under the isomorphism $\mathrm{Hom}(G, \mathbb{C}^\times) \to H^2(G, \mathbb{Z})$ (see [19, Chapter 6]).

Remark 3.1. For a cyclic group G of order m generated by γ, the cohomology class $v_G \in H^{2\phi(m)}(G, \mathbb{Z})$ in Theorem 1.1 is the one defined by $v_G := c(\gamma)^{\phi(m)}$.

§3.3. Algebraic Mumford–Morita–Miller classes

For each element $\rho = \langle\hat{\gamma}_1, \ldots, \hat{\gamma}_q\rangle$ of $\mathcal{M}_G$, define a sequence of cohomology classes $e_k^{\mathrm{alg}}(\rho) \in H^{2k}(G, \mathbb{Z})$ $(k \geq 1)$ by

$$e_k^{\mathrm{alg}}(\rho) := \sum_{i=1}^{q} \mathrm{Cor}_{\langle\gamma_i\rangle}^{G}(c(\gamma_i)^k) \in H^{2k}(G, \mathbb{Z}).$$

Since the corestriction is invariant under conjugation, $e_k^{\mathrm{alg}}(\rho)$ is well-defined for ρ. We call $e_k^{\mathrm{alg}}(\rho)$ the *k-th algebraic Mumford–Morita–Miller class* of ρ (algebraic MMM class for short). The definition of $e_k^{\mathrm{alg}}(\rho)$ is inspired by a result of Kawazumi and Uemura [13, Theorem B] concerning MMM classes for surface symmetries. In terms of Grieder monoids and algebraic MMM classes, their result can be restated as follows:

Theorem 3.2 (Kawazumi–Uemura). *For any surface symmetry (G, C), one has $e_k(G, C) = e_k^{\mathrm{alg}}(\delta(G, C)) \in H^{2k}(G, \mathbb{Z})$ where $\delta(G, C) \in \mathcal{M}_G$ is the ramification data of (G, C).*

The assignment $\rho \mapsto e_k^{\mathrm{alg}}(\rho)$ defines a homomorphism $\mathcal{M}_G \to H^{2k}(G, \mathbb{Z})$ of commutative monoids, which commutes with Cor_H^G and Res_H^G:

Proposition 3.3. *For any subgroup H of G, the following two diagrams are commutative:*

$$\begin{array}{ccc} \mathcal{M}_H & \xrightarrow{\mathrm{Cor}_H^G} & \mathcal{M}_G \\ {\scriptstyle e_k^{\mathrm{alg}}}\downarrow & & \downarrow{\scriptstyle e_k^{\mathrm{alg}}} \\ H^{2k}(H, \mathbb{Z}) & \xrightarrow{\mathrm{Cor}_H^G} & H^{2k}(G, \mathbb{Z}) \end{array} \qquad \begin{array}{ccc} \mathcal{M}_G & \xrightarrow{\mathrm{Res}_H^G} & \mathcal{M}_H \\ {\scriptstyle e_k^{\mathrm{alg}}}\downarrow & & \downarrow{\scriptstyle e_k^{\mathrm{alg}}} \\ H^{2k}(G, \mathbb{Z}) & \xrightarrow{\mathrm{Res}_H^G} & H^{2k}(H, \mathbb{Z}) \end{array}$$

Proof. The first diagram is commutative by the definitions of $\mathrm{Cor}_H^G : \mathcal{M}_H \to \mathcal{M}_G$ and algebraic MMM classes. Given $\rho \in \mathcal{M}_G$, choose a surface symmetry (G, C) with $\rho = \delta(G, C)$. Then $\mathrm{Res}_H^G(e_k^{\mathrm{alg}}(\rho)) = \mathrm{Res}_H^G(e_k(G, C)) = e_k(H, C)$ by Theorem 3.2 and the equation (3.1), while $e_k^{\mathrm{alg}}(\mathrm{Res}_H^G(\rho)) = e_k^{\mathrm{alg}}(\delta(H, C)) = e_k(H, C)$ by the definition of $\mathrm{Res}_H^G : \mathcal{M}_G \to \mathcal{M}_H$. Hence the second diagram is commutative. □

§4. Proof of the main result

Throughout this section, p is a fixed prime number, V_G is the regular representation of a finite group G, and $\nu_p(G) := p^m - p^{m-1}$ where p^m is the highest power of p dividing the order of G as in Theorem 1.2.

