A Numerical Characterization of Polarized Manifolds $(X, \mathcal{L})$ with $K_X = -(n-i)\mathcal{L}$ by the ith Sectional Geometric Genus and the ith Δ-genus

by

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Abstract

Let $(X, \mathcal{L})$ be a polarized manifold of dimension n. By using the ith sectional geometric genus and the ith Δ-genus, we will give a numerical characterization of $(X, \mathcal{L})$ with $K_X = -(n-i)\mathcal{L}$ for the following cases: (i) $i = 2$, (ii) $i = 3$ and $n \geq 5$, (iii) $\max\{2, \dim \mathrm{Bs}|\mathcal{L}| + 2\} \leq i \leq n - 1$.

§1. Introduction

Let X be a projective variety with $\dim X = n$ defined over the field of complex numbers, and let $\mathcal{L}$ be an ample line bundle on X. Then $(X, \mathcal{L})$ is called a *polarized variety*. If X is smooth, then we say that $(X, \mathcal{L})$ is a *polarized manifold*. The main purpose of this paper is to give a numerical characterization of $(X, \mathcal{L})$ with $K_X = -(n-i)\mathcal{L}$. The following is well-known:

Proposition 1.1. *Let $(X, \mathcal{L})$ be a polarized manifold of dimension $n \geq 2$.*

(1) $K_X = -(n+1)\mathcal{L}$ *(resp. $K_X = -n\mathcal{L}$) if and only if $2g(X, \mathcal{L}) - 2 = -2\mathcal{L}^n$ (resp. $2g(X, \mathcal{L}) - 2 = -\mathcal{L}^n$).*

(2) *([6, (1.9) Theorem]) $K_X = -(n-1)\mathcal{L}$ (then $(X, \mathcal{L})$ is called a* Del Pezzo *manifold) if and only if $2g(X, \mathcal{L}) - 2 = 0$ and $\Delta(X, \mathcal{L}) = 1$.*

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Here $g(X, \mathcal{L})$ (resp. $\Delta(X, \mathcal{L})$) denotes the sectional genus (resp. the Δ-genus) of $(X, \mathcal{L})$.

As the next step, we want to give a numerical characterization of polarized manifolds with $K_X = -(n-i)\mathcal{L}$ for $i \geq 2$ by using some invariants of $(X, \mathcal{L})$. In [14] and [16], we define the ith sectional geometric genus $g_i(X, \mathcal{L})$ and the ith Δ-genus $\Delta_i(X, \mathcal{L})$ of $(X, \mathcal{L})$ for every integer i with $0 \leq i \leq n$. The ith sectional geometric genus (resp. the ith Δ-genus) is a generalization of the sectional genus (resp. Δ-genus), namely, $g_1(X, \mathcal{L}) = g(X, \mathcal{L})$ (resp. $\Delta_1(X, \mathcal{L}) = \Delta(X, \mathcal{L})$). By looking at Proposition 1.1 above, the author thought that maybe one could give a numerical characterization of polarized manifolds $(X, \mathcal{L})$ with $K_X = -(n-i)\mathcal{L}$ by using the ith sectional geometric genus and the ith Δ-genus.

In this paper, as the main results, we prove the following:

Theorem 1.1 (see Theorems 4.2.1, 4.3.1 and 4.4.1 below). *Let $(X, \mathcal{L})$ be a polarized manifold of dimension $n \geq 3$. Assume that one of the following cases holds:*

(a) $i = 2$.

(b) $i = 3$ *and* $n \geq 5$.

(c) $\max\{2, \dim \mathrm{Bs}|\mathcal{L}| + 2\} \leq i \leq n - 1$.

Then the following are equivalent to one another:

$C(i, 1)$: $K_X + (n - i)\mathcal{L} = \mathcal{O}_X$.
$C(i, 2)$: $\Delta_i(X, \mathcal{L}) = 1$ *and* $2g_1(X, \mathcal{L}) - 2 = (i - 1)\mathcal{L}^n$.
$C(i, 3)$: $\Delta_i(X, \mathcal{L}) > 0$ *and* $2g_1(X, \mathcal{L}) - 2 = (i - 1)\mathcal{L}^n$.
$C(i, 4)$: $g_i(X, \mathcal{L}) = 1$ *and* $2g_1(X, \mathcal{L}) - 2 = (i - 1)\mathcal{L}^n$.
$C(i, 5)$: $g_i(X, \mathcal{L}) > 0$ *and* $2g_1(X, \mathcal{L}) - 2 = (i - 1)\mathcal{L}^n$.

Notation and conventions

We say that X is a *variety* if X is an integral separated scheme of finite type. In particular X is irreducible and reduced if X is a variety. Varieties are always assumed to be defined over the field of complex numbers. In this article, we shall study mainly smooth projective varieties. The words "line bundles" and "Cartier divisors" are used interchangeably. The tensor products of line bundles are denoted additively.

- $\mathcal{O}(D)$: invertible sheaf associated with a Cartier divisor D on X.
- $\mathcal{O}_X$: the structure sheaf of X.
- $\chi(\mathcal{F})$: the Euler–Poincaré characteristic of a coherent sheaf $\mathcal{F}$.

- $h^i(\mathcal{F}) := \dim H^i(X, \mathcal{F})$ for a coherent sheaf $\mathcal{F}$ on X.
- $h^i(D) := h^i(\mathcal{O}(D))$ for a Cartier divisor D.
- $q(X)$ $(= h^1(\mathcal{O}_X))$: the irregularity of X.
- $h^i(X, \mathbb{C}) := \dim H^i(X, \mathbb{C})$.
- $b_i(X) := h^i(X, \mathbb{C})$.
- K_X: the canonical divisor of X.
- $\mathbb{P}^n$: the projective space of dimension n.
- $\mathbb{Q}^n$: a quadric hypersurface in $\mathbb{P}^{n+1}$.
- $\sim$ (or $=$): linear equivalence.
- $\det(\mathcal{E}) := \bigwedge^r \mathcal{E}$, where $\mathcal{E}$ is a vector bundle of rank r on X.
- $\mathbb{P}_X(\mathcal{E})$: the projective space bundle associated with a vector bundle $\mathcal{E}$ on X.
- $H(\mathcal{E})$: the tautological line bundle on $\mathbb{P}_X(\mathcal{E})$.
- $\mathcal{F}^\vee := \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.
- $c_i(\mathcal{E})$: the i-th Chern class of a vector bundle $\mathcal{E}$.
- $c_i(X) := c_i(\mathcal{T}_X)$, where $\mathcal{T}_X$ is the tangent bundle of a smooth projective variety X.

For a real number m and a non-negative integer n, let

$$[m]^n := \begin{cases} m(m+1)\cdots(m+n-1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0, \end{cases}$$

$$[m]_n := \begin{cases} m(m-1)\cdots(m-n+1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Then for n fixed, $[m]^n$ and $[m]_n$ are polynomials in m of degree n.

For any non-negative integer n,

$$n! := \begin{cases} [n]_n & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Assume that m and n are integers with $n \geq 0$. Then we put

$$\binom{m}{n} := \frac{[m]_n}{n!}.$$

We note that $\binom{m}{n} = 0$ if $0 \leq m < n$, and $\binom{m}{0} = 1$.

§2. Preliminaries

Here we list up some facts which will be used later.

Definition 2.1. (1) Let X (resp. Y) be an n-dimensional projective manifold, and let $\mathcal{L}$ (resp. $\mathcal{A}$) be an ample line bundle on X (resp. Y). Then $(X, \mathcal{L})$ is called a *simple blowing up of* $(Y, \mathcal{A})$ if there exists a birational morphism $\pi : X \to Y$ such that π is a blowing up at a point of Y and $\mathcal{L} = \pi^*(\mathcal{A}) - E$, where E is the exceptional divisor.

(2) Let X (resp. M) be an n-dimensional projective manifold, and let $\mathcal{L}$ (resp. $\mathcal{A}$) be an ample line bundle on X (resp. M). Then we say that $(M, \mathcal{A})$ is a *reduction of* $(X, \mathcal{L})$ if $(X, \mathcal{L})$ is obtained by a composite of simple blowing ups of $(M, \mathcal{A})$, and $(M, \mathcal{A})$ is not obtained by a simple blowing up of any polarized manifold. The morphism $\mu : X \to M$ is called the *reduction map*.

Definition 2.2. Let $(X, \mathcal{L})$ be a polarized manifold of dimension n. We say that $(X, \mathcal{L})$ is a *scroll* (resp. *quadric fibration*, *Del Pezzo fibration*) *over a normal projective variety* Y *with* $\dim Y = m$ if there exists a surjective morphism with connected fibers $f : X \to Y$ such that $K_X + (n-m+1)\mathcal{L} = f^*\mathcal{A}$ (resp. $K_X + (n-m)\mathcal{L} = f^*\mathcal{A}$, $K_X + (n-m-1)\mathcal{L} = f^*\mathcal{A}$) for some ample line bundle $\mathcal{A}$ on Y.

Remark 2.1. If $(X, \mathcal{L})$ is a scroll over a smooth curve C (resp. a smooth projective surface S) with $\dim X = n \geq 3$, then by [5, (3.2.1) Theorem] and [4, Proposition 3.2.1 and Theorem 14.1.1] there exists an ample vector bundle $\mathcal{E}$ of rank n (resp. $n-1$) on C (resp. S) such that $(X, \mathcal{L}) \cong (\mathbb{P}_C(\mathcal{E}), H(\mathcal{E}))$ (resp. $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$).

Theorem 2.1. Let $(X, \mathcal{L})$ be a polarized manifold with $\dim X = n \geq 3$. Then $(X, \mathcal{L})$ is one of the following types.

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

(2) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(3) A scroll over a smooth projective curve.

(4) $K_X \sim -(n-1)\mathcal{L}$, that is, $(X, \mathcal{L})$ is a Del Pezzo manifold.

(5) A quadric fibration over a smooth curve.

(6) A scroll over a smooth projective surface.

(7) Let $(M, \mathcal{A})$ be a reduction of $(X, \mathcal{L})$.

    (7.1) $n = 4$, $(M, \mathcal{A}) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$.

    (7.2) $n = 3$, $(M, \mathcal{A}) = (\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$.

    (7.3) $n = 3$, $(M, \mathcal{A}) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$.

    (7.4) $n = 3$, M is a $\mathbb{P}^2$-bundle over a smooth curve C and for any fiber F' of it, $(F', \mathcal{A}|_{F'}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.

    (7.5) $K_M \sim -(n-2)\mathcal{A}$, that is, $(M, \mathcal{A})$ is a Mukai manifold.

