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Homotopical Presentations and Calculations of Algebraic K_0-Groups for Rings of Continuous Functions

by

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Abstract

Let $K_0(C_{\mathbb{F}}(X)) = K_0 \circ C_{\mathbb{F}}(X)$ be the K_0-group of the ring $C_{\mathbb{F}}(X)$ of $\mathbb{F}$-valued continuous functions on a topological space X, where $\mathbb{F}$ is the field of real or complex numbers or the quaternion algebra. It is known that the functor $K_0 \circ C_{\mathbb{F}}$ is representable on the category of compact Hausdorff spaces. It is a homotopy functor which is not representable on the category of topological spaces. With the use of the notion of a compactly-bounded homotopy set, which is a variant of a homotopy set, the functor $K_0 \circ C_{\mathbb{F}}$ has a homotopical presentation by means of the product of the ring of integers $\mathbb{Z}$ and the infinite Grassmannian $G_\infty(\mathbb{F})$. This presentation makes it possible to calculate the groups $K_0(C_{\mathbb{F}}(X))$ explicitly for some infinite-dimensional complexes X by using the results of H. Miller on Sullivan's conjecture.

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§1. Introduction and main results

Let $\mathbb{F}$ be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, the field of real or complex numbers or the quaternion algebra, respectively. For a topological space X let $C_{\mathbb{F}}(X)$ denote the ring of $\mathbb{F}$-valued continuous functions on X. In this paper we study a homotopical presentation of $K_0(C_{\mathbb{F}}(X)) = K_0 \circ C_{\mathbb{F}}(X)$, the algebraic K_0-group of $C_{\mathbb{F}}(X)$, for any topological space X and calculate the group $K_0(C_{\mathbb{F}}(X))$ explicitly for a certain class of infinite-dimensional complexes X, making use of Miller's result on Sullivan conjecture [13].

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For compact Hausdorff spaces X, the Serre–Swan theorem (see Theorem A in Section 2) enables us to identify the group $K_0(C_{\mathbb{F}}(X))$ with the topological K-group $K_{\mathbb{F}}(X)$, which is represented by the product of the ring of integers $\mathbb{Z}$ and the infinite Grassmannian $G_\infty(\mathbb{F})$ (cf. Section 2.1). Thus we need a generalization of the Serre–Swan theorem to study the group $K_{\mathbb{F}}(X)$ for any topological space.

The Serre–Swan theorem was generalized to paracompact Hausdorff spaces X by Goodearl [5, Theorem 1.1] and to any topological spaces X by Vaserstein [16, Theorems 1 and 2]. We begin by adopting the following definition which seems simpler than that of [16].

Definition 1.1. A vector bundle ξ over a topological space X is *of finite type* if it is realized as a subbundle of the product bundle ε^N of sufficiently large rank N.

The notion of vector bundle of finite type is necessary to generalize the Serre–Swan theorem, since a vector bundle over a noncompact space X does not always correspond to a finitely generated projective $C_{\mathbb{F}}(X)$-module as we see by Proposition 1.2(ii) below. Making use of our definition above, the results of Vaserstein are stated as follows:

Proposition 1.2. *Let X, X', Y be any topological spaces.*

(i) (Homotopy invariance) *Let ξ be a vector bundle of finite type over Y. Then the pull-back bundle $f^\sharp\xi$ is of finite type over X for any map $f : X \to Y$. If $f, g : X \to Y$ are homotopic maps, then the pull-back bundles $f^\sharp\xi$ and $g^\sharp\xi$ are isomorphic.*

(ii) (Equivalence) *The category of vector bundles of finite type over X is equivalent to the category of finitely generated projective modules over $C_{\mathbb{F}}(X)$ via the global section functor.*

(iii) (Functoriality) *Let $f : X \to X'$ be a continuous map. Then under the equivalence of* (ii)*, the pull-back of vector bundles by f corresponds to the extension of scalars $\otimes_{C_{\mathbb{F}}(X')}C_{\mathbb{F}}(X)$ by the induced ring homomorphism $f^\sharp : C_{\mathbb{F}}(X') \to C_{\mathbb{F}}(X)$.*

Assertion (i) of Proposition 1.2 shows the homotopy invariance of vector bundles of finite type with respect to pull-backs; (ii) is a generalization of the Serre–Swan theorem by Vaserstein [16, Theorem 1]; (iii) shows that the generalization of the Serre–Swan theorem is functorial with respect to spaces X. It follows that the group $K_0(C_{\mathbb{F}}(X))$ is determined by the category of vector bundles of finite type over X and it is a homotopy invariant of X.

In order to investigate $K_0(C_{\mathbb{F}}(X))$ for general spaces X, in particular for infinite-dimensional complexes X, a homotopical presentation of $K_0 \circ C_{\mathbb{F}}$ on the category of topological spaces is crucial. However, the functor $K_0 \circ C_{\mathbb{F}}$ is not representable (cf. Definition 3.1) on the category of topological spaces, as can be seen from Examples 3.2 and 3.3 in Section 3; and these examples lead us to introduce the notion of *compactly-bounded homotopy sets*:

Definition 1.3. Let X and Y be topological spaces. A continuous map $f : X \to Y$ is called a *compactly-bounded map* or a *CB-map* if the image of f is contained in a compact subset of Y. Compactly-bounded maps $f, g : X \to Y$ are said to be *CB-homotopic* if there exists a compactly-bounded homotopy $H : X \times I \to Y$ between f and g. Since this relation, denoted by $\simeq_{CB}$, is an equivalence relation; the *compactly-bounded homotopy set* $[X, Y]_{CB}$ is defined by

$$[X, Y]_{CB} = \{\text{compactly-bounded maps from } X \text{ to } Y\}/\simeq_{CB}.$$

This set $[X, Y]_{CB}$ coincides with the ordinary homotopy set $[X, Y]$ if X or Y is compact. If $A \xrightarrow{\alpha} X \xrightarrow{i} C_\alpha \xrightarrow{j} \Sigma A \xrightarrow{\Sigma\alpha} \Sigma X \to \cdots$ is a cofiber sequence, then the induced sequence of pointed compactly-bounded homotopy sets

$$\cdots \to [\Sigma X, Z]_{CB*} \xrightarrow{(\Sigma\alpha)^\sharp} [\Sigma A, Z]_{CB*} \xrightarrow{j^\sharp} [C_\alpha, Z]_{CB*} \xrightarrow{i^\sharp} [X, Z]_{CB*} \xrightarrow{\alpha^\sharp} [A, Z]_{CB*}$$

is exact for any based space Z (Proposition 3.10).

