Publ. RIMS Kyoto Univ. **48** (2012), 183–213
DOI 10.2977/PRIMS/66

Unitary Representations of the Group of Diffeomorphisms via Restricted Product Measures with Infinite Mass II

by

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Abstract

This paper concerns the problem of irreducibly decomposing unitary representations of the group $\mathrm{Diff}_0(M)$ of diffeomorphisms with compact support on the smooth manifold M. As was shown in [19], these representations are decomposable under a fairly mild condition. In this paper, we consider a specific example of unitary representations $(T, \mathrm{Diff}_0(M))$ that has been considered by [4]. $(T, \mathrm{Diff}_0(M))$ is already a factor representation of type II_∞; in addition, it may be decomposed into irreducible components through the left regular representation of the group $\mathfrak{S}_\infty$ of finite permutations. We describe concrete realization of these irreducible components. The results obtained herein bear some resemblance to the finite-dimensional case of [20] with the exception of the factor representation. In addition in the Appendix, we will give another proof of the irreducibility and equivalence that were obtained in [4].

2010 Mathematics Subject Classification: Primary 58D05; Secondary 22E65.
Keywords: group of diffeomorphisms, unitary representation, irreducible component, restricted product measure.

§1. Introduction

Let $M = M^d$ be a paracompact C^∞ manifold, and let $\mathrm{Diff}_0(M)$ be the group of all C^∞ diffeomorphisms with compact support on M with the natural topology τ. Various authors have studied and constructed many interesting unitary representations of $(\mathrm{Diff}_0(M), \tau)$, as well as their linear versions, most of which are irreducible. However, as far as we know, the important problem of whether it is possible to decompose a given unitary representation of an infinite-dimensional group into irreducible ones has not been considered.

Communicated by S. Mukai. Received April 6, 2006. Revised July 18, 2006 and May 31, 2011.

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So we have recently investigated the above problem in the context of the group $\mathrm{Diff}_0(M)$, and of more general groups, including some infinite-dimensional groups [19]. In brief, this previous work may be summarized as follows: As is well-known, $\mathrm{Diff}_0(M)$ is a nuclear group, or an inductive limit of such groups according as M is compact or non-compact. Therefore, we expected to find infinite-dimensional versions of the results on irreducible decompositions of unitary representations of locally compact groups due to Mautner [8, 9]. After checking his results carefully, we found that the existence of (finite) measures with quasi-invariance under dense translations was crucial (for example, right shifts, like the Shavgulidze measure [15, 17] on the group of diffeomorphisms).

We first worked with pairs of groups, (G, H), $H \subseteq G$, which have a probability measure μ on G that is right H-quasi-invariant, and studied the irreducible decompositions of unitary representations of G restricted to H. We then proceeded to the inductive limit. Of course, we were interested in the case that (H, τ_H) is dense in (G, τ_G). We showed that the abstract form of the decomposition problem has an affirmative answer.

Next, we considered the group $\mathrm{Diff}_0(M)$, where M is compact, and showed that the decomposition problem has an affirmative answer under the fairly mild condition that a given unitary representation $T(\phi)$, $\phi \in \mathrm{Diff}_0(M)$, is continuous with respect to the topology of uniform convergence of the maps ϕ, together with their derivatives of order less than or equal to some k. As mentioned above, the Shavgulidze measure played a crucial role in these arguments. Applying inductive limit methods, we arrived at a similar result in the non-compact case. This is the summary of the previous paper [19].

In contrast, the present paper concerns unitary representations of $\mathrm{Diff}_0(M)$ which have already been considered in [4, 18]; however, realizations of their irreducible decompositions were left for further work. We now provide a concrete description through the decompositions of the left regular representation of the finite permutation group $\mathfrak{S}_\infty$. We first recall the notation used not in [4] but in [18] because it is convenient to present a new result: Let M be a connected, non-compact but σ-compact, smooth manifold with $d := \dim(M) \geq 3$, let $\mathrm{Diff}_0(M)$ be the group of all smooth diffeomorphisms on M with compact support, and μ be a smooth locally Euclidean measure on M with infinite mass. Take a restricted product measure ν_E of countably many copies of μ depending on a family $E = \{E_n\}_n$ of disjoint Borel sets in M which satisfies $0 < \mu(E_n) < \infty$ and $\sum_{n=1}^{\infty} |1 - \mu(E_n)| < \infty$ as in the context of Moore [10] (the details will be given in the next section). ν_E is quasi-invariant under the diagonal action of $\mathrm{Diff}_0(M)$. It follows that we have a natural representation $T(\phi)$ of $\phi \in \mathrm{Diff}_0(M)$ on the representation Hilbert space

$L^2_{\nu_E}(M^\infty)$ (the formulations used here are an extension of, and variation on the work described in [20] on finite direct product spaces). Notice that ν_E is invariant under finite permutations of coordinates, and the group $\mathfrak{S}_\infty$ acts on $L^2_{\nu_E}(M^\infty)$ as another unitary representation $R(\sigma), \sigma \in \mathfrak{S}_\infty$.

Note that, as we show in Section 3, the representation $T(\phi)R(\sigma)$ of $\mathrm{Diff}_0(M) \times \mathfrak{S}_\infty$ is irreducible. In other words, $(T, L^2_{\nu_E}(M^\infty))$ and $(R, L^2_{\nu_E}(M^\infty))$ are factor representations of type II_∞ and of type II_1, respectively. It is interesting that these representations have different characteristics when compared with the natural representations described by [20] on the finite-dimensional space M^n, $n \in \mathbb{N}$.