Lemma 4.1. *Let G be a cyclic group of order p^m generated by γ. Then*

$$c_\bullet(V_G) = 1 - c(\gamma)^{\nu_p(G)} \in H^*(G, \mathbb{F}_p).$$

Proof. Since $V_G = \bigoplus_{k=0}^{p^m-1} L_\gamma^{\otimes k}$ and $c_\bullet(L_\gamma^{\otimes k}) = 1 + kc(\gamma)$, we have

$$\begin{aligned} c_\bullet(V_G) &= \prod_{k=1}^{p^m-1} c_\bullet(L_\gamma^{\otimes k}) = \prod_{k=1}^{p^m-1} (1 + kc(\gamma)) = \Big(\prod_{k=1}^{p-1}(1 + kc(\gamma))\Big)^{p^{m-1}} \\ &= (1 - c(\gamma)^{p-1})^{p^{m-1}} = 1 - c(\gamma)^{p^m - p^{m-1}}. \end{aligned}$$

As $\nu_p(G) = p^m - p^{m-1}$, the lemma is verified. □

Lemma 4.2. *Let G be a p-group and H a cyclic subgroup generated by γ. Then*

$$\mathrm{Res}_H^G(c_\bullet(V_G)) = c_\bullet(\mathrm{Res}_H^G(V_G)) = 1 - c(\gamma)^{\nu_p(G)} \in H^*(H, \mathbb{F}_p),$$

where $\mathrm{Res}_H^G(V_G)$ is the restriction of the regular representation V_G to H.

Proof. Set $|G| = p^n$ and $|H| = p^m$. Then $\mathrm{Res}^G_H(V_G) = (V_H)^{\oplus(G:H)} = (V_H)^{\oplus p^{n-m}}$, where $(G : H)$ is the index of H in G. Hence

$$\begin{aligned}\mathrm{Res}^G_H(c_\bullet(V_G)) &= c_\bullet((V_H)^{\oplus p^{n-m}}) = (1 - c(\gamma)^{p^m - p^{m-1}})^{p^{n-m}} \\ &= 1 - c(\gamma)^{(p^m-p^{m-1})p^{n-m}} = 1 - c(\gamma)^{p^n - p^{n-1}} \\ &= 1 - c(\gamma)^{\nu_p(G)}\end{aligned}$$

as claimed. □

In view of the discussions in Sections 2 and 3, Theorem 1.2 in the Introduction is equivalent to the following theorem which we now prove:

Theorem 4.3. *Let G be a finite group whose order is divisible by p. There exists a cohomology class $u_G \in H^{2\nu_p(G)}(G, \mathbb{F}_p)$ such that, for any $\rho \in \mathcal{M}_G$,*

$$e^{\mathrm{alg}}_{k+\nu_p(G)}(\rho) = e^{\mathrm{alg}}_k(\rho) \cup u_G \in H^{2(k+\nu_p(G))}(G, \mathbb{F}_p) \quad \textit{for all } k \geq 1.$$