Compactly-bounded homotopy sets enable us to present the Grothendieck group $K_0(C_{\mathbb{F}}(X))$ for any topological space X by the following theorem, which generalizes the representation of the functor $K_0 \circ C_{\mathbb{F}}$ on the category of compact Hausdorff spaces (see Corollary C in Section 2). Let $\mathbb{Z}$ be the ring of integers and $G_\infty(\mathbb{F})$ the infinite Grassmannian.

Theorem 1.4. *The functor $K_0 \circ C_{\mathbb{F}}$ can be presented in the form*

$$K_0 \circ C_{\mathbb{F}}\,(X) \cong [X, \mathbb{Z} \times G_\infty(\mathbb{F})]_{CB}$$

for any space X as a group-valued functor.

Note that topological spaces and compactly-bounded homotopy sets do not form a category, since 1_X is not always a CB-map. So it should be pointed out that our presentation is not a representation in the terminology of category theory; we use the term “presentation” and not “representation” in this paper. Nevertheless Theorem 1.4 shows that the functor $K_0 \circ C_{\mathbb{F}}$ depends only on the homotopy type of $\mathbb{Z} \times G_\infty(\mathbb{F})$. Using Theorem 1.4 we calculate the group $K_0(C_{\mathbb{F}}(X))$ for a certain class of infinite-dimensional complexes.

Theorem 1.5. *Assume that X is a connected CW-complex and the pointed mapping space* $\mathrm{map}_*(X, \widehat{Y})$ *from X to the profinite completion $\widehat{Y}$ of any nilpotent finite complex Y is weakly contractible. Then there exists a natural group isomorphism*

$$K_0(C_{\mathbb{F}}(\Sigma^m X)) \cong \mathbb{Z} \oplus \bigoplus_{sl>1} H^{sl-1}(\Sigma^m X; \widehat{\mathbb{Z}}/\mathbb{Z}) \quad \textit{for } m > 0,$$

where $\widehat{\mathbb{Z}}$ is the profinite completion of the integers $\mathbb{Z}$, and $s = 2$ if $\mathbb{F} = \mathbb{C}$, while $s = 4$ if $\mathbb{F} = \mathbb{R}, \mathbb{H}$.

The Sullivan conjecture was solved in 1984 by H. Miller [13]. Miller's result has been generalized in several directions; assertions (i)–(iii) in Example 1.6 below are due to Zabrodsky [18, 5.1 Lemma (a)], Friedlander–Mislin [3, (3.1) Theorem] and McGibbon [12, Theorem 3], respectively. From their results we see that many spaces X satisfy the condition in Theorem 1.5.

Example 1.6. Let $\mathcal{S}$ denote the class of connected CW-complexes X such that $\mathrm{map}_*(X, \widehat{Y})$ is weakly contractible for any nilpotent finite complex Y. Then the following hold:

(i) $\mathcal{S}$ contains connected CW-complexes X whose fundamental group is locally finite and $\pi_n(X) = 0$ for n sufficiently large.

(ii) $\mathcal{S}$ contains classifying spaces BG of all Lie groups G with only a finite number of path components.

(iii) $\mathcal{S}$ contains connected infinite loop spaces whose fundamental group is a torsion group.

Remark 1.7. The explicit calculation in Theorem 1.5 presents a striking contrast to the case of finite complexes. In particular we note the following aspects.

(i) $K_0(C_{\mathbb{F}}(A))$ can be a very large group, of the cardinality of the real numbers, since $\widehat{\mathbb{Z}}/\mathbb{Z}$ is a $\mathbb{Q}$-module of that cardinality. If A is a finite complex, then $K_0(C_{\mathbb{F}}(A))$ is a finitely generated abelian group.

(ii) Any generator ξ (a vector bundle of finite type over X) of $K_0(C_{\mathbb{F}}(X))$ for X satisfying the condition in Theorem 1.5 should be called a *phantom bundle*, as its restriction to any finite subcomplex is trivial. This is proved in the last part of Section 5.

We prove Proposition 1.2 and Theorems 1.4 and 1.5 in Sections 2.2, 4 and 5, respectively.

§2. Vector bundles of finite type

§2.1. Vector bundles over compact Hausdorff spaces

For a ring A let $K_0(A)$ denote the algebraic K_0-group of A, which is defined to be the Grothendieck group of the abelian semigroup of isomorphism classes of finitely generated projective (right) A-modules. The following Serre–Swan theorem (see Swan [15, Theorems 1 and 2] or Berrick [1, p. 18]) enables us to identify $K_0(C_{\mathbb{F}}(X))$ with the topological K-group of X for any compact Hausdorff space X.

Theorem A (The Serre–Swan theorem [15]). *Let X be a compact Hausdorff space. Then the category of vector bundles over X is equivalent to the category of finitely generated projective modules over $C_{\mathbb{F}}(X)$ via the global section functor.*

We first recall the representation of the topological $K_{\mathbb{F}}$-group on the category of compact Hausdorff spaces. Let $G_n(\mathbb{F}^N)$ denote the finite Grassmannian of n-dimensional subspaces in $\mathbb{F}^N$, endowed with the standard CW-complex structure. Moreover, the spaces $G_n(\mathbb{F})$ and $G_\infty(\mathbb{F})$ are defined by

$$G_n(\mathbb{F}) = \varinjlim_N G_n(\mathbb{F}^N), \qquad G_\infty(\mathbb{F}) = \varinjlim_n G_n(\mathbb{F}),$$

which also have the standard CW-complex structures. Then the product space $\mathbb{Z}\times G_\infty(\mathbb{F})$ has the standard H-group structure which corresponds to the Whitney sum operation. Using the H-group structure, one can prove the following representation theorem, where $[X, Y]$ denotes the unpointed homotopy set for unpointed spaces X and Y. The reader is referred to Karoubi [9, 7.14 Theorem in Chapter I; 1.33 Theorem in Chapter II] for compact Hausdorff spaces or Husemoller [6, Section 4 in Chapter 8] for finite CW-complexes.

Theorem B. *The topological $K_{\mathbb{F}}$-group is represented in the form*

$$K_{\mathbb{F}}(\ ) \cong [\ , \mathbb{Z} \times G_\infty(\mathbb{F})]$$

as a group-valued functor on the category of compact Hausdorff spaces.

Theorems A and B immediately imply the following representation of $K_0 \circ C_{\mathbb{F}}$ on the category of compact Hausdorff spaces.

Corollary C. *The functor $K_0 \circ C_{\mathbb{F}}$ is represented in the form*

$$K_0 \circ C_{\mathbb{F}}(\quad) \cong [\quad , \mathbb{Z} \times G_\infty(\mathbb{F})]$$

as a group-valued functor on the category of compact Hausdorff spaces.