Given $\mu, E = \{E_n\}_n$ and an irreducible unitary representation (Π, H) of $\mathfrak{S}_\infty$, consider the Hilbert space $\mathcal{H}(\Sigma)$, $\Sigma := (\mu, E, \Pi)$, of all Borel measurable H-valued functions f on M^∞ such that

$$f(x\sigma) = \Pi(\sigma)^{-1} f(x) \qquad \text{for all } \sigma \in \mathfrak{S}_\infty,$$

and

$$\|f\|_E^2 := \int_{D_E} \|f(x)\|_H^2 \, \nu_E(dx) < \infty,$$

where D_E is a Borel set such that $D_E\sigma \cap D_E = \emptyset$ if $\sigma \neq \mathrm{id}$, and the complement of $\bigcup_{\sigma \in \mathfrak{S}_\infty} D_E\sigma$ is ν_E-null, where $D_E\sigma := \{x\sigma \mid x \in D_E\}$.

It is easy to see that $\mathrm{Diff}_0(M)$ acts on the Hilbert space $\mathcal{H}(\Sigma)$, and hence we have its natural representations that are all irreducible by [4]. As the last fact is fundamental, we will give another proof of it in the Appendix. What was left for further work in [4] is the question of whether these are actually irreducible components of the natural representation $(T, L^2_{\nu_E}(M^\infty))$. We are now in a position to prove that this is the case (the details will be seen in the final section), and the successive steps of this proof are as follows:

$$\begin{aligned} L^2_{\nu_E}(M^\infty) &\simeq \ell^2(\mathfrak{S}_\infty) \otimes L^2_{\nu_E}(D_E) \\ &\simeq \int^\oplus H_\lambda \otimes L^2_{\nu_E}(D_E) \sqrt{d\sigma(\lambda)} \\ &\simeq \int^\oplus L^2_{\nu_E}(D_E, H_\lambda) \sqrt{d\sigma(\lambda)} \\ &\simeq \int^\oplus \mathcal{H}(\Sigma_\lambda) \sqrt{d\sigma(\lambda)}. \end{aligned}$$

The first line is due to the choice of the set D_E, the second line is a consequence of an irreducible decomposition of the left regular representation L of $\mathfrak{S}_\infty$ with spectral measure σ, and the fourth line is derived from the third line by the natural map with $\Sigma_\lambda := (\mu, E, \Pi_\lambda)$, where Π_λ is an irreducible component of L. Notice that the corresponding maps at each stage have no connections with the representations of $\mathrm{Diff}_0(M)$, but that the composition of all of the maps is an intertwining unitary operator of their natural representations.

Throughout, we assume that M is connected, σ-compact with $\dim(M) \geq 3$, but not compact; we must also impose a fairly mild condition (mcc) (stated just before Theorem 2.3) on M, for the technical reasons explained in the previous paper [18]. One might try proving the same results without (mcc), but this would most probably require much longer arguments.

As a rule, we follow the notation and terminology used by [18], and we recall these briefly in the next section.

§2. Notation and basic arguments

§2.1. Restricted product measure with infinite mass

We begin by introducing the notion of restricted product measure. Suppose that we are given, for each n, a σ-finite measure space $(X_n, \mathfrak{B}_n, \mu_n)$ and a set $E_n \in \mathfrak{B}_n$ with $0 < \mu_n(E_n) < \infty$. Denoting the restriction of μ_n to E_n by $\mu_n|E_n$, we form the product measure

$$\hat{\nu}_n := \frac{\mu_1 \times \cdots \times \mu_n}{\mu_1(E_1) \cdots \mu_n(E_n)} \times \prod_{k=n+1}^{\infty} \frac{\mu_k|E_k}{\mu_k(E_k)}.$$

As $\hat{\nu}_n$ is increasing in n, we then obtain a σ-finite measure $\hat{\nu}_E := \lim_{n\to\infty} \hat{\nu}_n$ on the product measurable space $(X^\infty, \mathfrak{B}^\infty)$ of $(X_n, \mathfrak{B}_n)$.

In what follows, all the measure spaces $(X_n, \mathfrak{B}_n, \mu_n)$ are identical: $(M, \mathfrak{B}(M), \mu)$, where M is a connected, non-compact but σ-compact, manifold of class C^∞, $\mathfrak{B}(M)$ is the Borel field and μ is a smooth, locally Euclidean measure whose total mass is infinite.

Definition 2.1 (Unital sequence). A sequence $E = \{E_n\}_n$ of Borel sets in M is said to be *μ-unital* if it satisfies the following two conditions:

(1) for all n, $0 < \mu(E_n) < \infty$,

(2) $\sum_{n=1}^\infty |1 - \mu(E_n)| < \infty$.

In addition, if the E_n are mutually disjoint, we call it a *disjoint μ-unital sequence*.

Without confusion, let us denote the product $\prod_{n=1}^\infty E_n$, which is called a *unital product set*, by the same letter E.

Definition 2.2 (Cofinality). Two μ-unital sequences, $E = \{E_n\}_n$ and $F = \{F_n\}_n$, are said to be *cofinal*, written $E \sim F$, if they satisfy $\sum_{n=1}^\infty \mu(E_n \ominus F_n) < \infty$.

In addition, if $E_n = F_n$ for sufficiently large n, then these sequences are said to be *strongly cofinal*, written $E \approx F$.

Take a μ-unital sequence $E = \{E_n\}_n$ and form the restricted product measure $\hat{\nu}_E$. As the infinite product of $\{\mu(E_n)\}_n$ absolutely converges, $\nu_E := \prod_{n=1}^{\infty} \mu(E_n)\hat{\nu}_E$ makes sense as a measure on M^∞. Moreover, it depends only on the cofinality class of E.

It is easy but important to observe that $\nu_E(M_E^c) = 0$, where $M_E := \bigcup_{n=1}^{\infty}(M^n \times \prod_{k=n+1}^{\infty} E_k)$, and that the action of $\sigma \in \mathfrak{S}_\infty$ on M^∞, $r(\sigma)\colon x \mapsto x\sigma$ leaves M_E invariant. It follows that we get the following theorem, which is basic for later discussions.

