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Asymptotic Property of Divergent Formal Solutions in Linearization of Singular Vector Fields

by

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Abstract

We study asymptotic properties of divergent formal solutions appearing in the linearization problem of a singular vector field without a Diophantine condition or existence of additional first integrals. We give an asymptotic meaning to divergent formal solutions constructed from a singular perturbative solution (cf. [6]).

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§1. Introduction

A linearizing transformation of a singular vector field satisfies a certain semilinear Fuchsian system of equations of several variables (cf. (2.2)). The system has a formal power series solution under a general nonresonance condition, while formal solutions are divergent in general (cf. [3] and Proposition 3.1 of [7]). The convergence of the series can be proved under a Diophantine condition or existence of additional first integrals. In this paper we study equations of two independent variables, and we shall give an asymptotic meaning to a formal solution without any Diophantine condition or existence of additional first integrals (cf. [4]).

In [6], we constructed a singular perturbative solution with respect to a singular perturbative parameter ε by resumming a singular perturbative formal solution. If the so-called Poincaré condition and the nonresonance condition are satisfied, then by analytic continuation with respect to ε up to $\varepsilon = 1$ we obtain the classical Poincaré solution. In this paper we are interested in the case where the Poincaré condition or a Diophantine condition is not satisfied. By the same method as in [6]

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we can construct a singular perturbative solution and make an analytic continuation with respect to ε to a sector with vertex at $\varepsilon = 1$ as well. On the other hand the analytic continuation of the resummed singular perturbative solution does not necessarily converge as $\varepsilon \to 1$.

Our goal in this paper is to show that the analytic continuation of the resummed singular perturbative solution is an asymptotic expansion of a certain analytic solution in a multisector of the space variables uniformly with respect to ε in a sector with vertex at $\varepsilon = 1$. More precisely, we can show the assertion for equations with nonlinear part satisfying certain support conditions (cf. (2.19) and (2.20)) for which a small denominator may appear. (See also [7].) We hope that our new approach to the linearization problem via an equation with a singular perturbative parameter may be generalized to the case of general independent variables. We also remark that our proof does not use the so-called Newton method in constructing a solution, which makes the proof simpler than the one based on the Newton method.

This paper is organized as follows. In Section 2 we state our results. In Section 3 we prepare a necessary lemma. In the last section we prove our main theorem.

§2. Statement of results

Let $x = (x_1, x_2) \in \mathbb{C}^2$. For a 2×2 constant matrix Λ, we denote by L_Λ the Lie derivative of the linear vector field $x\Lambda \cdot \partial_x$,

$$L_\Lambda := [x\Lambda\, \partial_x, \cdot\,] = \langle x\Lambda, \partial_x \rangle - \Lambda, \tag{2.1}$$

where $\langle x\Lambda, \partial_x \rangle = \sum_{j=1}^{2} (x\Lambda)_j (\partial/\partial x_j)$, with $(x\Lambda)_j$ being the j-th component of $x\Lambda$. It is well known that the following system of equations is the linearizing equation of the singular vector field $x\Lambda \cdot \partial_x + R(x)\partial_x$:

$$L_\Lambda u = R(x + u(x)), \tag{2.2}$$

where $u = {}^t(u_1, u_2)$ is an unknown vector function and the function

$$R(y) = {}^t(R_1(y), R_2(y)) \tag{2.3}$$

is holomorphic in some neighborhood of $y = 0$ in $\mathbb{C}^2$ such that $R(y) = O(|y|^2)$ as $|y| \to 0$. In order to study (2.2) we consider the following equation with parameter ε:

$$L^\varepsilon_\Lambda u \equiv \varepsilon \langle x\Lambda, \partial_x \rangle u - u\Lambda = R(x + u(x)), \tag{2.4}$$

and then we let $\varepsilon \to 1$.

In the following we assume that Λ is a diagonal matrix with diagonal entries 1 and $-\tau < 0$, where $\tau > 0$ is an irrational number. Hence we have

$$\langle x\Lambda, \partial_x\rangle = x_1\partial_1 - \tau x_2\partial_2. \tag{2.5}$$

We first construct a formal solution $u^W(x,\varepsilon)$ of (2.4) as a formal power series in ε,

$$u^W(x,\varepsilon) = \sum_{\nu=0}^{\infty} \varepsilon^\nu u_\nu^W(x) = u_0^W(x) + \varepsilon u_1^W(x) + \cdots, \tag{2.6}$$

where the coefficients $u_\nu^W(x)$ $(\nu = 0, 1, \dots)$ are holomorphic vector functions of x in some open set independent of ν. We substitute the expansion (2.6) into (2.4). We first note that

$$\varepsilon\langle x\Lambda, \partial_x\rangle u^W - u^W\Lambda = \sum_{\nu=0}^{\infty}(\varepsilon\langle x\Lambda, \partial_x\rangle u_\nu^W(x) - u_\nu^W(x)\Lambda)\varepsilon^\nu, \tag{2.7}$$

$$\begin{aligned} R(x+u^W) &= R(x + u_0^W + u_1^W\varepsilon + u_2^W\varepsilon^2 + \cdots) \\ &= R(x+u_0^W) + \varepsilon u_1^W(\nabla R)(x+u_0^W) + O(\varepsilon^2). \end{aligned} \tag{2.8}$$

By comparing the coefficients of $\varepsilon^0 = 1$ and ε, we obtain

$$u_0^W(x)\Lambda + R(x+u_0^W) = 0, \tag{2.9}$$

$$\langle x\Lambda, \partial_x\rangle u_0^W = u_1^W\Lambda + u_1^W(\nabla R)(x+u_0^W). \tag{2.10}$$

Because Λ is invertible and $u_0^W(x) = O(|x|^2)$ as $x \to 0$, we can determine u_0^W as a holomorphic vector function in some neighborhood of the origin $x = 0$ from (2.9). On the other hand, by noting that $\Lambda + (\nabla R)(x + u_0^W)$ is an invertible matrix in some neighborhood of $x = 0$ by the assumption $R(x) = O(|x|^2)$, we can determine u_1^W as a holomorphic function in some neighborhood of $x = 0$ from (2.10). In order to determine u_ν^W $(\nu \geq 2)$ we compare the coefficients of ε^ν on both sides of (2.4). Namely, we differentiate (2.4) with respect to ε, ν times, and we put $\varepsilon = 0$. Then we obtain

$$\begin{aligned} \langle x\Lambda, \partial_x\rangle u_{\nu-1}^W = {} & u_\nu^W\Lambda + u_\nu^W(\nabla R)(x+u_0^W) \\ & + (\text{terms involving } u_i^W,\ i \leq \nu - 1). \end{aligned} \tag{2.11}$$

Clearly from (2.11) we can determine u_ν^W as a holomorphic function in some neighborhood of $x = 0$. Hence we can determine u^W. We note that u_ν^W's are holomorphic in some neighborhood of the origin independent of ν in view of the above argument (cf. [5]).

By expanding $u_\nu^W(x)$ $(\nu = 0, 1, \ldots)$ into a power series in x, $u_\nu^W(x) = \sum_\alpha u_{\nu,\alpha}^W x^\alpha$, and summing up with respect to ν, we obtain the formal expansion of $u^W(x, \varepsilon)$,

$$u^W(x,\varepsilon) = \sum_{\alpha \in \mathbb{Z}_+^2} u_\alpha^W(\varepsilon) x^\alpha \tag{2.12}$$

with u_α^W being a formal power series in ε. In [6] we proved that, if τ is irrational, then the formal series $u_\alpha^W(\varepsilon)$ converges in some neighborhood of $\varepsilon = 1$ independent of α such that $u^W(x, \varepsilon)$ coincides with the unique formal power series solution of (2.4), a classical Poincaré series. Hence we can construct the solution of (2.2) from $u^W(x, \varepsilon)$ by setting $\varepsilon = 1$ in the class of formal power series. Note that we do not use any Diophantine condition in the argument.

In order to give an analytical meaning to this argument, we begin with the resummation of $u^W(x, \varepsilon)$ when ε is in some sector. We define $\tilde{u}^W(x, \varepsilon) = u^W(x, \varepsilon) - u_0^W(x)$. Then the (formal) Borel transform of $\tilde{u}^W$ is defined by

$$B(\tilde{u}^W)(x,\zeta) := \sum_{\nu=1}^{\infty} u_\nu^W(x) \frac{\zeta^{\nu-1}}{(\nu-1)!}. \tag{2.13}$$

Because $u_\nu^W(x)$ is holomorphic in some neighborhood of the origin $x = 0$ independent of ν, the expansion $u_\nu^W(x) = \sum_\alpha u_{\nu,\alpha}^W x^\alpha$ converges in a common neighborhood of the origin independent of ν. By substituting the expansion into (2.13) we obtain

$$B(\tilde{u}^W)(x,\zeta) = \sum_{\nu=1}^{\infty} \sum_\alpha u_{\nu,\alpha}^W x^\alpha \frac{\zeta^{\nu-1}}{(\nu-1)!}. \tag{2.14}$$

Let us assume that the right-hand side of (2.14) absolutely converges in some neighborhood of $(x, \zeta) = (0, 0)$. (For the rigorous proof of this fact we refer to [6].) Then, by changing the order of the summations we obtain

$$B(\tilde{u}^W)(x,\zeta) = \sum_\alpha \sum_{\nu=1}^{\infty} u_{\nu,\alpha}^W \frac{\zeta^{\nu-1}}{(\nu-1)!} x^\alpha. \tag{2.15}$$

We define the Laplace transform $\tilde{U}^W(x, \varepsilon)$ of $B(\tilde{u}^W)(x, \zeta)$ by

$$\tilde{U}^W(x,\varepsilon) := \sum_\alpha L\left(\sum_{\nu=1}^{\infty} u_{\nu,\alpha}^W \frac{\zeta^{\nu-1}}{(\nu-1)!}\right) x^\alpha, \tag{2.16}$$

where the operator L is given by

$$Lf(\varepsilon) = \int_0^\infty e^{-\zeta/\varepsilon} f(\zeta)\, d\zeta.$$

Here we assume an appropriate growth condition on $f(\zeta)$. We define

$$U^W(x,\varepsilon) := \tilde{U}^W(x,\varepsilon) + u_0^W(x).$$

If we recall that the Borel transform is the inverse of the Laplace transform, $U^W(x,\varepsilon)$ gives a holomorphic function of ε in a sectorial domain with the asymptotic expansion $u^W(x,\varepsilon)$. We call $U^W(x,\varepsilon)$ a *resummation* of a singular perturbative solution u^W. For a direction ξ $(0 \le \xi < 2\pi)$ and an opening $\theta > 0$ we define the sector $S_{\xi,\theta}$ by

$$S_{\xi,\theta} = \{\varepsilon \in \mathbb{C};\ |\arg \varepsilon - \xi| < \theta/2,\ \varepsilon \neq 0\}. \tag{2.17}$$

The following theorem was proved in [6, Theorem 2].

