

Algebraic Local Cohomology Classes Attached to Unimodal Singularities

by

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Abstract

Algebraic local cohomology classes and holonomic systems attached to non-quasihomogeneous isolated unimodal singularities are considered in the context of algebraic analysis. Holonomic systems and their algebraic local cohomology solution spaces attached to a unimodal singularity are studied in a constructive manner. The holonomic system constructed from linear partial differential operators of order at most two that annihilate the given algebraic local cohomology is proven to be simple for the case of non-quasihomogeneous unimodal singularities.

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Introduction

Let X be a neighborhood of the origin O in \mathbb{C}^n . Let $\mathcal{H}_{[O]}^n(\Omega_X^n)$ be the algebraic local cohomology group supported at the origin, where Ω_X^n is the sheaf on X of holomorphic differential n -forms. Let f be a holomorphic function on X with an isolated singularity at the origin. Denote by \mathcal{W}_f the set of algebraic local cohomology classes in $\mathcal{H}_{[O]}^n(\Omega_X^n)$ that are annihilated by the Jacobi ideal $\mathcal{I} \subset \mathcal{O}_{X,O}$ of f , and let ω be a generator of \mathcal{W}_f over $\mathcal{O}_{X,O}$, where $\mathcal{O}_{X,O}$ is the stalk at the origin of the sheaf \mathcal{O}_X of holomorphic functions on X .

Inspired by a result obtained by K. Saito [7], we previously studied algebraic local cohomology classes attached to quasihomogeneous hypersurface isolated singularities in the category of left \mathcal{D}_X -modules [8]. We showed in particular that

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the quasihomogeneity of hypersurface isolated singularities can be characterized in terms of the holonomic \mathcal{D}_X -modules. To be more precise, let $\text{Ann}_{\mathcal{D}_{X,O}}(\omega)$ be the annihilator ideal of $\omega \in \mathcal{W}_f$ in the stalk $\mathcal{D}_{X,O}$ at the origin of the sheaf \mathcal{D}_X of rings of linear partial differential operators. Let $\text{Ann}_{\mathcal{D}_{X,O}}^{(1)}(\omega)$ denote the right ideal in $\mathcal{D}_{X,O}$ generated by differential operators of order at most 1 that annihilate ω . Then the result can be stated in terms of right $\mathcal{D}_{X,O}$ -modules as follows.

Theorem 0.1 (cf. [8]). *Let X be an open neighborhood of the origin in \mathbb{C}^n . Let*

$$f = f(x_1, \dots, x_n) \in \mathcal{O}_{X,O}$$

be a holomorphic function with an isolated singularity at the origin and denote by ω a generator of \mathcal{W}_f over $\mathcal{O}_{X,O}$. The following conditions are equivalent:

- (i) $\mathcal{O}_{X,O}(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}) = \mathcal{O}_{X,O}(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$.
- (ii) $\text{Ann}_{\mathcal{D}_{X,O}}(\omega)$ is generated by differential operators of order at most 1, i.e., $\text{Ann}_{\mathcal{D}_{X,O}}(\omega) = \text{Ann}_{\mathcal{D}_{X,O}}^{(1)}(\omega)$.
- (iii) The holonomic $\mathcal{D}_{X,O}$ -module $\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(1)}(\omega)$ is simple.

Indeed, the above theorem can be proved, as done previously in [8], by showing in the category of right \mathcal{D}_X -modules that if a given hypersurface isolated singularity is non-quasihomogeneous, then the dimension of the algebraic local cohomology solution space

$$\text{Hom}_{\mathcal{O}_X}(\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(1)}(\omega), \mathcal{H}_{[O]}^n(\Omega_X^n))$$

to the holonomic system $\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(1)}(\omega)$ is greater than or equal to 2. However, the structure of the holonomic system $\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(1)}(\omega)$ and its algebraic local cohomology solution space associated with the non-quasihomogeneous isolated singularity have not yet been determined, even for the most typical case.

In this paper, we study the case of non-quasihomogeneous hypersurface isolated singularities. We consider the right ideal $\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega)$, $k = 0, 1, 2, \dots$, generated by differential operators of order at most k and holonomic systems $\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega)$ attached to non-quasihomogeneous hypersurface isolated singularities. We investigate in particular unimodal singularities, the most typical case among hypersurface singularities, that comprise hyperbolic singularities and 14 exceptional families [1]. By extending the algorithms introduced in earlier work [9], [10] to the parametric case for computing algebraic local cohomology and annihilators, we explicitly determine the algebraic local cohomology solution space

$$\text{Hom}_{\mathcal{D}_{X,O}}(\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega), \mathcal{H}_{[O]}^n(\Omega_X^n))$$

for the non-quasihomogeneous unimodal singularity case. We show as an application that the multiplicity of the holonomic system $\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(1)}(\omega)$ is equal to 2 and that of $\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(2)}(\omega)$ is equal to 1.

In Section 1, after introducing some basic notions, we associate a series of holonomic systems $\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega)$, $k = 0, 1, \dots$, to hypersurface isolated singularities. We show that the multiplicities of the holonomic systems are analytical invariants of the hypersurface isolated singularities. In Section 2, we study basic properties of algebraic local cohomology solution spaces for $\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega)$ and state the main results. In Section 3, we consider hyperbolic singularities and prove the main result by explicitly providing \mathcal{W}_f and annihilators. In Section 4, we study exceptional singularities by adopting a constructive approach. Instead of presenting the full set of data on \mathcal{W}_f , a generator ω and annihilators for each case, we examine the Z_{13} singularity and give the results of computations to illustrate the proof.

§1. Holonomic systems attached to hypersurface isolated singularities

Let X be an open neighborhood of the origin O in \mathbb{C}^n . Denote by $\mathcal{D}_{X,O}$ the stalk at O of the sheaf \mathcal{D}_X of holomorphic linear partial differential operators on X . Let $\mathcal{H}_{[O]}^n(\Omega_X^n)$ be the algebraic local cohomology group supported at O , where Ω_X^n is the sheaf of holomorphic differential n -forms on X . Note that, since Ω_X^n has the structure of the right $\mathcal{D}_{X,O}$ -module, differential operators in $\mathcal{D}_{X,O}$ act on algebraic local cohomology classes in $\mathcal{H}_{[O]}^n(\Omega_X^n)$ from the right.

Definition 1.1. For an algebraic local cohomology class $\zeta \in \mathcal{H}_{[O]}^n(\Omega_X^n)$, let $\mathcal{L}_{\mathcal{D}_{X,O}}^{(k)}(\zeta)$ be the set of germs of linear partial differential operators of order at most k that annihilate ζ . We define $\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\zeta)$ to be the right ideal in $\mathcal{D}_{X,O}$ generated by $\mathcal{L}_{\mathcal{D}_{X,O}}^{(k)}(\zeta)$:

$$\mathcal{L}_{\mathcal{D}_{X,O}}^{(k)}(\zeta) = \{P \in \mathcal{D}_{X,O} \mid \zeta P = 0, \text{ord } P \leq k\}, \quad \text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\zeta) = \mathcal{L}_{\mathcal{D}_{X,O}}^{(k)}(\zeta)\mathcal{D}_{X,O}.$$

Denote by $\mathcal{O}_{X,O}$ the stalk at O of the sheaf \mathcal{O}_X of holomorphic functions on X . Let $f = f(x_1, \dots, x_n)$ be a holomorphic function on X , defining an isolated singularity at the origin, and \mathcal{I} the Jacobi ideal of f in $\mathcal{O}_{X,O}$:

$$\mathcal{I} = \mathcal{O}_{X,O} \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right).$$

Let \mathcal{W}_f be the set of algebraic local cohomology classes annihilated by the Jacobi ideal \mathcal{I} :

$$\mathcal{W}_f = \{\eta \in \mathcal{H}_{[O]}^n(\Omega_X^n) \mid g\eta = 0 \text{ for all } g \in \mathcal{I}\}.$$

Then \mathcal{W}_f is a μ -dimensional vector space, where μ is the Milnor number $\dim_{\mathbb{C}} \mathcal{O}_{X,O}/\mathcal{I}$.

\mathcal{W}_f is generated, over $\mathcal{O}_{X,O}$, by one algebraic local cohomology class. Though the choice of a generator of \mathcal{W}_f over $\mathcal{O}_{X,O}$ is not unique, we have the following.

Lemma 1.1. *Let ω and ω' be two generators of \mathcal{W}_f over $\mathcal{O}_{X,O}$, that is,*

$$\mathcal{W}_f = \mathcal{O}_{X,O}\omega = \mathcal{O}_{X,O}\omega'.$$

Then the $\mathcal{D}_{X,O}$ -modules $\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega)$ and $\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega')$ are isomorphic.

Proof. There is a holomorphic function $g = g(x) \in \mathcal{O}_{X,O}$ with $g(0) \neq 0$ such that $\omega = g\omega'$. Let $\phi : \mathcal{D}_{X,O} \rightarrow \mathcal{D}_{X,O}$ be a $\mathcal{D}_{X,O}$ -linear homomorphism defined by $\phi(R) = gR$ for $R \in \mathcal{D}_{X,O}$. Then ϕ induces an isomorphism from $\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega)$ to $\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega')$. \square

It is known that the right $\mathcal{D}_{X,O}$ -module $\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\zeta)$ is a holonomic system with support at the origin, for any $\zeta \in \mathcal{H}_{[O]}^n(\Omega_X^n)$. The next statement immediately follows from Lemma 1.1.

Lemma 1.2. *Let ω and ω' be two generators of \mathcal{W}_f over $\mathcal{O}_{X,O}$. Then the multiplicity of the holonomic system $\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega)$ is equal to the multiplicity of $\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega')$.*

That is, the multiplicity of the holonomic system $\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega)$ is independent of the choice of a generator of \mathcal{W}_f . We thus arrive at the next definition.

Definition 1.2. We denote by $\mu_f^{(k)}$ the multiplicity of the holonomic system $\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega)$, where ω is a generator of \mathcal{W}_f over $\mathcal{O}_{X,O}$.

Note that if we denote the conormal space of the origin $O \in X$ by $T_{\{O\}}^*X$, then the characteristic cycle of the holonomic system $\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega)$ is given by $\mu_f^{(k)} T_{\{O\}}^*X$.