Theorem 2.1 (cf. [18]). *Given a disjoint μ-unital sequence E, there exists a Borel set D_E such that*

$$\forall \sigma \neq \mathrm{id}, \quad D_E\sigma \cap D_E = \emptyset \quad \textit{and} \quad M^\infty = \sum_{\sigma \in \mathfrak{S}_\infty} D_E\sigma \bmod \nu_E.$$

Sketch of proof. Take a sequence $\{F^n\}_n$ of disjoint μ-unital sequences that are all cofinal to E such that $M_E \subseteq \bigcup_{n=1}^{\infty} F^n \bmod \nu_E$. Now put $D_E := \sum_{n=1}^{\infty} D^n$, where $D^1 := F^1 \cap M_E$, $D^n := (F^n \cap M_E) \setminus \bigcup_{\sigma \in \mathfrak{S}_\infty,\, k=1}^{n-1} (F^k \cap M_E)$ $(n > 1)$. □

§2.2. Action of $\mathrm{Diff}_0(M)$ from the left and of $\mathfrak{S}_\infty$ from the right

Let $g \in \mathrm{Diff}_0(M)$ and $\sigma \in \mathfrak{S}_\infty$ act on M^∞ in the following manner:

$$g(x_1, x_2, \dots) = (gx_1, gx_2, \dots), \qquad (x_1, x_2, \dots)\sigma = (x_{\sigma(1)}, x_{\sigma(2)}, \dots).$$

Clearly the actions of g and of σ are mutually commutative, and they lead to transformations of ν_E, which will be denoted by $g\nu_E$ and $\sigma\nu_E$ respectively.

Theorem 2.2 (cf. [4]). *Given a disjoint μ-unital sequence $E = \{E_n\}_n$,*

(1) *ν_E is $\mathfrak{S}_\infty$-invariant, and*

(2) *ν_E is $\mathrm{Diff}_0(M)$-quasi-invariant.*

More precisely, the Radon–Nikodym derivative has the form

$$\frac{dg\nu_E}{d\nu_E}(x) = \prod_{n=1}^{\infty} \frac{dg\mu}{d\mu}(x_n),$$

where the infinite product converges in the $\mathrm{L}^1_{\nu_E}$-sense on each set $B \times \prod_{k=n_0+1}^{\infty} E_k$ ($B \in \mathfrak{B}(M^{n_0})$, $(\mu \times \cdots \times \mu)(B) < \infty$, and n_0 is arbitrary). (The above convergence is, of course, equivalent to $\mathrm{L}^2_{\nu_E}$-convergence for the square roots of the corresponding functions.)

By the above theorem, we have two unitary representations, $T(g)$, $g \in \mathrm{Diff}_0(M)$, and $R(\sigma)$, $\sigma \in \mathfrak{S}_\infty$, on $\mathrm{L}^2_{\nu_E}(M^\infty)$, such that

$$T(g) : f(x) \mapsto \sqrt{\frac{dg\nu_E}{d\nu_E}}(x) f(g^{-1}x), \qquad R(\sigma) : f(x) \mapsto f(x\sigma).$$

The representations T and R commute; moreover, they form a dual pair, as will be proved in the next section.

§2.3. Representation space $\mathcal{H}(\Sigma)$

Using the same notation for μ and E as before, and taking an irreducible unitary representation (Π, H) of $\mathfrak{S}_\infty$, where H is the separable representation Hilbert space, we put $\Sigma := (\mu, E, \Pi)$. Next, take a Borel measurable H-valued function f on M^∞ having the following property:

$$f(x\sigma) = \Pi(\sigma)^{-1} f(x) \quad \text{for } \sigma \in \mathfrak{S}_\infty. \tag{2.1}$$

Put

$$\|f\|^2 := \int_{D_E} \|f(x)\|_H^2 \, \nu_E(dx). \tag{2.2}$$

Then the space $\mathcal{H}(\Sigma)$ of those functions f such that $\|f\| < \infty$ forms a Hilbert space with the above norm.

It is useful to note that, for each $f \in \mathcal{H}(\Sigma)$, the integration set D_E may be replaced by an arbitrary Borel set D with the following two properties:

(1) $D\sigma \cap D = \emptyset$ if σ $(\in \mathfrak{S}_\infty) \neq \mathrm{id}$.

(2) $f = 0$ on $(\sum_{\sigma \in \mathfrak{S}_\infty} D\sigma)^c \bmod \nu_E$.

It follows that the following action T (denoted by the same letter, since no confusion can arrise) of $\mathrm{Diff}_0(M)$ on $\mathcal{H}(\Sigma)$ is well-defined and $(T, \mathcal{H}(\Sigma))$ is a unitary representation:

$$T(g) : f(x) \mapsto \sqrt{\frac{dg\nu_E}{d\nu_E}}(x) f(g^{-1}x).$$

We introduce a technical condition (cc) in the next lemma, which asserts the irreducibility of the natural representation $(T, \mathcal{H}(\Sigma))$.

Lemma 2.1 (cf. [18]). *Assume that $d := \dim(M) \geq 3$ and let M satisfy the following condition:*

(cc) *There exists a sequence $\{U_n\}_n$ of relatively compact, open sets $U_n \uparrow M$ such that $(\overline{U_n})^c$ is connected for every n.*

Given a disjoint unital sequence $E = \{E_n\}_n$, there exists a disjoint μ-unital sequence $G = \{G_n\}_n$ that is cofinal to E and has the following properties:

(1) *for all n, G_n is a relatively compact, open, connected set and $\mu(\overline{G_n} \setminus G_n) = 0$,*
(2) *for all n, $(\overline{\sum_{k=n+1}^{\infty} G_k})^c$ is connected,*
(3) *$\overline{G}_n \cap \overline{G}_m = \emptyset$ if $n \neq m$,*
(4) *given a compact set K, there exists $N_K \in \mathbb{N}$ such that $K \cap G_n = \emptyset$ for all $n \geq N_K$.*

Moreover given any $\epsilon > 0$, we may take G such that $\sum_{n=1}^{\infty} \mu(E_n \ominus G_n) < \epsilon$.