(7.6) $(M, \mathcal{A})$ *is a Del Pezzo fibration over a smooth curve.*

(7.7) $(M, \mathcal{A})$ *is a quadric fibration over a normal surface.*

(7.8) $n \geq 4$ *and* $(M, \mathcal{A})$ *is a scroll over a normal projective variety of dimension* 3. *In this case,* $(X, \mathcal{L}) \cong (M, \mathcal{A})$ *by* [4, Corollary 7.6.7].

(7.9) $K_M + (n-2)\mathcal{A}$ *is nef and big.*

Proof. See [4, Proposition 7.2.2, and Theorems 7.2.4, 7.3.2, 7.3.4, and 7.5.3]. See also [9, Chapter II, (11.2), (11.7), and (11.8)]. $\qquad\square$

Remark 2.2. Let $(X, \mathcal{L})$ be a polarized manifold with $\dim X = n \geq 3$.

(1) $\kappa(K_X + (n-2)\mathcal{L}) = -\infty$ if and only if $(X, \mathcal{L})$ is one of the types from (1) to (7.4) in Theorem 2.1.

(2) $\kappa(K_X + (n-2)\mathcal{L}) = 0$ if and only if $(X, \mathcal{L})$ is (7.5) in Theorem 2.1.

(3) $\kappa(K_X + (n-2)\mathcal{L}) \geq 1$ if and only if $(X, \mathcal{L})$ is one of the types from (7.6) to (7.9) in Theorem 2.1.

Remark 2.3. Let $(X, \mathcal{L})$ be a polarized manifold with $\dim X = n \geq 3$ and let $(M, \mathcal{A})$ be a reduction of $(X, \mathcal{L})$. Then by [4, Corollary 7.6.7], we see that $(X, \mathcal{L}) \cong (M, \mathcal{A})$ if $(M, \mathcal{A})$ is a scroll. See also Theorem 2.1, case (7.8).

Definition 2.3 ([4, 7.5.7 Definition-Notation]). Let $(X, \mathcal{L})$ be a polarized manifold of dimension $n \geq 3$, and let $(M, \mathcal{A})$ be a reduction of $(X, \mathcal{L})$. Assume that $K_M + (n-2)\mathcal{A}$ is nef and big. Then for large $m \gg 0$ the morphism $\varphi : M \to W$ associated to $|m(K_M + (n-2)\mathcal{A})|$ has connected fibers and normal image W. Note that there exists an ample line bundle $\mathcal{K}$ on W such that $K_M + (n-2)\mathcal{A} = \varphi^*(\mathcal{K})$. Let $\mathcal{D} := (\varphi_* \mathcal{A})^{\vee\vee}$, where $^{\vee\vee}$ denotes the double dual. Then the pair $(W, \mathcal{D})$ together with φ is called the *second reduction* of $(X, \mathcal{L})$.

Remark 2.4. (1) If $K_M + (n-2)\mathcal{A}$ is nef and big but not ample, then φ is equal to the nef value morphism of $\mathcal{A}$.

(2) If $K_M + (n-2)\mathcal{A}$ is ample, then φ is an isomorphism.

(3) If $n \geq 4$, then W has isolated terminal singularities and is 2-factorial. Moreover if n is even, then X is Gorenstein (see [4, Proposition 7.5.6]).

We note that $\kappa(K_X + (n-1)\mathcal{L}) = -\infty$ (resp. $\kappa(K_X + (n-2)\mathcal{L}) = -\infty$) if and only if $(X, \mathcal{L})$ is one of the types from (1) to (3) (resp. from (1) to (7.4)) in Theorem 2.1. Now we consider the case where $\kappa(K_X + (n-3)\mathcal{L}) = -\infty$. If $(X, \mathcal{L})$ is one of the types from (1) to (7.8) in Theorem 2.1, then $\kappa(K_X + (n-3)\mathcal{L}) = -\infty$. So we assume that $K_M + (n-2)\mathcal{A}$ is nef and big. Then there exist a normal projective variety W with only 2-factorial isolated terminal singularities, a bira-

tional morphism $\phi_2 : M \to W$ and an ample line bundle $\mathcal{K}$ on W such that $K_M + (n-2)\mathcal{A} = (\phi_2)^*(\mathcal{K})$. Let $\mathcal{D} := (\phi_2)_*(\mathcal{A})^{\vee\vee}$. Then $\mathcal{D}$ is a 2-Cartier divisor on W and $\mathcal{K} = K_W + (n-2)\mathcal{D}$ (see [4, Lemma 7.5.8]). Thus $(W, \mathcal{D})$ is the second reduction of $(X, \mathcal{L})$ (see Definition 2.3). We note that if $K_M + (n-2)\mathcal{A}$ is ample, then $(W, \mathcal{K}) \cong (M, K_M + (n-2)\mathcal{A})$.

Then the following properties hold:

(1) $\kappa(K_X + (n-3)\mathcal{L}) = \kappa(K_W + (n-3)\mathcal{K})$ [4, Corollary 7.6.2].
(2) $(n-2)(K_W + (n-3)\mathcal{D}) = K_W + (n-3)\mathcal{K}$ and $K_M + (n-3)\mathcal{A} = \phi_2^*(K_W + (n-3)\mathcal{D}) + \Delta$ for an exceptional $\mathbb{Q}$-effective divisor Δ of ϕ_2. Therefore

$$
\begin{aligned}
m(n-2)(K_X + (n-3)\mathcal{L}) &= m(n-2)\phi_1^*(K_M + (n-3)\mathcal{A}) + E_1 \\
&= m(n-2)\phi_1^* \circ \phi_2^*(K_W + (n-3)\mathcal{D}) + E_1 + m(n-2)\Delta \\
&= m\phi_1^* \circ \phi_2^*(K_W + (n-3)\mathcal{K}) + E_1 + m(n-2)\Delta.
\end{aligned}
$$

(Here $\phi_1 : X \to M$ is a reduction of $(X, \mathcal{L})$ and E_1 is a ϕ_1-exceptional effective divisor.)
(3) $h^0((n-2)m(K_X + (n-3)\mathcal{L})) = h^0(m(K_W + (n-3)\mathcal{K}))$ for every integer m with $m \geq 1$.

If $\tau(\mathcal{K}) \leq n-3$, then by the above we see that $\kappa(K_X + (n-3)\mathcal{L}) \geq 0$. (Here $\tau(\mathcal{K})$ denotes the nef value of $\mathcal{K}$.) So we may assume that $\tau(\mathcal{K}) > n-3$.

If $n \geq 5$, then $(W, \mathcal{K})$ with $\tau(\mathcal{K}) > n-3$ is one of some special types by [4, Theorems 7.7.2, 7.7.3 and 7.7.5]. So we can get the following:

Proposition 2.1. *Let $(X, \mathcal{L})$ be a polarized manifold of dimension $n \geq 5$, $(M, \mathcal{A})$ a reduction of $(X, \mathcal{L})$, and $(W, \mathcal{K})$ the second reduction of $(X, \mathcal{L})$. Then $\kappa(K_X + (n-3)\mathcal{L}) = -\infty$ if and only if $(X, \mathcal{L})$ satisfies one of the following:*

(1) *$(X, \mathcal{L})$ is one of the types (1)–(6), (7.5)–(7.8) in Theorem 2.1.*
(2) *$K_M + (n-2)\mathcal{A}$ is nef and big, and $(W, \mathcal{K})$ is one of the following:*

   (2.1) *$(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1))$.*
   (2.2) *1, 2 or 3 in [4, Theorem 7.7.5].*

Proof. See [4, Theorems 7.7.2, 7.7.3, 7.7.5 and Proposition 7.7.9]. $\qquad\square$

Furthermore we need the following two lemmas:

Lemma 2.1. *Let $(X, \mathcal{L})$ be a polarized manifold of dimension $n \geq 5$. If $\kappa(K_X + (n-3)\mathcal{L}) = -\infty$, then $h^j(\mathcal{O}_X) = 0$ for any $j \geq 3$ unless $(X, \mathcal{L})$ is a scroll over a normal projective variety of dimension 3, in which case $h^j(\mathcal{O}_X) = 0$ for all $j \geq 4$.*

Proof. By assumption and Proposition 2.1, $(X, \mathcal{L})$ satisfies either (1), (2.1) or (2.2) in Proposition 2.1. Note that since $h^j(\mathcal{O}_X) = h^j(\mathcal{O}_M) = h^j(\mathcal{O}_W)$, we only have to prove that $h^j(\mathcal{O}_W) = 0$. But by Proposition 2.1 this is easy and left to the reader. $\qquad\square$

Remark 2.5. By using a result of Fujita [10, (4.∞)], we can see that Lemma 2.1 holds for $n = 4$.

Lemma 2.2. *Let $(X, \mathcal{L})$ be a polarized manifold of dimension $n \geq 3$, and let $(M, \mathcal{A})$ be a reduction of $(X, \mathcal{L})$. Assume that $K_M + (n-2)\mathcal{A}$ is nef and big. Let $(W, \mathcal{D})$ be the second reduction of $(X, \mathcal{L})$ and $\varphi : M \to W$ its morphism. (Here we use the notation of Definition 2.3.) Then $h^j(\mathcal{A}) = h^j(\mathcal{D})$ for every integer $j \geq 3$.*

Proof. As in Definition 2.3, there exists an ample line bundle $\mathcal{K}$ on Y such that $K_M + (n-2)\mathcal{A} = \varphi^*(\mathcal{K})$. By [24, Theorem 1-2-5] we have $R^j\varphi_*(K_M + (n-1)\mathcal{A}) = 0$. On the other hand $R^j\varphi_*\mathcal{O}(K_M + (n-1)\mathcal{A}) = R^j\varphi_*(\varphi^*(\mathcal{K}) \otimes \mathcal{A}) = \mathcal{K} \otimes R^j\varphi_*(\mathcal{A})$. Therefore $R^j\varphi_*(\mathcal{A}) = 0$ and $h^j(\mathcal{A}) = h^j(\varphi_*(\mathcal{A}))$ for every positive integer j.