Let $\tilde{f}(x,t)$ be a function on $\tilde{X} = X \times T$ of the form $\tilde{f}(x,t) = f(x) + t_1^2 + \cdots + t_\ell^2$ with $t = (t_1, \dots, t_\ell) \in T$, where T is an open neighborhood of the origin in \mathbb{C}^ℓ . Let $\mathcal{W}_{\tilde{f}} \subset \mathcal{H}_{[O]}^{n+\ell}(\Omega_{\tilde{X}}^{n+\ell})$ be the set of algebraic local cohomology classes with support at the origin O of \tilde{X} annihilated by the Jacobi ideal $\mathcal{O}_{\tilde{X},O}(\partial f/\partial x_1, \dots, \partial f/\partial x_n, t_1, \dots, t_\ell)$ of $\tilde{f}(x,t)$.

Lemma 1.3. *Let $\tilde{\omega}$ be a generator of $\mathcal{W}_{\tilde{f}}$ over $\mathcal{O}_{\tilde{X},O}$. Then there exists an algebraic local cohomology class $\omega \in \mathcal{W}_f$ such that $\tilde{\omega} = \omega \otimes \delta_T$, where δ_T is the delta function with support at the origin of T . Furthermore,*

$$\text{Ann}_{\mathcal{D}_{\tilde{X},O}}^{(k)}(\tilde{\omega}) = \text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega)\mathcal{D}_{\tilde{X},O} + (t_1, \dots, t_\ell)\mathcal{D}_{\tilde{X},O}.$$

Theorem 1.1. *Let $\tilde{f} = \tilde{f}(x, t)$ be a holomorphic function defined on $\tilde{X} = X \times T$. Assume that \tilde{f} is stably equivalent to f . Then $\mu_{\tilde{f}}^{(k)} = \mu_f^{(k)}$ for every $k \in \mathbb{N}$, and $\nu_{\tilde{f}} = \nu_f$.*

Proof. If two holomorphic functions defined on \tilde{X} are right equivalent, then the multiplicities of the corresponding holonomic systems defined by k th-order annihilators are equal. Therefore, to prove the theorem, it suffices to consider \tilde{f} of the form $\tilde{f}(x, t) = f(x) + t_1^2 + \dots + t_\ell^2$. Let $\tilde{\omega}$ be a generator of $\mathcal{W}_{\tilde{f}}$ over $\mathcal{O}_{\tilde{X},O}$. Then $\tilde{\omega}$ can be written in the form $\tilde{\omega} = \omega \otimes \delta_T$ with some generator $\omega \in \mathcal{H}_{[O]}^n(\Omega_X^n)$ of \mathcal{W}_f . Lemma 1.3 also implies that

$$\mathcal{D}_{\tilde{X},O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\tilde{\omega}) \cong \mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega) \otimes_{\mathcal{D}_{\tilde{X},O}} \mathcal{D}_{T,O}/(t_1, \dots, t_\ell)\mathcal{D}_{T,O},$$

which yields the equality $\mu_{\tilde{f}}^{(k)} = \mu_f^{(k)}$. \square

By Theorem 1.1, $\mu_f^{(k)}$, $k = 1, 2, \dots$, are analytical invariants of a given singularity defined by the function f .

Since $\{\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega)\}_k$ is an increasing sequence of right ideals, $\{\mu_f^{(k)}\}_k$ becomes a decreasing sequence of positive integers:

$$\mu_f^{(0)} \geq \mu_f^{(1)} \geq \dots \geq \mu_f^{(k)} \geq \dots \geq 1.$$

There exists an integer m such that $\text{Ann}_{\mathcal{D}_{X,O}}^{(m)}(\omega)$ is equal to the right ideal $\text{Ann}_{\mathcal{D}_{X,O}}(\omega)$ of $\mathcal{D}_{X,O}$ consisting of linear partial differential operators that annihilate ω . Since the holonomic system $\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}(\omega)$ is simple, we have $\mu_f^{(m)} = 1$.

Example. Since $\text{Ann}_{\mathcal{D}_{X,O}}^{(0)}(\omega) = \mathcal{I}\mathcal{D}_{X,O}$, $\mu_f^{(0)}$ is the Milnor number $\dim_{\mathbb{C}} \mathcal{O}_{X,O}/\mathcal{I}$ at the origin of the isolated hypersurface singularity f .

Example. As seen in Theorem 0.1, $\mu_f^{(k)} = 1$ for $k \geq 1$ in quasihomogeneous cases and $\mu_f^{(1)} \geq 2$ in non-quasihomogeneous cases.

Let B be the μ -dimensional vector space spanned by a monomial basis of the quotient space $\mathcal{O}_{X,O}/\mathcal{I}$. If we set

$$L_{\mathcal{D}_X}^{(1)}(\omega) = \left\{ P = \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i} + a_0(x) \mid \omega P = 0, a_i(x) \in B, i = 0, 1, \dots, n \right\},$$

then

$$\text{Ann}_{\mathcal{D}_{X,O}}^{(1)}(\omega) = L_{\mathcal{D}_X}^{(1)}(\omega)\mathcal{D}_{X,O} + \mathcal{I}\mathcal{D}_{X,O}.$$

If we set

$$L_{\mathcal{D}_X}^{(2)}(\omega) = \left\{ P = \sum_{i,j=1}^n a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum a_i(x) \frac{\partial}{\partial x_i} + a_0(x) \mid \right. \\ \left. \omega P = 0, a_{i,j}(x) \in B, a_i(x) \in B, i, j = 1, \dots, n, a_0(x) \in B \right\},$$

then also

$$\text{Ann}_{\mathcal{D}_{X,O}}^{(2)}(\omega) = L_{\mathcal{D}_X}^{(2)}(\omega)\mathcal{D}_{X,O} + L_{\mathcal{D}_X}^{(1)}(\omega)\mathcal{D}_{X,O} + \mathcal{I}\mathcal{D}_{X,O}$$

(see [4]). Accordingly, generators of $\text{Ann}_{\mathcal{D}_{X,O}}^{(1)}(\omega)$ and of $\text{Ann}_{\mathcal{D}_{X,O}}^{(2)}(\omega)$ can be constructed by utilizing bases of the finite-dimensional vector spaces $L_{\mathcal{D}_X}^{(1)}(\omega)$ and $L_{\mathcal{D}_X}^{(2)}(\omega)$. Based on this observation, we have recently derived an efficient algorithm for computing the right ideal $\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega)$ generated by annihilating differential operators of order at most k (see [9]). We also implemented the resulting algorithm in the computer algebra system Risa/Asir [6].

§2. Main result

Let $\text{Sol}_{f,\omega}^{(k)}$ denote the algebraic local cohomology solution space of the system of partial differential equations defined by the ideal $\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega)$:

$$\text{Sol}_{f,\omega}^{(k)} = \{ \zeta \in \mathcal{H}_{[O]}^n(\Omega_X^n) \mid \zeta P = 0, \forall P \in \text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega) \}.$$

Proposition 2.1. *Let f be a holomorphic function with a hypersurface isolated singularity at the origin. Let ω be a generator of \mathcal{W}_f . Then*

$$\mu_f^{(k)} = \dim_{\mathbb{C}} \text{Sol}_{f,\omega}^{(k)}.$$

Proof. Since the holonomic $\mathcal{D}_{X,O}$ -module $\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega)$ is supported at the origin, we have $\text{Ext}_{\mathcal{D}_{X,O}}^j(\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega), \mathcal{H}_{[O]}^n(\Omega_X^n)) = 0$ for all $j \geq 1$, which implies

$$\mu_f^{(k)} = \dim_{\mathbb{C}} \text{Hom}_{\mathcal{D}_{X,O}}(\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega), \mathcal{H}_{[O]}^n(\Omega_X^n)).$$

The isomorphism

$$\text{Hom}_{\mathcal{D}_{X,O}}(\mathcal{D}_{X,O}/\text{Ann}_{\mathcal{D}_{X,O}}^{(k)}(\omega), \mathcal{H}_{[O]}^n(\Omega_X^n)) \cong \text{Sol}_{f,\omega}^{(k)}$$

yields the result. □

Lemma 2.1. *Let f be a holomorphic function with an isolated singularity at the origin. Let ω be a generator of \mathcal{W}_f . Then*

- (i) $Sol_{f,\omega}^{(0)} = \mathcal{W}_f$.
- (ii) $Sol_{f,\omega}^{(k)} \subset Sol_{f,\omega}^{(k-1)}$ for any $k \geq 1$.

Proof. By the definition of $Ann_{\mathcal{D}_{X,O}}^{(k)}(\omega)$, we have

$$Sol_{f,\omega}^{(k)} = \{\zeta \in \mathcal{H}_{[O]}^n(\Omega_X^n) \mid \zeta P = 0, \forall P \in \mathcal{L}_{\mathcal{D}_{X,O}}^{(k)}(\omega)\}.$$

- (i) Since $\mathcal{L}_{\mathcal{D}_{X,O}}^{(0)}(\omega) = \mathcal{I} = \mathcal{O}_{X,O}(\partial f/\partial x_1, \dots, \partial f/\partial x_n)$, we have

$$Sol_{f,\omega}^{(0)} = \left\{ \zeta \in \mathcal{H}_{[O]}^n(\Omega_X^n) \mid \frac{\partial f}{\partial x_1} \zeta = \dots = \frac{\partial f}{\partial x_n} \zeta = 0 \right\},$$

which is equal to \mathcal{W}_f .

- (ii) From $Ann_{\mathcal{D}_{X,O}}^{(k-1)}(\omega) \subset Ann_{\mathcal{D}_{X,O}}^{(k)}(\omega)$, we have $Sol_{f,\omega}^{(k)} \subset Sol_{f,\omega}^{(k-1)}$. \square

The lemma above immediately yields the following description:

$$Sol_{f,\omega}^{(k)} = \{\zeta \in Sol_{f,\omega}^{(k-1)} \mid \zeta P = 0, \forall P \in \mathcal{L}_{\mathcal{D}_{X,O}}^{(k)}(\omega)\}$$

and

$$(2.1) \quad Sol_{f,\omega}^{(k)} = \{\zeta \in \mathcal{W}_f \mid \zeta P = 0, \forall P \in \mathcal{L}_{\mathcal{D}_{X,O}}^{(k)}(\omega)\}.$$

Since $\omega \in Sol_{f,\omega}^{(k)}$ for any k , we have the following.