Proof. We will proceed in several steps.

Step 1. Let $\alpha := \inf_n \mu(E_n)$. For each E_n, we take a compact set $K_n^{(1)}$ satisfying

$$K_n^{(1)} \subseteq E_n \quad \text{and} \quad \mu(E_n \setminus K_n^{(1)}) < \epsilon_n,$$

where $\{\epsilon_n\}_n$ is a positive sequence such that $\epsilon_n < \alpha/4$ and $\sum_{n=1}^{\infty} \epsilon_n < \epsilon/4$. It is obvious that

$K^{(1)} := \{K_n^{(1)}\}_n \sim E$ is a disjoint μ-unital sequence, and $\mu(K_n^{(1)}) > \alpha/4$ for all n.

Step 2. Take a sequence $\{U_n\}_n$ as in condition (cc). Further, take an increasing sequence $\{k_n\}_n$ with

$$\sum_{i=k_n}^{\infty} \mu(K_i \cap \overline{U}_n) < \epsilon_n.$$

Next for $k_n \leq i < k_{n+1}$ we take a compact subset $K_i^{(2)}$ in $K_i^{(1)} \cap (\overline{U}_n)^c$ such that

$$\mu(K_i^{(1)} \cap (\overline{U}_n)^c \setminus K_i^{(2)}) < \min\left(\frac{\alpha}{4} - \epsilon_n, \frac{\epsilon_n}{k_{n+1} - k_n}\right),$$

and for $i \leq k_1 - 1$ we set $K_i^{(2)} := K_i^{(1)}$. It is easy to see that $\mu(K_i^{(2)}) > \alpha/4$ and

$$\sum_{i=1}^{\infty} \mu(K_i^{(1)} \setminus K_i^{(2)}) = \sum_{n=1}^{\infty} \sum_{i=k_n}^{k_{n+1}-1} \mu(K_i^{(1)} \setminus K_i^{(2)}) < 2 \sum_{n=1}^{\infty} \epsilon_n < \frac{\epsilon}{2}.$$

In addition, for a given compact set L, we have $K_n^{(2)} \cap L = \emptyset$ for all $n \geq k_{n_0}$, where n_0 is such that $L \subset U_{n_0}$. Therefore, $\sum_{i \in I} K_i^{(2)}$ is closed for any index set $I \subseteq \mathbb{N}$.

Step 3. Take open sets O_1 and O_2' that satisfy

$$O_1 \cap O_2' = \emptyset, \quad K_1^{(2)} \subset O_1, \quad \text{and} \quad \sum_{i=2}^{\infty} K_i^{(2)} \subset O_2'.$$

Of course we may assume that $\overline{O}_1$ is compact. By induction, suppose that we have already chosen relatively compact, open sets O_i $(i = 1, \dots, n-1)$ that satisfy

$$K_i^{(2)} \subset O_i, \quad \overline{O}_i \cap \overline{O}_j = \emptyset \quad \text{if } i \neq j, \quad \Big(\bigcup_{i=1}^{n-1} \overline{O}_i\Big) \cap \Big(\bigcup_{i=n}^{\infty} K_i^{(2)}\Big) = \emptyset.$$

Then there exist open sets O_n and O'_{n+1} that satisfy

$$K_n^{(2)} \subset O_n, \quad \Big(\bigcup_{i=1}^{n-1} \overline{O}_i\Big) \cup \Big(\bigcup_{i=n+1}^{\infty} K_i^{(2)}\Big) \subset O'_{n+1}, \quad O_n \cap O'_{n+1} = \emptyset.$$

Now the induction may proceed to the next stage. Moreover since $K_i^{(2)} \subset (\overline{U}_n)^c$ for all $k_n \leq i < k_{n+1}$, we may assume that

$$K_i^{(2)} \subset O_i \subset \overline{O}_i \subset (\overline{U}_n)^c \quad \text{and} \quad \sum_{i=1}^{\infty} \mu(O_i \setminus K_i^{(2)}) < \frac{\epsilon}{8},$$

and finally, that each O_i is a finite union of open sets that are diffeomorphic to disks in $\mathbb{R}^d$. In addition, we use regularity of μ and Riemann's method of quadrature. Then within small μ-mass gaps, we can deform O_i to a finite disjoint union of connected, open sets $O_{i,j}$, where $O_{i,j}$ is included in a neighbourhood diffeomorphic to $\mathbb{R}^d$ and it is a union of sets diffeomorphic to cubes in $\mathbb{R}^d$. Thus, we have checked all the properties listed in this lemma, except for connectedness.

Step 4. We connect the components $O_{i,j}$ by some polygonal lines included in $(\overline{U}_n)^c$. Next, we enlarge these curves to slim open tubes and get a connected open tubular neighbourhood G_i by adding those to $O_{i,j}$ or removing them. By property (4) which has already been shown, we can achieve that $\overline{G}_n$ $(n = 1, 2, \dots)$ are mutually disjoint and that $(\sum_{k=n+1}^{\infty} \overline{G}_n)^c$ is connected for each n.

When we connect $O_{i,j}$ by slim open tubes, it might happen that the tubes contact to G_k $(k < i)$ that have already been constructed. However, if $d \geq 3$, we can go through G_k by the slim tubes or by slimmer ones without loss of connectedness. This is the reason why we require the condition $d \geq 3$.