Since $\mathcal{A}$ is a line bundle on M, we see that $\varphi_*(\mathcal{A})$ is a torsion free coherent sheaf on W. Then there exists an injective homomorphism $\mu : \varphi_*(\mathcal{A}) \to (\varphi_*(\mathcal{A}))^{\vee\vee}$. Hence we get the exact sequence

$$0 \to \varphi_*(\mathcal{A}) \to (\varphi_*(\mathcal{A}))^{\vee\vee} \to \operatorname{Coker} \mu \to 0.$$

Note that $\dim \operatorname{Supp}(\operatorname{Coker}\mu) \leq 1$ because there exists a closed subset Z on W such that $\dim Z \leq 1$ and $M \backslash \varphi^{-1}(Z) \cong W \backslash Z$. Therefore $h^j(\operatorname{Coker}\mu) = 0$ for every $j \geq 2$ by [21, Theorem 2.7 in Chapter III] or [22, Theorem 4.6*]. Hence $h^j(\varphi_*(\mathcal{A})) = h^j((\varphi_*(\mathcal{A}))^{\vee\vee})$ for $j \geq 3$. Since $\mathcal{D} = (\varphi_*(\mathcal{A}))^{\vee\vee}$, we get the assertion. $\qquad\square$

Definition 2.4. Let X be a smooth projective variety and let $\mathcal{F}$ be a vector bundle on X. Then for every integer $j \geq 0$, the *j-th Segre class $s_j(\mathcal{F})$ of $\mathcal{F}$* is defined by the equation $c_t(\mathcal{F}^\vee)s_t(\mathcal{F}) = 1$, where $c_t(\mathcal{F}^\vee)$ is the Chern polynomial of $\mathcal{F}^\vee$ and $s_t(\mathcal{F}) = \sum_{j \geq 0} s_j(\mathcal{F})t^j$.

Remark 2.6. (1) Let X be a smooth projective variety and let $\mathcal{F}$ be a vector bundle on X. Let $\tilde{s}_j(\mathcal{F})$ be the Segre class which is defined in [20, Chapter 3]. Then $s_j(\mathcal{F}) = \tilde{s}_j(\mathcal{F}^\vee)$.

(2) For every $i \geq 1$, $s_i(\mathcal{F})$ can be written by using the Chern classes $c_j(\mathcal{F})$ with $1 \leq j \leq i$. (For example, $s_1(\mathcal{F}) = c_1(\mathcal{F})$, $s_2(\mathcal{F}) = c_1(\mathcal{F})^2 - c_2(\mathcal{F})$, and so on.)

§3. Review on the ith sectional geometric genus and the ith Δ-genus of polarized varieties

Here we review the ith sectional geometric genus and the ith Δ-genus of polarized varieties $(X, \mathcal{L})$ for every integer i with $0 \leq i \leq \dim X$. There have been many investigations of $(X, \mathcal{L})$ via the sectional genus and the Δ-genus. In order to analyze $(X, \mathcal{L})$ more deeply, the author extended these notions. In [14, Definition 2.1] we defined an invariant called the *ith sectional geometric genus* which is intended to be a generalization of the sectional genus. First we recall its definition.

Notation 3.1. Let $(X, \mathcal{L})$ be a polarized variety of dimension n, and let $\chi(t\mathcal{L})$ be the Euler–Poincaré characteristic of $t\mathcal{L}$. Then $\chi(t\mathcal{L})$ is a polynomial in t of degree n, and we can write it as

$$\chi(t\mathcal{L}) = \sum_{j=0}^{n} \chi_j(X, \mathcal{L}) \binom{t+j-1}{j}.$$

Definition 3.1 ([14, Definition 2.1]). Let $(X, \mathcal{L})$ be a polarized variety of dimension n. Then for any integer i with $0 \leq i \leq n$ the *ith sectional geometric genus* $g_i(X, \mathcal{L})$ *of* $(X, \mathcal{L})$ is defined by

$$g_i(X, \mathcal{L}) = (-1)^i(\chi_{n-i}(X, \mathcal{L}) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-i}(-1)^{n-i-j}h^{n-j}(\mathcal{O}_X).$$

Remark 3.1. (1) Since $\chi_{n-i}(X, \mathcal{L}) \in \mathbb{Z}$, the invariant $g_i(X, \mathcal{L})$ is an integer by definition.

(2) If $i = \dim X = n$, then $g_n(X, \mathcal{L}) = h^n(\mathcal{O}_X)$ and $g_n(X, \mathcal{L})$ is independent of $\mathcal{L}$.

(3) If $i = 0$, then $g_0(X, \mathcal{L}) = \mathcal{L}^n$.

(4) If $i = 1$, then $g_1(X, \mathcal{L}) = g(X, \mathcal{L})$, where $g(X, \mathcal{L})$ is the sectional genus of $(X, \mathcal{L})$. If X is smooth, then $g_1(X, \mathcal{L}) = 1 + (1/2)(K_X + (n-1)\mathcal{L})\mathcal{L}^{n-1}$, where K_X denotes the canonical line bundle on X.

(5) Let $(X, \mathcal{L})$ be a polarized manifold of dimension n and let $(M, \mathcal{A})$ be a reduction of $(X, \mathcal{L})$. Then $g_i(X, \mathcal{L}) = g_i(M, \mathcal{A})$ for every integer i with $1 \leq i \leq n$.

The following are main problems about the ith sectional geometric genus.

Problem 3.1. (i) *Does the ith sectional geometric genus have properties similar to those of the sectional genus? In particular:*

(i.1) *Does $g_i(X, \mathcal{L}) \geq 0$ hold? More strongly, does $g_i(X, \mathcal{L}) \geq h^i(\mathcal{O}_X)$ hold?*

(i.2) *Can we get the ith sectional geometric genus version of the theory of sectional genus?*

(ii) *Are there any relationships between $g_i(X, \mathcal{L})$ and $g_{i+1}(X, \mathcal{L})$?*

(iii) *Classify $(X, \mathcal{L})$ by the value of the ith sectional geometric genus.*

(iv) *What is the geometric meaning of the ith sectional geometric genus?*

Remark 3.2. (1) First we consider Problem 3.1(i.1). At present we can prove the non-negativity of $g_i(X, \mathcal{L})$ if (a) $i = 0$, (b) $i = 1$, (c) $i = 2$ and $n = 3$, (d) $i = n$. But in general it is unknown whether $g_i(X, \mathcal{L})$ is non-negative or not. Next we consider the stronger inequality. Of course, if it holds, then also $g_i(X, \mathcal{L}) \geq 0$. If $i = 0$ or n, then the stronger inequality holds. But it is unknown whether it holds or not in general. If $i = 1$, then this is a conjecture proposed by Fujita [9, (13.7) Remark], and this case has been studied under several assumptions (see, for example, [11]–[13]). In [15, Corollary 2.8], we showed that the stronger inequality holds if $\dim \mathrm{Bs}|\mathcal{L}| + 1 \leq i \leq n - 1$.

(2) As for Problem 3.1(ii), if $\mathrm{Bs}|L| = \emptyset$, then $g_i(X, \mathcal{L}) = 0$ implies $g_{i+1}(X, \mathcal{L}) = 0$.

(3) Concerning Problem 3.1(iii), if $i = 1$, then the classification of polarized manifolds $(X, \mathcal{L})$ with $g_1(X, \mathcal{L}) \leq 2$ was obtained (see [7], [23], [3], and [8]).

If $i = 2$, then the classification of polarized manifolds $(X, \mathcal{L})$ is obtained in the following cases (see [14, Corollary 3.5 and Theorem 3.6] and [19]):

(ii.1) $\mathrm{Bs}|\mathcal{L}| = \emptyset$ and $g_2(X, \mathcal{L}) = h^2(\mathcal{O}_X)$.

(ii.2) $\mathcal{L}$ is very ample and $g_2(X, \mathcal{L}) = h^2(\mathcal{O}_X) + 1$.

(4) Finally we consider Problem 3.1(iv). First we give the following definition.

Definition 3.2. Let $(X, \mathcal{L})$ be a polarized variety of dimension n. Then a k-*ladder* of $\mathcal{L}$ is a sequence of irreducible and reduced subvarieties $X \supset X_1 \supset \cdots \supset X_k$ such that $X_i \in |\mathcal{L}_{i-1}|$ for $1 \leq i \leq k$, where $X_0 := X$, $\mathcal{L}_0 := \mathcal{L}$ and $\mathcal{L}_i := \mathcal{L}|_{X_i}$. Note that $\dim X_j = n - j$. Let $r_{p,q} : H^p(X_q, \mathcal{L}_q) \to H^p(X_{q+1}, \mathcal{L}_{q+1})$ be the natural map.

Theorem 3.1 ([15, Propositions 2.1 and 2.3, and Theorem 2.4]). *Let X be a projective variety of dimension $n \geq 2$ and let $\mathcal{L}$ be an ample line bundle on X. Assume that $h^t(-s\mathcal{L}) = 0$ for any integers t and s with $0 \leq t \leq n - 1$ and $1 \leq s$, and that $|\mathcal{L}|$ has an $(n - i)$-ladder for an integer i with $1 \leq i \leq n$. Then:*

(1) *$g_i(X_j, \mathcal{L}_j) = g_i(X_{j+1}, \mathcal{L}_{j+1})$ for every integer j with $0 \leq j \leq n - i - 1$. (Here we use the notation of Definition 3.2.)*

(2) *$g_i(X, \mathcal{L}) \geq h^i(\mathcal{O}_X)$.*

In particular, if $(X, \mathcal{L})$ is a polarized manifold with $\mathrm{Bs}|\mathcal{L}| = \emptyset$, then $\mathcal{L}$ has an $(n - i)$-ladder $X \supset X_1 \supset \cdots \supset X_{n-i}$ such that each X_j is smooth, and

from Theorem 3.1(1) and Remark 3.1(2) we see that $g_i(X, \mathcal{L}) = g_i(X_{n-i}, \mathcal{L}_{n-i}) = h^i(\mathcal{O}_{X_{n-i}}) = h^0(\Omega_{X_{n-i}})$, that is, the ith sectional geometric genus is the geometric genus of the i-dimensional projective variety X_{n-i}. That is why we call this invariant the ith sectional geometric genus.

From Theorem 3.1 we see that the ith sectional geometric genus is expected to have properties similar to those of the geometric genus of i-dimensional projective varieties. In particular, if $i = 2$, then $g_2(X, \mathcal{L})$ is expected to have properties similar to those of the geometric genus of projective surfaces and we can propose several problems which can be considered as generalizations of theorems in the theory of surfaces. See [18] for further details.

For other results concerning the ith sectional geometric genus, see, for example, [14], [15], [17] and [18].

The following result will be used later.

