

# Hutchinson–Weber Involutions Degenerate Exactly when the Jacobian is Comessatti

by

Hisanori OHASHI

## Abstract

We consider the Jacobian Kummer surface  $X$  of a genus two curve  $C$ . We prove that the Hutchinson–Weber involution on  $X$  degenerates if and only if the Jacobian  $J(C)$  is Comessatti. Also we give several conditions equivalent to this, which include the classical theorem of Humbert. The key notion is Weber hexads, which are special sets of 2-torsion points of the Jacobian. We include an explanation of them and discuss the dependence between the conditions of the main theorem for various Weber hexads. It results in “the dual six equivalence”. We also give a detailed description of relevant moduli spaces. As an application, we give a conceptual proof of the computation of the patching subgroup for generic Hutchinson–Weber involutions.

*2010 Mathematics Subject Classification:* 14J28, 14J50, 14K22, 20B30.

*Keywords:* Jacobian Kummer surface, Hessian model, Weber hexad, Hutchinson–Weber involution, degeneration, Comessatti surface, outer automorphisms of the symmetric group.

## §1. Introduction

Let  $J(C)$  be the Jacobian of a curve  $C$  of genus two and  $X$  the minimal desingularization of  $\bar{X} = J(C)/\iota$ ,  $\iota = -\text{id}$ . Here every variety we consider is over  $\mathbb{C}$ , and  $X = \text{Km}(J(C))$  is called the *Jacobian Kummer surface* of  $C$ , which is well-known to be a  $K3$  surface.

In [11] we classified fixed-point-free involutions on  $X$ , or equivalently Enriques surfaces whose covering  $K3$  surface is isomorphic to  $X$ , under the condition that  $X$  is Picard-general. They consist of 10 switches, 15 Hutchinson–Göpel involutions and 6 Hutchinson–Weber involutions. In this paper we focus on the Hutchinson–Weber (HW) involutions; the point of our discussion here is that we do not assume any kind of generality on the curve  $C$ .

---

Communicated by S. Mukai. Received May 27, 2010. Revised March 8, 2011.

H. Ohashi: Graduate School of Mathematics, Nagoya University, Nagoya 464-8602, Japan;  
e-mail: pioggia@kurims.kyoto-u.ac.jp

HW involutions are closely related to the classical notion of *Weber hexads*, which are special subsets of 2-torsion points of the Jacobian, and associated *Hessian models* of  $X$  as treated in [6]. We recall these notions in Section 3.1. Besides the definition itself, the “dual six” equivalence relation plays an important role in this paper. In Section 3.2, we study the singularities of Hessian models. We prove that the singularities of a Hessian model consist of 10 or 11 nodes (Corollary 3.9). In Section 3.3 we show that an 11th node occurs exactly when the associated HW involution acquires fixed loci (Proposition 3.10), namely when the HW involution degenerates.

On the other hand, an abelian surface  $A$  is called a *Comessatti surface* if it has real multiplication in the maximal order  $\mathcal{O}_{\mathbb{Q}(\sqrt{5})}$  of  $\mathbb{Q}(\sqrt{5})$  [8]. A classical theorem of Humbert characterizes Comessatti Jacobians in terms of the branch points  $p_1, \dots, p_6$  of the bicanonical map  $C \rightarrow (\text{a conic}) \subset \mathbb{P}^2$ , see for example [14].

**Theorem 1.1** (Humbert). *The Jacobian  $J(C)$  is Comessatti if and only if for a suitable labeling of branch points there exists a conic  $D$  which is inscribed to the pentagon  $p_1 \cdots p_5$  and passes through  $p_6$ .*

We will show that these Comessatti Jacobians are closely related to Weber hexads. Corresponding to the dual six equivalence, there are essentially 6 ways of different labelings in the Humbert theorem.

The projective dual of the six points  $p_1, \dots, p_6$  is the six branch lines of the double plane model of the Jacobian Kummer surface  $X = \text{Km}(J(C))$ . The dual of the conic  $D$  induces a new genus two curve on  $J(C)$  different from the (translations of) theta divisors. Equivalently, these curves are the pullbacks of the theta divisors by the automorphism  $(\pm 1 + \sqrt{5})/2$  ( $\Xi$  in (4) of the main theorem below). We show in fact that each of these curves passes through six 2-torsion points, which form a Weber hexad (Proposition 4.2). As is expected, this curve corresponds exactly to the 11th node of the Hessian model (Theorem 4.5, (1) $\Leftrightarrow$ (4)). Our main theorem is as follows.

**Theorem 1.2** (Theorem 4.5). *Let  $C$  be a curve of genus two and  $(X, W)$  its Jacobian Kummer surface and a Weber hexad on it. Then the following conditions are equivalent.*

- (1) *The Hessian model  $X_W$  acquires the 11th node.*
- (2) *The Hutchinson–Weber involution  $\sigma_W$  degenerates in the sense that it acquires fixed loci.*
- (3) *The unique twisted cubic  $\bar{E}$  passing through the nodes  $\{n_\alpha\}_{\alpha \in W}$  of  $\bar{X}$  lies on the Kummer quartic surface  $\bar{X}$ . (Here the strict transform  $E \subset X$  satisfies the relations in Proposition 3.7.)*

- (4) *The Jacobian  $J(C)$  is a Comessatti surface and one curve  $\Xi$  among (4.2) passes through the 2-torsion points corresponding to  $W$ .*
- (5) *In the double plane model (Proposition 2.2) projected from one node  $n_{w_0}$  ( $w_0 \in W$ ), there exists an additional conic  $E' \subset \mathbb{P}^2$  which passes through the vertices of the pentagon formed by five images of  $\{n_w \mid w \in W - \{w_0\}\}$  and tangent to the remaining branch line. For example, when  $W = \{0, 12, 23, 34, 45, 51\}$  and  $w_0 = 0$  as in Proposition 2.2, then the pentagon is formed by  $l_1, \dots, l_5$  and the last line is  $l_6$ .*

The equivalence between (4) and (5) is nothing but the above theorem of Humbert, stated in the dual projective space. But our theorem is a bit extended in the sense that we refer to the Weber hexads. There are six equivalence classes of Weber hexads, which we will call the “dual” six (Section 3), and we can show that for equivalent Weber hexads, the conditions in the theorem are equivalent (Proposition 4.7). Thus our theorem is more quantitative than known results, even considered as the extension of the theorem of Humbert, and the equivalence with conditions (1) and (2) is apparently new. This theorem explains the title of the article.

Section 2 reviews facts about Jacobian Kummer surfaces and fixes the notation. Section 3 explains Weber hexads, Hessian models and the HW involutions. Proposition 3.7 is the essential ingredient. Section 4 studies the relationship between Comessatti Jacobian surfaces and Weber hexads. The main theorem 4.5 is proved there.

In Section 5 we give a detailed description of the moduli space of Jacobian Kummer surfaces, Jacobian Kummer surfaces equipped with an equivalence class of Weber hexad and the locus of degenerate Hutchinson–Weber involutions. We use the theory of period maps for  $K3$  surfaces. We obtain the irreducibility of the moduli space of Comessatti Jacobian Kummer surfaces (Theorem 5.7).

In the last section, we give an application of this characterization to the computation of patching subgroups (see [11]) of HW involutions. It seems interesting to the author that we can derive consequences for Picard-general Jacobian Kummer surfaces by studying degenerations.

In this paper we restrict ourselves to genus two curves. Extension to reducible principally polarized abelian surfaces, that is, products of elliptic curves, is entirely left to further research.

## §2. Jacobian Kummer surfaces

Here we recall the construction of Jacobian Kummer surfaces and fix the notation. We use the same indexing of divisors as in [11].

Let  $C$  be a smooth projective curve of genus 2. Let  $J(C) = \text{Pic}^0(C)$  be its Jacobian variety. It has the inversion morphism  $\iota: x \mapsto -x$ . We denote by  $\overline{X} = \overline{\text{Km}}(J(C))$  the quotient surface  $J(C)/\iota$  and by  $X = \text{Km}(J(C))$  the minimal resolution,

$$J(C) \xrightarrow{\iota} \overline{X} \xleftarrow{\text{min. res'n}} X.$$

Then  $X$  is a K3 surface associated to  $C$  and called the *Jacobian Kummer surface* of  $C$ .