Proposition 2.2. *The following conditions are equivalent:*

- (i) $\mu_f^{(k)} = 1$.
- (ii) $Ann_{\mathcal{D}_{X,O}}^{(k)}(\omega) = Ann_{\mathcal{D}_{X,O}}(\omega)$.
- (iii) $Sol_{f,\omega}^{(k)} = \text{Span}_{\mathbb{C}}\{\omega\}$.

Here, we study holonomic systems and their algebraic local cohomology solutions attached to non-quasihomogeneous unimodal isolated singularities: hyperbolic singularities and exceptional families. Our main results are the following theorems.

Theorem 2.1. *Let f be a holomorphic function with a non-quasihomogeneous unimodal isolated singularity at the origin O .*

- (i) $Sol_{f,\omega}^{(1)} = \text{Span}_{\mathbb{C}}\{\omega, \delta\}$, where δ is the delta function in $\mathcal{H}_{[O]}^n(\Omega_X^n)$.
- (ii) $Sol_{f,\omega}^{(2)} = \text{Span}_{\mathbb{C}}\{\omega\}$.

Remark. It has been shown that $Sol_{f,\omega}^{(1)} \supseteq \text{Span}_{\mathbb{C}}\{\omega, f\omega\}$ for any non-quasihomogeneous hypersurface isolated singularity [8]. Since the ideal quotient $\mathcal{I} : f$ for the non-quasihomogeneous unimodal function f is the maximal ideal in $\mathcal{O}_{X,O}$, $f\omega$ coincides with a constant multiple of the delta function δ . Therefore, the inclusion $Sol_{f,\omega}^{(1)} \supseteq \text{Span}_{\mathbb{C}}\{\omega, \delta\}$ was implicitly given in [8].

Theorem 2.2. *Let f be a holomorphic function with a non-quasihomogeneous unimodal isolated singularity at the origin O .*

- (i) *The multiplicity of the holonomic system $\mathcal{D}_{X,O}/Ann_{\mathcal{D}_{X,O}}^{(1)}(\omega)$ is equal to 2.*
- (ii) *The multiplicity of the holonomic system $\mathcal{D}_{X,O}/Ann_{\mathcal{D}_{X,O}}^{(2)}(\omega)$ is equal to 1.*

Theorem 2.2 is a consequence of Theorem 2.1 in combination with Proposition 2.1. By Proposition 2.2, the second statement of Theorem 2.2 can be rephrased as follows.

Corollary 2.1. *Let $f \in \mathcal{O}_{X,O}$ be a holomorphic function that defines a non-quasihomogeneous unimodal isolated singularity at the origin. Then*

$$Ann_{\mathcal{D}_{X,O}}^{(2)}(\omega) = Ann_{\mathcal{D}_{X,O}}(\omega).$$

In other words, the annihilator ideal $Ann_{\mathcal{D}_{X,O}}(\omega)$ can be generated by annihilating operators of order at most 2 for any generator ω of the dual space \mathcal{W}_f of the Milnor algebra of unimodal singularities.

§3. Hyperbolic singularities

The normal form of the hyperbolic singularity T_{pqr} is given by the polynomial

$$f = x^p + y^q + z^r + axyz,$$

with a parameter a ($a \neq 0$) and $p, q, r \in \mathbb{N}$ satisfying the condition $1/p + 1/q + 1/r < 1$. It is known that the Milnor number μ of the hyperbolic singularity is $p + q + r - 1$ (cf. [2, 3]). Note that if $1/p + 1/q + 1/r = 1$, the singularity defined by f , called the parabolic singularity, is quasihomogeneous. In this section, we study hyperbolic singularities.

To prove our results, we give ω , $Ann_{\mathcal{D}_{X,O}}^{(1)}(\omega)$, $Ann_{\mathcal{D}_{X,O}}^{(2)}(\omega)$ in an explicit manner. We shall deal with the case where one of the exponents p, q, r is 2 separately from the generic case that all exponents p, q, r are greater than or equal to 3.

§3.1. The cases $p, q, r \geq 3$

Assume $p, q, r \geq 3$. The next result provides a basis for the dual space of the Milnor algebra for a hyperbolic singularity.

Lemma 3.1. *For the function $f = x^p + y^q + z^r + axyz$ with a non-zero parameter a and $p, q, r \geq 3$, $1/p + 1/q + 1/r < 1$, the vector space \mathcal{W}_f is spanned by $p + q + r - 1$ algebraic local cohomology classes $\zeta = \psi dx \wedge dy \wedge dz$, where ψ represents the following algebraic local cohomology classes in $\mathcal{H}_{[\mathcal{O}]}^3(\mathcal{O}_X)$:*

$$\begin{aligned} & \left[\frac{1}{xyz} \right], \quad \left[\frac{1}{x^j yz} \right] \quad (2 \leq j \leq p-1), \\ & \left[\frac{1}{xy^k z} \right] \quad (2 \leq k \leq q-1), \quad \left[\frac{1}{xyz^\ell} \right] \quad (2 \leq \ell \leq r-1), \\ & \left[\frac{1}{x^p yz} - \frac{p}{a} \frac{1}{xy^2 z^2} \right], \quad \left[\frac{1}{xy^q z} - \frac{q}{a} \frac{1}{x^2 yz^2} \right], \quad \left[\frac{1}{xyz^r} - \frac{r}{a} \frac{1}{x^2 y^2 z} \right], \\ & \left[\frac{1}{xyz^{r+1}} + \frac{r}{q} \frac{1}{xy^{q+1}z} + \frac{r}{p} \frac{1}{x^{p+1}yz} - \frac{r}{a} \frac{1}{x^2 y^2 z^2} \right]. \end{aligned}$$

Proof. Since the $p+q+r-1$ cohomology classes $\psi \in \mathcal{H}_{[\mathcal{O}]}^3(\mathcal{O}_X)$ in the statement are linearly independent, it suffices to prove that they are annihilated by the partial derivatives $f_x = \partial f / \partial x$, $f_y = \partial f / \partial y$ and $f_z = \partial f / \partial z$. It is obvious that the cohomology classes $[\frac{1}{xyz}]$, $[\frac{1}{x^j yz}]$ ($2 \leq j \leq p-1$), $[\frac{1}{xy^k z}]$ ($2 \leq k \leq q-1$), $[\frac{1}{xyz^\ell}]$ ($2 \leq \ell \leq r-1$) are annihilated by f_x , f_y and f_z . Since $px^{p-1}[\frac{1}{x^p yz} - \frac{p}{a} \frac{1}{xy^2 z^2}] = p[\frac{1}{xyz}]$ and $ayz[\frac{1}{x^p yz} - \frac{p}{a} \frac{1}{xy^2 z^2}] = -p[\frac{1}{xyz}]$, we have $f_x[\frac{1}{x^p yz} - \frac{p}{a} \frac{1}{xy^2 z^2}] = 0$. It is easy to see that the class $[\frac{1}{x^p yz} - \frac{p}{a} \frac{1}{xy^2 z^2}]$ is also annihilated by f_y and f_z . One can verify in the same manner that the classes $[\frac{1}{xy^q z} - \frac{q}{a} \frac{1}{x^2 yz^2}]$ and $[\frac{1}{xyz^r} - \frac{r}{a} \frac{1}{x^2 y^2 z}]$ are annihilated by f_x , f_y and f_z . Since $px^{p-1}[\frac{1}{xyz^{r+1}} + \frac{r}{q} \frac{1}{xy^{q+1}z} + \frac{r}{p} \frac{1}{x^{p+1}yz} - \frac{r}{a} \frac{1}{x^2 y^2 z^2}] = [r \frac{1}{x^2 yz}]$ and $ayz[\frac{1}{xyz^{r+1}} + \frac{r}{q} \frac{1}{xy^{q+1}z} + \frac{r}{p} \frac{1}{x^{p+1}yz} - \frac{r}{a} \frac{1}{x^2 y^2 z^2}] = -r[\frac{1}{x^2 yz}]$, f_x annihilates the class $[\frac{1}{xyz^{r+1}} + \frac{r}{q} \frac{1}{xy^{q+1}z} + \frac{r}{p} \frac{1}{x^{p+1}yz} - \frac{r}{a} \frac{1}{x^2 y^2 z^2}]$. One can also see that f_y and f_z annihilate that class. Thus, all $p + q + r - 1$ classes of the form $\psi dx \wedge dy \wedge dz$ given above belong to \mathcal{W}_f and constitute a basis of the vector space \mathcal{W}_f . \square

Corollary 3.1. *The algebraic local cohomology class*

$$(3.1) \quad \left[\frac{1}{xyz^{r+1}} + \frac{r}{q} \frac{1}{xy^{q+1}z} + \frac{r}{p} \frac{1}{x^{p+1}yz} - \frac{r}{a} \frac{1}{x^2 y^2 z^2} \right] dx \wedge dy \wedge dz$$

in Lemma 3.1 is a generator of \mathcal{W}_f over $\mathcal{O}_{X, \mathcal{O}}$.

Proof. Denote by ω the cohomology class (3.1). For each class ω' in Lemma 3.1, one can find a function $h(x, y, z) \in \mathcal{O}_{X, \mathcal{O}}$ such that $\omega' = h(x, y, z)\omega$. \square

The notions of algebraic local cohomology classes and the use of Grothendieck local residues provide an answer to the membership problem for the Jacobi ideal $\mathcal{I} \subset \mathcal{O}_{X, \mathcal{O}}$ (see [9]). As an application, we have the next result.

Lemma 3.2. *The monomials*

$$(3.2) \quad 1, x^j \ (1 \leq j \leq p-2), \quad y^k \ (1 \leq k \leq q-1), \quad z^\ell \ (1 \leq \ell \leq r), \quad \text{and} \quad zy$$

constitute a basis of the quotient space $\mathcal{O}_{X,O}/\mathcal{I}$.

Proof. Let g and g' be two holomorphic functions defined in a neighborhood of O . Then the Grothendieck local duality implies that the difference $g - g'$ is in the Jacobi ideal \mathcal{I} if and only if

$$\text{res}_O(g\psi dx \wedge dy \wedge dz) = \text{res}_O(g'\psi dx \wedge dy \wedge dz)$$

for every algebraic local cohomology classes ψ given in Lemma 3.1, where $\text{res}_O(*)$ denotes the Grothendieck local residue at O . It follows that any element in the Milnor algebra $\mathcal{O}_{X,O}/\mathcal{I}$ can be uniquely represented as a linear combination of the monomials (3.2). This completes the proof. \square

The next result gives a characterization of annihilators in $L_{\mathcal{D}_X}^{(1)}(\omega)$.