Theorem 3.2. *Let X be a projective variety with $\dim X = n$ and let $\mathcal{L}$ be a nef and big line bundle on X.*

(1) *For any integer i with $0 \le i \le n-1$, we have*

$$g_i(X, \mathcal{L}) = \sum_{j=0}^{n-i-1} (-1)^{n-j} \binom{n-i}{j} \chi(-(n-i-j)\mathcal{L}) + \sum_{k=0}^{n-i} (-1)^{n-i-k} h^{n-k}(\mathcal{O}_X).$$

(2) *Assume that X is smooth. Then for any integer i with $0 \le i \le n-1$, we have*

$$g_i(X, \mathcal{L}) = \sum_{j=0}^{n-i-1} (-1)^j \binom{n-i}{j} h^0(K_X + (n-i-j)\mathcal{L}) + \sum_{k=0}^{n-i} (-1)^{n-i-k} h^{n-k}(\mathcal{O}_X).$$

Proof. (1) By the same argument as in the proof of [14, Theorem 2.2], we obtain

$$\chi_{n-i}(X, \mathcal{L}) = \sum_{j=0}^{n-i} (-1)^{n-i-j} \binom{n-i}{j} \chi(-(n-i-j)\mathcal{L})$$

$$= \sum_{j=0}^{n-i-1} (-1)^{n-i-j} \binom{n-i}{j} \chi(-(n-i-j)\mathcal{L}) + \chi(\mathcal{O}_X).$$

Hence by Definition 3.1, we get the assertion.

(2) By using the Serre duality and the Kawamata–Viehweg vanishing theorem, we get the assertion from (1).   □

Proposition 3.1. *Let $(X, \mathcal{L})$ be a polarized manifold of dimension $n \ge 5$. If $\kappa(K_X + (n-3)\mathcal{L}) = -\infty$, then $g_j(X, \mathcal{L}) = 0$ for every $j \ge 3$ unless $(X, \mathcal{L})$ is a scroll over a normal projective variety of dimension 3.*

Proof. Assume that $(X, \mathcal{L})$ is not a scroll over a normal projective variety of dimension 3. Then by Lemma 2.1 we have $h^j(\mathcal{O}_X) = 0$ for every $j \geq 3$. By assumption, $h^0(K_X + t\mathcal{L}) = 0$ for $1 \leq t \leq n - 3$. Hence by Theorem 3.2(2) we get $g_j(X, \mathcal{L}) = 0$ for all $j \geq 3$. $\qquad\square$

Remark 3.3. If $(X, \mathcal{L})$ is a scroll over a normal projective variety of dimension 3, then by [14, Example 2.10(8)] we have $g_j(X, \mathcal{L}) = 0$ for all $j \geq 4$ and $g_3(X, \mathcal{L}) = h^3(\mathcal{O}_X)$.

Remark 3.4. By Lemma 2.1 and Remark 2.5, we can prove that the conclusion of Proposition 3.1 holds for $n = 4$.

As the next step, we want to generalize the notion of the Δ-genus. Several generalizations can be considered from various points of view. Our point of view comes from the following result.

Theorem 3.3 (see e.g. [9, §3 in Chapter I]). *Let X be a projective variety of dimension $n \geq 2$ and let $\mathcal{L}$ be an ample line bundle on X. We use the notation of Definition 3.2. If $|\mathcal{L}|$ has an $(n-1)$-ladder and $h^0(\mathcal{L}_{n-1}) > 0$, then*

$$\Delta(X, \mathcal{L}) = \sum_{j=0}^{n-1} \dim \operatorname{Coker} r_{0,j}.$$

In particular, $\Delta(X, \mathcal{L}) \geq \Delta(X_1, \mathcal{L}_1) \geq \cdots \geq \Delta(X_{n-1}, \mathcal{L}_{n-1}) \geq 0$.

We now give the definition of the ith Δ-genus which satisfies a generalization of Theorem 3.3.

Definition 3.3 ([16, Definition 2.1]). Let $(X, \mathcal{L})$ be a polarized variety of dimension n. For every integer i with $0 \leq i \leq n$, the *ith Δ-genus $\Delta_i(X, \mathcal{L})$ of $(X, \mathcal{L})$* is

$$\Delta_i(X, \mathcal{L}) = \begin{cases} 0 & \text{if } i = 0, \\ g_{i-1}(X, \mathcal{L}) - \Delta_{i-1}(X, \mathcal{L}) \\ \quad + (n - i + 1)h^{i-1}(\mathcal{O}_X) - h^{i-1}(\mathcal{L}) & \text{if } 1 \leq i \leq n. \end{cases}$$

Remark 3.5. (1) If $i = 1$, then $\Delta_1(X, \mathcal{L})$ is equal to the Δ-genus of $(X, \mathcal{L})$.

(2) If $i = n$, then $\Delta_n(X, \mathcal{L}) = h^n(\mathcal{O}_X) - h^n(\mathcal{L})$ (see [16, Proposition 2.4]).

(3) For $1 \leq i \leq n$, by the definition of the ith Δ-genus, we have the following equality which will be used later:

$$\Delta_{i-1}(X, \mathcal{L}) = g_{i-1}(X, \mathcal{L}) - \Delta_i(X, \mathcal{L}) + (n - i + 1)h^{i-1}(\mathcal{O}_X) - h^{i-1}(\mathcal{L}).$$

(4) Let $(X, \mathcal{L})$ be a polarized manifold of dimension n and let $(M, \mathcal{A})$ be a reduction of $(X, \mathcal{L})$. Then $\Delta_i(X, \mathcal{L}) = \Delta_i(M, \mathcal{A})$ for $2 \le i \le n$ (see [16, Corollary 2.11]).

Theorem 3.4 (see [16, Theorem 2.8 and Corollary 2.9] and [15, Proposition 2.1]). *Let X be a projective variety of dimension $n \ge 2$ and let $\mathcal{L}$ be an ample line bundle on X. We use the notation of Definition 3.2. Assume that $h^t(-s\mathcal{L}) = 0$ for any integers t and s with $0 \le t \le n-1$ and $1 \le s$. If $|\mathcal{L}|$ has an $(n-i)$-ladder and $h^0(\mathcal{L}_{n-i}) > 0$ for an integer i with $1 \le i \le n$, then*

$$\Delta_i(X, \mathcal{L}) = \sum_{j=0}^{n-i} \dim \operatorname{Coker} r_{i-1,j}.$$

In particular, $\Delta_i(X, \mathcal{L}) \ge \Delta_i(X_1, \mathcal{L}_1) \ge \cdots \ge \Delta_i(X_{n-i}, \mathcal{L}_{n-i}) \ge 0$.

The definition of the ith Δ-genus is so complicated that a lot of things about it are unknown. The following questions are worth investigating.

Problem 3.2.   (i) *Does the ith Δ-genus have properties similar to those of the Δ-genus? In particular:*

(i.1) *Does $\Delta_i(X, \mathcal{L}) \ge 0$ hold?*

(i.2) *Can we get the ith Δ-genus version of the Fujita theory of Δ-genus?*

(ii) *Are there any relationships between $g_i(X, \mathcal{L})$ and $\Delta_i(X, \mathcal{L})$?*

(iii) *Are there any relationships between $\Delta_i(X, \mathcal{L})$ and $\Delta_{i+1}(X, \mathcal{L})$?*

(iv) *Classify $(X, \mathcal{L})$ by the value of the ith Δ-genus.*

(v) *What is the geometric meaning of the ith Δ-genus?*

Remark 3.6. If X is smooth and $\mathcal{L}$ is ample, then the following facts on Problem 3.2 are known.

(1) First we consider Problem 3.2(i.1). If $i = 1$, then $\Delta_1(X, \mathcal{L}) \ge 0$ (see [9, (4.2) Theorem]). Moreover if $\mathcal{L}$ is base point free, then $\Delta_i(X, \mathcal{L}) \ge 0$ for $0 \le i \le n$. But unfortunately, there exists an example of $(X, \mathcal{L})$ such that $\Delta_i(X, \mathcal{L}) < 0$ (see [16, Section 4]).

(2) As for Problem 3.2(ii), if $i = 1$ and $\mathcal{L}$ is merely ample, then $g_1(X, \mathcal{L}) = 0$ if and only if $\Delta_1(X, \mathcal{L}) = 0$ (see [9, (12.1) Theorem]). If $i \ge 2$, then under the assumption that $\operatorname{Bs}|\mathcal{L}| = \emptyset$ we have $g_i(X, \mathcal{L}) = 0$ if and only if $\Delta_i(X, \mathcal{L}) = 0$ (see [16, Theorem 3.13]).

(3) Concerning Problem 3.2(iii), under the assumption that $\mathcal{L}$ is base point free, we for example get the following: If $\Delta_i(X, \mathcal{L}) \le i-1$, then $\Delta_{i+1}(X, \mathcal{L}) = 0$ (see

[16, Proposition 3.9]). In particular, if $\Delta_i(X, \mathcal{L}) = 0$, then $\Delta_{i+1}(X, \mathcal{L}) = 0$. There may be other similar relationships between $\Delta_i(X, \mathcal{L})$ and $\Delta_{i+1}(X, \mathcal{L})$.

(4) For Problem 3.2(iv), there exist the following classifications of $(X, \mathcal{L})$ by the value of $\Delta_2(X, \mathcal{L})$:

    (4.1) The classification of polarized manifolds $(X, \mathcal{L})$ such that $\mathrm{Bs}|\mathcal{L}| = \emptyset$ and $\Delta_2(X, \mathcal{L}) = 0$ (see [16, Theorem 3.13 and Remark 3.13.1]).

    (4.2) The classification of polarized manifolds $(X, \mathcal{L})$ such that $\mathcal{L}$ is very ample and $\Delta_2(X, \mathcal{L}) = 1$ (see [16, Theorem 3.17] and [19, Remark 2]).

(5) At present, we do not know much about Problem 3.2(v). It seems to be the most difficult among the above problems even in the case where $\mathcal{L}$ is base point free or very ample.

In this paper, we consider Problems 3.1(i.2) and 3.2(i.2). In [6, (1.9) Theorem], Fujita proved that $(X, \mathcal{L})$ is a Del Pezzo manifold (i.e. $K_X = -(n-1)\mathcal{L}$) if and only if $g(X, \mathcal{L}) = 1$ and $\Delta(X, \mathcal{L}) = 1$, that is, $g_1(X, \mathcal{L}) = 1$ and $\Delta_1(X, \mathcal{L}) = 1$. In this paper, we consider an analogous characterization of $(X, \mathcal{L})$ with $K_X = -(n-i)\mathcal{L}$ by using $g_i(X, \mathcal{L})$ and $\Delta_i(X, \mathcal{L})$.

§4. Main theorems

§4.1. A conjecture

First we make the following conjecture which is the main theme of this paper.