Theorem 1. *There exist a direction ξ, an opening $\theta > 0$ and a neighborhood Ω_0 of the origin $x = 0$ such that $U^W(x,\varepsilon)$ is holomorphic in $(x,\varepsilon) \in \Omega_0 \times S_{\xi,\theta}$ and satisfies (2.4). The formal solution $u^W(x,\varepsilon)$ given by (2.12) is an asymptotic expansion of $U^W(x,\varepsilon)$ in $\Omega_0 \times S_{\xi,\theta}$ with respect to $\varepsilon \in S_{\xi,\theta}$.*

We note that one can take for ξ any direction such that $\xi \neq 0, \pi$. Suppose $\tau < 0$, that is, the *Poincaré condition* is satisfied. By Theorem 4 of [6], $U^W(x,\varepsilon)$ can be analytically continued with respect to ε to $\varepsilon = 1$ when x is in some neighborhood of the origin independent of ε.

We now consider the case of $\tau > 0$ irrational. By Theorem 4 of [6], $U^W(x,\varepsilon)$ can be analytically continued with respect to ε up to a neighborhood of $\varepsilon = 1$ such that $\operatorname{Im}\varepsilon > 0$ (or $\operatorname{Im}\varepsilon < 0$) when x is in some neighborhood of the origin which may depend on ε. By well known results on the divergence of the linearizing transformation in the non-Diophantine case we cannot expect the convergence of $u^W(x,\varepsilon)|_{\varepsilon=1}$ as a formal power series in x (cf. [7]). In the following we study the asymptotic meaning of the series.

Let η_1 and η_2 be such that $\eta_1 > 0$, $0 < \eta_2 < \pi/2$ and $\eta_1 + \eta_2/\tau < \pi/2$. Let $S_1 \subset \mathbb{C}$ and $S_2 \subset \mathbb{C}$ be sectors with openings η_1 and η_2, respectively, namely $S_j := \{x_j \in \mathbb{C};\ |\arg x_j| < \eta_j/2\}$ $(j = 1,2)$. For $0 < \rho \le 1$ we define $S_{j,\rho} := S_j \cap \{|x_j| < \rho\}$. Let $0 < \theta < \pi$ be given. We denote by $\mathcal{C}_{\pm,\theta}$ the cone with vertex at $\varepsilon = 1$ with opening θ,

$$\mathcal{C}_{\pm,\theta} := \{\varepsilon \in \mathbb{C};\ |\arg(\varepsilon - 1) \mp \pi/2| < \theta/2\}. \tag{2.18}$$

We define $\mathcal{C}_{\pm,\theta,\rho} = \mathcal{C}_{\pm,\theta} \cap \{|\varepsilon| < \rho\}$. For $\alpha = (\alpha_1,\alpha_2) \in \mathbb{Z}_+^2$ we set $|\alpha| = \alpha_1 + \alpha_2$.

We assume that $R(x)$ is holomorphic in some neighborhood of the origin with the Taylor expansion given either by

$$R(x) = \sum_{\alpha=(\alpha_1,\alpha_2)\in\mathbb{Z}_+^2,\ \alpha_1 - \tau\alpha_2 < -2\tau} R_\alpha x^\alpha, \tag{2.19}$$

or

$$R(x) = \sum_{\alpha=(\alpha_1,\alpha_2)\in\mathbb{Z}_+^2,\, \alpha_1-\tau\alpha_2>2\tau} R_\alpha x^\alpha. \tag{2.20}$$

Our main result in this paper is the following

Theorem 2. *Suppose that either (2.19) or (2.20) is satisfied. Let* $0 < \theta < \pi$. *Then there exists* $\rho > 0$ *such that (2.4) has a solution* $u_\pm(x, \varepsilon)$ *holomorphic in* $S_{1,\rho} \times S_{2,\rho} \times \mathcal{C}_{\pm,\theta,\rho}$ *such that, for every* $\varepsilon \in \mathcal{C}_{\pm,\theta,\rho}$ *and* $\nu = 0, 1, \dots,$

$$u_\pm(x,\varepsilon) - \sum_{|\alpha|\le\nu} u_\alpha^W(\varepsilon)x^\alpha = O(|x|^{\nu+1}) \quad \text{as } x \to 0,\ x \in S_{1,\rho} \times S_{2,\rho}. \tag{2.21}$$

Remark 1. If $\tau < 0$, that is, the Poincaré condition is satisfied, then we may take $u_\pm(x, \varepsilon)$ in Theorem 2 as an analytic continuation of $U^W(x, \varepsilon)$ up to $\varepsilon = 1$ (cf. [6]). Theorem 2 ensures the existence of a similar function in the case $\tau > 0$. We expect that our argument here also works for a resonant case after appropriate modifications, which is left for a future research.

§3. Preliminary lemma

In this section we prove the solvability of (2.4) modulo flat functions. We define

$$S_\rho := S_1 \times S_2 \cap \{(x_1, x_2) \in \mathbb{C}^2;\ |x_1|\,|x_2|^{1/\tau} < \rho,\ |x_2| < \rho\}. \tag{3.1}$$

For every $n \ge 1$ we choose the smallest positive integer k_n such that $n - \tau k_n < 0$. Namely, k_n is determined by the relation $-\tau < n - \tau k_n < 0$. We set $\alpha_n = (n, k_n)$. Let $U^W(x, \varepsilon) = \sum_{\alpha\in\mathbb{Z}_+^2} u_\alpha^W(\varepsilon)x^\alpha$ be as given in Theorem 1. Then we have

Lemma 3. *Suppose that (2.19) is satisfied. Let* $0 < \theta < \pi$. *Then there exist* $\rho > 0$ *and a function* $V(x, \varepsilon)$ *holomorphic in* $S_\rho \times \mathcal{C}_{\pm,\theta,\rho}$ *and continuous up to the boundary such that for every* $n = 0, 1, \dots$ *there exists* $\tilde{g}_n(x, \varepsilon)$ *holomorphic in* $S_\rho \times \mathcal{C}_{\pm,\theta,\rho}$ *and continuous up to the boundary such that, for every* $\varepsilon \in \mathcal{C}_{\pm,\theta,\rho}$,

$$R(x + V) - L_\Lambda^\varepsilon V = x^{\alpha_n}\tilde{g}_n(x, \varepsilon), \qquad x \in S_\rho, \tag{3.2}$$

$$V(x,\varepsilon) - \sum_{|\alpha|\le n} u_\alpha^W(\varepsilon)x^\alpha = O(|x|^{n+1}) \qquad \text{as } x \to 0,\ x \in S_\rho. \tag{3.3}$$

Moreover there exist infinitely many α_{n_ν} $(\nu = 1, 2, \dots)$ *and* $0 < \theta' < 1$ *independent of* α_{n_ν} *such that* $x_2^{-1-\theta'}\tilde{g}_n(x, \varepsilon)$ *is holomorphic and bounded in* $S_\rho \times \mathcal{C}_{\pm,\theta,\rho}$.

Remark 2. If (2.20) is satisfied, then we interchange the roles of x_1 and x_2. Then the conclusion of Lemma 3 also holds true, with the same proof.

Proof of Lemma 3. We divide the proof into 12 steps.

Step 1. For the sake of simplicity we denote $\mathcal{C}_{\pm,\theta}$ and $\mathcal{C}_{\pm,\theta,\rho}$ by $\mathcal{C}$ and $\mathcal{C}_\rho$, respectively. We will look for $U \equiv U(x,\varepsilon)$ in the form

$$U = a_0 + b_0 + \sum_{j=1}^{\infty} x^{\alpha_j}(a_j + b_j), \tag{3.4}$$

with $a_j \equiv a_j(x_1,\varepsilon)$ and $b_j \equiv b_j(x_2,\varepsilon)$ holomorphic and bounded in $S_1 \times \mathcal{C}_\rho$ and $S_{2,\rho} \times \mathcal{C}_\rho$, respectively, and

$$a_0 = O(x_1^2), \qquad b_j = O(x_2^2), \qquad j = 0, 1, \ldots, \tag{3.5}$$

such that the functions

$$U_{n-1} := a_0 + b_0 + \sum_{j=1}^{n-1} x^{\alpha_j}(a_j + b_j) \qquad (n \geq 1) \tag{3.6}$$

satisfy

$$\mathcal{R}_{n-1} := L_\Lambda^\varepsilon U_{n-1} - R(x + U_{n-1}) = x^{\alpha_n}\tilde{\mathcal{R}}_{n-1}(x,\varepsilon) \tag{3.7}$$

for some $\tilde{\mathcal{R}}_{n-1}(x,\varepsilon)$ holomorphic in $S_\rho \times \mathcal{C}_\rho$ and continuous up to the boundary such that $\tilde{\mathcal{R}}_{n-1} = O(x_2^2)$ as $x_2 \to 0$.