Proposition 3.1 ([5, 8, 9]). *Let A be a linear partial differential operator of order 1 with coefficients in B . The following conditions are equivalent:*

- (1) *There exists $h \in \mathcal{O}_{X,O}$ such that $A + h \in L_{\mathcal{D}_X}^{(1)}(\omega)$.*
- (2) *$Ag \in \mathcal{I}$ for any $g \in \mathcal{I}$.*

Proposition 3.1 provides an effective method [9] for constructing first order annihilators in $L_{\mathcal{D}_X}^{(1)}(\omega)$. We fix a term ordering $\frac{\partial}{\partial x} \succ \frac{\partial}{\partial y} \succ \frac{\partial}{\partial z}$.

Lemma 3.3. *Let $f = x^p + y^q + z^r + axyz$, $a \neq 0$, and let ω be the generator (3.1). Then $L_{\mathcal{D}_X}^{(1)}(\omega)$ is spanned by the following $p+q+r$ first-order differential operators:*

- $x^2 \frac{\partial}{\partial x} - \frac{(p-1)q}{a} y^{q-1} \frac{\partial}{\partial z} - (p-1)x \quad (p \geq 4),$
- $x^j \frac{\partial}{\partial x} - (p+1-j)x^{j-1}, \quad j = 3, \dots, p-2 \quad (p \geq 4),$
- $y^{q-1} \frac{\partial}{\partial x} - \frac{a}{q(r-1)} z^2 \frac{\partial}{\partial z} + \frac{a}{q} z,$
- $p \geq 4: yz \frac{\partial}{\partial x} + 2 \frac{p}{a} x^{p-2},$
 $p = 3: yz \frac{\partial}{\partial x} + \frac{6q}{a^2} y^{q-1} \frac{\partial}{\partial z} + \frac{6}{a} x,$
- $z^{r-1} \frac{\partial}{\partial x} + \frac{a}{r} yz \frac{\partial}{\partial z},$
- $z^r \frac{\partial}{\partial x} + \frac{a}{r} yz,$

- $y^2 \frac{\partial}{\partial y} + (q-1)yz \frac{\partial}{\partial z} - (q-1)y$,
- $y^k \frac{\partial}{\partial y} - (q+1-k)y^{k-1}$, $k = 3, \dots, q-1$ ($q \geq 4$),
- $yz \frac{\partial}{\partial y} + \frac{1}{r-1}z^2 \frac{\partial}{\partial z} - z$,
- $z^{r-1} \frac{\partial}{\partial y} - \frac{q}{r}y^{q-1} \frac{\partial}{\partial z}$,
- $z^r \frac{\partial}{\partial y} - \frac{q}{r}y^{q-1}$,
- $z^\ell \frac{\partial}{\partial z} - (r+1-\ell)z^{\ell-1}$, $\ell = 3, \dots, r$.

Proof. It is easy to see by direct computations that these operators annihilate ω . Let M be the set of monomials (3.2). Let M_x, M_y, M_z be the sets of residual monomials in $M \setminus \{1\}$ of the coefficients of the leading terms of the above operators:

$$\begin{aligned} M_x &= \{x, y^k (1 \leq k \leq q-1), z^\ell (1 \leq \ell \leq r)\}, \\ M_y &= \{y, x^j (1 \leq j \leq p-2), z^\ell (1 \leq \ell \leq r)\}, \\ M_z &= \{z, z^2, x^j (1 \leq j \leq p-2), y^k (1 \leq k \leq q-1), yz\}. \end{aligned}$$

Set

$$P = \left(\sum_{\lambda \in E_x} \alpha_\lambda \mathbf{x}^\lambda \right) \frac{\partial}{\partial x} + \left(\sum_{\lambda \in E_y} \beta_\lambda \mathbf{x}^\lambda \right) \frac{\partial}{\partial y} + \left(\sum_{\lambda \in E_z} \gamma_\lambda \mathbf{x}^\lambda \right) \frac{\partial}{\partial z} + \sum_{\lambda \in E} c_\lambda \mathbf{x}^\lambda,$$

where E_x, E_y, E_z and E are the sets of exponents of monomials in M_x, M_y, M_z and M , respectively, $\alpha_\lambda, \beta_\lambda, \gamma_\lambda$ and c_λ are undetermined coefficients, and $\mathbf{x}^\lambda = x^j y^k z^\ell$ for $\lambda = (j, k, \ell)$. Assume that $\omega P = 0$. Then

$$\begin{cases} p\alpha_{100} + c_{000} = 0, \\ q\beta_{010} + c_{000} = 0, \\ r\gamma_{001} + c_{000} = 0, \\ \alpha_{100} + \beta_{010} + \gamma_{001} + c_{000} = 0, \end{cases}$$

and $\alpha_{jkl} = 0$ for $(j, k, \ell) \neq (1, 0, 0)$, $\beta_{jkl} = 0$ for $(j, k, \ell) \neq (0, 1, 0)$, $\gamma_{jkl} = 0$ for $(j, k, \ell) \neq (0, 0, 1)$, and $c_{jkl} = 0$ for $(j, k, \ell) \neq (0, 0, 0)$.

Thus $(\alpha_{100}, \beta_{010}, \gamma_{001}, c_{000}) \neq (0, 0, 0, 0)$ only when $1/p + 1/q + 1/r = 1$, which contradicts the hyperbolicity. \square

Notice that for instance the second operator in Lemma 3.3 can be constructed from the first one by multiplying with x^{j-2} from the right. Thus, to compute

$Sol_{f,\omega}^{(1)}$, one does not need to solve the differential equations $\zeta P = 0$ for all operators in $L_{\mathcal{D}_X}^{(1)}(\omega)$. Actually, by solving the system $\zeta P = 0$ for the third, seventh and first ($p \geq 4$) or fourth ($p = 3$) operators, we obtain for $p, q, r \geq 3$ the following result, which is the first statement of Theorem 2.1 for the hyperbolic case.

Theorem 3.1. *Let f be a hyperbolic singularity and ω be a generator of \mathcal{W}_f . Then $Sol_{f,\omega}^{(1)} = \text{Span}_{\mathbb{C}}\{\delta, \omega\}$.*

Proof. We give the proof for $p, q, r \geq 4$. The case $p = 3$ can be proved in a similar manner.

Let P_1, P_3, P_7 be the first, third and seventh operators in Lemma 3.3. We verify that

$$\text{Span}_{\mathbb{C}}\{\zeta \in \mathcal{W}_f \mid \zeta P_1 = \zeta P_3 = \zeta P_7 = 0\} = \text{Span}_{\mathbb{C}}\{\delta, \omega\}.$$

Since the action of a differential operator P from the right on a differential form $\psi dx \wedge dy \wedge dz$ is given by $(P^*\psi)dx \wedge dy \wedge dz$, we compute $P^*\psi$ for each ψ in Lemma 3.1 where P^* is the formal adjoint operator of P .

For

$$P_1 = x^2 \frac{\partial}{\partial x} - \frac{(p-1)q}{a} y^{q-1} \frac{\partial}{\partial z} - (p-1)x,$$

we have

$$P_1^* = \left(-\frac{\partial}{\partial x}\right)x^2 - \frac{(p-1)q}{a} \left(-\frac{\partial}{\partial z}\right)y^{q-1} - (p-1)x.$$

$P_1^*\psi$ for each ψ in Lemma 3.1 is given below:

- $P_1^* \left[\frac{1}{xyz} \right] = 0,$
- $P_1^* \left[\frac{1}{x^2yz} \right] = -(p-1) \left[\frac{1}{xyz} \right],$
- $P_1^* \left[\frac{1}{x^jyz} \right] = \left(-\frac{\partial}{\partial x}\right) \left[\frac{1}{x^{j-2}yz} \right] - (p-1) \left[\frac{1}{x^{j-2}yz} \right] = (j-p-1) \left[\frac{1}{x^{j-1}yz} \right]$
 $(3 \leq j \leq p-2),$
- $P_1^* \left[\frac{1}{xy^kz} \right] = 0 \quad (2 \leq k \leq p-1),$
- $P_1^* \left[\frac{1}{xyz^\ell} \right] = 0 \quad (2 \leq \ell \leq r-1),$
- $P_1^* \left(\left[\frac{1}{x^p yz} - \frac{p}{a} \frac{1}{x^2 y^2 z^2} \right] \right) = (p-1) \left[\frac{1}{x^{p-1} yz} \right] - (p-1) \left[\frac{1}{x^{p-1} yz} \right] = - \left[\frac{1}{x^{p-1} yz} \right],$
- $P_1^* \left(\left[\frac{1}{xy^q z} - \frac{q}{a} \frac{1}{x^2 y^2 z^2} \right] \right) = - \frac{(p-1)q}{a} \left[\frac{1}{xyz^2} \right] + \frac{(p-1)q}{a} \left[\frac{1}{xyz^2} \right] = 0,$

$$\begin{aligned}
& \bullet P_1^* \left(\left[\frac{1}{xyz^r} - \frac{r}{a} \frac{1}{x^2y^2z} \right] \right) = \frac{r(p-1)}{a} \left[\frac{1}{xy^2z} \right], \\
& \bullet P_1^* \left(\left[\frac{1}{xyz^{r+1}} + \frac{r}{q} \frac{1}{xy^{q+1}z} + \frac{r}{p} \frac{1}{x^{p+1}yz} - \frac{r}{a} \frac{1}{x^2y^2z^2} \right] \right) \\
& = \frac{(p-1)r}{p} \left[\frac{1}{x^p yz} \right] - \frac{(p-1)r}{a} \left[\frac{1}{xy^2z^2} \right] - \frac{(p-1)r}{p} \left[\frac{1}{x^p yz} \right] + \frac{(p-1)r}{a} \left[\frac{1}{xy^2z^2} \right] = 0.
\end{aligned}$$

It follows from the above computations that the solution space for the operator P_1 in \mathcal{W}_f is spanned by ω , δ and $\psi dx \wedge dy \wedge dz$ where ψ represents the following algebraic local cohomology classes:

$$(3.3) \quad \left[\frac{1}{xy^k z} \right] \quad (2 \leq k \leq q-1), \quad \left[\frac{1}{xyz^\ell} \right] \quad (2 \leq \ell \leq r-1), \quad \left[\frac{1}{xy^q z} \right] - \frac{q}{a} \left[\frac{1}{x^2y^2z^2} \right].$$

Computing $P_3^* \psi$ for each ψ in (3.3) in the same manner, we find that the solution space $\{\zeta \in \mathcal{W}_f \mid \zeta P_1 = \zeta P_3 = 0\}$ is spanned by ω , δ and $\psi dx \wedge dy \wedge dz$ with ψ being the following algebraic local cohomology classes:

$$(3.4) \quad \left[\frac{1}{xy^k z} \right] \quad (2 \leq k \leq q-1), \quad \left[\frac{1}{xy^q z} \right] - \frac{q}{a} \left[\frac{1}{x^2y^2z^2} \right].$$

Finally, computing $P_7^* \psi$ for each ψ in (3.4), we find that the solution space of the system of differential equations $\zeta P_1 = \zeta P_3 = \zeta P_7 = 0$ is spanned by ω and δ .

