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Multidegrees of Tame Automorphisms in Dimension Three

by

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Abstract

We discuss when a sequence of positive integers can be the multidegree of some tame automorphism in dimension three, and we also relate these investigations to the problem of whether there exists a tame automorphism admitting a reduction of type II or type III.

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§1. Introduction

Throughout this paper, k is a field of characteristic zero and $\mathbb{N}$ is the set of non-negative integers. A map $F = (F_1, \dots, F_n) : k^n \to k^n$ of the form $\alpha \mapsto (F_1(\alpha), \dots, F_n(\alpha))$ is called a *polynomial map* if $F_i \in k[X_1, \dots, X_n], 1 \leq i \leq n$. A polynomial map is called an *automorphism* if it has an inverse which is also a polynomial map.

An automorphism of the form $(X_1, \dots, X_{i-1}, cX_i + a, X_{i+1}, \dots, X_n)$ is called *elementary* if $0 \neq c \in k$ and a is a polynomial not containing X_i. A finite composition of elementary automorphisms is called *tame*. The famous Tame Generators Problem asks if every polynomial automorphism is tame. It has an affirmative answer in dimension 2 (known as the Jung–van der Kulk theorem, see [J, Kul] or [E, Section 5.1]) and has a negative answer in dimension 3 (Shestakov and Umirbaev [SU1, SU2]). It remains open for any dimension $n \geq 4$.

Define by $\operatorname{tdeg} F := \sum_{i=1}^{n} \deg F_i$ the *total degree* of a polynomial map F. An automorphism F is said to *admit an elementary reduction* if there exists an

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elementary automorphism E such that $\operatorname{tdeg}(E\circ F) < \operatorname{tdeg} F$, where $\circ$ denotes composition. In dimension 3, four types of non-elementary reductions, labeled I–IV, were defined by Shestakov and Umirbaev ([SU2, Definition 1–4]), who showed that every tame automorphism $F : k^3 \to k^3$ with $\operatorname{tdeg} F > 3$ admits an elementary reduction or a reduction of one of the types I–IV ([SU2, Theorem 2]). They observed that an automorphism given by Nagata [N] admits none of these reductions, and thus is not tame.

There exists a tame automorphism admitting a reduction of type I (see [SU2, Example 1], and [EMW] for more examples). But recently, Kuroda [Kur2, Theorem 7.1] showed that there does NOT exist a tame automorphism admitting a reduction of type IV. However it is still open whether there exists a tame automorphism admitting a reduction of type II or III.

Karaś [K2] proposed the following problem: define by $\operatorname{mdeg} F := (\deg F_1, \dots, \deg F_n)$ the *multidegree* of a polynomial map F and by $\operatorname{mdeg}(T(k^n))$ the set of multidegrees of tame automorphisms from k^n to k^n. Which sequences $(d_1, \dots, d_n)$ belong to $\operatorname{mdeg}(T(k^n))$?

It is well known that $(d_1, d_2) \in \operatorname{mdeg}(T(k^2))$ if and only if $d_1 \mid d_2$ or $d_2 \mid d_1$ (see for example [E, Section 5.1]). In dimension 3, some partial results were obtained by Karaś in [K1, K2] as follows.

Theorem 1.1 (Karaś). (i) ([K2, Theorem 1.1]) *Let $3 \le d_2 \le d_3$ be integers. Then $(3, d_2, d_3) \in \operatorname{mdeg}(T(k^3))$ if and only if $3 \mid d_2$ or $d_3 \in 3\mathbb{N} + d_2\mathbb{N}$.*

(ii) ([K1, Theorem 1]) *Let $3 \le p_1 \le d_2 \le d_3$ be integers. If p_1 and d_2 are prime numbers, then $(p_1, d_2, d_3) \in \operatorname{mdeg}(T(k^3))$ if and only if $d_3 \in p_1\mathbb{N} + d_2\mathbb{N}$.*

These investigations led to the following conjecture.

Conjecture 1.2 ([K2, Conjecture 4.1]). Let $3 \le p_1 \le d_2 \le d_3$ be integers with p_1 a prime number. Then $(p_1, d_2, d_3) \in \operatorname{mdeg}(T(k^3))$ if and only if $p_1 \mid d_2$ or $d_3 \in p_1\mathbb{N} + d_2\mathbb{N}$.