§2.2. Vector bundles over arbitrary topological spaces

If X is a compact Hausdorff space, then by the Serre–Swan theorem there exists an equivalence between the category of vector bundles over X and that of finitely generated projective $C_{\mathbb{F}}(X)$-modules. To generalize the Serre–Swan theorem to any topological space X, we have to consider vector bundles of *finite type* (Definition 1.1), since not all vector bundles over X correspond to finitely generated projective $C_{\mathbb{F}}(X)$-modules; for example, the canonical vector bundle γ^n over the Grassmannian $G_n(\mathbb{F})$ does not correspond to any finitely generated $C_{\mathbb{F}}(G_n(\mathbb{F}))$-module by Lemma 2.1. In this section we establish fundamental properties of vector bundles of finite type and prove Proposition 1.2. We note that Husemoller called a vector bundle ξ over a base space B of finite type if there exists a finite open covering $U_1, \dots, U_n$ of B such that $\xi|U_i$ is trivial for $1 \leq i \leq n$ and considered them over paracompact base spaces B (cf. [6, 5.7 Definition and 5.8 Proposition in Chapter 3]).

The fiber rank of an $\mathbb{F}$-vector bundle over X defines a locally constant function $\mathrm{rk}_{\mathbb{F}} : X \to \mathbb{Z}_+$, where $\mathbb{Z}_+$ is the nonnegative integers. Thus the subspace of X where the fiber rank of ξ is a given integer r is open and closed, which allows us to assume that the fiber rank of ξ is constant in most cases; a vector bundle ξ of constant rank n is often denoted by ξ^n.

In Lemma 2.1(vi) below, we consider numerable open coverings; three definitions of such coverings are quoted in the last section and it is shown that they are equivalent.

Lemma 2.1. *For a vector bundle ξ^n over X, the following are equivalent*:

(i) *ξ^n is a vector bundle of finite type.*

(ii) *There is a bundle epimorphism $\widetilde{\varphi} : \varepsilon^N \to \xi^n$ from the product bundle of sufficiently large rank N.*

(iii) *ξ^n has a complementary bundle.*

(iv) *There is a fiberwise-isomorphic bundle map $\widetilde{f} : \xi^n \to \gamma^n_N$ for sufficiently large N, where γ^n_N is the canonical vector bundle over the Grassmannian $G_n(\mathbb{F}^N)$.*

(v) (Vaserstein [16]) *There exists a finite partition of unity $\{u_1, \dots, u_k\}$ on X such that the restriction of ξ to the set $\{x \mid u_i(x) \neq 0\}$ is trivial for each i.*

(vi) *ξ^n has a finite trivialization open cover $\{U_i\}$ which is numerable* (in the sense of Husemoller [6]; cf. Section 6).

Proof. (i)⇒(ii). We regard ξ^n as a subbundle of ε^N for some N. We define a metric on $\mathbb{F}^N$ by

$$((a_k), (b_k)) = \sum_{k=1}^{N} \overline{a_k} \cdot b_k,$$

which determines a fiberwise metric on $\varepsilon^N = X \times \mathbb{F}^N$. Let $\tilde{\varphi} : \varepsilon^N \to \xi^n$ be the orthogonal projection onto ξ^n. Then $\tilde{\varphi}$ is a bundle epimorphism since the Gram–Schmidt process ensures its continuity.

(ii)⇒(iii). Consider the exact sequence of vector bundles

$$O \to \operatorname{Ker} \widetilde{\varphi} \to \varepsilon^N \xrightarrow{\widetilde{\varphi}} \xi^n \to O,$$

where $\operatorname{Ker} \widetilde{\varphi}$ is a bundle by [6, 8.2 Theorem in Chapter 3]. The orthogonal complementary bundle of $\operatorname{Ker} \widetilde{\varphi}$ is isomorphic to ξ^n in view of the same theorem, that is, $\operatorname{Ker} \widetilde{\varphi}$ is a complementary bundle of ξ^n.

(iii)⇒(iv). If η^{N-n} is a complementary bundle of ξ^n, then by definition we have an isomorphism

$$\xi^n \oplus \eta^{N-n} \xrightarrow{\cong} \varepsilon^N = X \times \mathbb{F}^N.$$

It follows that the fiber of ξ^n at each point $x \in X$ determines an n-dimensional subspace of $\mathbb{F}^N$, or a point of $G_n(\mathbb{F}^N)$. Thus a (fiberwise-isomorphic) bundle map $\widetilde{f} : \xi^n \to \gamma_N^n$ over $f : X \to G_n(\mathbb{F}^N)$ is defined by [6, 5.2 Theorem in Chapter 3].

(iv)⇒(v). Since $G_n(\mathbb{F}^N)$ is a compact Hausdorff space, γ_N^n has a finite trivialization open cover. Then there exists a finite partition of unity $\{u_1, \dots, u_k\}$ on X such that the restriction of ξ to the set $\{x \mid u_i(x) \neq 0\}$ is trivial for each i by James [8, Proposition (7.31)]. This means that the covering is numerable in the sense of [8, Definition (7.25)] (cf. Section 6).

(v)⇒(vi). By Proposition 6.1.

(vi)⇒(i). For the numerable trivialization cover $\{U_1, \dots, U_k\}$ of ξ^n, there is a partition of unity $\{u_1, \dots, u_k\}$ such that $\operatorname{supp} u_i \subset U_i$. Fix a local trivialization

$$\xi^n|_{U_i} \xrightarrow{\cong} U_i \times \mathbb{F}^n, \qquad \upsilon \mapsto (\pi(\upsilon), \tau_i(\upsilon)),$$

for each $i = 1, \dots, k$, where π is the projection of ξ^n. Then we can define a bundle map

$$\xi^n \to \varepsilon^n = X \times \mathbb{F}^n, \qquad \upsilon \mapsto (\pi(\upsilon), u_i(\pi(\upsilon))\tau_i(\upsilon)),$$

over X for each $i = 1, \dots, k$. Hence, combining them, we have a monomorphism

$$\xi^n \to \varepsilon^{kn} = X \times \mathbb{F}^n \oplus \dots \oplus \mathbb{F}^n. \qquad \square$$

Proof of Proposition 1.2(i). Since the pull-back of a product bundle is also a product bundle, we obtain the first assertion. Now, ξ is a numerable vector bundle over Y by Lemma 2.1(vi), and hence homotopy invariant by [6, 9.9 Theorem in Chapter 4]. □

As already mentioned, Proposition 1.2(ii) is Theorem 1 of Vaserstein [16]. We here give another proof:

Proof of Proposition 1.2(ii). Let $\mathcal{A}$ and $\mathcal{B}$ denote the category of $\mathbb{F}$-vector bundles of finite type over X and the category of finitely generated projective $C_{\mathbb{F}}(X)$-modules respectively. By Lemma 2.1, the global section functor $\Gamma(X,\ ) : \mathcal{A} \to \mathcal{B}$ is well defined. Note that $\mathcal{A}$ and $\mathcal{B}$ are additive categories and $\Gamma(X,\ )$ is an additive functor.