In the following we introduce several divisors on  $X$  whose configuration is called the  $(16)_6$ -*configuration* on  $X$ . Recall that the morphism associated to the canonical system  $|K_C|$  represents  $C$  as a double cover of  $\mathbb{P}^1$  ramified at six Weierstrass points  $p_1, \dots, p_6 \in C$ . Using them, the set of 2-torsion points of the Jacobian can be written as

$$J(C)_2 = \{\alpha \in \text{Pic}^0(C) \mid 2\alpha \sim 0\} = \{0\} \cup \{[p_i - p_j] \mid i \neq j\}.$$

2-torsion points naturally correspond to the nodes  $n_\alpha$  of  $\overline{X}$  and exceptional curves  $N_\alpha$  of  $X$ . On the other hand, the set of theta characteristics of  $C$  can be written as

$$\begin{aligned} S(C) &= \{\beta \in \text{Pic}^1(C) \mid 2\beta \sim K_C\} \\ &= \{[p_i] \mid i = 1, \dots, 6\} \cup \{[p_i + p_j - p_k] \mid i \neq j \neq k \neq i\}. \end{aligned}$$

They also correspond to smooth rational curves on  $\overline{X}$  and  $X$  called *tropes*; the tropes  $\overline{T}_\beta \subset \overline{X}$  and  $T_\beta \subset X$  are the strict transforms of the theta divisor

$$\Theta_\beta = \{[p - \beta] \in J(C) \mid p \in C\}.$$

The incidence relation between  $N_\alpha$  and  $T_\beta$  is given by

$$\begin{aligned} (N_\alpha, N_{\alpha'}) &= -2\delta_{\alpha, \alpha'}, & (T_\beta, T_{\beta'}) &= -2\delta_{\beta, \beta'}, \\ (N_\alpha, T_\beta) &= 1 \Leftrightarrow \alpha + \beta \in \{[p_1], [p_2], [p_3], [p_4], [p_5], [p_6]\}, \\ (N_\alpha, T_\beta) &= 0 \quad \text{otherwise,} \end{aligned}$$

where  $\delta$  is the Kronecker symbol. We will abbreviate  $N_{[p_i - p_j]}$  to  $N_{ij}$  and  $T_{[p_i + p_j - p_k]}$  to  $T_{ijk}$ , etc. We record the relation  $T_{ijk} = T_{lmn}$  for any permutation  $i, \dots, n$  of  $1, \dots, 6$ .

We will denote by  $H$  the divisor class of  $2T_1 + N_0 + \sum_{j=2}^6 N_{1j}$ ; note that any analogous divisor  $2T_\beta + \sum_{(T_\beta, N_\alpha)=1} N_\alpha$  gives the same divisor class as  $H$ . The following fact is classically known; it is the reason for calling  $\overline{X}$  the *Kummer quartic surface*.

**Proposition 2.1** (Kummer quartic model). *The linear system  $|H|$  induces an embedding of  $\overline{X}$  into  $\mathbb{P}^3$  as a quartic surface with sixteen nodes. The trope  $\overline{T}_\beta \subset \overline{X}$  is a smooth conic on  $\overline{X}$  and the unique hyperplane containing  $\overline{T}_\beta$  cuts  $\overline{X}$  doubly along  $\overline{T}_\beta$ .*

We usually regard  $\overline{X}$  as embedded in  $\mathbb{P}^3$ . Projecting  $\overline{X}$  from one of its nodes, say  $n_0$ , we obtain the following model.

**Proposition 2.2** (double plane model). *The linear system  $|H - N_0|$  induces a generically two-to-one morphism of  $X$  onto  $\mathbb{P}^2$ . It contracts the exceptional curves  $N_\alpha$  other than  $N_0$ . If we denote the images of  $T_i$  by  $l_i$  for  $i = 1, \dots, 6$ , then  $\overline{X}$  (with  $n_0$  blown up) is a double cover of  $\mathbb{P}^2$  branched along the union  $\bigcup_i l_i$  of six lines. The image of  $N_0$  is a conic to which all  $l_i$  are tangent.*

We introduce two kinds of basic automorphisms.

**Proposition 2.3.** *For each  $\alpha_0 \in J(C)_2$ , the translation automorphism in  $\alpha_0$  on  $J(C)$  induces on  $X$  an automorphism called a translation. It acts on  $H^2(X, \mathbb{Z})$  by  $H \mapsto H$ ,  $N_\alpha \mapsto N_{\alpha+\alpha_0}$  and  $x \mapsto x$  for  $x$  orthogonal to  $\{H, N_\alpha\}$ .*

*Similarly for each  $\beta_0 \in S(C)$  there exists an automorphism of  $X$  called a switch that acts on  $H^2(X, \mathbb{Z})$  by:  $H \mapsto 3H - \sum_{\alpha \in J(C)_2} N_\alpha$ ,  $N_\alpha \mapsto T_{\alpha+\beta_0}$  and  $x \mapsto -x$  for  $x$  orthogonal to  $\{H, N_\alpha\}$ .*

These automorphisms exist and have the same action on cohomology for any Jacobian Kummer surface  $X$ . Therefore we may say that translations and switches do not degenerate under specialization of Jacobian Kummer surfaces.

### §3. The Hessian model

Let  $X$  be a Jacobian Kummer surface associated to a curve  $C$  of genus 2. In this section we focus on the *Hessian model*  $X_W$  of  $X$ , treated for example in [6]. After giving a self-contained proof of Proposition 3.4, we consider singularities of  $X_W$ . The point is that we do *not* assume that  $C$  is general, in any sense.

#### §3.1. Weber hexads

The Hessian model  $X_W$  is associated to a Weber hexad  $W$ , which is a special subset of  $J(C)_2$ . We first explain this notion and the equivalence relation among them, which is essential to this paper.

Since Weber hexads have a rather complicated definition, following the suggestion of the referee, we at first do not use it, but instead we proceed with the handy definition below. Later in this subsection, we will explain the definition given

in [6], which we need in the proof of Proposition 4.2, and show the equivalence of the two definitions in Lemma 3.3.

**Definition 3.1.** A subset  $W \subset J(C)_2$  is called a *Weber hexad* if it has one of the forms

$$\{0, ij, jk, kl, lm, mi\} \quad \text{or} \quad \{ij, jk, ki, il, jm, kn\},$$

where  $\{i, \dots, n\}$  is some permutation of  $\{1, \dots, 6\}$ , and  $ij$  for example means the divisor class of  $p_i - p_j$  in the notation of the previous section.

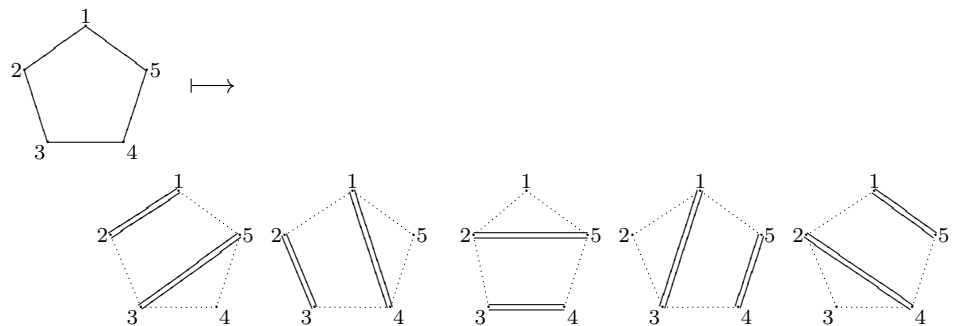
We see easily that there are 192 Weber hexads, 72 of which are in the first form and 120 in the second form.

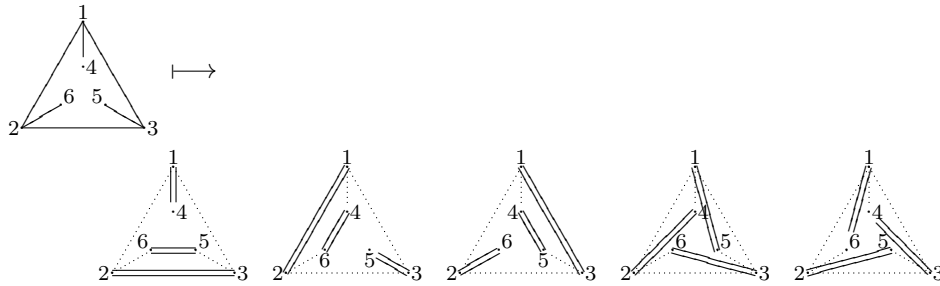
Weber hexads are essentially ways to express the “dual set” of  $\{1, \dots, 6\}$ . Recall that the symmetric group  $\mathcal{S}_6$  has two permutation representations. One is the natural representation on  $\{1, \dots, 6\}$  and the other is the one twisted by the outer automorphism.

In [11] we proved that if the curve  $C$  is generic, then the 192 Hutchinson–Weber involutions  $\sigma_W$  (constructed for each  $W$ , see Subsection 3.3) are divided into exactly six conjugacy classes in  $\text{Aut}(X)$ . Moreover we could see that the permutation on the labels of Weierstrass points of  $C$  and the permutation on these six conjugacy classes are related by an outer automorphism, hence these six conjugacy classes can be regarded as the dual set. In Remark (2) after Proposition 7.4 of [11] we have given one possible description of this correspondence. Let us here give a more visible one.