Of course, we can take $\{G_n\}_n$ such that $\sum_{n=1}^{\infty} \mu(G_n \ominus O_n) < \epsilon/8$. Hence,

$$\begin{aligned}\sum_{n=1}^{\infty} \mu(E_n \ominus G_n) &< \sum_{n=1}^{\infty} \mu(E_n \ominus K_n^{(1)}) + \sum_{n=1}^{\infty} \mu(K_n^{(1)} \ominus K_n^{(2)}) \\ &\quad + \sum_{n=1}^{\infty} \mu(K_n^{(2)} \ominus O_n) + \sum_{n=1}^{\infty} \mu(O_n \ominus G_n) \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{2} + \frac{\epsilon}{8} + \frac{\epsilon}{8} = \epsilon.\end{aligned}$$

□

Counter-example to (cc). Let B be an open set in $\mathbb{R}^d$ $(d \geq 3)$ surrounded by an outer large sphere and including an inner small sphere. The manifold B does not satisfy (cc).

Note that $M = \bigcup U_n$ is required in condition (cc), but it is actually sufficient to assume the following weaker (mcc) to obtain irreducibility:

(mcc) There exists a closed set S in M such that $\mu(S) = 0$, $M \setminus S$ is connected and satisfies condition (cc).

Thus, the following theorem concludes this section (cf. [4]).

Theorem 2.3. *Let E be a disjoint μ-unital sequence, and (Π, H) an irreducible unitary representation of $\mathfrak{S}_\infty$. Form a triplet $\Sigma = (E, \mu, \Pi)$ as before. If $\dim(M) \geq 3$ and M satisfies condition (mcc), then the unitary representation $(T, \mathcal{H}(\Sigma))$ is irreducible.*

As mentioned in the Introduction, another proof of the above theorem will be given in the Appendix (§A.1).

§3. Factor representations and dual pairs

§3.1. Factor representation

Let us begin with the following lemma.

Lemma 3.1. *Let M be a smooth manifold with $\dim(M) \geq 2$, and let μ be a locally Euclidean smooth σ-finite measure on M. Take the product measure μ^n on M^n $(n \in \mathbb{N})$, and define the natural representation T of $\mathrm{Diff}_0(M)$ and a representation R of $\mathfrak{S}_n$ on $\mathrm{L}^2_{\mu^n}(M^n)$ by*

$$T(g) : f(x_1, \dots, x_n) \mapsto \prod_{k=1}^{n} \sqrt{\frac{dg\mu}{d\mu}}(x_k) f(g^{-1}x_1, \dots, g^{-1}x_n),$$
$$R(\sigma) : f(x_1, \dots, x_n) \mapsto f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

These then form a dual pair. That is, the von Neumann algebra $(R(\sigma), \sigma \in \mathfrak{S}_n)''$ generated by $R(\sigma)$, $\sigma \in \mathfrak{S}_n$, is the commutant $(T(g),\ g \in \mathrm{Diff}_0(M))'$ of the von Neumann algebra generated by $T(g)$, $g \in \mathrm{Diff}_0(M)$.

Proof. Take disjoint, open, connected sets $U_1, \dots, U_n$ in M such that $\mu(\overline{U_k} \setminus U_k) = 0$ $(k = 1, \dots, n)$, and put $U_\infty := M \setminus \bigcup_{k=1}^n U_k$. Then

$$\mathrm{L}^2_{\mu^n}(M^n) = \sum\nolimits^{\oplus} \mathrm{L}^2_\mu(U_{q_1}) \otimes \cdots \otimes \mathrm{L}^2_\mu(U_{q_n}),$$

where $q_k = 1, \dots, n$ or ∞. Let P_q be the orthogonal projection from $\mathrm{L}^2_{\mu^n}(M^n)$ to $\mathrm{L}^2_\mu(U_{q_1}) \otimes \cdots \otimes \mathrm{L}^2_\mu(U_{q_n})$.

Given $A \in (T(g), g \in \mathrm{Diff}_0(M))'$, we put

$$A_U := A|\mathrm{L}^2_\mu(U_1) \otimes \cdots \otimes \mathrm{L}^2_\mu(U_n).$$

As the natural representation on $\bigotimes_{k=1}^n \mathrm{L}^2_\mu(U_k)$ of the group $\prod_{k=1}^n \mathrm{Diff}_0(U_k)$ is irreducible, and as $P_q A_U$ is an intertwining operator between the corresponding spaces, it follows that $P_q A_U = 0$ except for $q_k = \sigma(k)$ $(k = 1, \dots, n)$ for some $\sigma \in \mathfrak{S}_n$. Let us denote P_q by P_σ in this case only. Then

$$P_\sigma A_U = a_{\sigma,U} R(\sigma) \quad \text{ for some } a_{\sigma,U} \in \mathbb{C}$$

for each $\sigma \in \mathfrak{S}_n$, and it follows that

$$A_U = \sum_{\sigma \in \mathfrak{S}_n} a_{\sigma,U} R(\sigma) \quad \text{ on } \bigotimes_{k=1}^n \mathrm{L}^2_\mu(U_k).$$

If $\{U_k\}_k$ and $\{V_k\}_k$ are two sequences of open sets as above and $U_k \cap V_k \neq \emptyset$ for all k, then we have $a_{\sigma,U} = a_{\sigma,V}$ for all $\sigma \in \mathfrak{S}_n$. Finally, the connectedness assumption on M with $\dim(M) \geq 2$ leads to the independence of $a_{\sigma,U}$ from U, because we can connect any two points in $\tilde{M}^n := \{(x_1, \dots, x_n) \mid x_i \neq x_j \text{ for all } i \neq j\}$ with a continuous curve in $\tilde{M}^n$, and it follows that $a_\sigma := a_{\sigma,U}$. As $\bigotimes_{k=1}^n \mathrm{L}^2_\mu(U_k)$ generates the whole space, letting $\{U_1, \dots, U_n\}$ run through sequences of disjoint connected open sets, due to the regularity of μ and $\mu^n(M^n \setminus \tilde{M}^n) = 0$ (cf. [20]) we get

$$A = \sum_{\sigma \in \mathfrak{S}_n} a_\sigma R(\sigma),$$

and $(T(g), g \in \mathrm{Diff}_0(M))' \subseteq (R(\sigma), \sigma \in \mathfrak{S}_n)''$. The reverse inclusion is obvious. □