Step 2. We will construct a_j and b_j in (3.4) formally. We first rewrite $U^W(x,\varepsilon) = \sum_{\alpha \in \mathbb{Z}_+^2} u_\alpha^W(\varepsilon) x^\alpha$ in the form

$$U^W = \tilde{a}_0(x_1,\varepsilon) + \tilde{b}_0(x_2,\varepsilon) + \sum_{n=1}^{\infty} x^{\alpha_n}(\tilde{a}_n(x_1,\varepsilon) + \tilde{b}_n(x_2,\varepsilon)), \tag{3.8}$$

where the formal power series $\tilde{a}_n(x_1,\varepsilon)$ and $\tilde{b}_n(x_2,\varepsilon)$ $(n = 0, 1, \ldots)$ satisfy

$$\tilde{a}_0(x_1,\varepsilon) = O(x_1^2), \qquad \tilde{b}_n(x_2,\varepsilon) = O(x_2^2), \qquad n = 0, 1, \ldots. \tag{3.9}$$

We first consider the case $\tau > 1$. We note $k_j \leq j$ for every j. We determine $\tilde{a}_0(x_1,\varepsilon)$ and $\tilde{b}_0(x_2,\varepsilon)$ as the Taylor series in U^W consisting of powers of x_1 and x_2 only, respectively. By subtracting $\tilde{a}_0 + \tilde{b}_0$ from U^W we see that the resulting term is divisible by x_1x_2. Hence we can choose terms which are divisible by x^{α_1}. On determining $\tilde{a}_1$ and $\tilde{b}_1$ similarly to $\tilde{a}_0$ and $\tilde{b}_0$ we subtract $x^{\alpha_1}(\tilde{a}_1 + \tilde{b}_1)$ again and see that the remaining term is divisible by $x^{\alpha_1}x_1x_2$. Hence it is divisible by x^{α_2}. Repeating the argument, we can rearrange the series U^W in the above form. We note that because we may have $k_1 = \cdots = k_\ell$ for some $\ell > 1$, the expression is not unique in general.

Next we consider the case $0 < \tau < 1$. Because we have $k_j > j$ for some j, the situation is different from the case $\tau > 1$. We first show that the support of the Taylor expansion of U^W is contained in the convex cone $\Gamma_0 := \{(\alpha_1, \alpha_2) \in \mathbb{R}^2; \alpha_j \geq 0, \alpha_1 - \tau\alpha_2 < -2\tau\}$. To see this, we recall that U^W is the formal power series solution of (2.4) such that $U^W = O(|x|^2)$. On the other hand, by (2.19) the term of degree 2 in the expansion of R vanishes. Because L^ε_Λ preserves monomials, it follows that the term with degree 2 in U^W also vanishes. Next the term of degree 3 in R is a constant times x_2^3. Indeed, by the support condition on R, $\alpha_1 - \tau\alpha_2 < -2\tau$, we have $\alpha_2 \geq 3$ if $\alpha_1 \geq 1$. Because L^ε_Λ preserves monomials, it follows that the term with degree 3 in U^W has weight $\alpha_1 - \tau\alpha_2 < -2\tau$. Let us suppose that the assertion holds for every term x^α in U^W up to $|\alpha| \leq \nu$. Consider the monomial x^β, $\beta = (\beta_1, \beta_2)$, $|\beta| = \nu + 1$, appearing from $R(x + u)$. We may consider $(x_1 + u_1)^k (x_2 + u_2)^m$ for $k + m \leq \nu + 1$ instead of $R(x + u)$ without loss of generality. In order to estimate the weight $\beta_1 - \tau\beta_2$ from above for every x^β appearing from $(x_1 + u_1)^k (x_2 + u_2)^m$, it is sufficient to consider the terms which contain x_1^k because the weight of terms appearing from $(x_1 + u_1)^k$ is less than or equal to k. As for the weight of terms appearing from $(x_2 + u_2)^m$ it is largest when x_2^m appears because the weight of every monomial in u_2 is strictly smaller than -2τ by inductive assumption. Because $k - \tau m < -2\tau$ by (2.19), we see that every monomial x^β, $|\beta| = \nu + 1$, appearing from $R(x + u)$ has the desired property. Hence the support of the Taylor expansion of U^W is contained in Γ_0.

In order to write U^W in the form (3.8) we determine $\tilde{a}_0$ and $\tilde{b}_0$ similarly to the case $\tau > 1$. Subtracting $\tilde{a}_0 + \tilde{b}_0$ from U^W we see that the resulting term is divisible by $x_1 x_2$. Moreover, since k_1 satisfies $-\tau < 1 - \tau k_1 < 0$, it follows that $m \geq k_1 + 2$ if $(1, m)$ is in the support of U^W. Hence the resulting term is divisible by $x^{\alpha_1} x_2^2$. We now determine a_1 and b_1 as in the case $\tau > 1$ and consider $U^W - \sum_{j=0}^{1} x^{\alpha_j}(a_j + b_j)$, where $x^{\alpha_0} = 1$. It satisfies the same support condition as U^W. Hence we can proceed in the same way by noting that $m \geq k_n + 2$ if (n, m) is in the support of U^W. This proves that U^W can be expanded as in (3.8).

Step 3. We will determine a_0 and b_0 such that

$$\mathcal{R}_0 := L^\varepsilon_\Lambda(a_0 + b_0) - R(x + a_0 + b_0) = x^{\alpha_1}\tilde{\mathcal{R}}_0(x, \varepsilon) \tag{3.10}$$

for some holomorphic function $\tilde{\mathcal{R}}_0(x, \varepsilon)$ in $S_\rho \times \mathcal{C}_\rho$ continuous up to the boundary such that $\tilde{\mathcal{R}}_0 = O(x_2^2)$. Putting $x_2 = 0$ or $x_1 = 0$ in (3.10) we see that $w := a_0$ (resp. $w := b_0$), $w = (w_1, w_2)$, satisfies the system of equations

$$\varepsilon x_1 \partial_1 w_1 - w_1 = R_1(x_1 + w_1, w_2), \tag{3.11}$$

$$\varepsilon x_1 \partial_1 w_2 + \tau w_2 = R_2(x_1 + w_1, w_2), \tag{3.12}$$

respectively

$$(3.13)\qquad -\varepsilon\tau x_2\partial_2 w_1 - w_1 = R_1(w_1, x_2 + w_2),$$

$$(3.14)\qquad -\varepsilon\tau x_2\partial_2 w_2 + \tau w_2 = R_2(w_1, x_2 + w_2).$$

We note that $\tilde{a}_0$ (resp. $\tilde{b}_0$) is a formal solution of (3.11)–(3.12) (resp. (3.13)–(3.14)). We will show that $\tilde{a}_0 = 0$. By (2.19) we have $R(x_1, 0) \equiv 0$. It follows that the terms of order x_1^2 in $R(x_1+w_1, w_2)$ appear from the terms of the form $(x_1+w_1)w_2$ or w_2^2. By (3.9) these terms are $O(x_1^3)$. In order to see that the coefficients of x_1^2 in w_1 and w_2 vanish, we note that $\varepsilon\nu-1 \neq 0$ and $\varepsilon\nu+\tau \neq 0$ for all integers $\nu \geq 2$ and $\varepsilon \in \mathcal{C}_{\pm,\theta}$ because $\operatorname{Im}\varepsilon \neq 0$. Hence, the coefficient of x_1^2 in $\tilde{a}_0$ vanishes. Next, the coefficients of x_1^3 in the right-hand sides of (3.11)–(3.12) vanish by a similar argument because $\tilde{a}_0 = O(x_1^3)$. Hence the coefficient of x_1^3 in $\tilde{a}_0$ vanishes by (3.11) and (3.12). By induction we obtain $\tilde{a}_0 = 0$. By the condition $R_j(x_1, 0) \equiv 0$ $(j = 1, 2)$, we can put $a_0 = 0$.

We consider (3.13)–(3.14). By a similar argument to that in proving $\tilde{a}_0 = 0$ and (3.5) we see that (3.13)–(3.14) has a unique formal power series solution $\tilde{b}_0 = (\tilde{w}_1, \tilde{w}_2)$. By the well-known Briot–Bouquet theorem, $\tilde{b}_0$ converges in some neighborhood of the origin (cf. [2]). We set $b_0 := \tilde{b}_0$. By taking ρ sufficiently small we may assume that b_0 is holomorphic in $\{|x_2| < \rho\}$. We can easily see that b_0 is holomorphic with respect to ε in some neighborhood of $\varepsilon = 1$. By taking ρ sufficiently small we may asssume that b_0 is holomorphic in $\{|\varepsilon - 1| < \rho\}$.

We will estimate the remainder term $\tilde{\mathcal{R}}_0$ in (3.10). By (3.13) and (3.14) we have

$$L_\Lambda^\varepsilon b_0 = R((0, x_2) + b_0).$$

Hence, by setting $y_1 = (x_1, 0)$ and $y_2 = (0, x_2) + b_0(x_2, \varepsilon)$ and by recalling $R(0) = 0$ we have

$$(3.15)\qquad \mathcal{R}_0 = -R(y_1 + y_2) + R(y_2) = -\int_0^1 x_1(\partial_{x_1} R)(t_1 y_1 + y_2)\, dt_1.$$

By (2.19), if $(1, m)$ is in the support of R, then $m \geq k_1 + 2$. Hence $\mathcal{R}_0$ satisfies $\mathcal{R}_0 = x^{\alpha_1}\tilde{\mathcal{R}}_0$ for some $\tilde{\mathcal{R}}_0$ holomorphic and bounded when $x \in S_\rho$ and $\varepsilon \in \mathcal{C}_\rho$ and satisfying $\tilde{\mathcal{R}}_0 = O(x_2^2)$.