Let us recall the classical description of the dual set, found for example in [1]. An element in  $\mathcal{S}_6$  of the form  $(ij)$  is called a *duad*; similarly  $(ij)(kl)(mn)$  is called a *syntheme*; a five-element set is called a *total* if it consists of five synthemes that contain all fifteen duads. There are exactly six totals and this is the classical description of the dual set.

By definition, a Weber hexad is one of the two forms. The pictures below indicate the correspondence from a Weber hexad to a total.





The first picture indicates the correspondence

$$W_1 = \{0, 12, 23, 34, 45, 51\} \mapsto \{(12)(35)(46), (14)(23)(56), (16)(25)(34), (13)(26)(45), (15)(24)(36)\},$$

where the letter 6 is regarded as distinguished. The second picture indicates the correspondence

$$W_2 = \{12, 23, 31, 14, 26, 35\} \mapsto \{(14)(23)(56), (12)(35)(46), (13)(26)(45), (15)(24)(36), (16)(25)(34)\}.$$

In this example, since we obtain the same total, we see that  $\sigma_{W_1}$  and  $\sigma_{W_2}$  are conjugate in  $\text{Aut}(X)$  when  $C$  is generic. In the same way, for arbitrary  $C$ , we define the equivalence relation on Weber hexads by  $W_1 \sim W_2$  if and only if they correspond to the same total. In the following we refer to this as *the dual six equivalence relation*. Using the picture, it is easy to find six representatives of equivalence classes of Weber hexads. For example, the six Weber hexads of the form

$$\{0, 12, 2i, ij, jk, k1\},$$

where  $i, j, k$  runs over permutations of  $3, \dots, 5$ , constitute a set of representatives.

The geometric phenomenon which is the subject of this paper is related to this intrinsic equivalence relation on Jacobian Kummer surfaces.

Next we recall the definition of Weber hexads in [6] and show the equivalence of the two definitions. Let us define a symplectic form on  $J(C)_2$  by  $(\alpha, \alpha') = \#(\alpha \cap \alpha') \pmod 2 \in \mathbb{F}_2$ , where we identify  $\alpha$  with a two-element subset of  $\{1, \dots, 6\}$ . A 2-dimensional *affine* subspace of  $J(C)_2$  (that is, a translation of a 2-dimensional *linear* subspace) is called a *Göpel tetrad* if it is a translation of a totally isotropic 2-dimensional linear subspace. Other 2-dimensional affine subspaces are called *Rosenhain tetrads*; equivalently they are translations of nondegenerate 2-dimensional linear subspaces. We easily see that there are 60 (resp. 80) Göpel (resp. Rosenhain) tetrads.

We use the notation  $A \ominus B$  to denote the set-theoretic symmetric difference  $(A \cup B) - (A \cap B)$ . The original definition is as follows.

**Definition 3.2.** A six-element subset of  $J(C)_2$  is called a *Weber hexad* if it can be written in the form  $G \ominus R$ , where  $G$  is a Göpel tetrad and  $R$  is a Rosenhain tetrad such that  $\#G \cap R = 1$ .

To clarify, in the next lemma we will call a set  $W$  as in Definition 3.1 a  $W$ -set and use the name Weber hexad for a set as in Definition 3.2. The lemma asserts in fact they coincide.

**Lemma 3.3.** *Every  $W$ -set is a Weber hexad. Conversely any Weber hexad is a  $W$ -set.*

*Proof.* The former part is straightforward. We see that

$$\{0, ij, jk, kl, lm, mi\} = \{0, ij, jk, ki\} \ominus \{ki, kl, lm, mi\}$$

with  $\{0, ij, jk, ki\}$  a Rosenhain tetrad and  $\{ki, kl, lm, mi\} = \{0, il, jn, km\} + ki$  a Göpel tetrad. Similarly  $\{ij, jk, ki, il, jm, kn\} = \{0, ij, jk, ki\} \ominus \{0, il, jm, kn\}$ .

We prove the latter part. A permutation of letters  $1, \dots, 6$  induces an isometry of  $J(C)_2$ . This correspondence induces an isomorphism  $\mathcal{S}_6 \simeq \mathrm{Sp}(4, \mathbb{F}_2)$ , hence the affine isometry group of  $J(C)_2$  can be written as  $(\mathbb{Z}/2\mathbb{Z})^4 \cdot \mathrm{Sp}(4, \mathbb{F}_2) \simeq J(C)_2 \cdot \mathcal{S}_6 =: K$ .

First we show that  $K$  acts on the set of Weber hexads transitively. Given  $W$ , we translate it appropriately and can assume it is of the form  $G \ominus R$  where  $G \cap R = \{0\}$ . Then we easily check that the only possibility is  $G = \{0, ij, kl, mn\}$  and  $R = \{0, ik, km, mi\}$  for a suitable permutation  $i, \dots, n$  of  $1, \dots, 6$ . This shows the transitivity.

Next we compute the stabilizer subgroup  $H$  of  $W = \{ij, kl, mn, ik, km, mi\}$ . The intersection  $\mathcal{S}_6 \cap H$  consists of six elements  $\tau\sigma\tau\sigma^{-1}$  for  $\tau \in \mathcal{S}(\{i, k, m\})$  and  $\sigma = (ij)(kl)(mn)$ . On the other hand, for each  $\alpha \in J(C)_2 - W$  there exists a unique way of expressing  $W$  as  $G' \ominus R'$  with  $G' \cap R' = \{\alpha\}$ . Thus there exist six choices of  $\nu \in \mathcal{S}_6$  such that  $\nu\alpha \in K$  sends  $W$  onto itself. In this way we obtain  $6 \cdot 10 = 60$  elements in  $H$ . Thus there are at most  $2^4 6! / 60 = 192$  Weber hexads.

On the other hand we already saw that there are 192  $W$ -sets. Hence the lemma is proved.  $\square$

A final remark is on the generating relations of the dual six equivalence. In the generic case of [11], we saw that conjugacy relations between Hutchinson–Weber involutions are given by translations and switches (Proposition 2.3). Translations give the equivalence  $W \sim W + \alpha$  ( $\alpha \in J(C)_2$ ). When  $G \cap R = \{0\}$ , which we



can always assume after a translation, the equivalence by switches is expressed as  $W = G \ominus R \sim G \ominus R^\perp$ . Thus the dual six equivalence is generated by these relations.

### §3.2. The Hessian model

The Hessian model  $X_W$  is constructed for every Weber hexad  $W$ . In the following, we consider the pair  $(X, W)$  consisting of a Jacobian Kummer surface  $X$  and a Weber hexad  $W$ . The next proposition is known to experts (see [6] and its references), but our algebraic proof is more suited for what follows.

**Proposition 3.4** (The Hessian model). *The linear system  $|L| := |2H - \sum_{\alpha \in W} N_\alpha|$  maps  $X$  birationally to a quartic surface  $X_W$  whose equations are of the form*

$$s_1 + \cdots + s_5 = 0, \quad s_1 s_2 s_3 s_4 s_5 (\lambda_1/s_1 + \cdots + \lambda_5/s_5) = 0,$$

where  $\lambda_i$  are nonzero constants and  $s_i$  are homogeneous coordinates of  $\mathbb{P}^4$ .

*Proof.* As indicated above, Weber hexads are unique up to the affine symplectic group. The group  $(\mathbb{Z}/2\mathbb{Z})^4$  lifts to translation automorphisms of  $J(C)$  in the elements of  $J(C)_2$ , which commute with the quotient by  $\iota$ . The group  $\mathrm{Sp}(4, \mathbb{F}_2) \simeq \mathfrak{S}_6$  acts as permutations of the letters. So it is enough to check the proposition for a particular Weber hexad. Let us take  $W = \{12, 23, 31, 14, 25, 36\}$ .

Let us consider the divisors (cf. [6])

$$\begin{aligned} S_1 &= T_2 + T_3 + T_{124} + T_{134} + N_0 + N_{24} + N_{26} + N_{34} + N_{35} + N_{56}, \\ S_2 &= T_{123} + T_{145} + T_{134} + T_{125} + N_{15} + N_{26} + N_{34} + N_{45} + N_{46} + N_{56}, \\ S_3 &= T_1 + T_3 + T_{125} + T_{146} + N_0 + N_{15} + N_{16} + N_{34} + N_{35} + N_{46}, \\ S_4 &= T_{123} + T_{124} + T_{146} + T_{136} + N_{16} + N_{24} + N_{35} + N_{45} + N_{46} + N_{56}, \\ S_5 &= T_1 + T_2 + T_{136} + T_{145} + N_0 + N_{15} + N_{16} + N_{24} + N_{26} + N_{45}. \end{aligned}$$

It is easy to see that they belong to  $|L|$  and a careful check using them shows that  $|L|$  is base-point-free. Thus the associated map  $\varphi = \varphi_L$  is a morphism. By the Kawamata–Viehweg vanishing and Riemann–Roch we see that  $h^0(L) = 4$ . Hence the sections  $s_i \in H^0(L)$  corresponding to  $S_i$  are linearly dependent. On the other hand, by evaluating at general points of  $N_\alpha$  for several  $\alpha$ , we can check that any four among  $\{s_1, \dots, s_5\}$  are linearly independent. This shows that, up to adjusting the scalars, we can assume  $\sum_{i=0}^5 s_i = 0$ . By this equation, we regard the morphism  $\varphi$  as a morphism into  $\{\sum_{i=1}^5 s_i = 0\} \simeq \mathbb{P}^3$  in  $\mathbb{P}^4$ . We denote by  $X_W$  the image of  $\varphi$ .