Theorem 3.1. *Take the Hilbert space $\mathrm{L}^2_{\nu_E}(M^\infty)$ considered in Section 2. In addition, let M satisfy condition* (mcc) *and let* $\dim(M) \geq 3$. *Then the representation $(T(g)R(\sigma), \mathrm{L}^2_{\nu_E}(M^\infty))$, $g \in \mathrm{Diff}_0(M)$, $\sigma \in \mathfrak{S}_\infty$, is irreducible.*

Proof. Take a closed set S as in (mcc) and put $M' := M \cap S^c$. The unitary operator I defined by

$$I : f(x) \in \mathrm{L}^2_{\nu_E}(M'^\infty) \mapsto \prod_{k=1}^\infty \chi_{M'}(x_k) f(x) \in \mathrm{L}^2_{\nu_E}(M^\infty)$$

will be used to reduce the proof to the case of M', where $\chi_{M'}$ denotes the indicator function of M'. Thus, in what follows, we assume that condition (cc) on M holds, and apply Lemma 2.1. In addition, we replace the letter G in that lemma by E and set

$$E_{\infty,n} := \left(\overline{\sum_{k=n+1}^{\infty} E_k}\right)^c, \qquad \hat{E}^n := E_{\infty,n} \times \cdots \times E_{\infty,n} \times E_{n+1} \times E_{n+2} \times \cdots$$

($E_{\infty,n}$ appears n times in the above expression, and it is a connected, open set in M by condition (2) in Lemma 2.1), and

$$P_n : f \in \mathrm{L}^2_{\nu_E}(M^\infty) \mapsto \chi_{\hat{E}^n} \cdot f \in \mathrm{L}^2_{\nu_E}(M^\infty).$$

Then P_n is an increasing projection for each n and tends strongly to Id.

Given $A \in (T(g), g \in \mathrm{Diff}_0(M))'$ and $n \in \mathbb{N}$, define an operator $A_n \equiv A_n(u,v)$ on $\mathrm{L}^2_{\mu^n}(E^n_{\infty,n})$ for each $u, v \in \mathrm{L}^2_{\mu\times\mu\times\cdots}(\prod_{k=n+1}^{\infty} E_k)$ as

$$A_n(f) := \int_{\prod_{k=n+1}^{\infty} E_k} A(f \otimes u)(x_1, x_2, \ldots)\overline{v(x_{n+1}, x_{n+2}, \ldots)} \prod_{k=n+1}^{\infty} \mu(dx_k).$$

A_n is characterized as a bounded operator on $\mathrm{L}^2_{\mu^n}(E^n_{\infty,n})$ such that

$$\langle A_n(f), h\rangle_{\mathrm{L}^2_{\mu^n}(E^n_{\infty,n})} = \langle A(f \otimes u), h \otimes v\rangle_{\mathrm{L}^2_{\nu_E}(M^\infty)}$$

for all $h \in \mathrm{L}^2_{\mu^n}(E^n_{\infty,n})$. It follows that $T(g)A_n = A_n T(g)$ for all $g \in \mathrm{Diff}_0(E_{\infty,n})$, and in view of Lemma 3.1, there exist constants $\lambda_n(\sigma, u, v) \in \mathbb{C}$ such that

$$A_n(u,v) = \sum_{\sigma \in \mathfrak{S}_n} \lambda_n(\sigma, u, v) R(\sigma) \quad \text{ on } \mathrm{L}^2_{\mu^n}(E^n_{\infty,n}).$$

As $R(\sigma)$, $\sigma \in \mathfrak{S}_n$, are linearly independent, and as $A_n(u,v)$ is bounded for all u, v, we obtain a bounded operator B^σ_n on $\mathrm{L}^2_{\mu\times\mu\times\cdots}(\prod_{k=n+1}^{\infty} E_k)$ that satisfies

$$\langle B^\sigma_n u, v\rangle_{\mathrm{L}^2_{\mu\times\mu\times\cdots}(\prod_{k=n+1}^{\infty} E_k)} = \lambda_n(\sigma, u, v) \quad \text{ for all } \sigma \in \mathfrak{S}_n.$$

Therefore,

$$\langle A(f \otimes u), h \otimes v\rangle_{\mathrm{L}^2_{\nu_E(M^\infty)}} = \sum_{\sigma \in \mathfrak{S}_n} \langle R(\sigma)f, h\rangle_{\mathrm{L}^2_{\mu^n}(E^n_{\infty,n})} \langle B^\sigma_n u, v\rangle_{\mathrm{L}^2_{\mu\times\mu\times\cdots}(\prod_{k=n+1}^{\infty} E_k)}.$$

In other words,

$$P_n A = \sum_{\sigma \in \mathfrak{S}_n} R(\sigma) \otimes B^\sigma_n \quad \text{ on } P_n(\mathrm{L}^2_{\nu_E}(M^\infty)).$$

Now take another diffeomorphism g from the restricted product group $\prod_{k=n+1}^{\infty *} \mathrm{Diff}_0(E_k)$. It then follows from the trivial relation $P_n T(g) = T(g) P_n$ that

$$T(g) B^\sigma_n = B^\sigma_n T(g) \quad \text{ for all } \sigma \in \mathfrak{S}_n.$$