Step 4. We will determine a_1 and b_1. For $0 \leq t \leq 1$ we set

$$(3.16)\qquad u_t = b_0(x_2, \varepsilon) + t x^{\alpha_1}(a_1(x_1, \varepsilon) + b_1(x_2, \varepsilon)),$$

and we determine a_1 and b_1 ($b_1 = O(x_2^2)$) such that $\mathcal{R}_1 := L_\Lambda^\varepsilon(b_0 + x^{\alpha_1}(a_1+b_1)) - R(x + b_0 + x^{\alpha_1}(a_1+b_1))$ satisfies

$$\begin{aligned}\mathcal{R}_1 &= L_\Lambda^\varepsilon(x^{\alpha_1}(a_1+b_1)) + T_1 + \mathcal{R}_0 = x^{\alpha_2}\tilde{\mathcal{R}}_1(x,\varepsilon),\\ T_1 &:= R(x+u_0) - R(x+u_1),\end{aligned} \tag{3.17}$$

for some holomorphic function $\tilde{\mathcal{R}}_1(x,\varepsilon)$ in $S_\rho \times \mathcal{C}_\rho$ continuous up to the boundary such that $\tilde{\mathcal{R}}_1 = O(x_2^2)$.

We first show

$$\mathcal{R}_0 = -x^{\alpha_1}\beta_1(x_2,\varepsilon) + x^{\alpha_2}\Omega(x,\varepsilon) \tag{3.18}$$

for some holomorphic functions $\beta_1(x_2,\varepsilon)$ and $\Omega(x,\varepsilon)$ in $S_\rho \times \mathcal{C}_\rho$ continuous up to the boundary. Indeed, by Taylor's formula the integrand on the right-hand side of (3.15) can be written as

$$x_1(\partial_{x_1}R)(t_1y_1+y_2) = x_1(\partial_{x_1}R)(y_2) + \int_0^1 t_1x_1^2(\partial_{x_1}^2R)(t_1t_2y_1+y_2)\,dt_2.$$

Hence, by (3.15) we have

$$\begin{aligned}\mathcal{R}_0 &= -x_1(\partial_{x_1}R)(y_2) - \int_0^1 dt_1\int_0^1 t_1x_1^2(\partial_{x_1}^2R)(t_1t_2y_1+y_2)\,dt_2\\ &\equiv -x^{\alpha_1}\beta_1(x_2,\varepsilon) + x^{\alpha_2}\Omega(x,\varepsilon).\end{aligned} \tag{3.19}$$

By the support condition on R and (3.19) the function $\Omega(x,\varepsilon)$ is a bounded holomorphic function on $S_\rho \times \mathcal{C}_\rho$. Hence we obtain the desired decomposition of $\mathcal{R}_0$. We note that $\beta_1 = O(x_2^2)$ and $\Omega = O(x_2^2)$ by (2.19) and (3.19).

We consider T_1. By Taylor's formula we have

$$x^{-\alpha_1}T_1 = -\int_0^1 (a_1+b_1)\nabla R(x+u_t)\,dt. \tag{3.20}$$

We set

$$\Theta_1 := \nabla R(x_1,0), \qquad \Theta_2 := \nabla R((0,x_2)+b_0(x_2,\varepsilon)). \tag{3.21}$$

First we shall show that Θ_1 identically vanishes. Indeed, by (2.19) and $R(x) = O(|x|^2)$ we obtain $R(x) = O(x_2^3)$, from which we have the assertion. By letting $x_2 \to 0$ in (3.20) and by recalling $b_0(0,\varepsilon) \equiv b_1(0,\varepsilon) \equiv 0$ we see that the right-hand side of (3.20) tends to 0. Similarly, by letting $x_1 \to 0$ in the right-hand side of (3.20) we obtain $-(b_1 + a_1(0,\varepsilon))\Theta_2$. Therefore

$$T_1 + x^{\alpha_1}((b_1 + a_1(0,\varepsilon))\Theta_2) = x^{\alpha_1}x_1x_2\tilde{T}_1(x,\varepsilon) \tag{3.22}$$

for some $\tilde{T}_1(x,\varepsilon)$ holomorphic and bounded in $S_\rho \times \mathcal{C}_\rho$. Indeed, $x^{-\alpha_1}$ times the left-hand side of (3.22) is divisible by x_1x_2 by definition.

In order to obtain equations for a_1 and b_1, we note that, for U given by (3.4),

$$(x_1\partial_1 - \tau x_2\partial_2)(U - b_0) = \sum x^{\alpha_n}(x_1\partial_1 - \tau x_2\partial_2 + n - \tau k_n)(a_n + b_n). \tag{3.23}$$

By (3.17), (3.18) and (3.22) we have

$$\begin{aligned}\mathcal{R}_1 = x^{\alpha_1}(L_\Lambda^\varepsilon + \varepsilon - \varepsilon\tau k_1)a_1 + x^{\alpha_1}(L_\Lambda^\varepsilon + \varepsilon - \varepsilon\tau k_1)b_1 - x^{\alpha_1}\beta_1(x_2,\varepsilon)\\ - x^{\alpha_1}(b_1 + a_1(0,\varepsilon))\Theta_2 + x^{\alpha_1}x_1x_2\tilde{T}_1(x,\varepsilon) + x^{\alpha_2}\Omega(x,\varepsilon).\end{aligned} \tag{3.24}$$

Step 5. We will solve the equations for a_1 and b_1. By equating the coefficients of x^{α_1} in (3.24) which are functions of x_1 we obtain

$$(L_\Lambda^\varepsilon + \varepsilon - \varepsilon\tau k_1)a_1 = 0. \tag{3.25}$$

Clearly, $a_1 = \tilde{a}_1(x_1,\varepsilon) \equiv 0$ is the unique formal power series solution of (3.25) by assumption. Indeed, this follows from the assumption that $\operatorname{Im}\varepsilon \neq 0$. Hence we may set $a_1 = 0$.

As for b_1, we obtain

$$(L_\Lambda^\varepsilon + \varepsilon - \varepsilon\tau k_1)b_1 = b_1\Theta_2 + \beta_1(x_2,\varepsilon). \tag{3.26}$$

Let $\tilde{b}_1(x_2,\varepsilon) = \sum_{n=2}^\infty \gamma_n^{(0)}(\varepsilon)x_2^n$ be the unique formal power series solution of (3.26). Clearly, $\gamma_n^{(0)}(\varepsilon)$ is holomorphic in $\mathcal{C}_\rho$ and continuous up to the boundary. We define $\|\gamma_n^{(0)}\|$ as the maximum of $|\gamma_n^{(0)}(\varepsilon)|$ on the closure of $\mathcal{C}_\rho$. Let $0 < \delta < 1$, to be chosen later, and define, for $x_2 \in S_{2,\rho}$,

$$b_1^{(0)} = \sum_{n=2}^\infty \gamma_n^{(0)}(\varepsilon)\phi_n(x_2)^2x_2^n, \tag{3.27}$$

where

$$\phi_n(x_2) = \begin{cases} 1 - \exp\left(-\dfrac{\delta^n}{(\|\gamma_n^{(0)}\|+1)x_2(n-1)!}\right) & \text{if } \|\gamma_n^{(0)}\| \neq 0,\\ 1 & \text{if } \|\gamma_n^{(0)}\| = 0.\end{cases} \tag{3.28}$$

In order to show the convergence of (3.27) we recall the inequality

$$|1 - e^{-z}| < |z|, \qquad \operatorname{Re} z > 0. \tag{3.29}$$

Noting that $\operatorname{Re} x_2 > 0$ ($x_2 \in S_{2,\rho}$) and

$$\frac{\delta^n}{(\|\gamma_n^{(0)}\|+1)(n-1)!} \leq 1,$$

we find that, for $x_2 \in S_{2,\rho}$, $\gamma_n^{(0)} \neq 0$ and $n \geq 2$,

$$|\gamma_n^{(0)}(\varepsilon)|\,|x_2^n|\,|\phi_n(x_2)|^2 \leq |\gamma_n^{(0)}(\varepsilon)|\,|x_2^n|\left(\frac{\delta^n}{(\|\gamma_n^{(0)}\|+1)|x_2|(n-1)!}\right)^2 \leq \frac{|x_2|^{n-2}\delta^n}{(n-1)!}. \tag{3.30}$$

Hence the series in (3.27) converges uniformly on $S_{2,\rho}\times\mathcal{C}_\rho$, and the limit function is holomorphic in $(x,\varepsilon)\in S_{2,\rho}\times\mathcal{C}_\rho$ and bounded on its closure. Indeed, we have

$$\sum_{n\geq 2}|\gamma_n^{(0)}|\,|x_2^n|\,|\phi_n(x_2)|^2 \leq \delta^2\sum_{n\geq 2}\frac{|x_2|^{n-2}\delta^{n-2}}{(n-2)!} \leq \delta^2 e^{\delta|x_2|}. \tag{3.31}$$

If $x_2 \in S_{2,\rho}$ and $\delta > 0$ is sufficiently small, then the right-hand side can be made arbitrarily small. One can easily show that (cf. [1, p. 68]) $\tilde{b}_1$ is the asymptotic expansion of $b_1^{(0)}$ as $x_2 \to 0$, $x_2 \in S_{2,\rho}$. Moreover we can easily see that $b_1^{(0)}$ solves (3.26) asymptotically. Namely, for every $n = 0, 1, \ldots$, there exists $R_n^{(0)}(x_2,\varepsilon)$ holomorphic and bounded in $S_{2,\rho}\times\mathcal{C}_\rho$ such that, for every $\varepsilon\in\mathcal{C}_\rho$,

$$(L_\Lambda^\varepsilon + \varepsilon - \varepsilon\tau k_1)b_1^{(0)} - b_1^{(0)}\Theta_2 - \beta_1 = x_2^n R_n^{(0)}(x_2,\varepsilon), \quad x_2\in S_{2,\rho},\ x_2\to 0. \tag{3.32}$$

Step 6. For a holomorphic and bounded (vector) function $v = v(x_2,\varepsilon)$ in $S_{2,\rho}\times\mathcal{C}_\rho$, we define the norm of v by

$$\|v\| := \sup_{x_2\in S_{2,\rho},\,\varepsilon\in\mathcal{C}_\rho}|v(x_2,\varepsilon)|. \tag{3.33}$$

We similarly define the norm of a (vector) function $v = v(x_1,\varepsilon)$ on $S_1\times\mathcal{C}_\rho$.