Let us denote the hyperplane  $\{s_i = 0\}$  by  $H_i$ . Ten divisors  $T_\beta$  appearing in  $\bigcup S_i$  are mapped to lines on  $X_W$ . They appear with multiplicity two in  $\bigcup S_i$ , hence if  $T_\beta \subset S_i \cap S_j$  then we can write  $\varphi(T_\beta) = H_i \cap H_j =: L_{ij}$ . Similarly, the

ten divisors  $N_\alpha$  appearing in  $\bigcup S_i$  are contracted to a point on  $X_W$ . They appear exactly three times in  $\bigcup S_i$ , so we can write  $\varphi(N_\alpha) = H_i \cap H_j \cap H_k =: P_{ijk}$  if  $N_\alpha \subset S_i \cap S_j \cap S_k$ . In fact these ten  $N_\alpha$  are exactly those with  $\alpha \in J(C)_2 - W$ .

Let us look at the hyperplane section  $H_1 \cap X_W$  more closely. It contains four lines  $L_{1j}$ ,  $j = 2, \dots, 5$ , namely the images of  $T_{134}$ ,  $T_3$ ,  $T_{124}$  and  $T_2$ . General points of these four tropes are separated from each other by the divisors  $S_i$ . Thus the hyperplane  $H_1$  cuts  $X_W$  along four distinct lines. This implies that  $\deg X_W = 4$  and  $\varphi$  is birational. Let  $f(s_1, s_2, s_3, s_4)$  be the quartic equation of  $X_W$ ,  $s_5$  being replaced by  $-(s_1 + \dots + s_4)$ . The argument above shows that  $f(0, s_2, s_3, s_4)$  is a multiple of  $s_2 s_3 s_4 (s_2 + s_3 + s_4)$ . Similar consequences hold for  $s_2 = 0, s_3 = 0, s_4 = 0$ . In summary it follows that  $f$  is a linear combination of the terms

$$\begin{aligned} s_1 s_2 s_3 s_4, & \quad s_2 s_3 s_4 (s_2 + s_3 + s_4), & \quad s_1 s_3 s_4 (s_1 + s_3 + s_4), \\ s_1 s_2 s_4 (s_1 + s_2 + s_4), & \quad s_1 s_2 s_3 (s_1 + s_2 + s_3). \end{aligned}$$

Using  $s_5$ , these terms can be written as a linear combination of

$$s_1 s_2 s_3 s_4 s_5 / s_i, \quad i = 1, \dots, 5.$$

Thus we derived the equation. We have  $\lambda_i \neq 0$  because  $X_W$  is irreducible.  $\square$

Below we use the notation  $\varphi: X \rightarrow X_W$  of the previous proof.

**Corollary 3.5.**  *$X_W$  is normal.*

*Proof.* Let  $\psi: X \rightarrow Y$  be the morphism which contracts all the  $(-2)$ -curves on  $X$  orthogonal to  $L$ . Then  $Y$  is a normal surface with at most rational double points, and the canonical sheaf of  $Y$  is trivial. Since the exceptional sets of  $\psi$  and  $\varphi$  coincide,  $\varphi$  factors as  $\varphi = \nu\psi$ . By the adjunction formula  $K_{X_W}$  is also trivial, so  $\nu$  is étale in codimension one, hence  $X_W$  is regular in codimension one. Since  $X_W$  is a complete intersection, by Serre's criterion we see that  $X_W$  is normal.  $\square$

**Corollary 3.6.** *Each  $P_{ijk}$  is an ordinary node. (Recall that  $\{P_{ijk}\} = \{\varphi(N_\alpha) \mid \alpha \in J(C)_2 - W\}$ .)*

*Proof.* This follows from  $\varphi^{-1}(P_{ijk}) = S_i \cap S_j \cap S_k = N_\alpha$ .  $\square$

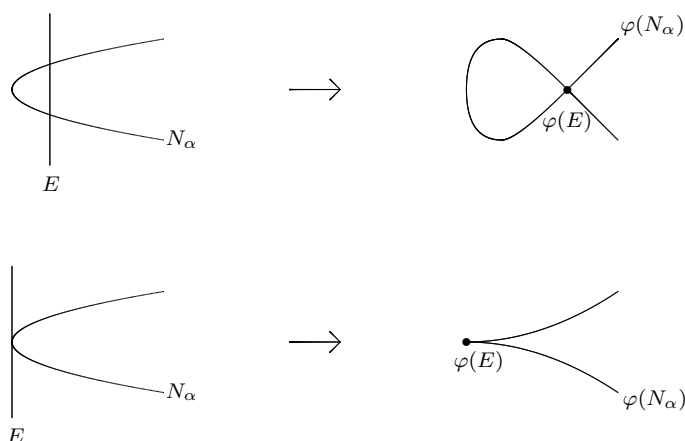
**Proposition 3.7.** *Suppose a  $(-2)$ -curve  $E$  different from  $\{N_\alpha\}$  is contracted by  $\varphi$ . Then  $E$  has to satisfy the relations*

$$\begin{cases} (E, N_\alpha) = 0, & \alpha \in J(C)_2 - W, \\ (E, N_\alpha) = 1, & \alpha \in W, \\ (E, H) = 3. \end{cases}$$

*Moreover, such a  $E$  is unique if it exists.*

*Proof.* By the previous corollary,  $E$  and the exceptional  $N_\alpha$  do not meet, otherwise the singularity is not a node. Hence  $(E, N_\alpha) = 0$  for  $\alpha \in J(C)_2 - W$ . Let us consider  $N_\alpha$  for  $\alpha \in W$ . By the projection formula  $(\varphi_*(N_\alpha), \mathcal{O}_{X_W}(1)) = (N_\alpha, L) = 2$ , hence we see that  $\varphi_*(N_\alpha)$  is a cycle of degree 2. It is irreducible and reduced by the Zariski main theorem, so  $\varphi_*(N_\alpha) = \varphi(N_\alpha)$  is a smooth conic. Hence  $\varphi$  induces the isomorphism  $N_\alpha \xrightarrow{\sim} \varphi(N_\alpha)$ .

If  $N_\alpha, \alpha \in W$ , intersects the exceptional  $E$  with intersection number  $\geq 2$ , then clearly  $\varphi(N_\alpha)$  acquires a singular point, a contradiction. See the picture below. It follows that  $(E, N_\alpha) = 0$  or 1. On the other hand  $(E, L) = (E, 2H - \sum_{\alpha \in W} N_\alpha) = 0$ , thus  $0 \leq (E, H) \leq 3$ .  $(E, H) = 0$  is prohibited by Proposition 2.1.



Let us denote by  $\overline{E}$  the corresponding curve on  $\overline{X} = J(C)/\iota$ . This is a smooth rational curve passing through  $2(H, E)$  nodes.

Assume  $(H, E) = 1$ . Then the inverse image of  $\overline{E}$  in  $J(C)$  is a double cover branched at two points of  $\overline{E}$ , hence a rational curve. Since an abelian surface does not contain any rational curve, this is a contradiction.

Assume  $(H, E) = 2$ . Then  $\overline{E}$  is an irreducible conic in  $\mathbb{P}^3$  passing through four nodes belonging to  $W$ . These nodes therefore must be contained in a hyperplane of  $\mathbb{P}^3$ , which contradicts the lemma below.

Assume  $(H, E) = 3$ . Then  $\overline{E}$  is a cubic curve passing through the six nodes of  $W$ . By the lemma below, it is exactly the twisted cubic determined by  $W$  and the uniqueness follows from the Steiner construction [7]. Thus the whole proposition is reduced to the next lemma.  $\square$

**Lemma 3.8.** *If we identify the Weber hexad  $W$  with the corresponding nodes  $n_\alpha$  of  $\overline{X}$ , then no four points of  $W$  are coplanar. That is, they are in general position with respect to  $\mathcal{O}(1)$ .*

*Proof.* We begin by showing that no three nodes of  $\overline{X}$  are collinear. Assume the contrary. Then since  $\overline{X}$  is a quartic surface, the line  $l$  containing them lies on  $\overline{X}$  and  $(l, H) = 1$  (the intersection numbers are computed on  $X$ , so we identify  $l$  with its strict transform on  $X$ ). By the relation

$$(3.1) \quad H \sim 2T_\beta + \sum_{(N_\alpha, T_\beta)=1} N_\alpha$$

we see that  $(l, T_\beta) = 0$ . On the other hand, clearly for (at least) three  $\alpha$  we have  $(l, N_\alpha) = 1$ . Summing up the relations (3.1) over  $\beta \in S(C)$ , we obtain  $16H \sim 2 \sum T_\beta + 6 \sum N_\alpha$ . The left-hand side intersects  $l$  in 16 elements but the right-hand side intersects  $l$  in at least  $6 \cdot 3 = 18$  elements, hence we obtain a contradiction.