Conjecture 4.1.1. *Let $(X, \mathcal{L})$ be a polarized manifold of dimension $n \geq 3$. Then, for every integer i with $2 \leq i \leq n - 1$, the following are equivalent to one another:*

$C(i, 1)$: $K_X = -(n - i)\mathcal{L}$.
$C(i, 2)$: $\Delta_i(X, \mathcal{L}) = 1$ and $2g_1(X, \mathcal{L}) - 2 = (i - 1)\mathcal{L}^n$.
$C(i, 3)$: $\Delta_i(X, \mathcal{L}) > 0$ and $2g_1(X, \mathcal{L}) - 2 = (i - 1)\mathcal{L}^n$.
$C(i, 4)$: $g_i(X, \mathcal{L}) = 1$ and $2g_1(X, \mathcal{L}) - 2 = (i - 1)\mathcal{L}^n$.
$C(i, 5)$: $g_i(X, \mathcal{L}) > 0$ and $2g_1(X, \mathcal{L}) - 2 = (i - 1)\mathcal{L}^n$.

Remark 4.1.1. If $i = 1$, then $C(1, 1)$ and $C(1, 2)$ are equivalent for *any* ample line bundle $\mathcal{L}$. Of course $C(1, 1)$ implies $C(1, 3)$ (resp. $C(1, 4)$, $C(1, 5)$). But $C(1, 3)$ (resp. $C(1, 4)$, $C(1, 5)$) does not imply $C(1, 1)$ because $(X, \mathcal{L})$ is possibly a scroll over an elliptic curve.

Remark 4.1.2. As a generalization of the case where $i = 1$, it is natural to expect that $C(i, 1)$ is equivalent to

$C(i, 6)$: $\Delta_i(X, \mathcal{L}) = 1$ and $g_i(X, \mathcal{L}) = 1$.

We can easily see that $C(i,1)$ implies $C(i,6)$. But from Examples 4.1.1 and 4.1.2 below we find that the converse is not true in general.

Example 4.1.1. Let $n \geq 3$, and let Y be a smooth projective variety of dimension m with $1 \leq m \leq n-2$. Let $\mathcal{H}$ be an ample line bundle on Y such that $K_Y \neq -(n-m-1)\mathcal{H}$ and $h^0(K_Y + (n-m-1)\mathcal{H}) = 1$. (There exists a polarized manifold $(Y, \mathcal{H})$ like this. For example, let $(Y, \mathcal{H})$ be a principally polarized abelian variety with $\dim Y = m = n-2$; then $\mathcal{H}$ is an ample line bundle on Y such that $\mathcal{H}^m = m!$. Then $K_Y + (n-m-1)\mathcal{H} = \mathcal{H}$ and $h^0(K_Y + (n-m-1)\mathcal{H}) = h^0(\mathcal{H}) = 1$.)

Next we take a Del Pezzo manifold $(F, \mathcal{A})$ of dimension $n-m$. Note that $K_F = -(n-m-1)\mathcal{A}$.

We set $X := Y \times F$ and $\mathcal{L} := p_1^*(\mathcal{H}) + p_2^*(\mathcal{A})$, where p_i denotes the ith projection map. Then $K_X + (n-m-1)\mathcal{L} = p_1^*(K_Y + (n-m-1)\mathcal{H})$. By [16, Lemma 1.6] we also get $h^j(\mathcal{O}_X) = 0$ and $h^j(\mathcal{L}) = 0$ for all $j \geq m+1$. Hence $\Delta_n(X, \mathcal{L}) = 0$ by Remark 3.5(2), and by Theorem 3.2(2) we see that $g_j(X, \mathcal{L}) = 0$ for every $j \geq m+2$ and $g_{m+1}(X, \mathcal{L}) = h^0(K_X + (n-m-1)\mathcal{L}) = 1$. Moreover by Remark 3.5(3) we deduce that $\Delta_j(X, \mathcal{L}) = 0$ for every $j \geq m+2$ and

$$\Delta_{m+1}(X, \mathcal{L}) = g_{m+1}(X, \mathcal{L}) - \Delta_{m+2}(X, \mathcal{L}) + (n-m-1)h^{m+1}(\mathcal{O}_X) - h^{m+1}(\mathcal{L}) = 1.$$

Therefore $g_{m+1}(X, \mathcal{L}) = \Delta_{m+1}(X, \mathcal{L}) = 1$. But $K_X \neq -(n-m-1)\mathcal{L}$.

Example 4.1.2. Let $k \geq 2$ and set $n := 2k+1$ and $i := (n-1)/2$. We consider $(M, \mathcal{A}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$. Then $K_M = -(k+1)\mathcal{A} = -(n-i)\mathcal{A}$. Moreover we see that $g_i(M, \mathcal{A}) = 1$ and $\Delta_i(M, \mathcal{A}) = 1$ (see (I) in the proof of Theorem 4.2.1 below). Let $\pi : X \to \mathbb{P}^n$ be the blowing up at a general point on $\mathbb{P}^n$ and let $\mathcal{L} := \pi^*(\mathcal{A}) - E$, where E is the exceptional divisor. Then by [1, Theorem 0.1], $(X, \mathcal{L})$ is a polarized manifold with $K_X + (n-i)\mathcal{L} = (i-1)E$. On the other hand, $(M, \mathcal{A})$ is a reduction of $(X, \mathcal{L})$ and $2 \leq i < n-1$. Hence by Remarks 3.1(5) and 3.5(4) we get $g_i(X, \mathcal{L}) = g_i(M, \mathcal{A}) = 1$ and $\Delta_i(X, \mathcal{L}) = \Delta_i(M, \mathcal{A}) = 1$.

§4.2. The case where $\max\{2, \dim \mathrm{Bs}|\mathcal{L}| + 2\} \leq i \leq n-1$

First we consider the case where $\max\{2, \dim \mathrm{Bs}|\mathcal{L}| + 2\} \leq i \leq n-1$.

Theorem 4.2.1. *Let $(X, \mathcal{L})$ be a polarized manifold of dimension $n \geq 3$, and let $m = \dim \mathrm{Bs}|\mathcal{L}|$. (If $\mathrm{Bs}|\mathcal{L}| = \emptyset$, then we set $m = -1$.) Assume that $\max\{2, m+2\} \leq i \leq n-1$. Then Conjecture 4.1.1 is true.*

Proof. By assumption and [15, Proposition 1.12(2)], the following hold:

(A_i) $\mathcal{L}$ has an $(n-i)$-ladder $X_{n-i} \subset \cdots \subset X_1 \subset X$.

(B_i) $h^0(\mathcal{L}_{n-i}) > 0$.

(C) $h^j(\mathcal{L}^{\otimes -t}) = 0$ for any j and t with $0 \leq j \leq n-1$ and $t > 0$.

(D_i) X_j is normal for any j with $0 \leq j \leq n-i$.

(E_i) X_j is Cohen–Macaulay for any j with $0 \leq j \leq n-i$.

(I) Assume that $C(i,1)$ holds. Then by Remark 3.1(4) we see that

$$2g_1(X, \mathcal{L}) - 2 = (K_X + (n-i)\mathcal{L} + (i-1)\mathcal{L})\mathcal{L}^{n-1} = (i-1)\mathcal{L}^n.$$

Note that $h^j(\mathcal{O}_X) = 0$ and $h^j(\mathcal{L}) = 0$ for every $j \geq 2$. Furthermore $h^0(K_X + (n-i)\mathcal{L}) = 1$ and $h^0(K_X + k\mathcal{L}) = 0$ for $1 \leq k \leq n-i-1$. Hence $g_i(X, \mathcal{L}) = 1$ and $g_k(X, \mathcal{L}) = 0$ for every $k \geq i+1$ by Theorem 3.2(2) (this means that $C(i,1)$ implies $C(i,4)$), and by Remark 3.5(2) and (3), we have $\Delta_k(X, \mathcal{L}) = 0$ for every $k \geq i+1$ and

$$\Delta_i(X, \mathcal{L}) = g_i(X, \mathcal{L}) - \Delta_{i+1}(X, \mathcal{L}) + (n-i)h^i(\mathcal{O}_X) - h^i(\mathcal{L}) = 1.$$

Therefore $C(i,1)$ implies $C(i,2)$ and $C(i,4)$ above.

(II) It is trivial that $C(i,2)$ implies $C(i,3)$, and $C(i,4)$ implies $C(i,5)$.

(III) Assume that $C(i,3)$ holds. We will prove that then $C(i,5)$ holds. It suffices to show that $g_i(X, \mathcal{L}) > 0$. Note that $g_i(X, \mathcal{L}) \geq 0$ by [15, Theorem 2.4]. In view of $C(i,3)$, the following shows that $g_i(X, \mathcal{L}) = 0$ is impossible:

Claim 4.2.1. *If* $g_i(X, \mathcal{L}) = 0$*, then* $\Delta_i(X, \mathcal{L}) = 0$*.*

Proof. Assume that $g_i(X, \mathcal{L}) = 0$. Then $0 = g_i(X, \mathcal{L}) = g_i(X_{n-i}, \mathcal{L}_{n-i}) = h^i(\mathcal{O}_{X_{n-i}})$ by Theorem 3.1 and Remark 3.1(2). Therefore $h^i(\mathcal{O}_X) = h^i(\mathcal{O}_{X_1}) = \cdots = h^i(\mathcal{O}_{X_{n-i-1}}) \leq h^i(\mathcal{O}_{X_{n-i}}) = 0$ by [15, Proposition 2.1(b)]. Hence $H^{i-1}(\mathcal{L}_j) \to H^{i-1}(\mathcal{L}_{j+1})$ is surjective for $0 \leq j \leq n-i$, so $\dim \operatorname{Coker} r_{i-1,j} = 0$ for $0 \leq j \leq n-i$. On the other hand by Theorem 3.4, we have

$$\Delta_i(X, \mathcal{L}) = \sum_{k=0}^{n-i} \dim \operatorname{Coker} r_{i-1,k}.$$

Therefore $\Delta_i(X, \mathcal{L}) = 0$. □

(IV) Assume that $C(i,5)$ holds. Then

$$1 + \frac{1}{2}(i-1)\mathcal{L}^n = g_1(X, \mathcal{L}) = 1 + \frac{1}{2}(K_X + (n-1)\mathcal{L})\mathcal{L}^{n-1}$$

$$= 1 + \frac{1}{2}(K_{X_{n-i}} + (i-1)\mathcal{L}_{n-i})\mathcal{L}_{n-i}^{i-1}$$

$$= 1 + \frac{1}{2}(i-1)\mathcal{L}^n + \frac{1}{2}K_{X_{n-i}}\mathcal{L}_{n-i}^{i-1}.$$

Hence $K_{X_{n-i}}\mathcal{L}_{n-i}^{i-1} = 0$. On the other hand, $g_i(X, \mathcal{L}) = h^i(\mathcal{O}_{X_{n-i}})$ by (A_i) and (C) (see also [15, Propositions 2.1 and 2.3]). Furthermore by (D_i), (E_i) and the Serre duality, we obtain $h^0(K_{X_{n-i}}) = h^i(\mathcal{O}_{X_{n-i}})$. Hence $0 < g_i(X, \mathcal{L}) = h^i(\mathcal{O}_{X_{n-i}}) = h^0(K_{X_{n-i}})$ and we see that $K_{X_{n-i}} = \mathcal{O}_{X_{n-i}}$.