In order to solve (3.26) in $S_{2,\rho}$ we define the approximate sequence $w^{(\nu)} = (w_1^{(\nu)}, w_2^{(\nu)})$ $(\nu = 0,1,\ldots)$ by $w^{(0)} = b_1^{(0)}$ and

$$(L_\Lambda^\varepsilon + \varepsilon - \varepsilon\tau k_1)w^{(1)} = \beta_1 + w^{(0)}\Theta_2 - (L_\Lambda^\varepsilon + \varepsilon - \varepsilon\tau k_1)w^{(0)}, \tag{3.34}$$

$$(L_\Lambda^\varepsilon + \varepsilon - \varepsilon\tau k_1)w^{(\nu)} = w^{(\nu-1)}\Theta_2, \quad \nu = 2,3,\ldots. \tag{3.35}$$

If we can show the uniform convergence of $b_1 := w^{(0)} + w^{(1)} + \cdots$ on $S_{2,\rho}\times\mathcal{C}_\rho$, then b_1 is the desired holomorphic solution of (3.26) in $S_{2,\rho}\times\mathcal{C}_\rho$.

We will estimate $w^{(j)}$. In order to solve (3.34)–(3.35) we recall that for every g holomorphic and bounded in $S_{2,\rho}$ with all derivatives vanishing at the origin and a complex number $\lambda\neq 0$, the solution of the equation $(x_2\partial_2 - \lambda)u = g$ is given by

$$u = (x_2\partial_2 - \lambda)^{-1}g = \int_{-\infty}^0 e^{-\lambda t}g(e^t x_2)\,dt, \tag{3.36}$$

where the integral converges by the assumption on g if $\operatorname{Re}\lambda \geq 0$. It follows that $w_1^{(1)}$ is well defined, holomorphic and bounded in $S_{2,\rho}$.

We shall prove that there exist constants $\eta_0 > 0$ and $0 < r_0 < 1$ such that

$$\|w_k^{(\nu)}\| \le \eta_0 r_0^{\nu}, \quad k = 1, 2;\ \nu = 0, 1, \ldots, \tag{3.37}$$

where $\eta_0 > 0$ can be chosen arbitrarily small if we take $\delta > 0$ sufficiently small. Clearly, if we can prove (3.37), then the limit $w_k := w_k^{(0)} + w_k^{(1)} + \cdots$ $(k = 1, 2)$ exists on $S_{2,\rho} \times \mathcal{C}_\rho$ and $b_1 := (w_1, w_2)$ gives the desired solution. We will estimate $w^{(1)}$ by (3.34) and (3.32). For simplicity, let us denote the right-hand side of (3.34) by h_0. We take n in (3.32) sufficiently large so that $(L_\Lambda^\varepsilon + \varepsilon - \varepsilon\tau k_1)^{-1} h_0$ is well defined. In view of the formula (3.36) we see that the norm of $w^{(1)}$ can be made arbitrarily small on $S_{2,\rho} \times \mathcal{C}_\rho$ by taking ρ sufficiently small because there appears a power x_2^n.

As for $w^{(\nu)}$, we can recursively estimate it in view of the recurrence relation (3.35) and the smallness of Θ_2. Indeed, Θ_2 vanishes up to order 2 by the assumption $R(x) = O(x_2^3)$.

Next we will show that $\tilde{b}_1$ is the asymptotic expansion of $b_1 := \sum_{\nu=0}^{\infty} w^{(\nu)}$. Because $\tilde{b}_1$ is the asymptotic expansion of $b_1^{(0)}$ we will show that $\sum_{\nu=1}^{\infty} w^{(\nu)} \sim 0$ as $x \to 0$. To see this, let $\ell \ge 2$ be a given integer and consider the sum $\sum_{\nu=1}^{\infty} \tilde{w}^{(\nu)}$, where $\tilde{w}^{(\nu)} = x_2^{-\ell} w^{(\nu)}$. If we can show the uniform convergence of $\sum_{\nu=1}^{\infty} \tilde{w}^{(\nu)}$ on $S_{2,\rho}$, then we see that $b_1 - b_1^{(0)}$ vanishes up to order ℓ as $x_2 \to 0$. Because $\ell \ge 2$ is arbitrary, this proves that the asymptotic expansion of b_1 is equal to $b_1^{(0)}$.

We define $\tilde{g}(z) := z^{-\ell} g(z)$. Then from (3.36) we get

$$\tilde{u}(x_2) := x_2^{-\ell} u(x_2) = \int_{-\infty}^{0} e^{-\lambda t + \ell t} \tilde{g}(e^t x_2)\, dt. \tag{3.38}$$

We note that $e^{-\lambda t + \ell t}$ is integrable if ℓ is sufficiently large. Hence we can estimate $\tilde{u}$ in terms of $\tilde{g}$. By (3.34) we can estimate $\tilde{w}^{(1)}$ in terms of the right-hand side of (3.34). By (3.35) we can similarly estimate $\tilde{w}^{(\nu)}$ in terms of $\tilde{g}$ with $g = w^{(\nu-1)}\Theta_2$. Because $\tilde{g}(z) = z^{-\ell} w^{(\nu-1)}\Theta_2 = \tilde{w}^{(\nu-1)}\Theta_2$, this proves the uniform convergence of $\sum_{\nu=1}^{\infty} \tilde{w}^{(\nu)}$.

Step 7. We will show (3.17) for some $\mathcal{R}_1 = O(x_2^2)$. We want to prove

$$T_1 + x^{\alpha_1} b_1 \Theta_2 = x^{\alpha_2} \tilde{T}_1 \tag{3.39}$$

for some holomorphic and bounded function $\tilde{T}_1(x, \varepsilon)$ on $S_\rho \times \mathcal{C}_\rho$ such that $\tilde{T}_1 = O(x_2^2)$. If we can prove this, then (3.18), (3.26) and (3.39) imply (3.17) for $\tilde{\mathcal{R}}_1 = \tilde{T}_1 + \Omega$.

We first show that $\alpha_j + \alpha_1 \ge \alpha_{j+1}$ for every $j \ge 1$. Indeed, by definition we have $-\tau < j - \tau k_j < 0$ for every j. Hence, by adding the inequalities for $j = j$ and $j = 1$ we obtain $-2\tau < j + 1 - \tau(k_j + k_1) < 0$. By the minimality of k_{j+1} we have $k_j + k_1 \ge k_{j+1}$.

In order to show (3.39) we first note, by (3.20) and $a_1 = 0$,

$$-x^{-\alpha_1}T_1 - \Theta_2 b_1 = \int b_1(\nabla R(x + b_0 + tx^{\alpha_1}b_1) - \Theta_2)\, dt. \tag{3.40}$$

By the definition of R and Θ_2 we can easily see that $\nabla R(x + b_0 + tx^{\alpha_1}b_1) - \Theta_2$ is divisible by x^{α_1} with the quotient holomorphic and bounded in $S_\rho \times \mathcal{C}_\rho$. In view of (3.40) and $2\alpha_1 \geq \alpha_2$ we have (3.39).

We easily see that the support of T_1 is contained in $\{(\alpha_1, \alpha_2) \in \mathbb{Z}_+^2;\ \alpha_1 - \tau\alpha_2 < -2\tau\}$ in view of (3.40). It follows that $\tilde{T}_1 = O(x_2^2)$. Because the support of $\mathcal{R}_0$ is contained in $\{(\alpha_1, \alpha_2) \in \mathbb{Z}_+^2;\ \alpha_1 - \tau\alpha_2 < -2\tau\}$, the same assertion holds for the support of $\mathcal{R}_1$.

Step 8. We will determine a_2 and b_2. We set $u_1 = b_0 + x^{\alpha_1}b_1$, and we determine $a_2(x_1, \varepsilon)$ and $b_2(x_2, \varepsilon)$ ($b_2(0, \varepsilon) \equiv 0$) such that

$$\mathcal{R}_2 := L_\Lambda^\varepsilon(x^{\alpha_2}(a_2 + b_2)) + T_2 + \mathcal{R}_1 = x^{\alpha_3}\tilde{\mathcal{R}}_2(x), \tag{3.41}$$

where $\tilde{\mathcal{R}}_2(x) = O(x_2^2)$ and

$$T_2 := -R(x + u_1 + x^{\alpha_2}(a_2 + b_2)) + R(x + u_1), \tag{3.42}$$

and $\mathcal{R}_1$ is given by (3.17) with $a_1 = 0$. In the following we do not indicate the dependence on ε explicitly if there is no risk of confusion. We will show that

$$T_2 = -x^{\alpha_2}b_2(x_2)\Theta_2 + x^{\alpha_2}x_1x_2\tilde{T}_2 \tag{3.43}$$

for some bounded holomorphic function $\tilde{T}_2$ in $S_\rho \times \mathcal{C}_\rho$. Indeed, by Taylor's formula and by similar calculations to those in the proof of (3.22) we can easily see that the term of order $O(x^{\alpha_2})$ in T_2 is given by $-x^{\alpha_2}(b_2(x_2) + a_2(0))\Theta_2$. Moreover,

$$T_2x^{-\alpha_2} + (b_2(x_2) + a_2(0))\Theta_2 = O(x_1x_2)$$

in view of the definition of the remainder term.