In this paper, we show that Conjecture 1.2 holds if additionally one of the following conditions is satisfied (i) $d_2/\gcd(d_2, d_3) \neq 2$; (ii) $d_3/\gcd(d_2, d_3) \neq 3$; (iii) $d_2 \ge 2p_1 - 5$. As corollaries, we show that Conjecture 1.2 holds in the following cases: (1) d_2 is odd; (2) $p_1 = 3$ or 5; (3) $p_1 = 7$ and $(d_2, d_3) \neq (8, 12)$. Furthermore, we relate the investigations with the problem of whether there exists a tame automorphism admitting a reduction of type II or III. We show that, if $(7, 8, 12) \in \operatorname{mdeg}(T(k^3))$, then there exists a tame automorphism admitting a reduction of type II, and if $(p_1, 2p_1 - 6, 3p_1 - 9) \in \operatorname{mdeg}(T(k^3))$, where $p_1 > 7$ is a prime number, then there exists one admitting a reduction of type III.

§2. Preliminaries

In this section, we recall some notions and results about the Poisson bracket and *-reduced pair; for details, see [SU1, SU2].

Let $L\langle X_1,\ldots,X_n\rangle$ be the free Lie algebra with free generators $X_1,\ldots,X_n$. Let $PL\langle X_1,\ldots,X_n\rangle$ be the free Poisson algebra with free generators $X_1,\ldots,X_n$, which is the k-algebra generated by a linear basis of $L\langle X_1,\ldots,X_n\rangle$ and the Poisson bracket of which is induced by the Lie bracket of $L\langle X_1,\ldots,X_n\rangle$. It becomes a graded algebra if we put $\deg X_i = 1, \deg\,[X_i,X_j] = 2, i \neq j$, etc. Imbedding the polynomial algebra $k[X_1,\ldots,X_n]$ into $PL\langle X_1,\ldots,X_n\rangle$, one can define the Poisson bracket of two polynomials f,g to be

$$[f,g] = \sum_{1\le i<j\le n}\left(\frac{\partial f}{\partial X_i}\frac{\partial g}{\partial X_j} - \frac{\partial f}{\partial X_j}\frac{\partial g}{\partial X_i}\right)[X_i,X_j].$$

Hence

$$\deg\,[f,g] = 2 + \max_{1\le i<j\le n}\deg\left(\frac{\partial f}{\partial X_i}\frac{\partial g}{\partial X_j} - \frac{\partial f}{\partial X_j}\frac{\partial g}{\partial X_i}\right).$$

Note that $\deg\,[f,g] \ge 2$ if f,g are algebraically independent.

Definition 2.1 ([SU1, Definition 1]). A pair of polynomials $f,g \in k[X_1,\ldots,X_n]$ is called **-reduced* if

(1) f,g are algebraically independent;

(2) $\overline{f},\overline{g}$ are algebraically dependent, where $\overline{h}$ denotes the highest homogeneous part of h;

(3) $\overline{f} \notin k[\overline{g}]$ and $\overline{g} \notin k[\overline{f}]$.

Let f,g be a *-reduced pair with $\deg f \le \deg g$ and let $p = \frac{\deg f}{\gcd(\deg f,\deg g)}$. Then f,g is also called a *p-reduced pair.*

Theorem 2.2 ([SU1, Theorem 3]). *Let f,g be a p-reduced pair and let $G(x,y) \in k[x,y]$ with $\deg_y G(x,y) = pq + r$, $0 \le r < p$. Then*

$$\deg G(f,g) \ge q(p\deg g - \deg g - \deg f + \deg\,[f,g]) + r\deg g.$$

Remark 2.3. Karaś observed that Theorem 2.2 is also true if the second condition of Definition 2.1 is not satisfied (see [K2, Proposition 2.4]).

We close this section by recalling several results which will also be used in the next section.

Lemma 2.4 (Brauer [B]). *If a,b are positive integers such that $\gcd(a,b) = 1$, then $l \in a\mathbb{N} + b\mathbb{N}$ for every integer $l \ge (a-1)(b-1)$.*

Lemma 2.5 ([K2, Proposition 2.2]). *Let* $d_1 \leq \cdots \leq d_n$ *be positive integers. If there exists some* i *such that* $d_i \in d_1\mathbb{N} + \cdots + d_{i-1}\mathbb{N}$*, then* $(d_1, \dots, d_n) \in \operatorname{mdeg}(T(k^n))$.

§3. Multidegrees of tame automorphisms of k^3

We start with a lemma.