Let $\mathcal{A}'$ and $\mathcal{B}'$ be the full subcategories of trivial vector bundles of finite rank over X and of finitely generated free $C_{\mathbb{F}}(X)$-modules respectively. Then it is easily seen that $\Gamma(X,\ )$ restricts to an equivalence of $\mathcal{A}'$ and $\mathcal{B}'$.

First, we show that $\Gamma(X,\ ) : \mathcal{A} \to \mathcal{B}$ is full and faithful. Note that an additive functor between additive categories preserves coproducts; and coproducts are also products in an additive category ([10, Section 2 in Chapter VIII] or [7, Section 3 in Chapter I]). Then we have natural decompositions

$$\begin{aligned} \Gamma(X, \xi_1 \oplus \xi_2) &\cong \Gamma(X, \xi_1) \oplus \Gamma(X, \xi_2), \\ \mathrm{Hom}_{\mathcal{A}}(\xi_1 \oplus \xi_2, \eta_1 \oplus \eta_2) &\cong \bigoplus_{i,j} \mathrm{Hom}_{\mathcal{A}}(\xi_i, \eta_j), \\ \mathrm{Hom}_{\mathcal{B}}(M_1 \oplus M_2, R_1 \oplus R_2) &\cong \bigoplus_{i,j} \mathrm{Hom}_{\mathcal{B}}(M_i, R_j). \end{aligned}$$

Let ξ_1 and ξ_2 be objects of $\mathcal{A}$. Then there exist objects ξ_1' and ξ_2' of $\mathcal{A}$ such that $\xi_i \oplus \xi_i' \cong \varepsilon^{N_i}$ for some N_i $(i = 1, 2)$. Since $\Gamma(X,\ ) : \mathcal{A}' \to \mathcal{B}'$ is an equivalence of categories,

$$\Gamma(X,\ ) : \mathrm{Hom}_{\mathcal{A}}(\varepsilon^{N_1}, \varepsilon^{N_2}) \to \mathrm{Hom}_{\mathcal{B}}(C_{\mathbb{F}}(X)^{N_1}, C_{\mathbb{F}}(X)^{N_2})$$

is an isomorphism. Then the natural decompositions above imply that the functor

$$\Gamma(X,\ ) : \mathrm{Hom}_{\mathcal{A}}(\xi_1, \xi_2) \to \mathrm{Hom}_{\mathcal{B}}(\Gamma(X, \xi_1), \Gamma(X, \xi_2))$$

is an isomorphism.

Next, let M be a finitely generated projective $C_{\mathbb{F}}(X)$-module. Then there exists M' such that $M \oplus M' \cong C_{\mathbb{F}}(X)^N$ for some N. Consider a morphism $P : C_{\mathbb{F}}(X)^N \xrightarrow{\text{proj.}} M \xrightarrow{\text{incl.}} C_{\mathbb{F}}(X)^N$ which is a projector, i.e. $P^2 = P$. Since $\Gamma(X,\ ) : \mathcal{A}' \to \mathcal{B}'$ is an equivalence of categories, the corresponding projector $\tilde{P} : \varepsilon^N \to \varepsilon^N$ is determined. We consider the decomposition $1_{\varepsilon^N} = \tilde{P} + (1 - \tilde{P})$. By elementary linear algebra, the decomposition

$$\mathbb{F}^N = \mathrm{Im}\, \tilde{P}_x \oplus \mathrm{Im}(1 - \tilde{P}_x)$$

holds in each fiber of ε^N. It is not difficult to see that $\mathrm{Im}\, \tilde{P}$ and $\mathrm{Im}(1 - \tilde{P})$ are subbundles of ε^N. Therefore $\mathrm{Im}\, \tilde{P} \in \mathcal{A}$ and $\Gamma(X, \mathrm{Im}\, \tilde{P}) \cong M$, which completes the proof. $\square$

Proof of Proposition 1.2(iii). The assertion is clear for equivalences of the full subcategory of product bundles $X \times \mathbb{F}^N$ and the full subcategory of free modules $C_{\mathbb{F}}(X)^N$. Since the pull-back $f^\sharp$ and the extension of scalars $\otimes_{C_{\mathbb{F}}(X')} C_{\mathbb{F}}(X)$ are also additive functors, they preserve direct sums. We remark that if $\mathbb{F} = \mathbb{H}$, then for any right $C_{\mathbb{F}}(X')$-module M and the *left* $C_{\mathbb{F}}(X')$-module $C_{\mathbb{F}}(X)$, we define $M \otimes_{C_{\mathbb{F}}(X')} C_{\mathbb{F}}(X)$, which is a right $C_{\mathbb{F}}(X)$-module. □

§3. A variant of homotopy set

By Proposition 1.2 we see that the functor $K_0 \circ C_{\mathbb{F}}$ is a homotopy functor on the category of topological spaces. In this section we give examples which show that $K_0 \circ C_{\mathbb{F}}$ is not representable and study some fundamental properties of the compactly-bounded homotopy set.

First we make the notion of representability precise. Note that we work in the unpointed setting.

Definition 3.1. A homotopy (contravariant) functor F on the category of topological spaces is said to be *representable* if F is naturally isomorphic to a functor of the form $[\quad, Z]$.

The following examples show that the functor $K_0 \circ C_{\mathbb{F}}$ is not representable.

Example 3.2. Let X be the set $\mathbb{N}$ of natural numbers with the discrete topology. Then $K_0 \circ C_{\mathbb{F}}(\mathbb{N}) = \{f \in \mathrm{map}(\mathbb{N}, \mathbb{Z}) \mid f \text{ is bounded}\}$ by Proposition 1.2(ii), since the vector bundle over $\mathbb{N}$ which corresponds to a finitely generated projective $C_{\mathbb{F}}(\mathbb{N})$-module has a finite-dimensional vector space of dimension, say, $f(n)$, over each $n \in \mathbb{N}$, and $f(n)$ is bounded by the condition that the vector bundle is of finite type. On the other hand, if $K_0 \circ C_{\mathbb{F}}$ is representable, then

$$K_0(C_{\mathbb{F}}(\mathbb{N})) = K_0(C_{\mathbb{F}}(\coprod_{n\in\mathbb{N}} \{n\})) = \prod_{n\in\mathbb{N}} K_0(C_{\mathbb{F}}(\{n\})) = \mathrm{map}(\mathbb{N}, \mathbb{Z}).$$

This implies that $K_0 \circ C_{\mathbb{F}}$ is not representable on the category of topological spaces.