Proof. Suppose first that G is a p-group. Define an element $u_G \in H^{2\nu_p(G)}(G, \mathbb{F}_p)$ by $u_G := -c_{\nu_p(G)}(V_G)$. Then, for any cyclic subgroup $\langle\gamma\rangle$ of G generated by γ, $\mathrm{Res}^G_{\langle\gamma\rangle}(u_G) = c(\gamma)^{\nu_p(G)}$ by Lemma 4.2. We have

$$c(\gamma)^{k+\nu_p(G)} = c(\gamma)^k \cup \mathrm{Res}^G_{\langle\gamma\rangle}(u_G) \quad \text{for all } k \geq 1$$

and hence

$$\begin{aligned}\mathrm{Cor}^G_{\langle\gamma\rangle}(c(\gamma)^{k+\nu_p(G)}) &= \mathrm{Cor}^G_{\langle\gamma\rangle}(c(\gamma)^k \cup \mathrm{Res}^G_{\langle\gamma\rangle}(u_G)) \\ &= \mathrm{Cor}^G_{\langle\gamma\rangle}(c(\gamma)^k) \cup u_G \quad \text{for all } k \geq 1.\end{aligned}$$

Setting $\rho = \langle\hat{\gamma}_1, \ldots, \hat{\gamma}_q\rangle$, we obtain

$$\begin{aligned}e^{\mathrm{alg}}_{k+\nu_p(G)}(\rho) &= \sum_{i=1}^q \mathrm{Cor}^G_{\langle\gamma_i\rangle}(c(\gamma_i)^{k+\nu_p(G)}) = \sum_{i=1}^q (\mathrm{Cor}^G_{\langle\gamma_i\rangle}(c(\gamma_i)^k) \cup u_G) \\ &= \Big(\sum_{i=1}^q \mathrm{Cor}^G_{\langle\gamma_i\rangle}(c(\gamma_i)^k)\Big) \cup u_G = e^{\mathrm{alg}}_k(\rho) \cup u_G.\end{aligned} \tag{4.1}$$

This proves the theorem for p-groups.

Now let G be an arbitrary finite group and P its Sylow p-subgroup. Observe first that $\mathrm{Res}^G_P(V_G) = (V_P)^{\oplus(G:P)}$ and hence $\mathrm{Res}^G_P(c_\bullet(V_G)) = c_\bullet(V_P)^{(G:P)}$. According to a result of Kahn [11, Théorème 0.1], $c_k(V_P) = 0 \in H^{2k}(P, \mathbb{F}_p)$ holds for $0 < k < \nu_p(P) = \nu_p(G)$, and hence

$$c_\bullet(V_P)^{(G:P)} = 1 + (G : P) \cdot c_{\nu_p(G)}(V_P) + \text{terms of higher degrees}$$

in $H^*(P, \mathbb{F}_p)$, which implies that

$$\mathrm{Res}_P^G(c_{\nu_p(G)}(V_G)) = (G : P) \cdot c_{\nu_p(G)}(V_P) = -(G : P) \cdot u_P.$$

Define a cohomology class $u_G \in H^{2\nu_p(G)}(G, \mathbb{F}_p)$ by

$$u_G := -(G : P)^{p-2} \cdot c_{\nu_p(G)}(V_G).$$

Then $\mathrm{Res}_P^G(u_G) = u_P$, for $(G : P)^{p-1} \equiv 1 \pmod p$ by Fermat's little theorem. We have

$$\begin{aligned}
\mathrm{Res}_P^G(e_{k+\nu_p(G)}^{\mathrm{alg}}(\rho)) &= e_{k+\nu_p(G)}^{\mathrm{alg}}(\mathrm{Res}_P^G(\rho)) && \text{(by Proposition 3.3)} \\
&= e_k^{\mathrm{alg}}(\mathrm{Res}_P^G(\rho)) \cup u_P && \text{(by (4.1))} \\
&= \mathrm{Res}_P^G(e_k^{\mathrm{alg}}(\rho) \cup u_G) && \text{(by Proposition 3.3).}
\end{aligned}$$

Since $\mathrm{Res}_P^G : H^*(G, \mathbb{F}_p) \to H^*(P, \mathbb{F}_p)$ is injective, this proves the theorem. □