Lemma 3.1. *Let* $F=(F_1,F_2,F_3)$ *be an automorphism with* $\operatorname{mdeg} F=(d_1,d_2,d_3)$*. If* $\deg[F_s,F_t]=2$*, then* $d_s \mid d_t$ *or* $d_t \mid d_s$*, where* $1 \leq s < t \leq 3$.

Proof. Let T be a linear automorphism and $F' = F \circ T$. It is easy to verify that $\deg[F'_s,F'_t] = \deg[F_s,F_t]$ for any $1 \leq s < t \leq 3$. Replacing F by some $F \circ T$ if necessary, we may assume that $F = (X_1 + H_1, X_2 + H_2, X_3 + H_3)$, where each H_i contains no linear terms.

Suppose that $\deg[F_s,F_t] = 2$ for some $1 \leq s < t \leq 3$, say $\deg[F_2,F_3] = 2$. Since

$$\deg[F_2,F_3] = 2 + \max_{1\leq i<j\leq 3} \deg c_{ij}, \quad \text{where} \quad c_{ij} = \frac{\partial F_2}{\partial X_i}\frac{\partial F_3}{\partial X_j} - \frac{\partial F_3}{\partial X_i}\frac{\partial F_2}{\partial X_j},$$

we have $c_{ij} \in k, 1 \leq i < j \leq 3$. It follows that

$$c_{12} = \frac{\partial H_2}{\partial X_1}\frac{\partial H_3}{\partial X_2} - \frac{\partial H_3}{\partial X_1}\left(1+\frac{\partial H_2}{\partial X_2}\right) = 0, \qquad c_{13} = \frac{\partial H_2}{\partial X_1}\left(1+\frac{\partial H_3}{\partial X_3}\right) - \frac{\partial H_3}{\partial X_1}\frac{\partial H_2}{\partial X_3} = 0.$$

Notice that $\partial H_2/\partial X_1 = 0$ if and only if $\partial H_3/\partial X_1 = 0$. Now suppose that $\partial H_2/\partial X_1 \neq 0$. Then $\partial H_3/\partial X_1 \neq 0$. Let u and v be the lowest homogeneous parts of $\partial H_2/\partial X_1$ and $\partial H_3/\partial X_1$ respectively. If $\deg u \leq \deg v$, then u is the lowest homogeneous part of c_{13}, which contradicts $c_{13} = 0$. Similarly, if $\deg v \leq \deg u$, then v is the lowest homogeneous part of c_{12}, which contradicts $c_{12} = 0$.

Therefore, $\partial H_2/\partial X_1 = 0$ and $\partial H_3/\partial X_1 = 0$. It follows that (F_2,F_3) is an automorphism in dimension 2, and thus $d_2 \mid d_3$ or $d_3 \mid d_2$. □

Lemma 3.2. *Let* $3 \leq p_1 \leq d_2 \leq d_3$ *be integers such that* p_1 *is a prime number,* $p_1 \nmid d_2$ *and* $d_3 \notin p_1\mathbb{N}+d_2\mathbb{N}$*. If* $(p_1,d_2,d_3) \in \operatorname{mdeg}(T(k^3))$*, then there exists a tame automorphism with multidegree* (p_1,d_2,d_3) *which admits an elementary reduction.*

Proof. If $(p_1,d_2,d_3) \in \operatorname{mdeg}(T(k^3))$, then there exists a tame automorphism F with $\operatorname{mdeg} F = (p_1,d_2,d_3)$. By [SU2, Theorem 2] and [Kur2, Theorem 7.1], F admits an elementary reduction or a reduction of one of the types I–III.

By [SU2, Definitions 1 and 2], if there exists a tame automorphism admitting a reduction of type I or II, then there exists a tame automorphism admitting an elementary reduction with the same multidegree.

Now suppose that F admits a reduction of type III. Then by [SU2, Definition 3] (through a permutation of indices), there exists some positive integer m such that one of the following is satisfied:

$$m < \deg F_1 = p_1 \le \tfrac{3}{2}m, \quad \deg F_2 = d_2 = 2m, \quad \deg F_3 = d_3 = 3m; \tag{3.1}$$

$$\deg F_1 = p_1 = \tfrac{3}{2}m, \quad \deg F_2 = d_2 = 2m, \quad \tfrac{5}{2}m < \deg F_3 = d_3 \le 3m. \tag{3.2}$$

In the case (3.2), the condition that p_1 is a prime number implies that $m = 2$, $p_1 = 3$ and $d_3 = 6$, which contradicts $d_3 \notin p_1\mathbb{N} + d_2\mathbb{N}$. So it suffices to consider the case (3.1).