Next we prove the following claim.

Claim 4.2.2. *The natural map* $\mathrm{Pic}(X_j) \to \mathrm{Pic}(X_{j+1})$ *is injective for* $0 \leq j \leq n - i - 1$.

Proof. From the exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{X_j} \to \mathcal{O}_{X_j}^* \to 0,$$

we get the commutative diagram

$$
\begin{array}{ccccccc}
H^1(X_j, \mathbb{Z}) & \longrightarrow & H^1(\mathcal{O}_{X_j}) & \longrightarrow & H^1(\mathcal{O}_{X_j}^*) & \longrightarrow & H^2(X_j, \mathbb{Z}) \\
\downarrow{\varphi_1} & & \downarrow{\varphi_2} & & \downarrow{\varphi_3} & & \downarrow{\varphi_4} \\
H^1(X_{j+1}, \mathbb{Z}) & \longrightarrow & H^1(\mathcal{O}_{X_{j+1}}) & \longrightarrow & H^1(\mathcal{O}_{X_{j+1}}^*) & \longrightarrow & H^2(X_{j+1}, \mathbb{Z})
\end{array}
$$

Because $\mathrm{Pic}(X_j) \cong H^1(\mathcal{O}_{X_j}^*)$ and $\mathrm{Pic}(X_{j+1}) \cong H^1(\mathcal{O}_{X_{j+1}}^*)$ we have to prove that φ_3 is injective. It suffices to show the following for $0 \leq j \leq n - i - 1$:

(a) $h^1(\mathcal{O}_{X_j}(-X_{j+1})) = 0$.
(b) $H^1(X_j, \mathbb{Z}) \cong H^1(X_{j+1}, \mathbb{Z})$.
(c) The map $H^2(X_j, \mathbb{Z}) \to H^2(X_{j+1}, \mathbb{Z})$ is injective.

By (C) we can prove $h^t(\mathcal{L}_j^{\otimes -s}) = 0$ for every j, t and s with $0 \leq j \leq n - i - 1$, $0 \leq t \leq n - j - 1$ and $1 \leq s$. Hence we get (a) since $\dim X_{n-i-1} = i + 1 \geq 3$.

Next we consider (b) and (c). In this case we need to choose an $(n-i)$-ladder carefully. Namely, we take general members $X_1 \in |\mathcal{L}|, X_2 \in |\mathcal{L}|_{X_1}, \ldots, X_{n-i} \in |\mathcal{L}|_{X_{n-i-1}}$. Then $X \supset X_1 \supset \cdots \supset X_{n-i}$ is an $(n-i)$-ladder such that $X_j - X_{j+1}$ is smooth for every j. Hence $X_j - X_{j+1}$ is locally a complete intersection. Here we use [4, Corollary 2.3.3]. Since X_{j+1} is an ample line bundle on X_j, we see that if $2 \leq i = \dim X_{n-i-1} - 1$, then $H^t(X_j, \mathbb{Z}) \to H^t(X_{j+1}, \mathbb{Z})$ is an isomorphism (resp. injective) for $t = 1$ (resp. $t = 2$) and every j with $0 \leq j \leq n - i - 1$.

Therefore we get the assertion of Claim 4.2.2. $\square$

By this claim we have $K_X + (n-i)\mathcal{L} = \mathcal{O}_X$. So we get $C(i, 1)$. This completes the proof of Theorem 4.2.1. $\square$

§4.3. The case of $i = 2$

Next we consider the case where $i = 2$ and $\mathcal{L}$ is ample. Then we can prove the following:

Theorem 4.3.1. *Let $(X, \mathcal{L})$ be a polarized manifold of dimension $n \geq 3$. Then Conjecture 4.1.1 is true for $i = 2$.*

Proof. (I) First we assume that $C(2,1)$ holds. Then by the same argument as in the proof of Theorem 4.2.1 we see that $C(2,2)$ and $C(2,4)$ hold.

(II) It is trivial that $C(2,2)$ (resp. $C(2,4)$) implies $C(2,3)$ (resp. $C(2,5)$).

(III) Assume that $C(2,3)$ holds. We will prove that $C(2,5)$ holds. It suffices to show that $g_2(X, \mathcal{L}) > 0$. By the assumption that $2g_1(X, \mathcal{L}) - 2 = \mathcal{L}^n$, we get $(K_X + (n-2)\mathcal{L})\mathcal{L}^{n-1} = 0$. Hence by [4, Lemma 2.5.9] we have $\kappa(K_X + (n-2)\mathcal{L}) \leq 0$.

(III.1) If $\kappa(K_X + (n - 2)\mathcal{L}) = -\infty$, then $(X, \mathcal{L})$ is one of the types (1) to (7.4) in Theorem 2.1 by Remark 2.2(1). By [14, Example 2.10] and [16, Example 2.12] we may assume that $(X, \mathcal{L})$ is a scroll over a smooth surface S because we assume that $\Delta_2(X, \mathcal{L}) > 0$. In this case, by [5, (3.2.1)] and [9, (11.8.6) in the proof of (11.8) Theorem], there exists an ample vector bundle $\mathcal{E}$ of rank $n-1$ on X such that $X = \mathbb{P}_S(\mathcal{E})$, $\mathcal{L} = H(\mathcal{E})$. Let $\pi : X \to S$ be its morphism. Then

$$(K_X + (n-2)\mathcal{L})\mathcal{L}^{n-1} = (-H(\mathcal{E}) + \pi^*(K_S + c_1(\mathcal{E})))H(\mathcal{E})^{n-1} = K_S c_1(\mathcal{E}) + c_2(\mathcal{E}).$$

If $h^2(\mathcal{O}_S) = 0$, then $h^2(\mathcal{O}_X) = h^2(\mathcal{O}_S) = 0$ and $\Delta_2(X, \mathcal{L}) = (n - 1)h^2(\mathcal{O}_X) - h^2(\mathcal{L}) = -h^2(\mathcal{L}) \leq 0$ and this contradicts the assumption. Hence $h^2(\mathcal{O}_S) \geq 1$ and by the Serre duality we have $h^0(K_S) \geq 1$. Since $\mathcal{E}$ is ample, we see that $K_S c_1(\mathcal{E}) \geq 0$ and $c_2(\mathcal{E}) > 0$. Therefore $(K_X + (n - 2)\mathcal{L})\mathcal{L}^{n-1} > 0$ and this contradicts the assumption. Therefore there does not exist any $(X, \mathcal{L})$ with $\kappa(K_X + (n - 2)\mathcal{L}) = -\infty$, $\Delta_2(X, \mathcal{L}) > 0$ and $2g_1(X, \mathcal{L}) - 2 = \mathcal{L}^n$.

(III.2) Assume that $\kappa(K_X + (n - 2)\mathcal{L}) = 0$. Let $(M, \mathcal{A})$ be a reduction of $(X, \mathcal{L})$. Then by Theorem 2.1, $(M, \mathcal{A})$ is a Mukai manifold and by [14, Proposition 2.6 and Example 2.10(7)] we see that $g_2(X, \mathcal{L}) = g_2(M, \mathcal{A}) > 0$. Hence by (III.1) and (III.2) we get $C(2,5)$.

(IV) Assume that $C(2,5)$ holds. Then by the same argument as in (III) above, we find that $\kappa(K_X + (n - 2)\mathcal{L}) \leq 0$. If $\kappa(K_X + (n - 2)\mathcal{L}) = -\infty$, then $(X, \mathcal{L})$ is a scroll over a smooth surface S since $g_2(X, \mathcal{L}) > 0$. On the other hand $h^2(\mathcal{O}_X) > 0$ because $g_2(X, \mathcal{L}) = h^2(\mathcal{O}_X)$. Hence by the same argument as in (III.1) above, we see that $(K_X + (n - 2)\mathcal{L})\mathcal{L}^{n-1} > 0$ and this is impossible. Therefore $\kappa(K_X + (n - 2)\mathcal{L}) = 0$ and $K_M + (n - 2)\mathcal{A} = \mathcal{O}_X$, where $(M, \mathcal{A})$ is a reduction of $(X, \mathcal{L})$. Now we prove that $(X, \mathcal{L}) \cong (M, \mathcal{A})$. Assume that $(X, \mathcal{L}) \not\cong (M, \mathcal{A})$. Then $(K_X + (n-2)\mathcal{L})\mathcal{L}^{n-1} > (K_M + (n-2)\mathcal{A})\mathcal{A}^{n-1}$. But since $(K_X + (n-2)\mathcal{L})\mathcal{L}^{n-1} = 0$

and $(K_M + (n-2)\mathcal{A})\mathcal{A}^{n-1} = 0$, this is impossible. Hence $(X, \mathcal{L}) \cong (M, \mathcal{A})$ and we get $C(2, 1)$.

This completes the proof of Theorem 4.3.1. $\hfill\square$

Corollary 4.3.1. *Let $(X, \mathcal{L})$ be a polarized manifold of dimension $n \geq 3$. Assume that $\dim \text{Bs}|\mathcal{L}| \leq 1$. Then Conjecture 4.1.1 is true.*

Proof. Since $\dim \text{Bs}|\mathcal{L}| \leq 1$, Conjecture 4.1.1 is true for $i \geq 3$ by Theorem 4.2.1. On the other hand, if $i = 2$, then Conjecture 4.1.1 is also true by Theorem 4.3.1. $\hfill\square$

§4.4. The case where $i = 3$ and $n \geq 5$

Next we consider the case where $i = 3$ and $n \geq 5$.

Theorem 4.4.1. *Let $(X, \mathcal{L})$ be a polarized manifold of dimension $n \geq 5$. Then Conjecture 4.1.1 is true for $i = 3$.*

Proof. (I) By the same argument as in the proof of Theorem 4.2.1, we see that $C(3, 1)$ implies $C(3, 2)$ and $C(3, 4)$.

(II) It is trivial that $C(3, 2)$ (resp. $C(3, 4)$) implies $C(3, 3)$ (resp. $C(3, 5)$).