Example 3.3. Let X be the wedge $\bigvee_{n>0} P^n(\mathbb{C})$ of complex projective spaces. Let $\alpha_n \in K_0(C_{\mathbb{C}}(P^n(\mathbb{C})))$ be the class of the canonical line bundle γ_n^1. If $K_0 \circ C_{\mathbb{C}}$ is representable, then there exists a class $\alpha \in K_0(C_{\mathbb{C}}(X))$ whose restriction to $P^n(\mathbb{C})$ is α_n for any n. Then $\alpha \in K_0(C_{\mathbb{C}}(X))$ can be expressed in the form $\alpha = \xi - \eta$ for some vector bundles ξ and η of finite type by Proposition 1.2(ii) and the definition of the Grothendieck group. Since η has a complementary vector bundle η' (Lemma 2.1(iii)), we have

$$\alpha = \xi + \eta' - \varepsilon^m$$

for some m. Hence Lemma 2.1(iv) implies $(c_1(\alpha))^N = 0 \in H^{2N}(X;\mathbb{Z})$ for sufficiently large N, where c_1 denotes the first Chern class. On the other hand, the restriction of $(c_1(\alpha))^N$ to $P^n(\mathbb{C})$ is $(c_1(\alpha_n))^N$, which is nonzero in $H^{2N}(P^n(\mathbb{C});\mathbb{Z})$ for $n > N$. It follows that $K_0 \circ C_{\mathbb{C}}$ cannot be expressed in the form of $[\ \ , Z]$ for any space Z, even on the category of connected CW-complexes.

The same argument applies to $K_0 \circ C_{\mathbb{R}}$ using $X = \bigvee_{n>0} P^n(\mathbb{R})$ and the first Stiefel–Whitney class; and to $K_0 \circ C_{\mathbb{H}}$ using $X = \bigvee_{n>0} P^n(\mathbb{H})$ and the first symplectic Pontryagin class.

We now prove some properties of the compactly-bounded homotopy set; some of them are necessary for the homotopical presentation of $K_0 \circ C_{\mathbb{F}}$.

Let **Toph** be the homotopy category of **Top**, the category of small topological spaces; and **Set** and **Grp** the categories of small sets and small groups, respectively (see MacLane [10, p. 12]).

Lemma 3.4. *The CB-homotopy set* $[\ \ ,\ \ ]_{CB}$ *defines a functor*

$$\mathbf{Top}^{\mathrm{op}} \times \mathbf{Top} \to \mathbf{Set}$$

which factors through $\mathbf{Toph}^{\mathrm{op}} \times \mathbf{Toph} \to \mathbf{Set}$.

Proof. Note that the image of a compact set under a continuous map is compact. This implies the assertion. □

Lemma 3.5. *If* Y *is an* H*-group, then*

$$[\ \ , Y]_{CB} : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Set}$$

has a canonical lifting to the category **Grp**.

Proof. Since the product of two compact sets is compact, there exists a natural isomorphism

$$[X, Y_1 \times Y_2]_{CB} \cong [X, Y_1]_{CB} \times [X, Y_2]_{CB}.$$

This natural isomorphism and Lemma 3.4 imply the assertion. □

Now, we have to introduce the notion of a pointed compactly-bounded homotopy set and establish its fundamental properties in order to calculate $K_0(C_{\mathbb{F}}(X))$ in Section 5.

Definition 3.6. Let X and Y be pointed topological spaces. Then the *pointed compactly-bounded homotopy set* $[X, Y]_{CB_*}$ is defined by

$$[X, Y]_{CB_*} = \{\text{pointed compactly-bounded maps from } X \text{ to } Y\}/{\simeq_{CB_*}}$$

where $\simeq_{CB*}$ is the equivalence relation defined just as $\simeq_{CB}$ but using compactly-bounded homotopies which preserve base-points.

Let $\mathbf{Top}_*$ denote the category of small pointed topological spaces, $\mathbf{Toph}_*$ its homotopy category, and $\mathbf{Set}_*$ the category of small pointed sets (see MacLane [10, p. 12]). Then the analogues of Lemmas 3.4 and 3.5 hold. Further we have the following result.

Lemma 3.7. *If X is a co-H-group, then the functor*

$$[X, \quad]_{CB*} : \mathbf{Top}_* \to \mathbf{Set}_*$$

has a canonical lifting to the category **Grp**.

Proof. Since the union of two compact sets is compact, there exists a natural isomorphism

$$[X_1 \vee X_2, Y]_{CB*} \cong [X_1, Y]_{CB*} \times [X_2, Y]_{CB*}.$$

This natural isomorphism and the analogue of Lemma 3.4 imply the assertion. □

Proposition 3.8. *If X is a co-H-group and Y is an H-group, then the group structures on $[X, Y]_{CB*}$ induced by X and Y coincide.*

Proof. This follows by an argument similar to that in Whitehead [17, Theorem 5.21 in Chapter III]. □

The following proposition describes the relation of $[X, Y]_{CB*}$ to $[X, Y]_{CB}$.

Proposition 3.9. *Let X be a CW-complex and Y a path connected space.*

(i) *The set $[X, Y]_{CB*}$ has a natural action of $\pi_1(Y)$ and the natural map $[X, Y]_{CB*} \to [X, Y]_{CB}$ induces a bijection*

$$[X, Y]_{CB*}/\pi_1(Y) \to [X, Y]_{CB}.$$

(ii) *If Y is a 1-connected space or an H-space, then $[X, Y]_{CB*} \to [X, Y]_{CB}$ is bijective.*

Proof. (i) This is obtained by an argument similar to that in [17, (1.11) in Chapter III].

(ii) If Y is 1-connected, the bijectivity is an immediate consequence of (i). The existence of an H-structure on Y implies that the action of $\pi_1(Y)$ on $[X, Y]_{CB*}$ is trivial. □

Finally, we show that any cofiber sequence implies exactness of the induced sequence of CB_*-homotopy sets.