Next, because the incidence relation between nodes and tropes is preserved under the affine symplectic group  $G$ , it suffices to prove the lemma in case  $W = \{12, 23, 31, 14, 25, 36\}$  for example. Choose four points  $\{12, 23, 14, 25\}$ . We see that the trope  $T_2$  passes through the points  $n_{12}, n_{23}, n_{25}$  but not through  $n_{14}$ . By Proposition 2.1 a trope is a conic and coincides with the hyperplane section. Thus the four points are not coplanar. Similarly for any four points from  $W$ , we can find a trope containing three but not all four points. Thus we obtain the lemma.  $\square$

**Corollary 3.9.** *The singularities of  $X_W$  consist of 10 or 11 ordinary nodes. If  $X$  is Picard-general, i.e., the Picard number of  $X$  is 17, then  $X_W$  has only 10 nodes.*

*Proof.* The former part follows from the previous proposition. The latter part is because for Picard-general  $X$ ,  $\text{NS}(X)$  is generated over  $\mathbb{Q}$  by the divisors  $\{H, N_\alpha\}$ , but the existence of  $E$  above imposes one more linearly independent element.  $\square$

### §3.3. Hutchinson–Weber involutions

We keep the notation as before. Let us consider the Hessian model  $X_W : \{\sum s_i = \sum \lambda_i/s_i = 0\}$  defined in  $\mathbb{P}^4$ . We consider the Hutchinson–Weber involution defined by  $\sigma_W : (s_1, \dots, s_5) \mapsto (\lambda_1/s_1, \dots, \lambda_5/s_5)$ . It induces a biregular involution on  $X$  (since  $X$  is a minimal surface), which we also denote by  $\sigma_W$ . As is well-known, an automorphism of a K3 surface degenerates if and only if there is a  $(-2)$ -class in the Néron–Severi group which is perpendicular to the invariant sublattice. The degeneration of  $\sigma_W$  can be described precisely as follows.

**Proposition 3.10.** *The following are equivalent:*

- (1) *There exists one more node other than the 10 nodes of Corollary 3.6.*
- (2)  *$\sigma_W$  is not fixed-point-free.*
- (3) *For some choice of signs, we have  $\pm\sqrt{\lambda_1} \pm \dots \pm \sqrt{\lambda_5} = 0$ .*

*Proof.* (1) $\Leftrightarrow$ (3): The 11th node  $p$  corresponds to the rational curve  $E$  of Proposition 3.7. The hyperplane  $\{s_i = 0\}$  cuts  $X_W$  along four lines in  $\mathbb{P}^2$  in general position. Its singularities are the six nodes appearing in Corollary 3.6. Hence  $p$  is located inside the open set  $\{s_1 \cdots s_5 \neq 0\}$ . By the Jacobian criterion of smoothness, we easily deduce that the 11th node should satisfy the relation

$$(3.2) \quad \text{rank} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \lambda_1/s_1^2 & \lambda_2/s_2^2 & \lambda_3/s_3^2 & \lambda_4/s_4^2 & \lambda_5/s_5^2 \end{pmatrix} \leq 1.$$

Thus its existence is equivalent to the condition (3).

(2) $\Leftrightarrow$ (3): First we notice that  $\sigma_W$  sends the line  $L_{ij} = \{s_i = s_j = 0\}$  to the point  $P_{klm} = \{s_k = s_l = s_m = 0\}$ , where  $\{i, \dots, m\}$  is an arbitrary permutation of  $\{1, \dots, 5\}$ . Vice versa,  $P_{klm}$  is sent to  $L_{ij}$  since  $\sigma_W$  is an involution. Thus a fixed point can occur only inside the open set  $\{s_1 \cdots s_5 \neq 0\}$ . Here clearly the fixed point is given by the further condition

$$\frac{\lambda_1}{s_1^2} = \frac{\lambda_2}{s_2^2} = \dots = \frac{\lambda_5}{s_5^2},$$

which is equivalent to (3.2). Thus it is equivalent to (3). □

By the above proof, the fixed point of  $\sigma_W$  corresponds to the 11th node of  $X_W$ . In this case since  $\sigma_W$  is non-symplectic, it fixes the whole exceptional curve  $E$ .

**Remark 3.11.** The equation

$$s_1 + \dots + s_5 = \frac{s_1^3}{\lambda_1} + \dots + \frac{s_5^3}{\lambda_5} = 0,$$

which defines a cubic surface, is called the *Sylvester form* of the cubic. It is known that a generic cubic surface can be written in the Sylvester form in a unique way up to permutations and homothety, so this equation is well-studied in connection with the moduli problem for cubic surfaces. Our  $X_W$  is exactly of the form of “Hessian surface” of this cubic, hence the name. We note that there are four parameters for cubic surfaces, while there are only three parameters for Jacobian Kummer surfaces. Hence general Hessian  $K3$  surfaces cannot be obtained as the Hessian model of Jacobian Kummer surfaces. In fact Rosenberg [12] characterizes this locus inside the moduli of Hessians.

It is known that condition (3) in the preceding proposition represents the locus of singular cubic surfaces (see for example [5]). Genus two curves and singular cubics constitute the Kummer divisor and the boundary divisor inside the four-dimensional moduli space of cubic surfaces, respectively. Thus our object, the degenerations of Hutchinson–Weber involutions, corresponds to the intersection of these divisors.

#### §4. Comessatti abelian surfaces and the main theorem

We begin with a definition.

**Definition 4.1.** An abelian surface  $A$  is called a *Comessatti surface* if it has real multiplication in the maximal order  $\mathcal{O}_{\mathbb{Q}(\sqrt{5})}$  of  $\mathbb{Q}(\sqrt{5})$ , i.e., if  $\mathcal{O}_{\mathbb{Q}(\sqrt{5})} = \mathbb{Z}[(1 + \sqrt{5})/2] \subset \text{End}(A)$ .

Let us suppose that the Jacobian  $J(C) =: A$  is at the same time Comessatti. We take a theta divisor  $\Theta_\beta$  as in Section 2 and let  $\varphi \mapsto \varphi'$  be the Rosati involution on  $\text{End}(A)$  associated to  $\mathcal{O}_A(\Theta_\beta)$  which is independent of  $\beta$ . We note that by the positivity of the Rosati involution, it acts on  $\mathcal{O}_{\mathbb{Q}(\sqrt{5})}$  trivially. The endomorphism  $\varepsilon = (1 + \sqrt{5})/2$  is in fact an automorphism whose inverse is  $\eta = \varepsilon^{-1} = (-1 + \sqrt{5})/2$ . By [9, Section 21], we get

$$(4.1) \quad (\Theta_\beta, \varepsilon^* \Theta_\beta) = \text{tr}_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}(\varepsilon \varepsilon') = 3$$

and similarly for  $\eta$ . Since  $\Theta_\beta$  contains six 2-torsion points  $[\beta - p_i]$  ( $i = 1, \dots, 6$ ),  $\Xi := \varepsilon^* \Theta_\beta$  also contains six 2-torsion points  $w_i = \varepsilon^{-1}([\beta - p_i])$ .

**Proposition 4.2.** *The subset  $W = \{w_1, \dots, w_6\}$  of  $J(C)_2$  is a Weber hexad, whose definition is in Section 3.1.*

*Proof.* Clearly the sum  $\sum w_i$  is zero. Hence the partial sums  $w_1 + w_2 + w_3$  and  $w_4 + w_5 + w_6$  are equal. We denote this element by  $x$ . It is easy to see that  $x \notin W$ . Then  $I = \{x, w_1, w_2, w_3\}$  and  $J = \{x, w_4, w_5, w_6\}$  are affine 2-dimensional subspaces of  $J(C)_2$  which satisfy  $I \ominus J = W$ . Recall that any 2-dimensional affine subspace is either a Rosenhain tetrad or a Göpel tetrad. Therefore as to the types of these subspaces, only three possibilities occur. Up to translation we can assume  $x = 0$  without loss of generality.

Assume that  $I, J$  are both Rosenhain tetrads. We can assume  $I = \{0, 12, 23, 31\}$  as permutations. Since  $w_i$  are distinct,  $J$  is either  $\{0, 14, 45, 51\}$  or  $\{0, 45, 56, 64\}$  up to permutations which preserve  $\{1, 2, 3\}$ . For each form of  $J$  we deduce that  $(\Xi, \Theta_{123}) \geq 4$  and get a contradiction to (4.1).