(III) Assume that $C(3, 3)$ holds. We will prove that $C(3, 1)$ holds. Since $2g_1(X, \mathcal{L}) - 2 = 2\mathcal{L}^n$, we have $(K_X + (n-3)\mathcal{L})\mathcal{L}^{n-1} = 0$. Hence by an argument similar to (III) in the proof of Theorem 4.3.1, we see that $\kappa(K_X + (n-3)\mathcal{L}) \leq 0$.

(III.1) Assume that $\kappa(K_X + (n-3)\mathcal{L}) = 0$. Then since $\kappa(K_X + (n-3)\mathcal{L}) = 0$ and $(K_X + (n-3)\mathcal{L})\mathcal{L}^{n-1} = 0$, there exists a positive integer t such that $t(K_X + (n-3)\mathcal{L}) = \mathcal{O}_X$. But by [4, Lemma 3.3.2] we have $K_X + (n-3)\mathcal{L} = \mathcal{O}_X$.

(III.2) Assume that $\kappa(K_X + (n-3)\mathcal{L}) = -\infty$.

Lemma 4.4.1. *There does not exist any $(X, \mathcal{L})$ with $\kappa(K_X + (n-3)\mathcal{L}) = -\infty$, $\Delta_3(X, \mathcal{L}) > 0$ and $2g_1(X, \mathcal{L}) - 2 = 2\mathcal{L}^n$.*

Proof. If $\kappa(K_X + (n-3)\mathcal{L}) = -\infty$, then $(X, \mathcal{L})$ satisfies either (1), (2.1) or (2.2) in Proposition 2.1.

(i) If $(X, \mathcal{L})$ satisfies (1) in Proposition 2.1, then by using [16, Example 2.12] we see that $\Delta_3(X, \mathcal{L}) = 0$ unless $(X, \mathcal{L})$ is a scroll over a normal projective variety of dimension 3.

We consider the case where $(X, \mathcal{L})$ is a scroll over a normal projective variety Y with $\dim Y = m \geq 2$. Then by [5, (3.2.1) Theorem] and [2, Proposition 2.5], we get the following:

Proposition 4.4.1. *Let $(X, \mathcal{L})$ be a scroll over a 3-dimensional normal projective variety Y. If $\dim X \geq 5$, then Y is smooth and $(X, \mathcal{L})$ is a classical scroll over Y.*

So we see that $(X, \mathcal{L})$ is a classical scroll, that is, Y is smooth and $(X, \mathcal{L}) = (\mathbb{P}_Y(\mathcal{E}), H(\mathcal{E}))$, where $\mathcal{E}$ is an ample vector bundle on Y. Then the following general claim holds.

Claim 4.4.1. *Let $(X, \mathcal{L})$ be a polarized manifold of dimension n. Assume that there exists a smooth projective variety Y of dimension $m \geq 2$ and an ample vector bundle $\mathcal{E}$ on Y of rank $n - m + 1$ such that $X = \mathbb{P}_Y(\mathcal{E})$ and $\mathcal{L} = H(\mathcal{E})$.*

(1) *If $g_m(X, \mathcal{L}) > 0$, then $2g_1(X, \mathcal{L}) - 2 > (m-1)\mathcal{L}^n$.*
(2) *If $\Delta_m(X, \mathcal{L}) > 0$, then $2g_1(X, \mathcal{L}) - 2 > (m-1)\mathcal{L}^n$.*

Proof. Assume that $h^m(\mathcal{O}_Y) \geq 1$. Then $\kappa(Y) \geq 0$. On the other hand,

$$
\begin{aligned}
2g_1(X, \mathcal{L}) - 2 - (m-1)\mathcal{L}^n &= (K_X + (n-m)\mathcal{L})\mathcal{L}^{n-1} \\
&= (-H(\mathcal{E}) + f^*(K_Y + \det \mathcal{E}))H(\mathcal{E})^{n-1} \\
&= -s_m(\mathcal{E}) + (K_Y + \det \mathcal{E})s_{m-1}(\mathcal{E}) \\
&= s_{m-1}(\mathcal{E})s_1(\mathcal{E}) - s_m(\mathcal{E}) + K_Y s_{m-1}(\mathcal{E}).
\end{aligned}
$$

(Here $f : X \to Y$ denotes the projection.) Since $\mathcal{E}$ is ample, $m \geq 2$ and $\kappa(Y) \geq 0$, we have $s_{m-1}(\mathcal{E})s_1(\mathcal{E}) - s_m(\mathcal{E}) > 0$ and $K_Y s_{m-1}(\mathcal{E}) \geq 0$ by [20, Example 12.1.7 and Lemma 14.5.1]. Therefore $2g_1(X, \mathcal{L}) - 2 > (m-1)\mathcal{L}^n$.

(1) If $g_m(X, \mathcal{L}) > 0$, then $h^m(\mathcal{O}_Y) > 0$ because $g_m(X, \mathcal{L}) = h^m(\mathcal{O}_Y)$ by [14, Example 2.10(8)]. Hence by the above argument we get assertion (1).

(2) Assume that $\Delta_m(X, \mathcal{L}) > 0$. Note that $g_j(X, \mathcal{L}) = 0$, $h^j(\mathcal{O}_X) = 0$ and $h^j(\mathcal{L}) = 0$ for every $j \geq m + 1$ and $g_m(X, \mathcal{L}) = h^m(\mathcal{O}_X)$ by [14, Example 2.10(8)] and [16, Lemma 1.6]. Therefore we get $\Delta_j(X, \mathcal{L}) = 0$ for every $j \geq m + 1$, and by using Remark 3.5(3) we have $\Delta_m(X, \mathcal{L}) = (n - m + 1)h^m(\mathcal{O}_X) - h^m(\mathcal{L})$. If $h^m(\mathcal{O}_X) = 0$, then $\Delta_m(X, \mathcal{L}) = -h^m(\mathcal{L}) \leq 0$ and this contradicts the assumption. Therefore $h^m(\mathcal{O}_X) > 0$. Hence by the above argument we get assertion (2). $\square$

Since $\dim Y = 3$ and we assume that $2g_1(X, \mathcal{L}) - 2 = 2\mathcal{L}^n$, we have $\Delta_3(X, \mathcal{L}) \leq 0$ by Claim 4.4.1 if $(X, \mathcal{L})$ is a scroll over a normal projective variety of dimension 3.

Therefore we have $\Delta_3(X, \mathcal{L}) \leq 0$ if $(X, \mathcal{L})$ satisfies (1) in Proposition 2.1.

(ii) Next we assume that $(X, \mathcal{L})$ satisfies (2.1) or (2.2) in Proposition 2.1. Assume that $(W, \mathcal{K})$ is of type 3 in [4, Theorem 7.7.5]. Then $2g_1(X, \mathcal{L}) - 2 = 2\mathcal{L}^5 - 1 \neq 2\mathcal{L}^5$. So we may assume that $(W, \mathcal{K})$ is not of this type.

By Proposition 3.1 and the assumption, $h^j(\mathcal{O}_X) = h^j(\mathcal{O}_M) = 0$ and $g_j(X, \mathcal{L}) = g_j(M, \mathcal{A}) = 0$ for every $j \geq 3$. By Remark 3.5(2) we have $\Delta_n(X, \mathcal{L}) = \Delta_n(M, \mathcal{A}) = h^n(\mathcal{O}_M) - h^n(\mathcal{A}) = -h^n(\mathcal{A})$. Moreover by Remark 3.5(3) & (4),

$$
\Delta_j(X, \mathcal{L}) = \Delta_j(M, \mathcal{A}) = g_j(M, \mathcal{A}) - \Delta_{j+1}(M, \mathcal{A}) + (n-j)h^j(\mathcal{O}_M) - h^j(\mathcal{A}).
$$

So in order to calculate $\Delta_3(X, \mathcal{L})$, we have to calculate $h^j(\mathcal{A})$ for $j \geq 3$. By Lemma 2.2 we see that $h^j(\mathcal{A}) = h^j(\mathcal{D})$ for every $j \geq 3$.

(ii.1) If $(W, \mathcal{K}) \cong (\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1))$, then $\mathcal{D} = \mathcal{O}_{\mathbb{P}^6}(2)$ and we see that $h^j(\mathcal{D}) = 0$ for every $j \geq 2$.

(ii.2) Assume that $(W, \mathcal{K})$ is of type 1 in [4, Theorem 7.7.5], that is, $(W, \mathcal{K}) \cong (\mathbb{Q}^5, \mathcal{O}_{\mathbb{Q}^5}(1))$. Then K_W is a Cartier divisor and $\mathcal{K} = K_W + 3\mathcal{D}$. Hence $3\mathcal{D}$ is also Cartier. On the other hand $2\mathcal{D}$ is Cartier by [4, Lemma 7.5.8]. Hence $\mathcal{D} = 3\mathcal{D} - 2\mathcal{D}$ is also Cartier and $\mathcal{D} = \mathcal{O}_{\mathbb{Q}^5}(2)$. Therefore by the Kawamata–Viehweg vanishing theorem [24, Theorem 1-2-5], we have $h^j(\mathcal{D}) = h^j(\mathcal{O}_{\mathbb{Q}^5}(2)) = h^j(K_W + \mathcal{O}_{\mathbb{Q}^5}(7)) = 0$ for every $j \geq 1$.

(ii.3) Assume that $(W, \mathcal{K})$ is of type 2 in [4, Theorem 7.7.5]. Let $\pi : W \to C$ be the $\mathbb{P}^4$-bundle over a smooth curve C. Then by [4, Proposition 3.2.1], there exists an ample vector bundle $\mathcal{E}$ on C such that $W \cong \mathbb{P}_C(\mathcal{E})$ and $H(\mathcal{E}) = \mathcal{K}$. Since W is smooth in this case, $\mathcal{D}$ is a Cartier divisor and $\mathcal{D} = 2H(\mathcal{E}) + \pi^*(\mathcal{B})$ for $\mathcal{B} \in \mathrm{Pic}(C)$.

Claim 4.4.2. $h^j(2H(\mathcal{E}) + \pi^*(\mathcal{B})) = 0$ *for every* $j \geq 2$.