Proposition 3.10. *Let* $A \xrightarrow{\alpha} X \xrightarrow{i} C_\alpha \xrightarrow{j} \Sigma A \xrightarrow{\Sigma\alpha} \Sigma X \to \cdots$ *be a cofiber sequence. Then the induced sequence of pointed compactly-bounded homotopy sets*

$$\cdots \to [\Sigma X, Z]_{CB*} \xrightarrow{(\Sigma\alpha)^\sharp} [\Sigma A, Z]_{CB*} \xrightarrow{j^\sharp} [C_\alpha, Z]_{CB*} \xrightarrow{i^\sharp} [X, Z]_{CB*} \xrightarrow{\alpha^\sharp} [A, Z]_{CB*}$$

is exact for any pointed space Z.

Proof. Let $g : X \to Z$ be a CB_*-map such that $g \circ \alpha \simeq_{CB*} 0 : A \to Z$. Then there exists a CB_*-homotopy $H : A \times I \to Z$ such that $H(x, 1) = g \circ \alpha(x)$ and $H(x, 1) = *$ for any $x \in A$. Hence we can construct a CB_*-map $\overline{g} : C_\alpha \to Z$ which satisfies $\overline{g} \circ i = g$. $\square$

§4. Proof of Theorem 1.4

The fiber rank defines an epimorphism $K_0 \circ C_{\mathbb{F}}(X) \to [X, \mathbb{Z}]_{CB}$. Let $\widetilde{K}_0 \circ C_{\mathbb{F}}(X)$ be its kernel. Then using the standard splitting, we have the following isomorphism of abelian groups:

$$K_0 \circ C_{\mathbb{F}}(X) \cong [X, \mathbb{Z}]_{CB} \oplus \widetilde{K}_0 \circ C_{\mathbb{F}}(X).$$

Let us interpret the abelian group $\widetilde{K}_0 \circ C_{\mathbb{F}}(X)$ in geometric language. Let $\mathbf{Vect}^{\mathrm{ft}}(X)$ be the set of isomorphism classes of vector bundles of finite type over X, which is an abelian semigroup. An equivalence relation $\sim_s$ on $\mathbf{Vect}^{\mathrm{ft}}(X)$ is defined by

$$\xi_1 \sim_s \xi_2 \;\Leftrightarrow\; \xi_1 \oplus \varepsilon_1 \cong \xi_2 \oplus \varepsilon_2 \text{ for some untwisted bundles } \varepsilon_1 \text{ and } \varepsilon_2.$$

Here, a vector bundle ε of finite type is said to be *untwisted* if, for each $r \in \mathbb{Z}_+$, ε is trivial on the open set $X_{(r)}$ on which the fiber rank of ε is r; this equivalence relation is called *stable equivalence*. Then the set $\mathbf{Vect}^{\mathrm{ft}}(X)/\!\sim_s$ of stable equivalence classes is an abelian group and we have a natural isomorphism

$$\widetilde{K}_0 \circ C_{\mathbb{F}}(X) \cong \mathbf{Vect}^{\mathrm{ft}}(X)/\!\sim_s.$$

First we define a map $\Phi : \widetilde{K}_0 \circ C_{\mathbb{F}}(X) \to [X, G_\infty(\mathbb{F})]_{CB}$ as follows. For a given element of $\widetilde{K}_0 \circ C_{\mathbb{F}}(X)$ we can take a representative ξ^n of constant rank, so we obtain a map

$$f_\xi : X \to G_n(\mathbb{F}^N) \hookrightarrow G_\infty(\mathbb{F})$$

by choosing an embedding $\xi^n \subset \varepsilon^N$ into the product bundle. The same argument as in topological K-theory shows that another choice of a representative $\xi^{n'}$ and of an embedding $\xi^{n'} \subset \varepsilon^{N'}$ determines a map

$$f_{\xi'} : X \to G_{n'}(\mathbb{F}^{N'}) \hookrightarrow G_\infty(\mathbb{F})$$

which is homotopic to f_ξ in a sufficiently large finite Grassmannian $G_m(\mathbb{F}^M)$ (see Husemoller [6, 6.2 Theorem in Chapter 3]). The map f_ξ is uniquely determined up to CB-homotopy, since a finite Grassmannian is compact. Thus the map Φ is well defined.

Secondly we define a map $\Psi : [X, G_\infty(\mathbb{F})]_{CB} \to \widetilde{K}_0 \circ C_{\mathbb{F}}(X)$. For a given class $[f] \in [X, G_\infty(\mathbb{F})]_{CB}$, a representative f factors through a sufficiently large Grassmannian $G_n(\mathbb{F}^N)$. Then the class $[f^\sharp \gamma^n_N] \in \widetilde{K}_0 \circ C_{\mathbb{F}}(X)$ is independent of the representative f by Proposition 1.2(i). Thus the map Ψ is well defined.

Obviously the two maps Φ and Ψ constructed above are mutually inverse, which establishes the natural isomorphism $K_0 \circ C_{\mathbb{F}}(X) \cong [X, \mathbb{Z} \times G_\infty(\mathbb{F})]_{CB}$ as abelian groups. This completes the proof of Theorem 1.4.

§5. Proof of Theorem 1.5

The symbol $[\ ,\ ]_*$ stands for the pointed homotopy set.

Lemma 5.1. *Let A be a CW-complex. $[\Sigma A, G_\infty(\mathbb{F})]_{CB*}$ and $[\Sigma A, G_n(\mathbb{F}^{2n})]_*$ are endowed with group structures induced by the H-group structure on $G_\infty(\mathbb{F})$ and the co-H-group structure on ΣA, respectively. Then the group $[\Sigma A, G_\infty(\mathbb{F})]_{CB*}$ is isomorphic to the group $\varinjlim_n [\Sigma A, G_n(\mathbb{F}^{2n})]_*$.*

Proof. By the definition of the pointed compactly-bounded homotopy set and the filtration $G_\infty(\mathbb{F}) = \bigcup_n G_n(\mathbb{F}^{2n})$, the group $[\Sigma A, G_\infty(\mathbb{F})]_{CB*}$ is naturally identified with the group $\varinjlim [\Sigma A, G_n(\mathbb{F}^{2n})]_*$ as sets. We see that this identification is the one in the category of groups by Proposition 3.8. □

Let F_Y be the homotopy fiber of the profinite completion $\phi_Y : Y \to \widehat{Y}$ of Y.