Since $Sol_{f,\omega}^{(1)} \supseteq \text{Span}_{\mathbb{C}}\{\delta, \omega\}$, this completes the proof. \square

Lemma 3.4. *Let $p, q, r \geq 3$. Then the second-order partial differential operator*

$$(3.5) \quad R = z^2 \frac{\partial^2}{\partial z^2} - cqx \frac{\partial}{\partial x} - cpy \frac{\partial}{\partial y} - c(pq - p - q)z \frac{\partial}{\partial z} + cpq$$

with $c = \frac{r(r-1)}{pqr - pq - qr - pr}$ is in $L_{\mathcal{D}_X}^{(2)}(\omega)$.

Proof. It is easy to see by direct computations that R annihilates ω . \square

For $p, q, r \geq 3$, we have the following theorem.

Theorem 3.2. *Let f be a hyperbolic singularity and ω be a generator of \mathcal{W}_f . Then $Sol_{f,\omega}^{(2)} = \text{Span}_{\mathbb{C}}\{\omega\}$.*

Proof. Since the operator (3.6) gives $\delta R = cpq\delta$, we have $Sol_{f,\omega}^{(2)} = \text{Span}_{\mathbb{C}}\{\omega\}$. \square

§3.2. The case $p = 2$

To finish the proof of Theorems 3.1 and 3.2, we now examine the case where one of the exponents p, q, r is equal to 2. We may assume without loss of generality that $p = 2$ and $3 \leq q \leq r$. We give \mathcal{W}_f , ω , $Ann_{\mathcal{D}_X}^{(1)}(\omega)$, $Ann_{\mathcal{D}_X}^{(2)}(\omega)$ in an explicit manner for seven cases of q and r .

3.2.1. The case $q \geq 5$. When $p = 2$ and $q \geq 5$, \mathcal{W}_f , $\text{Ann}_{\mathcal{D}_X}^{(1)}(\omega)$ and $\text{Ann}_{\mathcal{D}_X}^{(1)}(\omega)$ are given in the following results.

Lemma 3.5. *For the function $f = x^2 + y^q + z^r + axyz$ with a non-zero parameter a , and $5 \leq q \leq r$, the vector space \mathcal{W}_f is spanned by $q+r+1$ algebraic local cohomology classes $\zeta = \psi dx \wedge dy \wedge dz$, where ψ represents the algebraic local cohomology classes in $\mathcal{H}_{[O]}^3(\mathcal{O}_X)$ given below:*

$$\begin{aligned} & \left[\frac{1}{xyz} \right], \quad \left[\frac{1}{xy^k z} \right] \quad (2 \leq k \leq q-1), \quad \left[\frac{1}{xyz^\ell} \right] \quad (2 \leq \ell \leq r-1), \\ & \left[\frac{1}{x^2 y z} - \frac{2}{a} \frac{1}{xy^2 z^2} \right], \quad \left[\frac{1}{xy^q z} - \frac{q}{a} \frac{1}{x^2 y z^2} + \frac{2q}{a^2} \frac{1}{xy^2 z^3} \right], \\ & \left[\frac{1}{xyz^r} - \frac{r}{a} \frac{1}{x^2 y^2 z} + \frac{2r^2}{a^2} \frac{1}{xy^3 z^2} \right] \end{aligned}$$

and

$$\eta = \left[\frac{1}{xyz^{r+1}} + \frac{r}{2} \frac{1}{x^3 y z} + \frac{r}{q} \frac{1}{xy^{q+1} z} - \frac{r}{a} \frac{1}{x^2 y^2 z^2} + \frac{2r}{a^2} \frac{1}{xy^3 z^3} \right].$$

The algebraic local cohomology class $\eta dx \wedge dy \wedge dz$ is a generator of \mathcal{W}_f over $\mathcal{O}_{X,O}$.

Lemma 3.6. *Let $f = x^2 + y^q + z^r + axyz$, $a \neq 0$ ($5 \leq q \leq r$). For a generator*

$$\omega = \left[\frac{1}{xyz^{r+1}} + \frac{r}{2} \frac{1}{x^3 y z} + \frac{r}{q} \frac{1}{xy^{q+1} z} - \frac{r}{a} \frac{1}{x^2 y^2 z^2} + \frac{2r}{a^2} \frac{1}{xy^3 z^3} \right] dx \wedge dy \wedge dz$$

of \mathcal{W}_f , $L_{\mathcal{D}_X}^{(1)}(\omega)$ is spanned by the following $2+q+r$ first-order differential operators:

- $\theta_{(q,r),1} = yz^2 \frac{\partial}{\partial x} + \left(-\frac{2^2 q(r-2)}{(r-1)a^3} y^{q-2} - \frac{2}{(r-1)a} z^2 \right) \frac{\partial}{\partial z} + \frac{2}{a} z,$
- $\theta_{(*,r),2} = z^{r-1} \frac{\partial}{\partial x} - \frac{2^2}{a} z^{r-2} \frac{\partial}{\partial y} + \frac{1}{r} a y z \frac{\partial}{\partial z},$
- $\theta_{(*,r),3} = z^r \frac{\partial}{\partial x} - \frac{a}{r} y z^2 \frac{\partial}{\partial z} + \frac{a}{r} y z,$
- $\theta_{(q,r),4} = \left(y^2 - \frac{2qr}{a^2} z^{r-2} \right) \frac{\partial}{\partial y} + (q-1) y z \frac{\partial}{\partial z} - (q-1) y,$
- $\theta_{(q,r),5} = y^3 \frac{\partial}{\partial y} - (q-2) y^2 + \frac{2(q-2)r}{a^2} z^{r-2},$
- $\theta_{(q,*) ,6}^{(k)} = y^k \frac{\partial}{\partial y} - (q+1-k) y^{k-1}, \quad k = 4, \dots, q-2 \quad (q \geq 6),$
- $\theta_{(q,r),7} = yz \frac{\partial}{\partial y} + \left(-\frac{2qr}{(r-1)a^2} y^{q-2} + \frac{1}{r-1} z^2 \right) \frac{\partial}{\partial z} - z,$

- $\theta_{(q,*) ,8} = yz^2 \frac{\partial}{\partial y} - \frac{2^2 q}{a^2} y^{q-2},$
- $\theta_{(*,r) ,9} = z^{r-1} \frac{\partial}{\partial y} - \frac{1}{2r} a^2 yz^2 \frac{\partial}{\partial z},$
- $\theta_{(*,r) ,10} = z^r \frac{\partial}{\partial y} - \frac{1}{2r} a^2 yz^2,$
- $\theta_{(q,r) ,11} = z^3 \frac{\partial}{\partial z} + \frac{2q(r-2)}{a^2} y^{q-2} - (r-2)z^2,$
- $\theta_{(*,r) ,12}^{(\ell)} = z^\ell \frac{\partial}{\partial z} - (r+1-\ell)z^{\ell-1}, \quad \ell = 4, \dots, r.$

The coefficients of the operators in the above lemma are in

$$B = \text{Span}_{\mathbb{C}}\{1, y, y^2, \dots, y^{q-2}, yz, yz^2, z, z^2, \dots, z^r\}.$$

Note that the solution space $Sol_{f,\omega}^{(1)}$ can be determined by solving only two equations, $\zeta\theta_{(q,r),1} = 0$ and $\zeta\theta_{(q,r),4} = 0$.

Lemma 3.7. *Let $f = x^2 + y^q + z^r + axyz$, $a \neq 0$ ($q \geq 5$, $r \geq q$). The second-order partial differential operator*

$$(3.6) \quad S = \left(y^{q-2} - \frac{1}{2qa^2} z^2 \right) \frac{\partial^2}{\partial z^2} - \frac{ca^3}{2^2} yz \frac{\partial}{\partial x} + \frac{ca^2}{q} y \frac{\partial}{\partial y} + \frac{(q-2)ca^2}{2q} z \frac{\partial}{\partial z} - ca^2$$

with $c = \frac{r(r-1)}{qr-2(q+r)}$ is in $L_{\mathcal{D}_x}^{(2)}(\omega)$.