As has already been noted, by the irreducibility of the natural representation of $\prod_{k=n+1}^{\infty *} \mathrm{Diff}_0(E_k)$ on $\mathrm{L}^2_{\mu\times\mu\times\cdots}(\prod_{k=n+1}^{\infty} E_k)$ (cf. [18]), B_n^σ is a scalar operator $\lambda_{n,\sigma}\,\mathrm{Id}$ for each $\sigma \in \mathfrak{S}_n$. Consequently,

$$P_n A P_n = \Big(\sum_{\sigma\in\mathfrak{S}_n} \lambda_{n,\sigma} R(\sigma)\Big) P_n, \tag{3.1}$$

where the symbol $R(\sigma)$ is used for both $\mathrm{L}^2_{\mu^n}(E^n_{\infty,n})$ and $\mathrm{L}^2_{\nu_E}(M^\infty)$, since no confusion can arise. Fix $N \in \mathbb{N}$ for a while. Further, take $n_0 \geq N$ and a Borel set B_0 of product type such that

$$\begin{aligned} &B_0 := U_1 \times \cdots \times U_{n_0} \times E_{n_0+1} \times E_{n_0+2} \times \cdots\,, \\ &B_0 \subset \hat{E}^N, \qquad U_i \cap U_j = \emptyset \qquad (\forall i \neq j \leq n_0), \\ &\forall i, \qquad 0 < \mu(U_i) < \infty. \end{aligned} \tag{3.2}$$

Consequently, $U_i \cap E_k = \emptyset$ for $1 \leq i \leq n_0$ and $k \geq n_0 + 1$, and it follows that $B_0\sigma \cap B_0 = \emptyset$ for σ ($\neq$ id) $\in \mathfrak{S}_\infty$, and $\chi_{B_0} \in P_n(\mathrm{L}^2_{\nu_E}(M^\infty))$ for all $n \geq n_0$. Now applying (3.1) to χ_{B_0}, we obtain

$$\|P_n A P_n \chi_{B_0}\|^2 = \sum_{\sigma\in\mathfrak{S}_n} |\lambda_{n,\sigma}|^2 \nu_E(B_0),$$

and this implies

$$\sum_{\sigma\in\mathfrak{S}_n} |\lambda_{n,\sigma}|^2 \leq \|A\|^2. \tag{3.3}$$

Next, from the assumption $A \in (R(\sigma),\ \sigma \in \mathfrak{S}_\infty))'$ and from the fact that $P_n R(\sigma) = R(\sigma) P_n$ for $\sigma \in \mathfrak{S}_n$, it follows that

$$\lambda_{n,\sigma} = \lambda_{n,\tau\sigma\tau^{-1}} \qquad \text{for all } \sigma, \tau \in \mathfrak{S}_n.$$

Take any $k \in \mathbb{N}$ and $\sigma_0 \in \mathfrak{S}_k$ not equal to id. As can be easily seen, for any $l \in \mathbb{N}$ greater than k, there exist $\tau_1, \ldots, \tau_m \in \mathfrak{S}_l$ such that $\tau_i^{-1}\sigma_0\tau_i$ are mutually distinct ($i = 1, \ldots, m$), where m is the greatest integer smaller than l/k. Thus,

$$|\lambda_{l,\sigma_0}|^2 \leq \|A\|^2/m \to 0 \qquad (l \to \infty).$$

Hence, for any $\sigma \neq \mathrm{id} \in \mathfrak{S}_\infty$,

$$\langle A\chi_{B_0}, \chi_{B_0\sigma}\rangle = \lim_{l\to\infty} \langle P_l A \chi_{B_0}, \chi_{B_0\sigma}\rangle = \lim_{l\to\infty} \lambda_{l,\sigma^{-1}} \nu_E(B_0) = 0,$$

while

$$\langle A\chi_{B_0}, \chi_{B_0}\rangle = \lim_{l\to\infty} \langle P_l A \chi_{B_0}, \chi_{B_0}\rangle = \lim_{l\to\infty} \lambda_{l,\mathrm{id}} \nu_E(B_0).$$

Thus, we have a limit of $\{\lambda_{l,\mathrm{id}}\}_l$ which is denoted by λ.

Note that the indicator functions of the Borel sets as in (3.2) generate the space $\mathrm{L}^2_{\nu_E}(\hat{E}^N)$. Take finitely many such Borel sets $B_1, \dots, B_s$ of $\hat{E}^n$ and complex constants $c_1, \dots, c_s$. Then

$$\begin{aligned}\Big\langle A\Big(\sum_{i=1}^s c_i\chi_{B_i}\Big), \sum_{i=1}^s c_i\chi_{B_i}\Big\rangle &= \lim_{l\to\infty}\Big\langle P_l A\Big(\sum_{i=1}^s c_i\chi_{B_i}\Big), \sum_{i=1}^s c_i\chi_{B_i}\Big\rangle \\ &= \lim_{l\to\infty}\sum_{\sigma\in\mathfrak{S}_l}\sum_{i,j}^s \lambda_{l,\sigma}c_i\overline{c_j}\langle\chi_{B_i\sigma^{-1}}, \chi_{B_j}\rangle.\end{aligned}$$

Since $\langle\chi_{B_i\sigma^{-1}}, \chi_{B_j}\rangle = 0$ except for $\sigma \in \mathfrak{S}_N$,

$$\begin{aligned}\Big\langle A\Big(\sum_{i=1}^s c_i\chi_{B_i}\Big), \sum_{i=1}^s c_i\chi_{B_i}\Big\rangle &= \sum_{\sigma\in\mathfrak{S}_N}\lim_{l\to\infty}\sum_{i,j}^s \lambda_{l,\sigma}c_i\overline{c_j}\langle\chi_{B_i\sigma^{-1}}, \chi_{B_j}\rangle \\ &= \lambda\sum_{i,j}^s c_i\overline{c_j}\langle\chi_{B_i}, \chi_{B_j}\rangle.\end{aligned}$$

Hence, $\langle Af, f\rangle = \lambda\langle f, f\rangle$ for any $f = \sum_{i=1}^s c_i\chi_{B_i}$. As N is arbitrary, this shows that A is a scalar operator. □

Corollary 3.1. *Both* $(T(g), \mathrm{L}^2_{\nu_E}(M^\infty))$, $g \in \mathrm{Diff}_0(M)$, *and* $(R(\sigma), \mathrm{L}^2_{\nu_E}(M^\infty))$, $\sigma \in \mathfrak{S}_\infty$, *are factor representations.*

Proof. In fact, $(T(g), \mathrm{Diff}_0(M))'' \subseteq (R(\sigma), \sigma \in \mathfrak{S}_\infty)'$ and $(R(\sigma), \sigma \in \mathfrak{S}_\infty)'' \subseteq (T(g), \mathrm{Diff}_0(M))'$. Thus, the conclusion follows from the above theorem. □

Theorem 3.1 and Corollary 3.1 indicate that T and R behave as in the finite-dimensional case (cf. [20]).