Also by [SU2, Definition 3], there exist $\alpha, \beta, \gamma \in k$ with $(\alpha, \beta, \gamma) \neq (0, 0, 0)$ such that the elements $G_2 := F_2 - \beta F_1$, $G_3 := F_3 - \gamma F_1 - \alpha F_1^2$ satisfy $\deg G_2 = 2m$, $\deg G_3 = 3m$ and $E \circ (F_1, G_2, G_3) = (G_1, G_2, G_3)$, for some elementary automorphism E, with $\deg G_1 \le \frac{3}{2}m$, $\deg [G_1, G_2] < 3m + \deg [G_2, G_3]$ and $\deg G_1 < m + \deg [G_2, G_3]$. By [SU2, Corollary 4], $\operatorname{tdeg}(G_1, G_2, G_3) < \operatorname{tdeg} F$. Since $\operatorname{tdeg} F = \operatorname{tdeg}(F_1, G_2, G_3)$, it follows that (F_1, G_2, G_3) admits an elementary reduction and $\operatorname{mdeg}(F_1, G_2, G_3) = (p_1, d_2, d_3)$. □

Theorem 3.3. *Let $3 \le p_1 \le d_2 \le d_3$ be integers with p_1 a prime number. If one of the following conditions is satisfied:* (i) $d_2/\gcd(d_2, d_3) \neq 2$*;* (ii) $d_3/\gcd(d_2, d_3) \neq 3$*;* (iii) $d_2 \ge 2p_1 - 5$*, then* $(p_1, d_2, d_3) \in \operatorname{mdeg}(T(k^3))$ *if and only if* $p_1 | d_2$ *or* $d_3 \in p_1\mathbb{N} + d_2\mathbb{N}$.

Proof. By Lemma 2.5, if $p_1 \mid d_2$ or $d_3 \in p_1\mathbb{N} + d_2\mathbb{N}$, then $(p_1, d_2, d_3) \in \operatorname{mdeg}(T(k^3))$. Now assume that $p_1 \nmid d_2$ and $d_3 \notin p_1\mathbb{N} + d_2\mathbb{N}$. By Lemma 3.2, we only need to show that, if one of the three conditions in the theorem is satisfied, then an automorphism F with $\operatorname{mdeg} F = (p_1, d_2, d_3)$ does not admit an elementary reduction.

Since $p_1 \nmid d_2$ and $d_3 \notin p_1\mathbb{N} + d_2\mathbb{N}$, we have $d_3 < (p_1 - 1)(d_2 - 1)$ due to Lemma 2.4. Moreover Lemma 3.1 yields $\deg [F_s, F_t] \ge 3$ for any $1 \le s < t \le 3$.

(1) Suppose that $(F_1, F_2, F_3 - g(F_1, F_2))$ is an elementary reduction of F, i.e., $\deg(F_3 - g(F_1, F_2)) < \deg F_3$, where $g \in k[x, y]$. Then $\deg g(F_1, F_2) = \deg F_3 = d_3$. Notice that $p_1/\gcd(p_1, d_2) = p_1$. Let $\deg_y g(x, y) = qp_1 + r$, where $0 \le r < p_1$. The pair F_1, F_2 satisfies the first and the third condition of Definition 2.1, since F_1, F_2 are algebraically independent and $p_1 \nmid d_2$. Then by Theorem 2.2 (and noticing Remark 2.3, similarly hereinafter), we have

$$\begin{aligned} d_3 = \deg g(F_1, F_2) &\ge q(p_1 d_2 - d_2 - p_1 + \deg [F_1, F_2]) + r d_2 \\ &\ge q(p_1 d_2 - d_2 - p_1 + 3) + r d_2 \ge q(p_1 - 1)(d_2 - 1) + r d_2. \end{aligned}$$

Since $d_3 < (p_1 - 1)(d_2 - 1)$, we have $q = 0$ and thus $\deg_y g(x, y) = r < p_1$.