Next, let $\mathcal{R}_1 = x^{\alpha_2}(\tilde{T}_1(x) + \Omega(x))$ be given by (3.17). Because the term $L_\Lambda^\varepsilon(x^{\alpha_2}(a_2+b_2))$ cancels with the corresponding terms in $T_2+\mathcal{R}_1$ of order $O(x^{\alpha_2})$, we look for a decomposition

$$\mathcal{R}_1 = -x^{\alpha_2}(\gamma_2(x_1, \varepsilon) + \beta_2(x_2, \varepsilon)) + x^{\alpha_3}\Omega_1(x, \varepsilon) \tag{3.44}$$

for some $\gamma_2(x_1, \varepsilon)$ and $\beta_2(x_2, \varepsilon)$, $\beta_2 = O(x_2^2)$ holomorphic in S_1 and $S_{2,\rho}$, respectively, and Ω_1 holomorphic in $S_\rho \times \mathcal{C}_\rho$. In order to compute γ_2 and β_2 we restrict $\tilde{T}_1(x, \varepsilon) + \Omega(x, \varepsilon)$ to $x_2 = 0$ or $x_1 = 0$. By the definition of $\Omega(x, \varepsilon)$ in (3.19) and the assumption (2.19) we have $\Omega(x_1, 0, \varepsilon) \equiv 0$. Next, by (3.40) and $b_1(0) = 0$ we have $\tilde{T}_1(x_1, 0, \varepsilon) \equiv 0$. Hence $\gamma_2 = 0$. By defining

$$\beta_2(x_2, \varepsilon) = -\tilde{T}_1(0, x_2, \varepsilon) - \Omega(0, x_2, \varepsilon)$$

we will show (3.44). In view of (3.40) and (2.19) we see that $\tilde{T}_1(x, \varepsilon) - \tilde{T}_1(0, x_2, \varepsilon)$ is divisible by x^{α_1}. From $\alpha_2 + \alpha_1 \geq \alpha_3$ we see that $x^{\alpha_2}(\tilde{T}_1(x, \varepsilon) - \tilde{T}_1(0, x_2, \varepsilon))$ is divisible by x^{α_3}. On the other hand, by (3.19) and (2.19), $x^{\alpha_2}(\Omega(x, \varepsilon) - \Omega(0, x_2, \varepsilon))$ is divisible by x^{α_3}. We also note that $\beta_2(x_2, \varepsilon) = O(x_2^2)$.

Therefore we will determine a_2 and b_2 from the equations

$$(L_\Lambda^\varepsilon + 2\varepsilon - \varepsilon\tau k_2)a_2 = 0, \tag{3.45}$$

$$(L_\Lambda^\varepsilon + 2\varepsilon - \varepsilon\tau k_2)b_2 = (b_2 + a_2(0))\Theta_2 + \beta_2. \tag{3.46}$$

We easily see that $\tilde{a}_2 = 0$ and we can take $a_2 = 0$. Because (3.46) has formal power series solution $\tilde{b}_2$, we define $b_2^{(0)}$ by a formula similar to (3.27). Then $b_2^{(0)}$ has asymptotic expansion $\tilde{b}_2$. We note that the modulus of $b_2^{(0)}$ can be taken arbitrarily small in a neighborhood of the origin by taking δ in (3.27) sufficiently small. In order to solve (3.46) we construct the approximate sequence $w^{(\nu)}$ $(\nu \geq 1)$ from relations like (3.34) and (3.35). We can easily see that $b_2 := w^{(0)} + w^{(1)} + \cdots$ converges in $S_{2,\rho} \times \mathcal{C}_\rho$ and gives a holomorphic solution of (3.46) with asymptotic expansion $\tilde{b}_2$. We can show that

$$T_2 = -x^{\alpha_2} b_2 \Theta_2 + x^{\alpha_3} \tilde{T}_2 \tag{3.47}$$

for a possibly different holomorphic function $\tilde{T}_2 = \tilde{T}_2(x, \varepsilon)$ in $S_\rho \times \mathcal{C}_\rho$ such that $\tilde{T}_2 = O(x_2^2)$. We can also prove that the support of T_2 lies in $\{(\alpha_1, \alpha_2) \in \mathbb{Z}_+^2;\ \alpha_1 - \tau\alpha_2 < -2\tau\}$. Indeed, these facts follow from the support conditions on R and b_2 by applying to (3.42) Taylor's formula in integral form.

Step 9. We will determine a_n and b_n. Suppose that we have determined $a_j = 0$ and $b_j = O(x_2^2)$ as holomorphic and bounded functions on $S_{2,\rho} \times \mathcal{C}_\rho$ for all $j \leq n-1$ satisfying (3.7) up to n in such a way that the support of $\mathcal{R}_{n-1}$ is contained in $\{(\alpha_1, \alpha_2) \in \mathbb{Z}_+^2;\ \alpha_1 - \tau\alpha_2 < -2\tau\}$. We will determine $a_n(x_1, \varepsilon)$ (resp. $b_n(x_2, \varepsilon)$) such that

$$U_n := U_{n-1} + x^{\alpha_n}(a_n(x_1, \varepsilon) + b_n(x_2, \varepsilon))$$

satisfies (3.7) with n replaced by $n+1$. Let x_1 and x_2 be so small that $R(x + U_n)$ is well defined. First we consider

$$\begin{aligned} \mathcal{R}_n &:= L_\Lambda^\varepsilon U_n - R(x + U_n) = L_\Lambda^\varepsilon U_{n-1} - R(x + U_{n-1}) \\ &\quad + L_\Lambda^\varepsilon(x^{\alpha_n}(a_n + b_n)) + R(x + U_{n-1}) - R(x + U_n) \\ &= \mathcal{R}_{n-1} + L_\Lambda^\varepsilon(x^{\alpha_n}(a_n + b_n)) + T_n, \end{aligned} \tag{3.48}$$

where $T_n = R(x + U_{n-1}) - R(x + U_n)$.

We want to write

$$\mathcal{R}_{n-1} = x^{\alpha_n}\tilde{\mathcal{R}}_{n-1}(x,\varepsilon) = -x^{\alpha_n}(\gamma_n(x_1,\varepsilon) + \beta_n(x_2,\varepsilon)) + x^{\alpha_{n+1}}\Omega_n(x,\varepsilon). \tag{3.49}$$

Indeed, by an appropriate choice of β_n and γ_n we have $\tilde{\mathcal{R}}_{n-1}(x,\varepsilon) + \beta_n + \gamma_n = O(x_1x_2)$. By the support property of $\mathcal{R}_{n-1}$ we may define $\gamma_n = 0$. Moreover, by (2.19), we have $\beta_n = O(x_2^2)$. We will show that the $O(x_1x_2x^{\alpha_n})$ term in $\mathcal{R}_{n-1}$ is $O(x^{\alpha_{n+1}})$. This is clear when $\tau > 1$ because $k_{n+1} = k_n$ or $k_{n+1} = k_n + 1$. On the other hand, if $0 < \tau < 1$, then in view of the support property of $\mathcal{R}_{n-1}$ the $O(x_1x_2x^{\alpha_n})$ term in $\mathcal{R}_{n-1}$ is $O(x^{\alpha_{n+1}})$ and consequently $O(x_2^2x^{\alpha_{n+1}})$ by the same condition.

In order to obtain equations for a_n and b_n we note that

$$\begin{aligned} T_n &= -x^{\alpha_n}\int_0^1 (a_n+b_n)\nabla R(x+U_{n-1}+tx^{\alpha_n}(a_n+b_n))\,dt \\ &= x^{\alpha_n}(b_n + a_n(0,\varepsilon))\Theta_2 + O(x_1x_2x^{\alpha_n}). \end{aligned} \tag{3.50}$$

Therefore, by dividing (3.7) with n replaced by $n+1$ by x^{α_n} and by setting $x_2 = 0$ we obtain, in view of (3.23) and (3.48),

$$(L_\Lambda^\varepsilon + n\varepsilon - k_n\tau\varepsilon)a_n = 0. \tag{3.51}$$

As in the previous case, the formal solution $\tilde{a}_n$ of (3.51) vanishes and we may define $a_n = 0$. Next we consider the equation for b_n. We divide (3.7) with n replaced by $n+1$ by x^{α_n}. Then by setting $x_1 = 0$ we obtain

$$(L_\Lambda^\varepsilon + n\varepsilon - k_n\tau\varepsilon)b_n = b_n\Theta_2 + \beta_n(x_2,\varepsilon). \tag{3.52}$$

By the same argument as for b_1 we can determine b_n as a bounded holomorphic function on $S_{2,\rho}\times\mathcal{C}_\rho$ such that $b_n = O(x_2^2)$. Therefore we can determine the formal solution U in (3.4).

We can see from (3.48) and the inductive assumption for $\mathcal{R}_{n-1}$ that the support of $\mathcal{R}_n$ is contained in $\{(\alpha_1,\alpha_2)\in\mathbb{Z}_+^2;\ \alpha_1 - \tau\alpha_2 < -2\tau\}$, because the support of T_n is contained in the same set. In order to prove (3.7), note that $O(x_1x_2x^{\alpha_n})$ terms in (3.50) are, indeed, $O(x_2^2x^{\alpha_{n+1}})$, which can be shown by the support condition on R.

Step 10. We apply a Borel–Ritt type argument to the formal series (3.4). By definition we have $\alpha_n = (n,k_n)$, $-\tau < n - \tau k_n < 0$. Hence $\lim_{n\to\infty}\alpha_n/n = (1,\tau^{-1})$. By the definition of S_ρ we can show that there exists $N \geq 1$ such that for any $n \geq N$ we have $\operatorname{Re} x^{\alpha_n/n} > 0$ on S_ρ. Indeed, by setting $x_j = r_je^{i\theta_j}$ for $j = 1,2$ with $0 < r_1 < \infty$, $0 < r_2 \leq \rho$, $|\theta_j| \leq \eta_j$, we obtain

$$x^{\alpha_n/n} = r_1r_2^{k_n/n}\exp(i(\theta_1 + \theta_2k_n/n)).$$

By the assumption $\eta_1 + \eta_2/\tau < \pi/2$ and the relation $k_n/n \to \tau^{-1}$, we see that there exists $N > 0$ such that for $n \geq N$, we have $|\theta_1 + \theta_2 k_n/n| < \pi/2$. This shows the assertion.