Let C be a closed oriented surface of genus $g \geq 2$ and Diff_+C the group of orientation preserving diffeomorphisms of C equipped with the C^∞-topology. The *mapping class group* Γ_g of genus g is defined to be the group of connected components of Diff_+C. Given a surface symmetry (G, C), the composition of canonical homomorphisms

$$G \hookrightarrow \mathrm{Diff}_+C \twoheadrightarrow \Gamma_g$$

is injective (see [16, Section 2] for instance). Regarding G as a subgroup of Γ_g via the above homomorphism, we have

$$e_k(G, C) = \mathrm{Res}_G^{\Gamma_g}(e_k) \in H^{2k}(G, \mathbb{Z}),$$

where $e_k \in H^{2k}(\Gamma_g, \mathbb{Z})$ is the k-th MMM class of Γ_g (see [17, Section 1] for the definition of MMM classes of Γ_g). Conversely, according to the affirmative solution of Nielsen's realization problem by Kerchoff [14], any finite subgroup of Γ_g can be realized as a surface symmetry in this way. Therefore we obtain the following corollary:

Corollary 4.4. *Under the notation of Theorem 4.3, for any finite subgroup G of Γ_g $(g \geq 2)$ whose order is divisible by p, we have*

$$\mathrm{Res}_G^{\Gamma_g}(e_{k+\nu_p(G)}) = \mathrm{Res}_G^{\Gamma_g}(e_k) \cup u_G \in H^{2(k+\nu_p(G))}(G, \mathbb{F}_p) \qquad \textit{for all } k \geq 1.$$

§5. Vanishing results for MMM classes of surface symmetries

In this section, we will prove a couple of vanishing results for mod p MMM classes of surface symmetries. Before doing so, we quote a result of Galatius, Madsen and

Tillmann [7, Theorem 1.2], which was an affirmative solution to the conjecture posed by the author [1, Conjecture 1]:

Theorem 5.1 (Galatius–Madsen–Tillmann). *Let $\pi : E \to B$ be an oriented surface bundle. Then the mod p MMM class $e_k(\pi) \in H^{2k}(B, \mathbb{F}_p)$ of π vanishes whenever $k \equiv -1 \pmod{p-1}$. In particular, the mod 2 MMM class $e_k(\pi) \in H^{2k}(B, \mathbb{F}_2)$ vanishes for all k.*

Note that [7, Theorem 1.2] was stated only for MMM classes of the stable mapping class group rather than for oriented surface bundles, but the argument is valid for arbitrary oriented surface bundles as well. An alternative proof of Theorem 5.1 can be found in [6].

Applying Theorem 5.1 to the oriented surface bundle $E \times_G C \to BG$ associated to a surface symmetry (G, C), we obtain the following corollary:

Corollary 5.2. *If $k \equiv -1 \pmod{p-1}$ then*

1. *$e_k(G, C) = 0 \in H^{2k}(G, \mathbb{F}_p)$ for any surface symmetry (G, C),*
2. *$e_k^{\mathrm{alg}}(\rho) = 0 \in H^{2k}(G, \mathbb{F}_p)$ for any $\rho \in \mathcal{M}_G$.*

Note that, for each prime $p \geq 3$ and the cyclic group G of order p, there exists an element $\rho \in \mathcal{M}_G$ satisfying $e_k^{\mathrm{alg}}(\rho) \neq 0 \in H^{2k}(G, \mathbb{F}_p)$ whenever $k \not\equiv -1 \pmod{p-1}$ (see [1, Proof of Theorem 3]).

Now we give two sufficient conditions for the vanishing of mod p MMM classes, both of which are independent of Corollary 5.2:

Proposition 5.3. *Let G a finite group whose Sylow p-subgroup is not cyclic. Then, for all $k \geq 1$,*

1. *$e_{k\nu_p(G)}(G, C) = 0 \in H^{2k\nu_p(G)}(G, \mathbb{F}_p)$ for any surface symmetry (G, C),*
2. *$e_{k\nu_p(G)}^{\mathrm{alg}}(\rho) = 0 \in H^{2k\nu_p(G)}(G, \mathbb{F}_p)$ for any $\rho \in \mathcal{M}_G$.*

Since $k\nu_p(G) \equiv 0 \pmod{p-1}$, Proposition 5.3 is independent of Corollary 5.2.