3.2.2. The case $q = 4$. Let $f = x^2 + y^4 + z^r + axyz$ with $r \geq 5$. A basis of the vector space \mathcal{W}_f is the same as for $q \geq 5$. The second-order partial differential operator (3.6) also belongs to $L_{\mathcal{D}_x}^{(2)}(\omega)$ in this case. We now give $L_{\mathcal{D}_x}^{(1)}(\omega)$:

Lemma 3.8. *Let $f = x^2 + y^4 + z^r + axyz$, $a \neq 0$ ($r \geq 5$). For the generator*

$$\omega = \left[\frac{1}{xyz^{r+1}} + \frac{r}{2} \frac{1}{x^3yz} + \frac{r}{4} \frac{1}{xy^5z} - \frac{r}{a} \frac{1}{x^2y^2z^2} + \frac{2r}{a^2} \frac{1}{xy^3z^3} \right] dx \wedge dy \wedge dz$$

of \mathcal{W}_f , $L_{\mathcal{D}_x}^{(1)}(\omega)$ is spanned by the following $6 + r$ first-order differential operators:

- $r \geq 6: \theta_{(4,r),1} + \frac{2^5 3(r-2)r}{(r-1)a^5} z^2,$
- $r = 5: \theta_{(4,5),1} - \frac{2^6 3^2 5}{a^7} y^2 + \frac{2^7 3^2 5^2}{a^9} z^3 + \frac{2^3 3^2 5}{a^5} z^2,$
- $\theta_{(*,r),2},$
- $\theta_{(*,r),3},$

- $\theta_{(4,r),4}$,
- $r \geq 6$: $\theta_{(4,r),7} + \frac{2^4 3 r^2}{(r-1)a^4} z^{r-3}$,
- $r = 5$: $\theta_{(4,5),7} - \frac{2^5 3 \cdot 5^2}{a^6} y^2 + \frac{2^6 3 \cdot 5^3}{a^8} z^3 + \frac{2^2 3 \cdot 5^2}{a^4} z^2$,
- $\theta_{(4,*) ,8} + \frac{2^5 r}{a^4} z^{r-2}$,
- $\theta_{(*,r),9}$,
- $\theta_{(*,r),10}$,
- $\theta_{(4,r),11} - \frac{2^4 (r-2)r}{a^4} z^{r-2}$,
- $\theta_{(*,r),12}^{(\ell)}$, $\ell = 4, \dots, r$.

The coefficients of the operators in the above lemma are in

$$B = \text{Span}_{\mathbb{C}}\{1, y, y^2, yz, yz^2, z, z^2, \dots, z^r\}.$$

Note that the solution space $\text{Sol}_{f,\omega}^{(1)}$ can be determined by solving only two equations, $\zeta_{\theta_{(q,r),1}} = 0$ and $\zeta_{\theta_{(q,r),4}} = 0$.

3.2.3. The case $q = 3$. Let $f = x^2 + y^3 + z^r + axyz$ with $r \geq 7$.

Lemma 3.9. *For the function $f = x^2 + y^3 + z^r + axyz$ with a non-zero parameter a , $r \geq 7$, the vector space \mathcal{W}_f is spanned by $r + 4$ algebraic local cohomology classes $\zeta = \psi dx \wedge dy \wedge dz$, where ψ represents the algebraic local cohomology classes in $\mathcal{H}_{[\mathcal{O}]}^3(\mathcal{O}_X)$ given below:*

$$\begin{aligned} & \left[\frac{1}{xyz} \right], \quad \left[\frac{1}{xy^2z} \right], \quad \left[\frac{1}{xyz^\ell} \right] \quad (2 \leq \ell \leq r-1), \\ & \left[\frac{1}{x^2yz} - \frac{2}{a} \frac{1}{xy^2z^2} \right], \quad \left[\frac{1}{xy^3z} - \frac{3}{a} \frac{1}{x^2yz^2} + \frac{6}{a^2} \frac{1}{xy^2z^3} \right], \\ & \left[\frac{1}{xyz^r} - \frac{r}{a} \frac{1}{x^2y^2z} + \frac{2r^2}{a^2} \frac{1}{xy^3z^2} - \frac{6r}{a^3} \frac{1}{x^2yz^3} + \frac{12r}{a^4} \frac{1}{xy^2z^4} \right] \end{aligned}$$

and

$$\eta = \left[\frac{1}{xyz^{r+1}} + \frac{r}{2} \frac{1}{x^3yz} + \frac{r}{3} \frac{1}{xy^4z} - \frac{r}{a} \frac{1}{x^2y^2z^2} + \frac{2r}{a^2} \frac{1}{xy^3z^3} - \frac{6r}{a^3} \frac{1}{x^2yz^4} + \frac{12r}{a^4} \frac{1}{xy^2z^5} \right].$$

The algebraic local cohomology class $\eta dx \wedge dy \wedge dz$ above is a generator of \mathcal{W}_f over $\mathcal{O}_{X,\mathcal{O}}$ in each case.

Lemma 3.10. *Let $f = x^2 + y^3 + z^r + axyz$, $a \neq 0$. For the generator*

$$\omega = \left[\frac{1}{xyz^{r+1}} + \frac{r}{2} \frac{1}{x^3yz} + \frac{r}{3} \frac{1}{xy^4z} - \frac{r}{a} \frac{1}{x^2y^2z^2} + \frac{2r}{a^2} \frac{1}{xy^3z^3} - \frac{6r}{a^3} \frac{1}{x^2yz^4} + \frac{12r}{a^4} \frac{1}{xy^2z^5} \right] dx \wedge dy \wedge dz$$

of \mathcal{W}_f , $L_{D_X}^{(1)}(\omega)$ is spanned by the following $5 + r$ first-order differential operators:

- $r \geq 10$: $\theta_{r,1} = yz^2 \frac{\partial}{\partial x} - \frac{2^2}{3a} yz \frac{\partial}{\partial y} + \left(-\frac{2^2(r-6)}{(r-1)a^3} y - \frac{2 \cdot 5}{3(r-1)a} z^2 \right) \frac{\partial}{\partial z}$,
 $+ \frac{2^4 3 \cdot 5 r(r-6)}{(r-1)a^7} z^{r-5} + \frac{2 \cdot 5}{3a} z$
- $r = 9$: $\theta_{9,1} = \frac{2^2 3^5 5}{a^9} yz^2 + \frac{2^4 3^8 5}{a^{13}} z^7$,
- $r = 8$: $\theta_{8,1} = \frac{2^9 3^2 5}{7a^9} yz + \frac{2^{14} 3^3 5}{7a^{13}} z^5$,
- $r = 7$: $\theta_{7,1} = \frac{2^{10} 3^5 5 \cdot 7^3}{a^{21}} yz^2 - \frac{2^7 3^3 5 \cdot 7^2}{a^{15}} yz - \frac{2^4 3 \cdot 5 \cdot 7}{a^9} y$
 $+ \frac{2^{12} 3^6 5 \cdot 7^4}{a^{25}} z^5 + \frac{2^9 3^4 5 \cdot 7^3}{a^{19}} z^4 + \frac{2^6 3^2 5 \cdot 7^2}{a^{13}} z^3$,
- $r \geq 10$: $\theta_{r,2} = z^{r-2} \frac{\partial}{\partial x} + \left(-\frac{2}{a} z^{r-3} - \frac{a^3}{2 \cdot 3^2 r} yz \right) \frac{\partial}{\partial y}$
 $+ \left(\frac{a}{3(r-1)} y - \frac{a^3}{2 \cdot 3^2 r(r-1)} z^2 \right) \frac{\partial}{\partial z} - \frac{2^2 \cdot 5r}{(r-1)a^3} z^{r-5} + \frac{a^3}{2 \cdot 3^2 r} z$,
- $r = 9$: $\theta_{9,2} = \frac{3^3 5}{a^5} yz^2 - \frac{2^2 3^6 5}{a^9} z^7$,
- $r = 8$: $\theta_{8,2} = \frac{2^6 3 \cdot 5}{7a^5} yz - \frac{2^{11} 3^2 5}{7a^9} z^5$,
- $r = 7$: $\theta_{7,2} = \frac{2^5 3^2 7}{a^7} z^5 \frac{\partial}{\partial y} - \frac{2^3 3}{a^3} z^3 \frac{\partial}{\partial z} - \frac{2^8 3^4 7^2 109}{a^{17}} yz^2 - \frac{2^5 3^2 7 \cdot 109}{a^{11}} yz - \frac{2^2 109}{a^5} y$
 $+ \frac{2^{10} 3^5 7^3 109}{a^{21}} z^5 + \frac{2^7 3^3 7^2 109}{a^{15}} z^4 + \frac{2^4 3 \cdot 7 \cdot 109}{a^9} z^3 + \frac{2^3 \cdot 3 \cdot 5}{a^3} z^2$,
- $r \geq 9$: $\theta_{r,3} = z^{r-1} \frac{\partial}{\partial x} - \frac{2^2}{a} z^{r-2} \frac{\partial}{\partial y} + \frac{1}{2 \cdot 3r} a^3 z^3 \frac{\partial}{\partial z}$
 $+ \frac{r-2}{r} ay - \frac{2^2 3(r-2)}{a^3} z^{r-4} - \frac{r-2}{2 \cdot 3r} a^3 z^2$,
- $r = 8$: $\theta_{8,3} = \frac{2^4 3^3}{a^5} yz^2 + \frac{2^9 3^4}{a^9} z^6$,
- $r = 7$: $\theta_{7,3} = \frac{2^6 3^4 5 \cdot 7}{a^{11}} yz^2 + \frac{2^3 3^2 5}{a^5} yz - \frac{2^8 3^5 5 \cdot 7^2}{a^{15}} z^5 - \frac{2^5 3^3 5 \cdot 7}{a^9} z^4$,