Suppose $\delta > 0$. Then we define

$$\gamma_n := \max_{x,\varepsilon}\{\|x_2^{-1}b_n\| + 1, \|x_2^{-1}\varepsilon(-\tau x_2\partial_{x_2} + n - \tau k_n)b_n)\|\}, \tag{3.53}$$

and define $V(x,\varepsilon)$ on $S_\rho \times \mathcal{C}_\rho$ by

$$V(x,\varepsilon) = \sum_{n=0}^{\infty} b_n(x_2,\varepsilon)\varphi_n(x)^2 x^{\alpha_n}, \tag{3.54}$$

where $\varphi_n = 1$ for $0 \leq n < N$, and for $n \geq N$,

$$\varphi_n(x) = 1 - \exp\Big(-\frac{\delta^n}{\gamma_n x^{\alpha_n/n}(n-1)!}\Big). \tag{3.55}$$

In order to show that $V(x,\varepsilon)$ is holomorphic in $S_\rho \times \mathcal{C}_\rho$ we use a similar argument to that for (3.27). Let $\operatorname{Re} x^{\alpha_n/n} > 0$ on S_ρ. Then, for $n \geq N$,

$$\begin{aligned} |b_n|\,|\varphi_n|^2|x^{\alpha_n}| &\leq \|x_2^{-1}b_n\|\,|x_2x^{\alpha_n}|\Big(\frac{\delta^n}{\gamma_n|x^{\alpha_n/n}|(n-1)!}\Big)^2 \\ &\leq \delta^{2n}|x_2x^{\alpha_n(1-2/n)}|((n-1)!)^{-2}. \end{aligned} \tag{3.56}$$

Because $\alpha_n(1-2/n) = (n-2)(1, k_n/n)$, we see that the sum

$$\sum \delta^{2n}|x_2x^{\alpha_n(1-2/n)}|((n-1)!)^{-2}$$

converges on $S_\rho \times \mathcal{C}_\rho$. Hence the series (3.54) converges on $S_\rho \times \mathcal{C}_\rho$.

Step 11. We will show (3.3). Take any positive integer $n \geq N$ and write

$$\begin{aligned} V(x,\varepsilon) = &\sum_{j=0}^{n} x^{\alpha_j}b_j(x_2,\varepsilon) + \sum_{j=0}^{n} x^{\alpha_j}b_j(x_2,\varepsilon)(\varphi_j(x)^2 - 1) \\ &+ \sum_{j=n+1}^{\infty} x^{\alpha_j}b_j(x_2,\varepsilon)\varphi_j(x)^2 \equiv V_1 + V_2 + V_3. \end{aligned} \tag{3.57}$$

First, we show that $V_2 = O(x_2^2x^{\alpha_{n+1}})$ as $x \to 0$, $x \in S_\rho$. Indeed, for $j \geq N$ we have

$$\varphi_j(x) - 1 = -\exp\Big(-\frac{\delta^j}{\gamma_j x^{\alpha_j/j}(j-1)!}\Big).$$

For every $\nu \geq 1$ the right-hand side is $O(|x^{\alpha_j\nu/j}|)$ on S_ρ as $x \to 0$. Hence, by taking ν sufficiently large, it is divisible by $x^{\alpha_{n+1}}$ with the quotient bounded and holomorphic in S_ρ. Because $b_j = O(x_2^2)$, we have $V_2 = O(x_2^2 x^{\alpha_{n+1}})$. Next we will show that V_3 is divisible by $x^{\alpha_{n+1}}$ with the quotient bounded and holomorphic in $S_\rho \times \mathcal{C}_\rho$. Because $\alpha_j \geq \alpha_{n+1}$ for every $j \geq n+1$ and $b_j = O(x_2^2)$ it is sufficient to prove that $|x_2 x^{\alpha_j - \alpha_{n+1}}| < \rho^{j-n-1}$ on S_ρ for every $j > n+1$.

Indeed, by definition we have $j - \tau k_j < 0$ and $-\tau < n+1-\tau k_{n+1} < 0$. It follows that

$$k_j - k_{n+1} > \tau^{-1}(j-n-1) - 1.$$

Therefore, since $|x_2| < 1$ and $|x_1|\,|x_2|^{1/\tau} < \rho$ on S_ρ, we have

$$\begin{aligned}|x_2 x^{\alpha_j-\alpha_{n+1}}| &= |x_1|^{j-n-1}|x_2|^{k_j-k_{n+1}+1} \leq |x_1|^{j-n-1}|x_2|^{\tau^{-1}(j-n-1)} \\ &\leq (|x_1|\,|x_2|^{1/\tau})^{j-n-1} \leq \rho^{j-n-1}.\end{aligned}$$

Therefore we have (3.3).

Step 12. We will prove (3.2). We set $g = R(x+V) - L_\Lambda^\varepsilon V$, where $R(x+V)$ is well defined for sufficiently small $\delta > 0$ in view of the definition of V. We write V in the form (3.57) and for x sufficiently small we write

$$g = R(x+W) - L_\Lambda^\varepsilon W + R(x+W+V_3) - R(x+W) - L_\Lambda^\varepsilon V_3, \tag{3.58}$$

where $V = W + V_3$ and $W := V_1 + V_2$.

We want to show that $L_\Lambda^\varepsilon V_3 = x^{\alpha_{n+1}} A_1(x,\varepsilon)$ for some bounded holomorphic function $A_1 = O(x_2)$ on $S_\rho \times \mathcal{C}_\rho$. Indeed, if a derivation in L_Λ^ε is applied to $\varphi_j(x)^2$, then, by the same argument as for the convergence of V, we see that the resulting series is convergent and divisible by x_2^2. We also note that every term in the series has a factor x^{α_j} with $\alpha_j \geq \alpha_{n+1}$. If L_Λ^ε is applied to the term $x^{\alpha_j} b_j(x_2,\varepsilon)$ in $x^{\alpha_j} b_j(x_2,\varepsilon)\varphi_j(x)^2$, then we have

$$L_\Lambda^\varepsilon(x^{\alpha_j} b_j) = x^{\alpha_j}(\varepsilon(-\tau x_2 \partial_{x_2} + j - \tau k_j) - \Lambda) b_j(x_2). \tag{3.59}$$

In view of (3.53) and the proof of the convergence of $V(x,\varepsilon)$ the sum of terms on the right-hand side (3.59) converges and is bounded on $S_\rho \times \mathcal{C}_\rho$.

In view of the estimate of V_3, we can see that A_1 is divisible by x_2. It is also easy to see that if $0 < \theta' < 1$ satisfies $-(1-\theta')\tau < n+1-\tau k_{n+1} < 0$, then $|x_2|^{1-\theta'}|x^{\alpha_j-\alpha_{n+1}}| \leq \rho^{j-n-1}$. In fact, for every $0 < \theta' < 1$ there exist infinitely many k_n such that $-(1-\theta')\tau < n+1-\tau k_{n+1} < 0$. For those n's we have $A_1 = O(|x_2|^{1+\theta'})$.

Next, by Taylor's formula we have

$$R(x+W+V_3) - R(x+W) = \int_0^1 V_3 \cdot \nabla R(x+W+tV_3)\,dt.$$

It follows that $R(x+W+V_3)-R(x+W)=x^{\alpha_{n+1}}A_2(x,\varepsilon)$ for some bounded holomorphic function A_2 in $S_\rho\times\mathcal{C}_\rho$. In view of the estimate of V_3 and since $\nabla R(x+W+tV_3)=O(x_2)$ we see that $A_2=O(x_2^2)$.

We consider

$$R(x+W)-L_\Lambda^\varepsilon W=R(x+W)-L_\Lambda^\varepsilon V_1-L_\Lambda^\varepsilon V_2.$$

It is easy to see that $L_\Lambda^\varepsilon V_2=x^{\alpha_{n+1}}A_3(x,\varepsilon)$ for some bounded holomorphic function A_3 in $S_\rho\times\mathcal{C}_\rho$ such that $A_3=O(x_2^2)$. Indeed, the functions $\varphi_j(x)^2-1$ in V_2 and $L_\Lambda^\varepsilon(\varphi_j(x)^2-1)$ can be divisible by an arbitrary power of $x^{\alpha_j/j}=x_1x_2^{d_j/j}$ such that the quotient is holomorphic and bounded in $S_\rho\times\mathcal{C}_\rho$. Because $d_j/j>\tau^{-1}$, we see that it is $O(x_2^2x^{\alpha_{n+1}})$.