Proof. Since the two statements are equivalent, we will prove the one for algebraic MMM classes. As the restriction $\mathrm{Res}_P^G : H^*(G, \mathbb{F}_p) \to H^*(P, \mathbb{F}_p)$ to the Sylow p-subgroup P is injective, it suffices to consider the case where G is a p-group. For any cyclic subgroup $\langle\gamma\rangle$ generated by $\gamma \in G$, we have

$$\mathrm{Res}_{\langle\gamma\rangle}^G(u_G) = c(\gamma)^{\nu_p(G)} \in H^{2\nu_p(G)}(\langle\gamma\rangle, \mathbb{F}_p)$$

as in the proof of Theorem 4.3. Since $c(\gamma)^{\nu_p(G)}$ is a generator of $H^{2\nu_p(G)}(\langle\gamma\rangle, \mathbb{F}_p) \cong \mathbb{Z}/p\mathbb{Z}$, the restriction $\mathrm{Res}_{\langle\gamma\rangle}^G : H^{2\nu_p(G)}(G, \mathbb{F}_p) \to H^{2\nu_p(G)}(\langle\gamma\rangle, \mathbb{F}_p)$ is surjective. It follows that the corestriction

$$\mathrm{Cor}^G_{\langle\gamma\rangle} : H^{2\nu_p(G)}(\langle\gamma\rangle, \mathbb{F}_p) \to H^{2\nu_p(G)}(G, \mathbb{F}_p)$$

is trivial, because $\mathrm{Cor}^G_{\langle\gamma\rangle} \circ \mathrm{Res}^G_{\langle\gamma\rangle}(u) = (G : \langle\gamma\rangle) \cdot u = 0$ for all $u \in H^{2\nu_p(G)}(G, \mathbb{F}_p)$. Here $(G : \langle\gamma\rangle)$ is divisible by p by assumption. Setting $\rho = \langle\hat{\gamma}_1, \ldots, \hat{\gamma}_q\rangle \in \mathcal{M}_G$, we have

$$e^{\mathrm{alg}}_{\nu_p(G)}(\rho) = \sum_{i=1}^{q} \mathrm{Cor}^G_{\langle\gamma_i\rangle}(c(\gamma_i)^{\nu_p(G)}) = 0 \in H^{2\nu_p(G)}(G, \mathbb{F}_p).$$

Now the proposition follows from Theorem 4.3. □

The second vanishing result is based on the principal ideal theorem concerning the transfer in group theory. Let G be a finite group and H a subgroup of index n. Choose a set $E = \{g_1, \ldots, g_n\}$ of representatives for the right cosets of H in G. For each $g \in G$, let $\bar{g} \in E$ be the representative of Hg. The transfer $\mathrm{Tr}^G_H : G_{\mathrm{ab}} \to H_{\mathrm{ab}}$ is the homomorphism defined by

$$g \bmod [G,G] \ \mapsto \prod_{i=1}^{n} g_i g (\overline{g_i g})^{-1} \bmod [H,H]$$

(see [10, Section 5A] or [5, Section III.10]). The transfer Tr^G_H induces a homomorphism $\mathrm{Hom}(H, \mathbb{C}^\times) \to \mathrm{Hom}(G, \mathbb{C}^\times)$, which fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(H, \mathbb{C}^\times) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}(G, \mathbb{C}^\times) \arrow[d, "\cong"] \\
H^2(H, \mathbb{Z}) \arrow[r, "\mathrm{Cor}^G_H"] & H^2(G, \mathbb{Z})
\end{tikzcd}$$

where the vertical arrows are the isomorphisms introduced in Section 3.2. The commutativity of the diagram follows from the fact that Tr^G_H can be identified with the transfer $H_1(G, \mathbb{Z}) \to H_1(H, \mathbb{Z})$ in group homology (see [5, Section III.10]).