- $r \geq 8$: $\theta_{r,4} = z^r \frac{\partial}{\partial x} - \frac{1}{2 \cdot 3r} a^3 z^4 \frac{\partial}{\partial z} - \frac{r-4}{r} ayz + \frac{2^2 3(r-4)}{a^3} z^{r-3} + \frac{r-3}{2 \cdot 3r} a^3 z^3$,
 $r = 7$: $\theta_{7,4} = \frac{2^3 3^3}{a^5} yz^2 + \frac{2^5 3^4 7}{a^9} z^5$,
- $r \geq 9$: $\theta_{r,5} = \left(yz^2 - \frac{2^2 3^2 r}{a^4} z^{r-2} \right) \frac{\partial}{\partial y} + 2z^3 \frac{\partial}{\partial z}$
 $+ \frac{2^2 3(r-3)}{a^2} y - \frac{2^4 3^2 (r-3)r}{a^6} z^{r-4} - 2(r-2)z^2$,
 $r = 8$: $\theta_{8,5} = \frac{2^8 3^3 5}{a^8} yz^2 - \frac{2^{13} 3^4 5}{a^{12}} z^6$,
 $r = 7$: $\theta_{7,5} = \frac{2^{10} 3^5 7^2}{a^{14}} yz^2 + \frac{2^7 3^3 7}{a^8} yz - \frac{2^{12} 3^6 7^3}{a^{18}} z^5 - \frac{2^9 3^4 7^2}{a^{12}} z^4$,
- $r \geq 8$: $\theta_{r,6} = z^{r-1} \frac{\partial}{\partial y} - \frac{1}{2^2 3r} a^4 z^4 \frac{\partial}{\partial z} - \frac{r-3}{2r} a^2 yz + \frac{2 \cdot 3(r-3)}{a^2} z^{r-3} + \frac{r-3}{2^2 3r} a^4 z^3$,
 $r = 7$: $\theta_{7,6} = \frac{2^4 3^2}{a^4} yz^2 + \frac{2^6 3^3 7}{a^8} z^5$,
- $\theta_{r,7} = z^r \frac{\partial}{\partial y} - \frac{1}{2r} a^2 yz^2 + \frac{2 \cdot 3}{a^2} z^{r-2}$,
- $r \geq 9$: $\theta_{r,8} = \left(yz - \frac{1}{2 \cdot 3} a^2 z^3 \right) \frac{\partial}{\partial z} - (r-2)y + \frac{2^2 3r(r-6)}{a^4} z^{r-4} + \frac{r-2}{2 \cdot 3} a^2 z^2$,
 $r = 8$: $\theta_{8,8} = \frac{2^7 3^2}{a^6} yz^2 + \frac{2^{12} 3^3}{a^{10}} z^6$,
 $r = 7$: $\theta_{7,8} = \frac{2^6 3^4 7^2}{a^{12}} yz^2 - \frac{2^3 3^2 7}{a^6} yz + \frac{2^8 3^5 7^3}{a^{16}} z^5 + \frac{2^5 3^3 7^2}{a^{10}} z^4$,
- $r \geq 8$: $\theta_{r,9} = \left(z^2 y - \frac{1}{2 \cdot 3} a^2 z^4 \right) \frac{\partial}{\partial z} - (r-3)yz + \frac{2^2 3r(r-6)}{a^4} z^{r-3} + \frac{r-3}{2 \cdot 3} a^2 z^3$,
 $r = 7$: $\theta_{7,9} = \frac{2^3 3^2 7}{a^6} yz^2 + \frac{2^5 3^3 7^2}{a^{10}} z^5$,
- $\theta_{r,10} = z^5 \frac{\partial}{\partial z} + \frac{2 \cdot 3(r-4)}{a^2} yz^2 - \frac{2^3 3^2 (r-4)r}{a^6} z^{r-2} - (r-4)z^4$,
- $\theta_{r,11}^{(\ell)} = z^\ell \frac{\partial}{\partial z} - (r+1-\ell)z^{\ell-1}$, $\ell = 6, \dots, r$.

The coefficients of the operators in the above lemma are in

$$B = \text{Span}_{\mathbb{C}}\{1, y, yz, yz^2, z, z^2, \dots, z^r\}.$$

Note that in order to check $\text{Sol}_{f,\omega}^{(1)} = \text{Span}_{\mathbb{C}}\{\omega, \delta\}$, it is enough to solve only two equations, $\zeta\theta_{r,1} = 0$ and $\zeta\theta_{r,5} = 0$.

Lemma 3.11. *For $q = 3$, the following second-order partial differential operator is in $L_{\mathcal{D}_X}^{(2)}(\omega)$:*

- For $f = x^2 + y^3 + z^r + axyz$ with $r \geq 9$,

$$\begin{aligned} & z^4 \frac{\partial^2}{\partial x \partial z} + \frac{2^3(r+3)(r-6)}{a^5} z^{r-2} \frac{\partial^2}{\partial y^2} - \frac{2(r^2-2r-12)}{3ra} z^3 \frac{\partial^2}{\partial y \partial z} \\ & \quad - \frac{2^3(r-3)}{ra^3} z^2 \frac{\partial^2}{\partial z^2} - (r-3)z^3 \frac{\partial}{\partial x} \\ & + \left(\frac{2^4 3r(r-6)(r-2)}{a^7} z^{r-4} + \frac{2(r-2)(r^2-2r-12)}{3ra} z^2 - \frac{2^2(r^3-6r^2-2r+24)}{ra^3} y \right) \frac{\partial}{\partial y} \\ & \quad + \frac{2^3(r-3)(r+2)}{ra^3} z \frac{\partial}{\partial z} - \frac{2^3 3(r-3)}{a^3}. \end{aligned}$$

- For $f = x^2 + y^3 + z^8 + axyz$,

$$\begin{aligned} & z^4 \frac{\partial^2}{\partial x \partial z} + \frac{2^5 23}{11a^5} z^6 \frac{\partial^2}{\partial y^2} + \left(-\frac{2 \cdot 7}{11a} z^3 - \frac{2^6}{11a^3} yz \right) \frac{\partial^2}{\partial y \partial z} - \frac{2 \cdot 3 \cdot 5}{11a^3} z^2 \frac{\partial^2}{\partial z^2} \\ & \quad + \left(-5z^3 + \frac{2 \cdot 3 \cdot 5^3}{11a^2} yz \right) \frac{\partial}{\partial x} + \left(-\frac{2^2 3 \cdot 7}{11a} z^2 - \frac{2^3 7 \cdot 19}{11a^3} y \right) \frac{\partial}{\partial y} + \frac{2^4 3 \cdot 5 \cdot 7}{11a^3} \end{aligned}$$

- For $f = x^2 + y^3 + z^7 + axyz$,

$$\begin{aligned} & \left(yz + \frac{1}{2 \cdot 11} a^2 z^3 \right) \frac{\partial^2}{\partial x \partial z} + \left(-\frac{2^7 3^2 5^2 7^3 23}{11a^{15}} z^6 + \frac{2^6 3^3 7^2}{a^9} z^5 - \frac{2^2 3 \cdot 7^2}{11a^3} z^4 - \frac{7}{3^2 11} ayz \right) \frac{\partial^2}{\partial y^2} \\ & \quad + \left(\frac{2^5 3 \cdot 5^2 7^2 23}{11a^{11}} z^4 - \frac{2^3 7^2}{3a^5} z^3 - \frac{2^5 5 \cdot 7^3}{11a^7} yz + \frac{7}{3^3 11} az^2 + \frac{2 \cdot 7}{3^2 a} y \right) \frac{\partial^2}{\partial y \partial z} - \frac{2^4 3^3 7}{11a^7} z^2 \frac{\partial^2}{\partial z^2} \\ & \quad + \left(-\frac{2^2 3^3 5 \cdot 7}{11a^4} z^3 + \frac{2^5 3^5 7}{11a^6} yz - \frac{5}{2 \cdot 11} a^2 z^2 + \frac{1}{11} y \right) \frac{\partial}{\partial x} - \frac{2 \cdot 7}{3^2 \cdot 11} az \frac{\partial}{\partial y} + \frac{2^5 3^4 7^2}{11a^7}. \end{aligned}$$

Theorems 3.1 and 3.2 can be proved for $p = 2$ by using the results presented in this subsection. This completes the proof of the main theorem for the hyperbolic case.

§4. Exceptional families

There are 14 exceptional families of unimodal singularities, E_{12} , E_{13} , E_{14} , Z_{11} , Z_{12} , Z_{13} , W_{12} , W_{13} , Q_{10} , Q_{11} , Q_{12} , S_{11} , S_{12} , and U_{12} . Theorem 2.2 is proved for these exceptional families by performing case-by-case computations. The proof involves the following five steps.

Step 1. Compute the basis of the vector space \mathcal{W}_f and find a generator ω of \mathcal{W}_f .

Step 2. Compute a monomial basis of $\mathcal{O}_{X,O}/\mathcal{I}$.

Step 3. Compute generators of $\text{Ann}_{\mathcal{D}_{X,O}}^{(1)}(\omega)$ by constructing a basis of $L_{\mathcal{D}_X}^{(1)}(\omega)$.

Step 4. Compute $\text{Sol}_{f,\omega}^{(1)}$ by solving the system of partial differential equations

$$\zeta P = 0, \quad \forall P \in L_{\mathcal{D}_X}^{(1)}(\omega), \text{ for } \zeta \in \mathcal{W}_f.$$

Step 5. Compute generators of $\text{Ann}_{\mathcal{D}_{X,O}}^{(2)}(\omega)$.

Since the defining function of the normal form of an exceptional family of the unimodal singularity in question contains a parameter, we have extended the previously developed algorithm [10] to handle such parametric cases. Step 1 is executed by the resulting algorithm implemented in the computer algebra system Risa/Asir. Steps 3 and 5 are performed by another algorithm described in [9].

By executing Steps 1 to 4 for each exceptional family of unimodal singularities, we obtain the following.

Theorem 4.1 (cf. [5]). *Let f be an exceptional family of unimodal singularities. Then $\text{Sol}_{f,\omega}^{(1)} = \text{Span}_{\mathbb{C}}\{\delta, \omega\}$.*

By executing Step 5 for each case, we have the following result.

Lemma 4.1. *Let f define an exceptional family of unimodal hypersurface isolated singularities at the origin. Let ω be a generator of \mathcal{W}_f . Then there exists in $L_{\mathcal{D}_X}^{(2)}(\omega)$ a second-order partial differential operator S such that*

$$S^{(0)} \notin \mathfrak{m},$$

where $S^{(0)}$ is the zeroth-order part of the differential operator S , and \mathfrak{m} is the maximal ideal in $\mathcal{O}_{X,O}$.

Since $\delta S \neq 0$ provided that $S^{(0)} \notin \mathfrak{m}$ for the delta function δ , we have the following.

Theorem 4.2. *Let f be an exceptional family of non-quasihomogeneous unimodal isolated singularities and ω be a generator of \mathcal{W}_f . Then $\text{Sol}_{f,\omega}^{(2)}$ is the one-dimensional vector space spanned by ω .*

Instead of providing proofs by presenting the full set of data for each exceptional family of unimodal singularities, we restrict ourselves to confirming the results for the Z_{13} singularity.