Lemma 5.2. *Assume that X satisfies the condition in Theorem 1.5.*

(i) *The bijection $[X, Y]_* \cong [X, F_Y]_*$ holds.*

(ii) *The suspension space $\Sigma^m X$ $(m > 0)$ satisfies the condition in Theorem 1.5.*

Proof. (i) Since $\mathrm{map}_*(X, \widehat{Y})$ is weakly contractible, the map $i_\sharp : \mathrm{map}_*(X, F_Y) \to \mathrm{map}_*(X, Y)$ induced by the fiber inclusion $i : F_Y \to Y$ is a weak homotopy equivalence. Hence we have the bijection $i_\sharp : [X, F_Y]_* \cong [X, Y]_*$.

(ii) If $\mathrm{map}_*(X, \widehat{Y})$ is weakly contractible, then also $\Omega^m \mathrm{map}_*(X, \widehat{Y}) \cong \mathrm{map}_*(\Sigma^m X, \widehat{Y})$ is weakly contractible. □

If Y is a nilpotent CW-complex of finite type with finite fundamental group, then the homotopy fiber F_Y is homotopy equivalent to the product of Eilenberg–MacLane spaces $\prod_{i>0} K(\pi_{i+1}(Y) \otimes \widehat{\mathbb{Z}}/\mathbb{Z}, i)$ by Roitberg–Touhey [14, Theorem 1.1].

However, the homotopy equivalence is not natural and it is not possible to naturally identify the homotopy set $[A, F_Y]_*$ with the cohomology group $\prod_{i>0} H^i(A; \pi_{i+1}(Y) \otimes \widehat{\mathbb{Z}}/\mathbb{Z})$ making use of the homotopy equivalence. Nevertheless, the homotopy group $[\Sigma A, F_Y]_*$ can be identified with

$$\prod_{i>0} H^i(\Sigma A; \pi_{i+1}(Y) \otimes \widehat{\mathbb{Z}}/\mathbb{Z})$$

naturally with respect to Y as we see in Proposition 5.4 below.

Lemma 5.3. *Let A be a CW-complex and let $\pi_1, \pi_2, \ldots,$ and $\pi'_1, \pi'_2, \ldots$ be $\mathbb{Q}$-modules. Let $\varphi : \prod_i K(\pi_i, i) \to \prod_i K(\pi'_i, i)$ be a pointed map between the products of Eilenberg–MacLane spaces. Then the induced map*

$$\varphi_\sharp : [\Sigma A, \prod_i K(\pi_i, i)]_* \to [\Sigma A, \prod_i K(\pi'_i, i)]_*$$

on the pointed homotopy sets is identified with $\prod_i H^i(\Sigma A; \pi_i(\varphi))$ under the standard identifications $[\Sigma A, \prod_i K(\pi_i, i)]_*$ with $\prod_i H^i(\Sigma A; \pi_i)$ and $[\Sigma A, \prod_i K(\pi_i{}', i)]_*$ with $\prod_i H^i(\Sigma A; \pi_i{}')$.

Proof. We may assume $\pi'_i = 0$ $(i \neq n)$. For an element $(\alpha_i) \in \prod_i H^i(\Sigma A; \pi_i)$, we have to compute its image $\alpha'_n \in H^n(\Sigma A; \pi'_n)$ under $\varphi_\sharp$, or equivalently, the image of the fundamental class $e'_n \in H^n(K(\pi'_n, n); \pi'_n)$ under the homomorphism induced by the composite

$$\Sigma A \xrightarrow{f} \prod_i K(\pi_i, i) \xrightarrow{\varphi} K(\pi'_n, n),$$

where f is the pointed map corresponding to (α_i). There exists a natural isomorphism

$$H^n(\prod_i K(\pi_i, i); \pi'_n) \cong H^n(K(\pi_n, n); \pi'_n) \oplus \pi'_n \otimes [\bigotimes_{i<n} H^*(K(\pi_i, i); \mathbb{Q})]^n,$$

where $[\ ]^n$ is the component of degree n. Any element of $[\bigotimes_{i<n} H^*(K(\pi_i, i); \mathbb{Q})]^n$ is a linear combination of products of elements of degree $< n$ and the cohomology product of $H^*(\Sigma A; \mathbb{Q})$ is trivial. Then any element of the second component of $H^n(\prod_i K(\pi_i, i); \pi'_n)$ is mapped to 0 by the homomorphism induced by $f : \Sigma A \to \prod_i K(\pi_i, i)$. Thus we have

$$(\varphi \circ f)^\sharp e'_n = f^\sharp(\varphi^\sharp e'_n) = f^\sharp(\text{the first component of } \varphi^\sharp e'_n),$$

which implies the assertion. $\square$

Proposition 5.4. *Let A be a CW-complex and Y a nilpotent CW-complex of finite type with finite fundamental group. Then there exists an isomorphism of groups*

$$[\Sigma A, F_Y]_* \cong \prod_{i>0} H^i(\Sigma A; \pi_{i+1}(Y) \otimes \widehat{\mathbb{Z}}/\mathbb{Z})$$

which is natural with respect to Y.

Proof. Take a homotopy equivalence h from F_Y to $\prod_{i>0} K(\pi_{i+1}(Y) \otimes \widehat{\mathbb{Z}}/\mathbb{Z}, i)$ which induces the identity on their homotopy groups. Then by Lemma 5.3 the bijection

$$[\Sigma A, F_Y]_* \cong \prod_{i>0} H^i(\Sigma A; \pi_{i+1}(Y) \otimes \widehat{\mathbb{Z}}/\mathbb{Z})$$

induced by h does not depend on the choice of h, and this bijection is natural with respect to Y. We see that the co-H-group structure on ΣA and the H-group structure on $\prod_{i>0} K(\pi_{i+1}(Y) \otimes \widehat{\mathbb{Z}}/\mathbb{Z}, i)$ determine the same group structure on $[\Sigma A, \prod_{i>0} K(\pi_{i+1}(Y) \otimes \widehat{\mathbb{Z}}/\mathbb{Z}, i)]_*$ (cf. Whitehead [17, Theorem 5.21 in Chapter III]). This shows that the bijection in question is an isomorphism of groups. □

Proof of Theorem 1.5. We calculate $K_0(C_{\mathbb{F}}(\Sigma^m X))$ $(m > 0)$ for a CW-complex X satisfying the condition in Theorem 1.5. The reader is referred to Chapter 8 of Husemoller [6] for the properties of $G_n(\mathbb{F}^N)$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