Theorem 5.4. *Let G be a finite group and $[G,G]$ the commutator subgroup. Then the transfer $\mathrm{Tr}^G_{[G,G]}$ is trivial. Consequently, the corestriction*

$$\mathrm{Cor}^G_{[G,G]} : H^2([G,G], \mathbb{Z}) \to H^2(G, \mathbb{Z})$$

is trivial.

Proof. The triviality of the transfer $\mathrm{Tr}^G_{[G,G]}$ is known as the principal ideal theorem. See [10, Chapter 10.C] for the proof. □

As an immediate consequence, we obtain the following corollary. For simplicity, we state the argument only for algebraic MMM classes:

Corollary 5.5. *Let G be a finite group and $\rho = \langle \hat{\gamma}_1, \ldots, \hat{\gamma}_q \rangle \in \mathcal{M}_G$ an element satisfying $\gamma_i \in [G,G]$ $(1 \leq i \leq q)$. Then*

$$e_1^{\mathrm{alg}}(\rho) = 0 \in H^2(G, \mathbb{Z}) \tag{5.1}$$

and hence

$$e_{1+k\nu_p(G)}^{\mathrm{alg}}(\rho) = 0 \in H^{2(1+k\nu_p(G))}(G, \mathbb{F}_p) \quad \text{for all } k \geq 0. \tag{5.2}$$

In particular, if G is perfect, then (5.1) and (5.2) hold for all $\rho \in \mathcal{M}_G$.

Proof. The corollary follows from the equality

$$\mathrm{Cor}_{\langle \gamma_i \rangle}^{G}(c(\gamma_i)) = \mathrm{Cor}_{[G,G]}^{G} \circ \mathrm{Cor}_{\langle \gamma_i \rangle}^{[G,G]}(c(\gamma_i)) = 0 \in H^2(G, \mathbb{Z}),$$

which holds for $1 \leq i \leq q$. □

Since $1 + k\nu_p(G) \equiv 1 \pmod{p-1}$, Corollary 5.5 is independent of Corollary 5.2. Finally, by applying the following proposition to Corollary 5.5, we can obtain other sufficient conditions for the vanishing of mod p MMM classes:

Proposition 5.6. *Let G be a finite group and p an odd prime. Then, for all $k \geq 1$,*

1. $e_k(G,C)^p = e_{kp}(G,C) \in H^{2kp}(G, \mathbb{F}_p)$ *for any surface symmetry* (G,C),
2. $e_k^{\mathrm{alg}}(\rho)^p = e_{kp}^{\mathrm{alg}}(\rho) \in H^{2kp}(G, \mathbb{F}_p)$ *for any* $\rho \in \mathcal{M}_G$.

Proof. Let $\pi : E \to B$ be an oriented surface bundle. It was proved in [2] that

$$\mathrm{P}^i(e_k(\pi)) = \binom{k}{i} e_{k+i(p-1)}(\pi) \in H^{2(k+i(p-1))}(B, \mathbb{F}_p),$$

where $\mathrm{P}^i : H^k(-, \mathbb{F}_p) \to H^{k+2i(p-1)}(-, \mathbb{F}_p)$ is the reduced power operation. Hence

$$e_k(\pi)^p = \mathrm{P}^k(e_k(\pi)) = \binom{k}{k} e_{k+k(p-1)}(\pi) = e_{kp}(\pi) \in H^{2kp}(B, \mathbb{F}_p).$$

By applying the last equality to the oriented surface bundle $EG \times_G C \to BG$ associated with a surface symmetry (G,C), the proposition follows. □

Acknowledgements

This research was partly supported by Grant-in-Aid for Scientific Research (B) (No. 20340008) from Japan Society for Promotion of Sciences (JSPS).