Let $g(x,y) = \sum_{i=0}^{p_1-1} g_i(x)y^i$. Since $\gcd(p_1, d_2) = 1$, the sets $p_1\mathbb{N}, d_2 + p_1\mathbb{N}, \dots,$ $(p_1-1)d_2 + p_1\mathbb{N}$ are disjoint. Hence

$$\begin{aligned} d_3 = \deg g(F_1, F_2) &= \deg \Big(\sum_{i=0}^{p_1-1} g_i(F_1)F_2^i \Big) \\ &= \max_{0\le i\le p_1-1} (\deg F_1 \deg g_i + i \deg F_2) = \max_{0\le i\le p_1-1} (p_1 \deg g_i + id_2), \end{aligned}$$

which contradicts $d_3 \notin p_1\mathbb{N} + d_2\mathbb{N}$.

(2) Suppose that $(F_1, F_2 - g(F_1, F_3), F_3)$ is an elementary reduction of F, where $g \in k[x,y]$. Then $\deg g(F_1, F_3) = \deg F_2 = d_2$. Notice that $p_1/\gcd(p_1, d_3)$ $= p_1$. Let $\deg_y g(x,y) = qp_1 + r$, where $0 \le r < p_1$. By Theorem 2.2,

$$\begin{aligned} d_2 = \deg g(F_1, F_3) &\ge q(p_1d_3 - d_3 - p_1 + \deg [F_1, F_3]) + rd_3 \\ &\ge q(p_1d_3 - 2d_3) + rd_3 \ge qd_3 + rd_3, \end{aligned}$$

which implies that $q = r = 0$. Then $\deg_y g(x,y) = 0$ and thus $d_2 = \deg g(F_1, F_3) =$ $\deg g(F_1) \in p_1\mathbb{N}$, which contradicts $p_1 \nmid d_2$.

(3) Suppose that $(F_1 - g(F_2, F_3), F_2, F_3)$ is an elementary reduction of F, where $g \in k[x,y]$. Then $\deg g(F_2, F_3) = \deg F_1 = p_1$. Let $p = d_2/\gcd(d_2, d_3)$ and let $\deg_y g(x,y) = qp + r$, where $0 \le r < p$. By Theorem 2.2 we obtain

$$p_1 = \deg g(F_2, F_3) \ge q(pd_3 - d_3 - d_2 + \deg [F_2, F_3]) + rd_3. \tag{3.3}$$

If $\deg_y g(x,y) = 0$, then $p_1 = \deg g(F_2, F_3) = \deg g(F_2) \in d_2\mathbb{N}$, a contradiction. So in what follows we always assume that $\deg_y g(x,y) > 0$.

We divide the following discussion into several subcases.

(a) $p = d_2/\gcd(d_2, d_3) \ne 2$ or $d_3/\gcd(d_2, d_3) \ne 3$.

If $p \ne 2$, noticing that $p > 1$, we have $p \ge 3$. Then

$$pd_3 - d_3 - d_2 + \deg [F_2, F_3] \ge 2d_3 - d_2 \ge d_2.$$

If $p = 2$, then $d_3/\gcd(d_2, d_3) \ne 3$. Let $\gcd(d_2, d_3) = m$. Then $d_2 = 2m, d_3 =$ lm, where $l \ge 4$, and thus $d_3 - d_2 \ge d_2$. We have also

$$pd_3 - d_3 - d_2 + \deg [F_2, F_3] \ge d_3 - d_2 \ge d_2.$$

Then it follows by (3.3) that $p_1 \ge qd_2 + rd_3$, which implies that $q = r = 0$. Hence $\deg_y g(x,y) = 0$, a contradiction.

(b) $d_2 \ge 2p_1 - 5$.

By (a) we may assume that $p = d_2/\gcd(d_2, d_3) = 2$ and $d_3/\gcd(d_2, d_3) = 3$, which implies that $d_3 - d_2 \ge d_2/2$. Due to Lemma 3.1, we have $\deg [F_2, F_3] \ge 3$.

Then by (3.3) we obtain

$$\begin{aligned} p_1 &\geq q(2d_3 - d_3 - d_2 + \deg\,[F_2, F_3]) + rd_3 \\ &\geq q\big(d_2/2 + 3\big) + rd_3 \geq q\big(p_1 + 1/2\big) + rd_3. \end{aligned}$$

It follows that $q = r = 0$, and thus $\deg_y g(x, y) = 0$, also a contradiction. □

It is easy to verify that Karaś's results (summarized in Theorem 1.1) are direct corollaries of our Theorem 3.3. And we also have the following corollaries.