Assume that  $I, J$  are both Göpel tetrads. We can assume  $I = \{0, 12, 34, 56\}$  as permutations. Then as in the former case,  $J$  can be only  $\{0, 23, 45, 61\}$  up to permutations preserving  $I$ . Again  $(\Xi, \Theta_{123}) \geq 4$  and we get a contradiction.

Thus  $W = I \ominus J$  with  $I, J$  Rosenhain and Göpel. Hence  $W$  is a Weber hexad by Lemma 3.3.  $\square$

Obviously this proposition is also true for  $\eta$ . We thus obtain the following set consisting of genus two curves on  $J(C)$ :

$$(4.2) \quad \mathcal{W} = \{\varepsilon^*\Theta_\beta \mid \beta \in S(C)\} \cup \{\eta^*\Theta_\beta \mid \beta \in S(C)\}.$$

Our observations in Lemma 4.4 and Proposition 4.7 below show that, if we look at the set of Weber hexads passed through by curves in  $\mathcal{W}$ , then it constitutes exactly a dual six equivalence class of Weber hexads. In this way Comessatti Jacobian surfaces and Weber hexads are closely related. The proof of this fact is completed in Proposition 4.7 after our main theorem 4.5.

To proceed, first we note that under the isomorphism

$$(4.3) \quad \text{NS}(J(C)) \xrightarrow{\sim} \text{End}^{\text{sym}}(J(C)) = \{\varphi \in \text{End}(J(C)) \mid \varphi' = \varphi\}$$

of [3, Chapter 5], we have  $c_1(\mathcal{O}(\Theta_\beta)) \mapsto \text{id}$  and  $c_1(\mathcal{O}(\varepsilon^*\Theta_\beta)) \mapsto \varepsilon^2$ . By the relation  $\varepsilon^4 - 3\varepsilon^2 + 1 = 0$ , we obtain the algebraic equivalence  $\eta^*\Theta_\beta \approx 3\Theta_\beta - \varepsilon^*\Theta_\beta$ .

Recall that  $\iota = -\text{id}$  is the inversion automorphism of  $J(C)$ .

**Lemma 4.3.** *Let  $F$  be a smooth genus two curve on  $J(C)$ . Then  $\iota^*F = F$  (as a set) if and only if  $F$  passes through six 2-torsion points.*

*Proof.* First assume  $\iota^*F = F$ . Then, since  $J(C) = \text{Pic}^0(F) = H^1(F, \mathcal{O}_F)/H^1(F, \mathbb{Z})$ ,  $\iota|_F$  acts as a hyperelliptic involution and it has six fixed points which are 2-torsion. Conversely suppose  $F$  contains six 2-torsion points. They are fixed by  $\iota$ . Since  $\iota$  acts on  $H^2(J(C), \mathbb{Z})$  trivially,  $(F, \iota^*F) = (F^2) = 2$  but  $\#F \cap \iota^*F \geq 6$  implies  $F = \iota^*F$ .  $\square$

**Lemma 4.4.** *Curves  $F \in \mathcal{W}$  (for the definition of  $\mathcal{W}$  see (4.2)) are characterized by the conditions*

$$\iota^*F = F \text{ (as sets) and } F \approx \varepsilon^*\Theta \text{ or } \eta^*\Theta,$$

where  $\approx$  is algebraic equivalence. Moreover different  $F \in \mathcal{W}$  pass through distinct Weber hexads. Hence we obtain 32 Weber hexads from  $\mathcal{W}$ .

*Proof.* It is clear that every  $F \in \mathcal{W}$  satisfies the conditions. Conversely let  $F$  satisfy the conditions. By the algebraic equivalence and  $h^0(\mathcal{O}(F)) = 1$ ,  $F$  is a translate of some pullback of a theta divisor:  $F = \varepsilon^*\Theta_\beta + \gamma$ ,  $\gamma \in J(C)$ . For any  $x \in \varepsilon^*\Theta_\beta$  we have  $-x \in \varepsilon^*\Theta_\beta$  and the former condition implies  $-(-x + \gamma) \in F$ , hence  $x \in \varepsilon^*\Theta_\beta + 2\gamma$ . Thus  $2\gamma = 0$ .

The last assertion follows from Proposition 3.7. In fact, since  $(\varepsilon^*\Theta_\beta, \eta^*\Theta_\beta) = (\varepsilon^*\Theta_\beta, 3\Theta_\beta - \varepsilon^*\Theta_\beta) = 7$ , there are 32 curves in  $\mathcal{W}$ . Let  $F \in \mathcal{W}$ . Then by the conditions, it corresponds to the unique twisted cubic curve in Proposition 3.7.

They are determined by the six nodes of  $\overline{X}$ . Hence  $F$  can be recovered from the Weber hexad.  $\square$

Now we arrive at the following theorem.

**Theorem 4.5.** *Let  $C$  be a curve of genus two and  $(X, W)$  its Jacobian Kummer surface and a Weber hexad on it. Then the following conditions are equivalent:*

- (1) *The Hessian model  $X_W$  acquires the 11th node.*
- (2) *The Hutchinson–Weber involution  $\sigma_W$  degenerates in the sense that it acquires fixed loci.*
- (3) *The unique twisted cubic  $\overline{E}$  passing through the nodes  $\{n_\alpha\}_{\alpha \in W}$  of  $\overline{X}$  lies on the Kummer quartic surface  $\overline{X}$ . (Here the strict transform  $E \subset X$  satisfies the relations in Proposition 3.7.)*
- (4) *The Jacobian  $J(C)$  is a Comessatti surface and one curve  $\Xi$  among the set  $\mathcal{W}$  of (4.2) passes through the 2-torsion points corresponding to  $W$ .*
- (5) *In the double plane model (Proposition 2.2) projected from one node  $n_{w_0}$  ( $w_0 \in W$ ), there exists an additional conic  $E' \subset \mathbb{P}^2$  which passes through the vertices of the pentagon formed by five images of  $\{n_w \mid w \in W - \{w_0\}\}$  and tangent to the remaining branch line. For example, when  $W = \{0, 12, 23, 34, 45, 51\}$  and  $w_0 = 0$  as in Proposition 2.2, the pentagon is formed by  $l_1, \dots, l_5$  and the last line is  $l_6$ .*

*Proof.* (1) $\Leftrightarrow$ (2) follows from Proposition 3.10. (2) $\Leftrightarrow$ (3) follows from Proposition 3.7.

(3) $\Rightarrow$ (4): The inverse image  $\Xi \subset J(C)$  of  $\overline{E}$  is a genus two curve with  $(\Xi, \Theta) = 3$  since  $\overline{E}$  is a cubic curve. Then the endomorphism  $\varphi$  corresponding to the divisor  $\Xi$  in the isomorphism (4.3) (which holds in general) satisfies the relation  $\varphi^2 - 3\varphi + 1 = 0$ , hence  $J(C)$  is Comessatti. By construction  $\Xi$  corresponds to some element in  $\mathcal{W}$ , by Lemma 4.4. (4) $\Rightarrow$ (3) is already mentioned in the proof of Lemma 4.4.

(3) $\Leftrightarrow$ (5): These correspond to each other as  $E'$  is the image of  $E$  under the projection  $X \rightarrow \mathbb{P}^2$ .  $\square$

**Remark 4.6.** The proof of Humbert’s theorem in [14] covers (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) except that it does not mention Weber hexads.

**Proposition 4.7.** *If  $W$  and  $W'$  are dual six equivalent, then the conditions of the previous theorem for  $W$  and  $W'$  are equivalent.*

*Proof.* Condition (3) is the easiest translated into this proposition. By using Proposition 2.3, we can easily see that the images  $\sigma_{\alpha_0}(E), \sigma_{\beta_0}(E)$  (by translations and switches) satisfy the conditions in Proposition 3.7 for other equivalent  $W$ ’s.  $\square$



§5. Periods

General HW involutions  $\sigma_W$  are fixed-point-free, hence they determine Enriques surfaces. The moduli space of Enriques surfaces obtained in this way is isomorphic to an open set of the moduli space of pairs  $(X, W)$  where  $X$  is a Jacobian Kummer surface and  $W$  is a Weber hexad, considered modulo dual six equivalence. By what we have studied, we can describe the boundary divisor consisting of Kummer surfaces of Comessatti Jacobians explicitly.

First we recall the periods of Jacobian Kummer surfaces. We fix a lattice  $T = U(2) \oplus U(2) \oplus \langle -4 \rangle$ , which is isomorphic to the transcendental lattice of Picard-general Jacobian Kummer surfaces. Recall that  $T$  has a unique embedding into a K3 lattice  $L_{K3}$ . We formally take a  $\mathbb{Z}$ -generator  $\{N_\alpha, T_\beta\}$  of the orthogonal complement NS of  $T$  analogous to that in Section 2. Let  $\Phi = \sum(N_\alpha + T_\beta)/4 \in L_{K3}$ . Under this notation, we have the following criterion.