We take $\rho'\le\rho$ so small that for every x with $|x_1|\,|x_2|^{1/\tau}<\rho'$ and $|x_2|<\rho$ the values $x+V_1$, $x+V_1+V_2$ are in the domain of R. Then $R(x+V_1+V_2)-R(x+V_1)=\int_0^1 V_2\cdot\nabla R(x+V_1+tV_2)\,dt$. Clearly, the right-hand side function can be written as $x^{\alpha_{n+1}}A_4(x,\varepsilon)$ for some bounded holomorphic function A_4 in $S_\rho\times\mathcal{C}_\rho$ with $|x_1|\,|x_2|^{1/\tau}<\rho'$ such that $A_4=O(x_2^2)$. Now we have

$$R(x+W)-L_\Lambda^\varepsilon W=R(x+V_1+V_2)-R(x+V_1)+R(x+V_1)-L_\Lambda^\varepsilon V_1-L_\Lambda^\varepsilon V_2.$$

By the definition of V_1 we see that $R(x+V_1)-L_\Lambda^\varepsilon V_1=x^{\alpha_{n+1}}A_5(x,\varepsilon)$ for some bounded holomorphic function A_5 in $x\in S_\rho$, $|x_1|\,|x_2|^{1/\tau}<\rho'$ such that $A_5=O(x_2^2)$. It follows that $F(x):=x^{-\alpha_{n+1}}(R(x+W)-L_\Lambda^\varepsilon W)$ is holomorphic and bounded in S_ρ such that $|x_1|\,|x_2|^{1/\tau}<\rho'$. Because $R(x+W)-L_\Lambda^\varepsilon W$ is holomorphic in S_ρ and $x^{\alpha_{n+1}}$ does not vanish in $\rho'\le|x_1|\,|x_2|^{1/\tau}\le\rho$, we see that $F(x)$ is also holomorphic in S_ρ. In order to prove the boundedness of $F(x)$ in S_ρ, we will show the boundedness of $F(x)$ when $\rho'\le|x_1|\,|x_2|^{1/\tau}\le\rho$. We may assume, without loss of generality, that $0<|x_2|<1$. We note

$$|x^{\alpha_{n+1}}|=(|x_1|\,|x_2|^{1/\tau}|x_2|^{\frac{k_{n+1}}{n+1}-\frac{1}{\tau}})^{n+1}\ge(\rho')^{n+1}|x_2|^{(\frac{k_{n+1}}{n+1}-\frac{1}{\tau})(n+1)}.$$

Because

$$\frac{k_{n+1}}{n+1}-\frac{1}{\tau}<\frac{1}{n+1}$$

it follows that

$$|x_2|^{(\frac{k_{n+1}}{n+1}-\frac{1}{\tau})(n+1)}>|x_2|.$$

On the other hand, $R(x+W)-L_\Lambda^\varepsilon W=O(x_2)$. This proves that $F(x)$ is bounded when $\rho'\le|x_1|\,|x_2|^{1/\tau}\le\rho$. Because n is arbitrary, we have proved (3.2). This completes the proof of the lemma.

§4. Proof of Theorem 2

We prove Theorem 2 in case (2.19) is satisfied. For (2.20) we can argue similarly by changing the roles of x_1 and x_2. Let V be given by Lemma 3. Let $\alpha_N = (N, k_N)$ be such that $x_2^{-1-\theta'}\tilde{g}_N(x,\varepsilon)$ is holomorphic and bounded in $S_\rho \times \mathcal{C}_{\pm,\theta,\rho}$ as in Lemma 3. In order to solve (2.4), set $u(x) = v(x) + V(x)$ and consider

$$(4.1)\qquad L_\Lambda^\varepsilon v = R(x+V+v) - L_\Lambda^\varepsilon V = R(x+V+v) - R(x+V) + g,$$

where $g := R(x+V) - L_\Lambda^\varepsilon V$.

Let $\rho > 0$ and $N \geq 1$ be an integer. For a bounded holomorphic (vector) function

$$h = (h_1, h_2) = x^{\alpha_N}\tilde{h}(x,\varepsilon) = x^{\alpha_N}(\tilde{h}_1, \tilde{h}_2)$$

in $S_\rho \times \mathcal{C}_\rho$ with $\tilde{h}(x,\varepsilon)$ holomorphic and bounded in $S_\rho \times \mathcal{C}_\rho$, we define the norm of h by

$$(4.2)\qquad \|h\|_N := \sup_{x\in S_\rho,\,\varepsilon\in\mathcal{C}_\rho} (|x^{-\alpha_N}h_1(x,\varepsilon)| + |x^{-\alpha_N}x_2^{-1-\theta'/2}h_2(x,\varepsilon)|).$$

Let X_N be the set of functions h holomorphic and bounded in $S_\rho \times \mathcal{C}_\rho$ such that $\|h\|_N < \infty$. Clearly, X_N is the Banach space with the norm (4.2). We choose a sequence $\alpha_N = (N, k_N)$, $N = N_\nu$ $(\nu = 1, 2, \ldots)$, such that for every pair α_N and α_ℓ in the sequence with $N > \ell$ we have

$$d_N - \frac{N}{\tau} \geq d_\ell - \frac{\ell}{\tau}.$$

Because $q - p/\tau$ is dense on $\mathbb{R}$ if p and q run in $\mathbb{Z}$, we can choose $\{\alpha_N\}$ satisfying the condition. We shall show that X_N is continuously embedded into X_ℓ. Indeed, for every $h = x^{\alpha_N}\tilde{h}_N \in X_N$ we have

$$x^{\alpha_N}\tilde{h}_N = x^{\alpha_\ell}x^{\alpha_N-\alpha_\ell}\tilde{h}_N = x^{\alpha_\ell}(x_1x_2^{1/\tau})^{N-\ell}x_2^{d_N-d_\ell-(N-\ell)/\tau}\tilde{h}_N.$$

Because $d_N - d_\ell - (N-\ell)/\tau > 0$ by assumption, we see that there exists $C > 0$ such that $\|h\|_\ell \leq C\|h\|_N$. This proves the assertion.

For $\|h\|_N < \infty$ we define

$$(4.3)\qquad v := -\frac{1}{\varepsilon}\int_0^\infty e^{-\Lambda t/\varepsilon}h(e^{t\Lambda}x,\varepsilon)\,dt.$$

Because $|x_1^N e^{Nt}x_2^{k_N}e^{-k_N\tau t}| = |x_1^N x_2^{k_N}e^{t(N-\tau k_N)}| \leq |x_1^N x_2^{k_N}|$ for all $t \geq 0$, we see that $h(e^{t\Lambda}x,\varepsilon)$ in the integrand is bounded if $x \in S_\rho$, $\varepsilon \in \mathcal{C}_\rho$, $t \geq 0$. In order to show that the integral (4.3) converges we may consider the second component. In the integrand the following factor appears:

$$e^{t\tau/\varepsilon}e^{-(1+\theta'/2)\tau t}, \quad t \geq 0.$$

Therefore, if ε is sufficiently close to 1, then the integral converges. We easily see that v is the solution of the equation $L_\Lambda^\varepsilon v = h$, namely $v = (L_\Lambda^\varepsilon)^{-1}h$, where $(L_\Lambda^\varepsilon)^{-1}$ has the expression (4.3). Moreover, $\|v\|_N < \infty$.

We want to define an approximate sequence $\{v^{(k)}\}$ by

$$
\begin{aligned}
(4.4)\qquad & v^{(0)} := (L_\Lambda^\varepsilon)^{-1}g, \qquad v^{(1)} := (L_\Lambda^\varepsilon)^{-1}(R(x+V+v^{(0)}) - R(x+V)),\\
& v^{(k)} := (L_\Lambda^\varepsilon)^{-1}(R(x+V+v^{(0)}+\cdots+v^{(k-1)})\\
& \qquad\qquad - R(x+V+v^{(0)}+\cdots+v^{(k-2)})), \qquad k = 2,3,\ldots.
\end{aligned}
$$

It is easy to see that if $v := \sum_{k=0}^\infty v^{(k)}$ converges, then v solves (4.1). In order to see that $v^{(k)}$'s are well defined, we note, from the definition of V in Lemma 3 and (2.19), that $g(x,\varepsilon) = x^{\alpha_N}\tilde g(x,\varepsilon)$ for some bounded holomorphic function $\tilde g$ in $S_\rho \times \mathcal{C}_\rho$ such that $\tilde g = O(x_2^{1+\theta'})$. In particular $\|g\|_N < \infty$. Hence $v^{(0)} \in X_N$. In order to estimate $v^{(0)}$ we obtain, in view of (4.3) and (4.4),

$$
\begin{aligned}
(4.5)\qquad \|v^{(0)}\|_N \le \sup \frac{1}{|\varepsilon|}\int_0^\infty & e^{t(N-\tau k_N)}\big(|e^{-t/\varepsilon}\tilde g_1(e^{t\Lambda}x,\varepsilon)|\\
& + |e^{t\tau/\varepsilon - \tau t - \theta'\tau t/2}(x_2^{-1-\theta'/2}\tilde g_2)(e^{t\Lambda}x,\varepsilon)|\big)\,dt \le C\rho^{\theta'/2}\|g\|_N
\end{aligned}
$$

for some constant $C > 0$ independent of N since $N - \tau k_N < 0$ and $|x_2| < \rho$. Indeed, there appears $(x_2 e^{-t\tau})^{\theta'/2}$ from $\tilde g_1(e^{t\Lambda}x,\varepsilon)$ and $(x_2^{-1-\theta'/2}\tilde g_2)(e^{t\Lambda}x,\varepsilon)$.

By (4.5) the function $R(x+V+v^{(0)}) - R(x+V)$ is well defined if $\delta > 0$ and $\rho > 0$ are sufficiently small, and it is divisible by x_2^2. Hence v_1 is well defined. Moreover

$$
\begin{aligned}
(4.6)\qquad \|v_1\|_N &\le C\Big\|\int_0^1 v^{(0)}\cdot\nabla R(\cdot + V + tv^{(0)})\,dt\Big\|_N\\
&\le C\int_0^1 \|v^{(0)}\|_N |||R|||\,dt = C\|v^{(0)}\|_N|||R|||,
\end{aligned}
$$

where $|||R||| = \sup_x \|\nabla R(x)\|$. Take $\rho > 0$ so that $C|||R||| < 1/2$. Then $\|v^{(1)}\|_N \le \|v^{(0)}\|_N 2^{-1}$. Hence $v^{(2)}$ is well defined and it has the same property as $v^{(1)}$ if $v^{(0)}$ is sufficiently small. Moreover $\|v^{(2)}\|_N \le \|v^{(0)}\|_N 2^{-2}$. In the same way, we can determine $v^{(k)}$ as bounded holomorphic functions in $S_\rho \times \mathcal{C}_\rho$ such that $\|v^{(k)}\|_N \le \|v^{(0)}\|_N 2^{-k}$ $(k = 1,2,\ldots)$. This proves that the limit $v := \sum_{k=0}^\infty v^{(k)}$ exists in $S_\rho \times \mathcal{C}_\rho$ in the $\|\cdot\|_N$-norm. By the definition of the norm we have $v(x) = O(x^{\alpha_N})$ as $x \to 0$. The limit function v is independent of N because X_N is continuously embedded into X_ℓ for every $N > \ell$. Because there exist infinitely many N, this proves Theorem 2.

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