Example 1 (Z_{13} singularity). Set $f = x^3y + y^6 + axy^5$ with a non-zero parameter a . Then f defines the Z_{13} singularity at the origin. \mathcal{W}_f is spanned by the

cohomology classes $\zeta = \psi dx \wedge dy$, where ψ represents the algebraic local cohomology classes in $\mathcal{H}_{[O]}^2(\mathcal{O}_X)$ given below:

$$\begin{aligned} & \left[\frac{1}{xy} \right], \left[\frac{1}{xy^2} \right], \left[\frac{1}{x^2y} \right], \left[\frac{1}{xy^3} \right], \left[\frac{1}{x^2y^2} \right], \left[\frac{1}{x^3y} \right], \left[\frac{1}{xy^4} \right], \left[\frac{1}{x^2y^3} \right], \\ & \left[\frac{1}{xy^5} \right], \left[\frac{1}{x^2y^4} \right], \left[\frac{1}{xy^6} - \frac{1}{3}a \frac{1}{x^3y^2} - 6 \frac{1}{x^4y} \right], \left[\frac{1}{x^2y^5} - 5a \frac{1}{x^4y} \right] \\ & \left[\frac{1}{x^2y^6} - 6 \frac{1}{x^5y} - \frac{7}{9}a \frac{1}{xy^7} - \frac{1}{3}a \frac{1}{x^4y^2} + \frac{7}{27}a^2 \frac{1}{x^3y^3} \right]. \end{aligned}$$

The algebraic local cohomology class

$$\omega = \left[\frac{1}{x^2y^6} - 6 \frac{1}{x^5y} - \frac{7}{9}a \frac{1}{xy^7} - \frac{1}{3}a \frac{1}{x^4y^2} + \frac{7}{27}a^2 \frac{1}{x^3y^3} \right] dx \wedge dy$$

is a generator of \mathcal{W}_f over $\mathcal{O}_{X,O}$.

The monomials

$$(4.1) \quad 1, y, x, y^2, xy, x^2, y^3, xy^2, y^4, xy^3, y^5, xy^4, y^6$$

constitute a basis of the quotient space $\mathcal{O}_{X,O}/\mathcal{I}$.

Let B be the vector space spanned by monomials (4.1). Taking coefficients from B , the algorithm described in [9] outputs the following 16 operators as a basis of the space $L_{\mathcal{D}_X}^{(1)}(\omega)$:

- $xy \frac{\partial}{\partial x} + \left(-\frac{7^2}{2^2 \cdot 3^3 \cdot 5} axy + \frac{23}{2^2 \cdot 3 \cdot 5} y^2 \right) \frac{\partial}{\partial y}$
 $- \frac{2^2 \cdot 7^5}{3^{13} \cdot 5} a^7 xy^4 - \frac{2^2 \cdot 7^4}{3^{11} \cdot 5} a^6 y^5 + \frac{2^2 \cdot 7^3}{3^8 \cdot 5} a^4 xy^2 + \frac{2^2 \cdot 7^2}{3^6 \cdot 5} a^3 y^3 - \frac{7}{2 \cdot 3^3 \cdot 5} ax - \frac{2 \cdot 19}{3 \cdot 5} y,$
- $ax^2 \frac{\partial}{\partial x} + \left(\frac{37}{2^2 \cdot 3 \cdot 5} axy + \frac{3}{2^2 \cdot 5} y^2 \right) \frac{\partial}{\partial y}$
 $- \frac{7^4 \cdot 11}{3^{11} \cdot 5} a^7 xy^4 - \frac{7^3 \cdot 11}{3^9 \cdot 5} a^6 y^5 + \frac{7^2 \cdot 11}{3^6 \cdot 5} a^4 xy^2 + \frac{7 \cdot 11}{3^4 \cdot 5} a^3 y^3 - \frac{89}{2 \cdot 3 \cdot 5} ax - \frac{3}{5} y,$
- $xy^2 \frac{\partial}{\partial x} + \frac{7^3}{3^7} a^4 xy^3 + \frac{2^3 \cdot 7}{3^5} a^3 y^4 + \frac{7}{3^4} a^2 x^2 - \frac{7}{3^2} axy - y^2,$
- $a^2 y^4 \frac{\partial}{\partial x} + \left(\frac{1}{2^2} axy + \frac{3^2}{2^2} y^2 \right) \frac{\partial}{\partial y}$
 $- \frac{2 \cdot 7^4}{3^{10}} a^7 xy^4 - \frac{2 \cdot 7^3}{3^8} a^6 y^5 + \frac{2 \cdot 7^2}{3^5} a^4 xy^2 + \frac{2 \cdot 7}{3^3} a^3 y^3 + \frac{1}{2} ax - 3^2 y,$
- $xy^3 \frac{\partial}{\partial x} + \frac{7^3}{3^7} a^4 xy^4 + \frac{7^2}{3^5} a^3 y^5 - \frac{7}{3^2} axy^2 - y^3,$
- $y^5 \frac{\partial}{\partial x} + \frac{1}{3} x^2 + \frac{7^2}{3^4} a^2 xy^3 + \frac{2^3}{3^2} ay^4,$

- $xy^4 \frac{\partial}{\partial x} - \frac{7}{3^2}axy^3 - y^4,$
- $y^6 \frac{\partial}{\partial x} + \frac{7^2}{3^4}a^2xy^4 + \frac{7}{3^2}ay^5,$
- $\left(a^2x^2 - \frac{3}{2^2}axy - \frac{3^3}{2^2}y^2\right) \frac{\partial}{\partial y}$
 $- \frac{2^2 \cdot 7^3}{3^9}a^7xy^4 - \frac{2^2 \cdot 7^2}{3^7}a^6y^5 + \frac{2^2 \cdot 7}{3^4}a^4xy^2 + \frac{13}{3^2}a^3y^3 - \frac{3}{2}ax + 3^3y,$
- $y^3 \frac{\partial}{\partial y} - \frac{7^3}{3^7}a^4xy^3 - \frac{7}{3^3}a^3y^4 - \frac{2 \cdot 7}{3^4}a^2x^2 + \frac{7}{3^2}axy - 3y^2,$
- $xy^2 \frac{\partial}{\partial y} + \frac{7^2}{3^4}a^3xy^3 + \frac{23}{3^3}a^2y^4 + \frac{2}{3^2}ax^2 - 4xy,$
- $y^4 \frac{\partial}{\partial y} - \frac{7^3}{3^7}a^4xy^4 + \frac{7}{3^2}axy^2x - \frac{7^2}{3^5}a^3y^5 - 2y^3,$
- $xy^3 \frac{\partial}{\partial y} + \frac{7^2}{3^4}a^3xy^4 - 3xy^2 + \frac{7}{3^2}a^2y^5,$
- $y^5 \frac{\partial}{\partial y} + \frac{7}{3^2}axy^3 - y^4,$
- $xy^4 \frac{\partial}{\partial y} - 2xy^3,$
- $y^6 \frac{\partial}{\partial y} + \frac{7}{3^2}axy^4.$

By solving the system of partial differential equations

$$\zeta P = 0, \quad \forall P \in L_{\mathcal{D}_x}^{(1)}(\omega)$$

for $\zeta \in \mathcal{W}_f$, we have

$$Sol_{f,\omega}^{(1)} = \text{Span}_{\mathbb{C}}\{\omega, \delta\}.$$

Actually, it is enough to consider the first and the ninth operators to determine the algebraic local cohomology solution space.

As for the second-order annihilators, the algorithm outputs five operators. Among them, we find the operator

$$S = y^2 \frac{\partial^2}{\partial y^2} + \left(\frac{3 \cdot 5}{2}x + \frac{7 \cdot 13}{2 \cdot 3^3}a^2y^3\right) \frac{\partial}{\partial x} + \left(\frac{7}{2 \cdot 3^2}ax + \frac{1}{2}y\right) \frac{\partial}{\partial y}$$

$$- \frac{5 \cdot 7^5}{3^{12}}a^7xy^3 - \frac{5 \cdot 7^4}{3^{10}}a^6y^4 + \frac{5 \cdot 7^3}{3^7}a^4xy + \frac{5 \cdot 7^2}{3^5}a^3y^2 - 2 \cdot 3 \cdot 5.$$

Since the zeroth-order part

$$S^{(0)} = -\frac{5 \cdot 7^5}{3^{12}} a^7 xy^3 + \frac{5 \cdot 7^3}{3^7} a^4 xy - \frac{5 \cdot 7^4}{3^{10}} a^6 y^4 + \frac{5 \cdot 7^2}{3^5} a^3 y^2 - 2 \cdot 3 \cdot 5$$

of S is not contained in the maximal ideal \mathfrak{m} , we have

$$\text{Sol}_{f,\omega}^{(2)} = \text{Span}_{\mathbb{C}}\{\omega\}.$$

References

- [1] V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of differentiable maps*, Volume I, Monogr. Math. 82, Birkhäuser, Boston, 1985. [Zbl 0554.58001](#) [MR 0777682](#)
- [2] W. Ebeling, *Functions of several complex variables and their singularities*, Grad. Stud. Math. 83, Amer. Math. Soc., 2007. [Zbl 1188.32001](#) [MR 2319634](#)
- [3] A. M. Gabrielov, Dynkin diagrams for unimodal singularities, *Functional Anal. Appl.* **8** (1973), no. 3, 192–196. [Zbl 0304.14010](#) [MR 367274](#)
- [4] Y. Nakamura and S. Tajima, A method for constructing holonomic systems for algebraic local cohomology classes with support on a zero dimensional variety, in *Mathematical Software* (Beijing, 2002), World Sci., 2002, 158–168. [Zbl 1059.14007](#) [MR 1932608](#)
- [5] ———, Unimodal singularities and differential operators, in *Singularités Franco-Japonaises*, Sémin. Congr. 10, Soc. Math. France, 2005, 191–208. [Zbl 1086.32024](#) [MR 2145955](#)
- [6] M. Noro and T. Takeshima, Risa/Asir—a computer algebra system, in *ISSAC 1992*, P. S. Wang (ed.), ACM Press, 1992, 387–396. [Zbl 0964.68597](#)
- [7] K. Saito, Quasihomogene isolierte Singularitäten von Hyperflächen, *Invent. Math.* **14** (1971), 123–142. [Zbl 0224.32011](#) [MR 0294699](#)
- [8] S. Tajima and Y. Nakamura, Algebraic local cohomology classes attached to quasi-homogeneous isolated hypersurface singularities, *Publ. RIMS Kyoto Univ.* **41** (2005), 1–10. [Zbl 1105.32020](#) [MR 2115965](#)
- [9] ———, Annihilating ideals for an algebraic local cohomology class, *J. Symbolic Comput.* **44** (2009), 435–448. [Zbl 1171.32020](#) [MR 2499921](#)
- [10] S. Tajima, Y. Nakamura and K. Nabeshima, Standard bases and algebraic local cohomology for zero dimensional ideals, in *Singularities—Niigata–Toyama 2007*, Adv. Stud. Pure Math. 56, Math. Soc. Japan, 2009, 341–361. [Zbl 1194.13020](#) [MR 2604090](#)