Proof. Since $R^k\pi_*(2H(\mathcal{E}) + \pi^*(\mathcal{B})) = R^k\pi_*(2H(\mathcal{E})) \otimes \mathcal{B} = 0$ for every positive integer k, we see that $h^j(2H(\mathcal{E}) + \pi^*(\mathcal{B})) = h^j(\pi_*(2H(\mathcal{E}) + \pi^*(\mathcal{B})))$ for every $j \geq 0$. Since $\dim C = 1$, we get $h^j(\pi_*(2H(\mathcal{E}) + \pi^*(\mathcal{B}))) = 0$ for every $j \geq 2$. Hence the assertion follows. $\square$

By the above argument we have $h^j(\mathcal{A}) = 0$ for every $j \geq 3$. Hence by Remark 3.5(4) we see that $\Delta_j(X, \mathcal{L}) = \Delta_j(M, \mathcal{A}) = 0$ for every $j \geq 3$. Therefore $\Delta_3(X, \mathcal{L}) = 0$ if $(X, \mathcal{L})$ satisfies (2.1) or (2.2) in Proposition 2.1.

From (i) and (ii) above, we see that $\Delta_3(X, \mathcal{L}) \leq 0$ if $\kappa(K_X + (n-3)\mathcal{L}) = -\infty$ and $2g_1(X, \mathcal{L}) - 2 = 2\mathcal{L}^n$. This yields the assertion of Lemma 4.4.1. $\square$

From (III.1) and (III.2) we see that $C(3, 3)$ implies $C(3, 1)$.

(IV) Assume that $C(3, 5)$ holds. We will prove that $C(3, 1)$ holds. First we see that $\kappa(K_X + (n-3)\mathcal{L}) \leq 0$ by the same reasoning of case (III) above.

(IV.1) Assume that $\kappa(K_X + (n-3)\mathcal{L}) = 0$. Then by the same argument as in (III.1) above, we get $K_X + (n-3)\mathcal{L} = \mathcal{O}_X$.

(IV.2) Assume that $\kappa(K_X + (n-3)\mathcal{L}) = -\infty$. By Proposition 3.1 we see that $g_3(X, \mathcal{L}) = 0$ unless $(X, \mathcal{L})$ is a scroll over a normal projective variety of dimension 3. Next we assume that $(X, \mathcal{L})$ is a scroll over a normal projective variety Y of dimension 3. Note that Y is smooth and $(X, \mathcal{L})$ is a classical scroll over Y by Proposition 4.4.1. Then $g_3(X, \mathcal{L}) = h^3(\mathcal{O}_X)$ (see Remark 3.3). If $h^3(\mathcal{O}_X) > 0$, then by Claim 4.4.1 we have $2g_1(X, \mathcal{L}) - 2 > 2\mathcal{L}^n$ and this contradicts the assumption $C(3, 5)$. Hence $h^3(\mathcal{O}_X) = 0$, and so $g_3(X, \mathcal{L}) = 0$ also in this case. Therefore

there does not exist any $(X, \mathcal{L})$ with $\kappa(K_X + (n-3)\mathcal{L}) = -\infty$, $g_3(X, \mathcal{L}) > 0$ and $2g_1(X, \mathcal{L}) - 2 = 2\mathcal{L}^n$.

By (IV.1) and (IV.2) we get $C(3,1)$. This completes the proof of Theorem 4.4.1. $\hfill\square$

By Theorems 4.2.1, 4.3.1 and 4.4.1 we get the following corollary.

Corollary 4.4.1. *Let $(X, \mathcal{L})$ be a polarized manifold of dimension n.*

(i) *Assume that* $\dim \mathrm{Bs}|\mathcal{L}| \leq 1$. *Then Conjecture 4.1.1 is true.*

(ii) *Assume that* $\dim \mathrm{Bs}|\mathcal{L}| = 2$ *and $n \geq 5$. Then Conjecture 4.1.1 is true.*

Remark 4.4.1. Next we consider the case where $n = 4$. By the same argument as in the proof of Theorem 4.4.1, we find that the following implications hold: $C(3.1) \Rightarrow C(3.2)$, $C(3.1) \Rightarrow C(3.4)$, $C(3.2) \Rightarrow C(3.3)$, and $C(3.4) \Rightarrow C(3.5)$.

Next we consider the implication $C(3.3) \Rightarrow C(3.1)$. By the same argument as in case (III) of Theorem 4.4.1 we have $\kappa(K_X + \mathcal{L}) \leq 0$.

If $\kappa(K_X + \mathcal{L}) = 0$, then we can prove that $K_X + \mathcal{L} = \mathcal{O}_X$. So we assume that $\kappa(K_X + \mathcal{L}) = -\infty$. Since $n = 4$, by using [10, (4.∞)] we see that $h^4(\mathcal{O}_X) = 0$ and $h^4(\mathcal{L}) = 0$ because $h^4(\mathcal{L}) = h^0(K_X - \mathcal{L})$ and $\kappa(X) = -\infty$. By Lemma 2.1, Proposition 3.1, and Remarks 2.5 and 3.4, we have $g_3(X, \mathcal{L}) = 0$ and $h^3(\mathcal{O}_X) = 0$ unless $(X, \mathcal{L})$ is a scroll over a normal projective variety of dimension 3. Therefore $\Delta_4(X, \mathcal{L}) = 0$ and $\Delta_3(X, \mathcal{L}) = -h^3(\mathcal{L}) \leq 0$ unless $(X, \mathcal{L})$ is a scroll over a normal 3-fold. But this contradicts the assumption. Therefore if $(X, \mathcal{L})$ is not a scroll over a normal 3-fold, then $C(3.3)$ implies $C(3.1)$. By the same argument as above, we see that $C(3.5)$ implies $C(3.1)$ if $(X, \mathcal{L})$ is not a scroll over a normal 3-fold.

If $(X, \mathcal{L})$ is a classical scroll over a smooth 3-fold, then by Claim 4.4.1 we also find that $C(3.3)$ and $C(3,5)$ cannot occur.

So in order to prove that Conjecture 4.1.1 for $n = 4$ and $i = 3$ is true, it suffices to consider the case where $(X, \mathcal{L})$ is a scroll over a normal projective variety Y with $\dim Y = 3$, but not a classical scroll over a smooth 3-fold Y.

§4.5. Some remarks

Finally we make some comments about Conjecture 4.1.1.

(a) We can easily see that $C(i,1)$ implies $C(i,2)$ and $C(i,4)$, and $C(i,2)$ (resp. $C(i,4)$) implies $C(i,3)$ (resp. $C(i,5)$) by the same argument as in the proof of Theorem 4.2.1.

(b) By looking at the proof of Theorems 4.2.1, 4.3.1 or 4.4.1, we can prove the following:

Proposition 4.5.1. *If there does not exist any polarized manifold $(X, \mathcal{L})$ with $\kappa(K_X + (n - i)\mathcal{L}) = -\infty$, $\Delta_i(X, \mathcal{L}) > 0$ (resp. $g_i(X, \mathcal{L}) > 0$) and $2g_1(X, \mathcal{L}) - 2 = (i - 1)\mathcal{L}^n$, then $C(i, 3)$ (resp. $C(i, 5)$) implies $C(i, 1)$.*

So it is important to know whether there exists an example of $(X, \mathcal{L})$ with $\kappa(K_X + (n - i)\mathcal{L}) = -\infty$, $\Delta_i(X, \mathcal{L}) > 0$ (resp. $g_i(X, \mathcal{L}) > 0$) and $2g_1(X, \mathcal{L}) - 2 = (i - 1)\mathcal{L}^n$.

(c) We can regard the following result as the case where $i = n$ in Conjecture 4.1.1.

Proposition 4.5.2. *Let X be a smooth projective variety of dimension n. Then the following are equivalent to one another:*

(i) $K_X \sim \mathcal{O}_X$.

(ii) $\Delta_n(X, \mathcal{L}) = 1$ *and* $2g(X, \mathcal{L}) - 2 = (n - 1)\mathcal{L}^n$ *for any ample line bundle* $\mathcal{L}$.

(iii) $\Delta_n(X, \mathcal{L}) > 0$ *and* $2g(X, \mathcal{L}) - 2 = (n - 1)\mathcal{L}^n$ *for any ample line bundle* $\mathcal{L}$.

(iv) $g_n(X, \mathcal{L}) = 1$ *and* $2g(X, \mathcal{L}) - 2 = (n - 1)\mathcal{L}^n$ *for any ample line bundle* $\mathcal{L}$.

(v) $g_n(X, \mathcal{L}) > 0$ *and* $2g(X, \mathcal{L}) - 2 = (n - 1)\mathcal{L}^n$ *for any ample line bundle* $\mathcal{L}$.

Proof. (i)$\Rightarrow$(ii): By Remark 3.5(2) we have $\Delta_n(X, \mathcal{L}) = h^n(\mathcal{O}_X) - h^n(\mathcal{L})$. By assumption we get $h^n(\mathcal{O}_X) = h^0(K_X) = 1$. Next we calculate $h^n(\mathcal{L})$. Since $h^n(\mathcal{L}) = h^0(K_X - \mathcal{L}) = h^0(-\mathcal{L})$, we see that $h^n(\mathcal{L}) = 0$ because $\mathcal{L}$ is ample. Therefore $\Delta_n(X, \mathcal{L}) = 1$. Of course $2g(X, \mathcal{L}) - 2 = (n - 1)\mathcal{L}^n$ since $K_X \sim \mathcal{O}_X$.

(ii)$\Rightarrow$(iii): This is trivial.

(iii)$\Rightarrow$(i): Since $0 < \Delta_n(X, \mathcal{L}) = h^n(\mathcal{O}_X) - h^n(\mathcal{L})$, we have $h^n(\mathcal{O}_X) \geq 1$. By the Serre duality we see that $h^0(K_X) \geq 1$. On the other hand, since $2g(X, \mathcal{L}) - 2 = (n - 1)\mathcal{L}^n$, we have $K_X \mathcal{L}^{n-1} = 0$. Therefore $K_X \sim \mathcal{O}_X$.

(i)$\Rightarrow$(iv): Since $K_X \sim \mathcal{O}_X$, we see that $2g(X, \mathcal{L}) - 2 = (K_X + (n-1)\mathcal{L})\mathcal{L}^{n-1} = (n - 1)\mathcal{L}^n$ and $h^0(K_X) = 1$. By the Serre duality we have $h^n(\mathcal{O}_X) = h^0(K_X) = 1$. Hence $g_n(X, \mathcal{L}) = h^n(\mathcal{O}_X) = 1$ by Remark 3.1(2).

(vi)$\Rightarrow$(v): This is trivial.

(v)$\Rightarrow$(i): By assumption, we have $h^n(\mathcal{O}_X) = g_n(X, \mathcal{L}) > 0$. We also note that $K_X \mathcal{L}^{n-1} = 0$ by the assumption that $2g(X, \mathcal{L}) - 2 = (n - 1)\mathcal{L}^n$. Hence $K_X \sim \mathcal{O}_X$ because $h^0(K_X) = h^n(\mathcal{O}_X) > 0$. $\square$

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