References

[1] T. Akita, Nilpotency and triviality of mod p Morita–Mumford classes of mapping class groups of surfaces, Nagoya Math. J. **165** (2002), 1–22. Zbl 1041.55011 MR 1892095

[2] ———, On mod p Riemann–Roch formulae for mapping class groups, in *Groups of diffeomorphisms*, Adv. Stud. Pure Math. 52, Math. Soc. Japan, Tokyo, 2008, 111–118. Zbl 1158.57302 MR 2509708

[3] T. Akita and N. Kawazumi, Integral Riemann–Roch formulae for cyclic subgroups of mapping class groups, Math. Proc. Cambridge Philos. Soc. **144** (2008), 411–421. Zbl 1159.55005 MR 2405898

[4] T. Akita, N. Kawazumi and T. Uemura, Periodic surface automorphisms and algebraic independence of Morita–Mumford classes, J. Pure Appl. Algebra **160** (2001), 1–11. Zbl 0980.57011 MR 1829309

[5] K. S. Brown, *Cohomology of groups*, Grad. Texts in Math. 87, Springer, New York, 1982. Zbl 0584.20036 MR 0672956

[6] S. Galatius, Secondary characteristic classes of surface bundles, Algebr. Geom. Topol. **9** (2009), 293–303. Zbl 1166.55004 MR 2482078

[7] S. Galatius, I. Madsen and U. Tillmann, Divisibility of the stable Miller–Morita–Mumford classes, J. Amer. Math. Soc. **19** (2006), 759–779. Zbl 1102.57015 MR 2219303

[8] R. Grieder, Geometric representation theory and G-signature, arXiv:math/9811102.

[9] ———, G-actions on Riemann surfaces and the associated group of singular orbit data, arXiv:math/9902048.

[10] I. M. Isaacs, *Finite group theory*, Grad. Stud. Math. 92, Amer. Math. Soc., Providence, 2008. Zbl 1169.20001 MR 2426855

[11] B. Kahn, Les classes de Chern modulo p d'une représentation régulière, Doc. Math. **4** (1999), 167–178. Zbl 0954.20028 MR 1701292

[12] N. Kawazumi, Weierstrass points and Morita–Mumford classes on hyperelliptic mapping class groups, Topology Appl. **125** (2002), 363–383. Zbl 1020.57005 MR 1934277

[13] N. Kawazumi and T. Uemura, Riemann–Hurwitz formula for Morita–Mumford classes and surface symmetries, Kodai Math. J. **21** (1998), 372–380. Zbl 0924.57031 MR 1664755

[14] S. P. Kerckhoff, The Nielsen realization problem, Ann. of Math. (2) **117** (1983), 235–265. Zbl 0528.57008 MR 0690845

[15] E. Y. Miller, The homology of the mapping class group, J. Differential Geom. **24** (1986), 1–14. Zbl 0618.57005 MR 0857372

[16] G. Mislin, Mapping class groups, characteristic classes, and Bernoulli numbers, in *The Hilton Symposium 1993*, CRM Proc. Lecture Notes 6, Amer. Math. Soc., Providence, 1994, 103–131. Zbl 0844.57012 MR 1290586

[17] S. Morita, Characteristic classes of surface bundles, Invent. Math. **90** (1987), 551–577. Zbl 0608.57020 MR 0914849

[18] D. Mumford, Towards an enumerative geometry of the moduli space of curves, in *Arithmetic and geometry* Vol. II, Birkhäuser, Boston, 1983, 271–328. Zbl 0554.14008 MR 0717614

[19] C. B. Thomas, *Characteristic classes and cohomology of finite groups*, Cambridge Stud. Adv. Math. 9, Cambridge Univ. Press, Cambridge, 1986. Zbl 0618.20036 MR 0878978

[20] T. Uemura, Morita–Mumford classes on finite cyclic subgroups of the mapping class group of closed surfaces, Hokkaido Math. J. **28** (1999), 597–611. Zbl 0946.57031 MR 1723456