**Proposition 5.1** ([10, Theorem 6.3]). *Let  $X$  be a K3 surface. Then  $X$  is isomorphic to a Jacobian Kummer surface if and only if there exists a marking  $H^2(X, \mathbb{Z}) \xrightarrow{\sim} L_{K3}$  inducing an embedding  $T_X \subset T$  such that under this marking, there exists no  $(-2)$ -element  $E$  in  $\text{NS}(X)$  which is orthogonal to  $\Phi$ .*

Let us compute the obstruction  $E$ . We put  $E = E_{\text{NS}} + E_T$  according to the decomposition  $L_{K3, \mathbb{Q}} = \text{NS}_{\mathbb{Q}} \oplus T_{\mathbb{Q}}$  (notice that NS is formally a sublattice of  $\text{NS}(X)$ , not equal to  $\text{NS}(X)$  here). After some computation, we obtain  $E_{\text{NS}} = \pm H/4 \pm (\sum_{\alpha \in R} N_\alpha)/2$ , where  $R$  is a Rosenhain tetrad. Correspondingly we have  $(E_T^2) = -1/4$ . Conversely, for any  $E_T \in T^*$  with  $(E_T^2) = -1/4$ , it is easy to see that there exists an element  $E_{\text{NS}} \in \text{NS}^*$  such that  $E_{\text{NS}} + E_T \in L_{K3}$  and  $((E_{\text{NS}} + E_T)^2) = -2$ . (In fact any  $(1/4)$ -element in the discriminant group  $\text{NS}^*/\text{NS}$  corresponds to a patching element of a switch of an even theta characteristic [11, Section 5].) Let

$$\mathcal{E} = \{e \in T^* \mid (e^2) = -1/4\}$$

and  $H_e \subset T_{\mathbb{C}}$  be the hyperplane orthogonal to  $e \in \mathcal{E}$ . Since  $T$  has a unique primitive embedding into  $L_{K3}$ , we obtain

**Proposition 5.2.** *The moduli space JKS of Jacobian Kummer surfaces is isomorphic to*

$$\left( \mathcal{D}(T) - \bigcup_{e \in \mathcal{E}} H_e \right) / O(T),$$

where  $\mathcal{D}(T) = \{[\omega] \in \mathbb{P}(T_{\mathbb{C}}) \mid (\omega^2) = 0, (\omega, \bar{\omega}) > 0\}$  is the period domain.

We remark that we can show  $O(T)$  acts on  $\mathcal{E}$  transitively, hence the removed divisor is irreducible in the moduli space. The proof is the same as that of Lemma 5.5 below.

Next we consider the Weber hexads. For the time being, suppose that  $\text{NS}$ ,  $T$  are identified with the Néron–Severi  $\text{NS}(X)$  and the transcendental lattice  $T_X$  of a Picard-general surface  $X$ . Recall that the discriminant group  $T_X^*/T_X$  has exactly six cyclic subgroups  $C_W$  of order 4, whose generators have the norm  $(3/4) \bmod 2\mathbb{Z}$ . These subgroups are exactly those arising as the patching subgroups of HW involutions [11, Section 7]. In other words they are one-to-one to the dual six. The correspondence is given by

$$(5.1) \quad (\text{the class of}) W \leftrightarrow C_W = \left\langle \frac{3}{4}H - \frac{1}{2} \left( \sum_{\alpha \in W} N_\alpha \right) \right\rangle \subset \text{NS}(X)^*/\text{NS}(X),$$

via the sign-reversing isometry  $\text{NS}(X)^*/\text{NS}(X) \simeq T_X^*/T_X$ .

We return to the general situation. Let us fix one subgroup  $C_0 \subset T^*/T$  as above once and for all.

**Definition 5.3.** For a pair  $(X, W)$  of a Jacobian Kummer surface and an equivalence class of Weber hexads, a *marking*  $\phi: H^2(X, \mathbb{Z}) \xrightarrow{\sim} L_{K3}$  is an isometry satisfying the following conditions:

- (Lattice polarization)  $\phi^{-1}(\text{NS})$  coincides with the sublattice of  $\text{NS}(X)$  generated by the  $(16)_6$  configuration. We denote this sublattice by  $\text{NS}(X)'$ .
- The subgroup  $C_W \subset \text{NS}(X)^*/\text{NS}(X)'$  defined by (5.1) corresponds to  $\phi^{-1}C_0$  via  $\text{NS}(X)^*/\text{NS}(X)' \simeq \phi^{-1}(T^*/T)$ .

Let  $\Gamma$  be the subgroup of  $O(T)$  whose induced action on  $T^*/T$  stabilizes  $C_0$ . Clearly  $\Gamma$  acts on the set of markings of a pair  $(X, W)$  and the moduli space of  $(X, W)$  is given by restricting the arithmetic group to  $\Gamma$ .

**Proposition 5.4.** *The moduli space  $\mathcal{JKS}_W$  of Jacobian Kummer surfaces equipped with a Weber hexad, considered modulo the dual six equivalence, is isomorphic to the period domain  $\mathcal{D}(T) - \bigcup_{e \in \mathcal{E}} H_e$  divided by the arithmetic subgroup  $\Gamma$ .*

By [11, Lemma 3.3] the natural projection  $\mathcal{JKS}_W \rightarrow \mathcal{JKS}$  is  $6 : 1$  and corresponds to the forgetful map  $(X, W) \mapsto X$ .

Let us compute the locus of degenerate HW involutions. The HW involution  $\sigma_W$  degenerates if and only if there exists a curve  $E \in H^2(X, \mathbb{Z})$  as in Proposition 3.7. From the relations there, the element  $E = E_{\phi^{-1}(\text{NS})} + E_{\phi^{-1}(T)}$  satisfies  $E_{\phi^{-1}(\text{NS})} = (3/4)H - (\sum_{\alpha \in W} N_\alpha)/2$ , where  $N_\alpha$  is the  $(16)_6$ -configuration on  $X$ . Hence  $e = \phi(E_{\phi^{-1}(T)})$  satisfies the conditions

$$(5.2) \quad e \in T^*, \quad (e^2) = -5/4, \quad e \text{ generates } C_0 \text{ in } T^*/T.$$

Conversely, if such an element  $e$  exists and is orthogonal to the period under a marking (as a pair  $(X, W)$ ), then by [11, Section 7] we obtain a  $(-2)$ -element

$E \in \text{NS}(X)$  satisfying the numerical conditions in Proposition 3.7. By Riemann–Roch, the nef and big property of  $L$  and the Cauchy–Schwarz inequality,  $E$  is a sum of  $(-2)$ -curves and then Proposition 3.7 shows that  $E$  is a class of irreducible  $(-2)$ -curve. Thus the above condition is also sufficient for the degeneration.

Let

$$\mathcal{E}' = \{e \in T^* \mid (e^2) = -5/4\}.$$

**Lemma 5.5.**  *$O(T)$  acts on  $\mathcal{E}'$  transitively.*

*Proof.* Instead of  $e \in T^*$  we consider the element  $f = 4e \in T$  which is primitive,  $(f^2) = -20$  and  $(f, T) = 4\mathbb{Z}$ . Clearly the transitivity for  $e$  follows from that for  $f$ .

The bilinear form of the lattice  $T$  is always even, hence the problem reduces to that in the lattice  $T(1/2) = U^2 \oplus \langle -2 \rangle$ . Because it contains two hyperbolic planes, [13, Proposition 3.7.3] concludes the proof.  $\square$

**Corollary 5.6.**  *$\Gamma$  acts transitively on the set*

$$\mathcal{E}'_0 = \{e \in T^* \mid (e^2) = -5/4, \text{ and } e \text{ generates } C_0 \text{ in } T^*/T\}.$$

*Proof.* This follows from the lemma by definition.  $\square$

Hence we obtain

**Theorem 5.7.** *The moduli space  $\mathcal{CJKS} = \{(X, W)\}$  of Comessatti Jacobian Kummer surfaces which satisfy the conditions of Theorem 4.5 is isomorphic to the quotient  $(\bigcup_{e \in \mathcal{E}'_0} H_e - (\bigcup_{e \in \mathcal{E}} H_e) \cap (\bigcup_{e \in \mathcal{E}'_0} H_e))/\Gamma$ , which is in fact irreducible by the previous corollary.*

*The moduli space of Enriques surfaces obtained by HW involutions is given by*

$$\left( \mathcal{D}(T) - \bigcup_{e \in \mathcal{E}} H_e - \bigcup_{e \in \mathcal{E}'_0} H_e \right) / \Gamma.$$

### §6. An application to the patching subgroups

This section aims at giving a better way of understanding [11, Proposition 7.3] and reproving it. We hope there are other cases to which our ideas will be applicable.