First suppose that $\mathbb{F} = \mathbb{C}$. We have

$$\begin{aligned} K_0(C_{\mathbb{C}}(\Sigma^m X)) &\cong [\Sigma^m X, \mathbb{Z} \times G_\infty(\mathbb{C})]_{CB} \cong \mathbb{Z} \oplus [\Sigma^m X, G_\infty(\mathbb{C})]_{CB} \\ &\cong \mathbb{Z} \oplus [\Sigma^m X, G_\infty(\mathbb{C})]_{CB*} \end{aligned}$$

by Theorem 1.4 and Proposition 3.9(ii). Further we have

$$\begin{aligned} K_0(C_{\mathbb{C}}(\Sigma^m X)) &\cong \mathbb{Z} \oplus \varinjlim_n [\Sigma^m X, G_n(\mathbb{C}^{2n})]_* \cong \mathbb{Z} \oplus \varinjlim_n [\Sigma^m X, F_{G_n(\mathbb{C}^{2n})}]_* \\ &\cong \mathbb{Z} \oplus \varinjlim_n \prod_{i>0} H^i(\Sigma^m X; \pi_{i+1}(G_n(\mathbb{C}^{2n})) \otimes \widehat{\mathbb{Z}}/\mathbb{Z}) \end{aligned}$$

by Lemmas 5.1 and 5.2, and Proposition 5.4. Since $G_n(\mathbb{C}^{2n})$ has nontrivial rational homotopy groups in only finitely many dimensions and $\varinjlim_n$ and $\bigoplus_{i>0}$ commute, we have

$$\begin{aligned} K_0(C_{\mathbb{C}}(\Sigma^m X)) &\cong \mathbb{Z} \oplus \varinjlim_n \bigoplus_{i>0} H^i(\Sigma^m X; \pi_{i+1}(G_n(\mathbb{C}^{2n})) \otimes \widehat{\mathbb{Z}}/\mathbb{Z}) \\ &\cong \mathbb{Z} \oplus \bigoplus_{i>0} H^i(\Sigma^m X; \pi_{i+1}(G_\infty(\mathbb{C})) \otimes \widehat{\mathbb{Z}}/\mathbb{Z}). \end{aligned}$$

By Bott periodicity (cf. [6, 5.2 Corollary in Chapter 8]), we have

$$K_0(C_{\mathbb{C}}(\Sigma^m X)) \cong \mathbb{Z} \oplus \bigoplus_{i>0} H^{2l-1}(\Sigma^m X; \widehat{\mathbb{Z}}/\mathbb{Z}).$$

An argument similar to the above proof applies to the case $\mathbb{F} = \mathbb{H}$, where $G_n(\mathbb{H}^{2n})$ is a 1-connected CW-complex of finite type for any $n \in \mathbb{N}$.

Let us now proceed to the case $\mathbb{F} = \mathbb{R}$. We use the filtration $G_\infty(\mathbb{R}) = \bigcup_{n \text{ odd}} G_n(\mathbb{R}^{2n})$. Then the same argument works since $G_n(\mathbb{R}^{2n})$ (n odd) is a nilpotent (in fact simple) CW-complex of finite type with finite fundamental group (see Glover and Homer [4, Proposition 2.2]). This completes the proof of Theorem 1.5. □

Proof of Remark 1.7(ii). Any vector bundle ξ of finite type over X has a classifying map with image contained in some finite Grassmannian $G_n(\mathbb{F}^N)$. By Lemma 5.2(i), we have $[X, G_n(\mathbb{F}^N)]_* \cong [X, F_{G_n(\mathbb{F}^N)}]_*$, which implies that every map from X to $G_n(\mathbb{F}^N)$ is a phantom map (see McGibbon [11, Theorem 5.1]). □

§6. Definitions of numerable coverings

Some different definitions of numerable coverings are known in the literature. In this section, to clarify the statements in Lemma 2.1, especially (vi), we quote three definitions of the numerable covering and prove that they are equivalent for open coverings.

- (Dold [2, 2.1 Definition]) A (not necessarily open) covering $\{V_\lambda\}_{\lambda\in\Lambda}$ of B is called *numerable* if it admits a refinement by a locally finite partition of unity, i.e. if there exists a locally finite partition of unity $\{\pi_\gamma : B \to [0,1]\}_{\gamma\in\Gamma}$ (a *numeration* of $\{V_\lambda\}$) such that every set $\pi_\gamma^{-1}(0,1]$ is contained in some V_λ.
- (Husemoller [6, 9.1 Definition in Chapter 4]) An open covering $\{U_i\}_{i\in S}$ of topological space B is *numerable* provided there exists a (locally finite) partition of unity $\{u_i\}_{i\in S}$ such that $\overline{u_i^{-1}(0,1]} \subset U_i$ for each $i \in S$.
- James [8, Definition (7.25)]) Let $\{X_j\}_{j\in J}$ be a covering of the space X. A *numeration* of $\{X_j\}$ is a locally finite partition of unity $\{\pi_j\}$ such that $\pi_j^{-1}(0,1] \subset X_j$ for each index j. If there exists a numeration, the covering is said to be *numerable*. A covering which is the family of cozero sets of a locally finite partition of unity is said to be *numerically defined*.

Proposition 6.1. *Let J be any index set. Let $\{U_j\}_{j\in J}$ be an open covering of a space X. Then the following statements are equivalent.*

(i) *$\{U_j\}_{j\in J}$ is numerable in the sense of Dold [2].*

(ii) *$\{U_j\}_{j\in J}$ is numerable in the sense of James [8].*

(iii) *$\{U_j\}_{j\in J}$ is numerable in the sense of Husemoller [6].*

Proof. (i)⇒(ii). We may assume that J is a *well-ordered* set. Let $\{\pi_\gamma : B \to [0,1]\}_{\gamma\in\Gamma}$ be a locally finite partition of unity such that each set $\pi_\gamma^{-1}(0,1]$ is contained in some U_j. We define a function

$$F : \{\pi_\gamma\}_{\gamma\in\Gamma} \to J$$

by $F(\pi_\gamma) = \min\{j \mid \pi_\gamma^{-1}(0,1] \text{ is contained in } U_j\}$. Let $\Gamma_j = \{\pi_\gamma \mid F(\pi_\gamma) = j\}$. It follows that

$$\Gamma_i \cap \Gamma_j = \emptyset \ (i \neq j) \quad \text{and} \quad \Gamma = \bigcup_{j=1}^{n} \Gamma_j.$$

We define $u_j = \sum_{\pi_\gamma\in\Gamma_j} \pi_\gamma$ if $\Gamma_j \neq \emptyset$ and $u_j = 0$ if $\Gamma_j = \emptyset$. Then we get a locally finite partition of unity $\{u_j\}_{j\in J}$ such that each set $u_j^{-1}(0,1]$ is contained in U_j.

(ii)⇒(iii). By Proposition (7.24) of James [8] (see a remark after Definition (7.25) of James [8]).

(iii)⇒(i). This is a consequence of the definitions. □

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