Corollary 3.4. *Let $3 \leq p_1 \leq d_2 \leq d_3$ be integers with p_1 is a prime number. If d_2 is odd, then $(p_1, d_2, d_3) \in \mathrm{mdeg}(T(k^3))$ if and only if $p_1 \,|\, d_2$ or $d_3 \in p_1\mathbb{N} + d_2\mathbb{N}$.*

Corollary 3.5. *Let $5 \leq d_2 \leq d_3$ be integers. Then $(5, d_2, d_3) \in \mathrm{mdeg}(T(k^3))$ if and only if $5 \,|\, d_2$ or $d_3 \in 5\mathbb{N} + d_2\mathbb{N}$.*

Corollary 3.6. (1) *Let $7 \leq d_2 \leq d_3$ be integers with $(d_2, d_3) \neq (8, 12)$. Then $(7, d_2, d_3) \in \mathrm{mdeg}(T(k^3))$ if and only if $7 \,|\, d_2$ or $d_3 \in 7\mathbb{N} + d_2\mathbb{N}$.*

(2) *Let $11 \leq d_2 \leq d_3$ be integers with $(d_2, d_3) \neq (12, 18), (14, 21), (16, 24)$. Then $(11, d_2, d_3) \in \mathrm{mdeg}(T(k^3))$ if and only if $11 \,|\, d_2$ or $d_3 \in 11\mathbb{N} + d_2\mathbb{N}$.*

Proof. By Theorem 3.3, we may assume that $d_2 < 2d_1 - 5$, $d_2/\gcd(d_2, d_3) = 2$ and $d_3/\gcd(d_2, d_3) = 3$. When $d_1 = 7$, this implies that $(d_2, d_3) = (8, 12)$, and when $d_1 = 11$ this implies that $(d_2, d_3) = (12, 18)$, $(14, 21)$ or $(16, 24)$. □

In what follows, we relate the research on Conjecture 1.2 to the problem of whether exists a tame automorphism admitting a reduction of type II or III. In fact, we have the following result.

Theorem 3.7. *Let $p_1 \geq 7$ be a prime number and assume that $(p_1, d_2, d_3) = (p_1, 2p_1 - 6, 3p_1 - 9) \in \mathrm{mdeg}(T(k^3))$. If $p_1 = 7$, i.e. $(p_1, d_2, d_3) = (7, 8, 12)$, then there exists a tame automorphism admitting a reduction of type II; if $p_1 > 7$ (in particular if $(p_1, d_2, d_3) = (11, 16, 24)$), then there exists a tame automorphism admitting a reduction of type III.*

Proof. Notice that $p_1 \nmid d_2$ and $d_3 \notin p_1\mathbb{N} + d_2\mathbb{N}$. By Lemma 3.2, there exists a tame automorphism F with $\mathrm{mdeg}\, F = (p_1, d_2, d_3)$ which admits an elementary reduction. By the proof of Theorem 3.3, there exists some $g(x, y) \in k[x, y]$ such that $\deg(F_1 - g(F_2, F_3)) < \deg F_1$, and thus $\deg g(F_2, F_3) = \deg F_1 = p_1$. Let $\deg_y g(x, y) = 2q + r$, where $0 \leq r < 2$. By Theorem 2.2,

$$\begin{aligned} p_1 = \deg g(F_2, F_3) &\geq q(2d_3 - d_3 - d_2 + \deg\,[F_2, F_3]) + rd_3 \\ &= q(d_2/2 + \deg\,[F_2, F_3]) + rd_3, \end{aligned} \tag{3.4}$$

which implies that $r = 0$ and $q \leq 1$. If $q = 0$, then $\deg_y g(x, y) = 0$, and thus $p_1 = \deg g(F_2) \in d_2\mathbb{N}$, a contradiction. Hence $q = 1$ and then by (3.4) we obtain $\deg [F_2, F_3] < p_1 < \deg F_2 + \deg F_3$. It follows that $\overline{F_2}, \overline{F_3}$ are algebraically dependent. In addition, F_2, F_3 are algebraically independent, and the condition $d_2 = 2p_1 - 6$ and $d_3 = 3p_1 - 9$ ensures that $\overline{F_2} \notin k[\overline{F_3}]$ and $\overline{F_3} \notin k[\overline{F_2}]$. Therefore, F_2, F_3 is a 2-reduced pair.