We fix a Weber hexad  $W$  once and for all. First we recall the situation of [11, Section 7]. Let  $X_1$  be a Picard-general Jacobian Kummer surface and  $\sigma_{W,1}$  be the HW involution. The problem is to determine the patching subgroup  $\Gamma_{\sigma_{W,1}}$  which was defined in [11, Definition 2.2]. To this end, we can use the degeneration of HW involutions we have studied in this paper.

We consider a one-dimensional smooth family of Jacobian Kummer surfaces  $f : X \rightarrow \Delta$  (in what follows the letters  $X$  and  $X_W$ ,  $N_\alpha$ ,  $\sigma_W$ , etc. represent

families of surfaces, divisors, automorphisms, etc.) and its associated Hessian model  $X_W \rightarrow \Delta$  with fibers

$$X_{W,t}: \sum s_i = \sum \lambda_i(t)/s_i = 0, \quad t \in \Delta,$$

where  $\Delta$  is a small disk. We can assume that the Hessian model  $X_{W,1}$  of  $X_1$  appears as some fiber (over  $t = 1$ , say) and the central fiber  $X_{W,0}$  has eleventh node  $p$  while the other fibers have exactly ten nodes. The HW involution  $\sigma_W = \{\sigma_{W,t}\}_{t \in \Delta}$  acts on  $X_W$  birationally and fiberwise. Blowing up the ten (families of) nodes corresponding to  $N_\alpha$  ( $\alpha \in J(C)_2 - W$ ), we obtain the family  $\tilde{X}_W \rightarrow \Delta$  whose fibers are smooth for  $t \in \Delta - \{0\}$  and  $\tilde{X}_{W,0}$  has one node  $p$ . This is the same situation as in [4, Section 7],

$$X \xrightarrow{\pi} \tilde{X}_W \rightarrow \Delta \quad (\pi \text{ small}).$$

On  $\tilde{X}_W$ ,  $\sigma_W$  acts biregularly and fiberwise. Denote by  $\Gamma \subset X \times_\Delta X$  the graph of  $\sigma_W$ ; since  $\pi$  is an isomorphism over  $\Delta^* = \Delta - \{0\}$ ,  $\Gamma$  is just the closure of  $\Gamma|_{\Delta^*}$ . Let  $\Gamma_t \subset X_t \times X_t$  be the fiber of  $\Gamma \rightarrow \Delta$ . We can think of  $\Gamma_0 = \lim_{t \rightarrow 0} \Gamma_t$  as the limit in the Barlet space of  $X \times_\Delta X$  as in [2, VIII, Lemma 10.3], [4, Theorem 2]. Hence the induced map on cohomology

$$[\Gamma_0]_* : H^2(X_0, \mathbb{Z}) \rightarrow H^2(X_0, \mathbb{Z})$$

is the same as that of  $[\Gamma_1]_* : H^2(X_1, \mathbb{Z}) \rightarrow H^2(X_1, \mathbb{Z})$  under the obvious trivialization of the local system  $R^2 f_* \mathbb{Z}_X$ .

Clearly  $\Gamma_t$  is the graph of the HW involution  $\sigma_{W,t}$  for  $t$  nonzero. But the point is that  $\Gamma_0$  does not give the graph of the HW involution  $\sigma_{W,0}$ , because  $\sigma_{W,0}$  has fixed points and therefore its action on cohomology cannot be the same as for other  $\sigma_{W,t}$ . By [2, VIII, Proposition 10.5],  $\Gamma_0$  is of the form  $\Lambda_0 + E \times E$ , where  $\Lambda_0$  is the graph of  $\sigma_{W,0}$  and  $E \subset X_0$  is the fixed curve of  $\sigma_{W,0}$ , that is, the exceptional curve for  $\pi$ . Therefore the induced map is of the form

$$[\Gamma_0]_* = [\Lambda_0 + E \times E]_* : x \mapsto \sigma_{W,0}^*(x) + (x, E)E.$$

Since  $E$  is the fixed curve of  $\sigma_{W,0}$ , it follows that  $[\Gamma_0]_* = \sigma_{W,0}^* \circ r_E = r_E \circ \sigma_{W,0}^*$  where  $r_E$  is the reflection in  $E$ .

Let us return to the computation of the patching subgroup  $\Gamma_{\sigma_{W,1}}$ . We have seen that the action of  $\sigma_{W,1}$  on cohomology is the same as

$$\sigma_{W,1}^* = [\Gamma_1]_* = [\Gamma_0]_* = \sigma_{W,0}^* \circ r_E,$$

where  $\sigma_{W,0}$  is the degenerate HW involution. In particular  $\sigma_{W,1}^*(E) = -E$  (this is not a contradiction since the cycle  $E$  is not an algebraic cycle at  $t = 1$ ). Let us

write  $E$  as  $E_{\text{NS}} + E_T$  according to the orthogonal decomposition over the rationals  $H^2(X_1, \mathbb{Q}) = \text{NS}(X_1)_{\mathbb{Q}} \oplus T_{X_1, \mathbb{Q}}$ . Using the relations in Proposition 3.7, it is easy to see that  $E_{\text{NS}} = (3/4)H_1 - (\sum_{\alpha \in W} N_{\alpha,1})/2$ . By the definition of the patching subgroup,  $E_{\text{NS}}$  is the patching element. Since we know that  $\Gamma_{\sigma_{W,1}}$  is of order 4, it is generated by the class of  $E_{\text{NS}}$ .

### Acknowledgements

The author is grateful to Shigeru Mukai for fruitful discussions. His suggestion on Comessatti surfaces was the starting point of this paper. Also the content of the last section was fixed in a discussion with him.

The author is supported by global COE program of Kyoto University and JSPS Grant-in-Aid (S), No. 22224001. This work was supported by KAKENHI 21840031.

### References

- [1] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of finite groups*, Clarendon Press, Oxford, 1985. [Zbl 0568.20001](#) [MR 0827219](#)
- [2] W. Barth, K. Hulek, C. Peters and A. Van de Ven, *Compact complex surfaces*, 2nd enlarged ed., *Ergeb. Math. Grenzgeb.* 4, Springer, 2004. [Zbl 1036.14016](#) [MR 2030225](#)
- [3] C. Birkenhake and H. Lange, *Complex abelian varieties*, *Grundlehren Math. Wiss.* 302, Springer, 1992. [Zbl 0779.14012](#) [MR 1217487](#)
- [4] D. Burns and M. Rapoport, On the Torelli problem for Kählerian  $K3$  surfaces, *Ann. Sci. École Norm. Sup.* **8** (1975), 235–274. [Zbl 0324.14008](#) [MR 0447635](#)
- [5] E. Dardanelli and B. van Geemen, Hessians and the moduli space of cubic surfaces, in *Algebraic geometry*, *Contemp. Math.* 422, Amer. Math. Soc., Providence, RI, 2007, 17–36. [Zbl 1117.14039](#) [MR 2296431](#)
- [6] I. V. Dolgachev and J. H. Keum, Birational automorphisms of quartic hessian surfaces, *Trans. Amer. Math. Soc.* **354** (2002), 3031–3057. [Zbl 0994.14007](#) [MR 1897389](#)
- [7] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Classics Library, Wiley, New York, 1994. [Zbl 0836.14001](#) [MR 1288523](#)
- [8] K. Hulek and H. Lange, The Hilbert modular surface for the ideal  $(\sqrt{5})$  and the Horrocks–Mumford bundle, *Math. Z.* **198** (1988), 95–116. [Zbl 0665.14015](#) [MR 0938032](#)
- [9] D. Mumford, *Abelian varieties*, Oxford Univ. Press, 1970. [Zbl 0223.14022](#) [MR 0282985](#)
- [10] V. V. Nikulin, An analogue of the Torelli theorem for Kummer surfaces of Jacobians, *Izv. Akad. Nauk SSSR Ser. Mat.* **38** (1974), 22–41 (in Russian). [Zbl 0305.14011](#) [MR 0357410](#)
- [11] H. Ohashi, Enriques surfaces covered by Jacobian Kummer surfaces, *Nagoya Math. J.* **195** (2009), 165–186.
- [12] J. Rosenberg, Hessian quartic surfaces that are Kummer surfaces, arXiv: math/9903037.
- [13] F. Scattone, On the compactification of moduli spaces for algebraic  $K3$  surfaces, *Mem. Amer. Math. Soc.* **70** (1987), no. 374, x+86 pp. [Zbl 0633.14019](#) [MR 0912636](#)
- [14] G. van der Geer, *Hilbert modular surfaces*, *Ergeb. Math. Grenzgeb.* 16, Springer, Berlin, 1988. [Zbl 0634.14022](#) [MR 0930101](#)