Now let $\theta = (f_1, f_2, f_3) = (F_1, F_1 + F_2, F_3)$. Then $\deg f_1 = \deg F_1 = p_1$, $\deg f_2 = \deg F_2 = 2m$ and $\deg f_3 = \deg F_3 = 3m$, where $m = p_1 - 3$.

(1) Suppose that $p_1 = 7$. Then $m = 4$ and $(p_1, d_2, d_3) = (7, 8, 12)$. It follows that $\frac{3}{2}m < \deg f_1 = p_1 \leq 2m$. Notice that $\overline{f_1}, \overline{f_2}$ are linearly independent. Let $(\alpha, \beta) = (1, 0)$ and let $g_2 := f_2 - \alpha f_1 = F_2$, $g_3 := f_3 - \beta f_1 = F_3$. Then g_2, g_3 is a 2-reduced pair and $\deg g_2 = 2m$, $\deg g_3 = 3m$. Moreover, $(f_1, g_2, g_3) = (F_1, F_2, F_3)$ admits an elementary reduction and $(g_1, g_2, g_3) = (f_1 - g(g_1, g_2), g_2, g_3) = (F_1 - g(F_2, F_3), F_2, F_3)$ is such a reduction.

Notice that $p_1 = m + 3$. By Lemma 3.1, we have $\deg [g_2, g_3] \geq 3$. Then

$$\begin{aligned} \deg [g_1, g_2] &\leq \deg g_1 + \deg g_2 \leq \deg f_1 - 1 + \deg g_2 \\ &= p_1 - 1 + 2m = 3m + 2 < 3m + \deg [g_2, g_3]. \end{aligned}$$

Therefore, by [SU2, Definition 2], θ admits a reduction of type II with the active element f_1.

(2) Suppose that $p_1 > 7$. Since p is a prime number, $p_1 \geq 11$, and noticing that $p_1 = m + 3$, we have $m < \deg f_1 = p_1 \leq \frac{3}{2}m$. Let $(\alpha, \beta, \gamma) = (0, 1, 0)$ and let $g_2 := f_2 - \beta f_1 = F_2$, $g_3 := f_3 - \gamma f_1 - \alpha f_1^2 = F_3$. Then g_2, g_3 is a 2-reduced pair and $\deg g_2 = 2m$, $\deg g_3 = 3m$. And $(f_1, g_2, g_3) = (F_1, F_2, F_3)$ admits an elementary reduction: $(g_1, g_2, g_3) = (F_1 - g(F_2, F_3), F_2, F_3)$. As proved in (1), $\deg [g_1, g_2] < 3m + \deg [g_2, g_3]$. Moreover, $\deg g_1 < \deg f_1 = p_1 = m + 3 \leq m + \deg [g_2, g_3]$. Therefore, by [SU2, Definition 3], θ admits a reduction of type III with the active element f_1. □

Remark 3.8. If we replace the condition that p_1 is a prime number by the condition that p_1, d_2 are relatively prime, then Conjecture 1.2 is not valid. In fact, we note that, among other things, Kuroda [Kur1] constructed tame automorphisms with multidegrees $(d_1, d_2, d_3) = (2m, 2pm + p + 1, (2p + 1)m)$, where $m = pq + p + q$ and $p, q \in \mathbb{N} - 0$. If we take $p = 2$, then $(d_1, d_2, d_3) = (2m, 4m + 3, 5m)$, where $m = 3q + 2$ and $q \in \mathbb{N} - 0$. And in this case, d_1, d_2 are relatively prime and $d_3 \notin d_1\mathbb{N} + d_2\mathbb{N}$.

Remark 3.9. The editor points out that our Corollaries 3.5 and 3.6(1) were independently obtained by Karaś in [K3, Sections 7.2–7.3], and it was shown that the existence of a tame automorphism of k^3 with multidegree $(p_1, d_2, d_3) := (p_1, 2(p_1-2), 3(p_1-2))$ for some prime number $p_1 > 35$ (in particular with multidegree $(37, 70, 105)$) would imply that the two-dimensional Jacobian conjecture is not true. However, by our Theorem 3.3, one can see easily that $(p_1, d_2, d_3) = (p_1, 2(p_1-2), 3(p_1-2)) \notin \mathrm{mdeg}(T(k^3))$ for any prime number $p_1 \geq 3$ because $d_2 = 2(p_1-2) \geq 2p_1 - 5$, $p_1 \nmid d_2$ and $d_3 \notin p_1\mathbb{N} + d_2\mathbb{N}$.

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