Theorem 3.2. *Using the same assumptions and notation as in Theorem 3.1, let F be another disjoint μ-unital sequence. The representations $(T(g)R(\sigma), \mathrm{L}^2_{\nu_E}(M^\infty))$ and $(T(g)R(\sigma), \mathrm{L}^2_{\nu_F}(M^\infty))$ are equivalent if and only if there exists a permutation a of $\mathbb{N}$ (maybe infinite) such that $E \sim Fa^{-1}$.*

Proof. This is an immediate consequence of Theorem B in the Appendix.

Theorem 3.3. $(R(\sigma), \mathrm{L}^2_{\nu_E}(M^\infty))$, $\sigma \in \mathfrak{S}_\infty$, *and* $(T(g), \mathrm{L}^2_{\nu_E}(M^\infty))$, $g \in \mathrm{Diff}_0(M)$, *are factor representations of type* II_1 *and* II_∞, *respectively.*

Proof. Let $\ell^2(\mathfrak{S}_\infty)$ be the representation space of the right regular representation of $\mathfrak{S}_\infty$, and denote the indicator function of the set $\{\sigma\}$ consisting of a single element $\sigma \in \mathfrak{S}_\infty$ by e_σ. It is well-known that the von Neumann algebra $\mathcal{M}$ generated by the right regular representation is of type II_1, and that its relative dimensionality function $d_{\mathcal{M}}$ is given by

$$d_{\mathcal{M}}(M) := \langle P_M e_{\mathrm{id}}, e_{\mathrm{id}} \rangle_{\ell^2(\mathfrak{S}_\infty)} \quad \text{for all } P_M \in \mathcal{M}.$$

Now, take the set D_E introduced in Section 2, take a c.o.n.s. $\{h_n\}_n$ in $\mathrm{L}^2_{\nu_E}(D_E)$, and finally, take the Hilbert space H_n spanned by $R(\sigma)h_n$, $\sigma \in \mathfrak{S}_\infty$. Then

$$\mathrm{L}^2_{\nu_E}(M^\infty) = \sum\nolimits^{\oplus} H_n,$$

and $\ell^2(\mathfrak{S}_\infty)$ and H_n are isomorphic through the following intertwining unitary operator η_n:

$$\eta_n : e_\sigma \in \ell^2(\mathfrak{S}_\infty) \mapsto R(\sigma)^{-1} h_n \in H_n \quad \text{for all } \sigma \in \mathfrak{S}_\infty.$$

Let Q_n be the orthogonal projection from $\mathrm{L}^2_{\nu_E}(M^\infty)$ to H_n. It is easy to see that

$$T \in \mathcal{R} := (R(\sigma), \sigma \in \mathfrak{S}_\infty)'' \;\Rightarrow\; \eta_n^{-1} Q_n T \eta_n \in \mathcal{M} \text{ for all } n.$$

Note that if $T = P_M$ is a projection, $Q_n P_M$ is also a projection in H_n. As a result, we can define a relative dimensionality function $d_{\mathcal{R}}$ on $\mathcal{R}$ by

$$d_{\mathcal{R}}(M) := \langle Q_n P_M h_n, h_n \rangle_{H_n} \quad \text{for all } P_M \in (R(\sigma), \sigma \in \mathfrak{S}_\infty)'',$$

which obviously does not depend on $n \in \mathbb{N}$. This demonstrates the first assertion.

Next, consider the factor representation T, and use the fact that $T(g)$, $g \in \mathrm{Diff}_0(M)$, and $R(\sigma)$, $\sigma \in \mathfrak{S}_\infty$, form a dual pair, as claimed in [5]. However, due to a mistake in the uniform estimate of a norm in [5, Lemma 5.5], we will give a corrected proof in the next subsection.

By the result on dual pairs,

$$Q_n \in (R(\sigma), \sigma \in \mathfrak{S}_\infty)' = (T(g), g \in \mathrm{Diff}_0(M))'' =: \mathcal{T}.$$

In addition, take a unitary operator $S_{n,m}$ for $n \neq m \in \mathbb{N}$ defined by

$$S_{n,m} := \begin{cases} R(\sigma)^{-1} h_n \mapsto R(\sigma)^{-1} h_m, \\ R(\sigma)^{-1} h_m \mapsto R(\sigma)^{-1} h_n, \\ R(\sigma)^{-1} h_l \mapsto R(\sigma)^{-1} h_l \quad (l \neq n, m). \end{cases}$$

It is clear that $S_{n,m} \in \mathcal{T}$ and $S_{n,m}(H_n) = H_m$.

As a well-known theorem on coupled factors (cf. [11]) guarantees that $\mathcal{T}$ is of type II, we have only to check that its relative dimensionality function $d_{\mathcal{T}}$ has

$$d_{\mathcal{T}}(\mathrm{L}^2_{\nu_E}(M^\infty)) = \infty,$$

which follows directly from the above arguments. □

§3.2. Dual pairs of $T(g)$, $g \in \mathrm{Diff}_0(M)$, and $R(\sigma)$, $\sigma \in \mathfrak{S}_\infty$

Theorem 3.4. *Under the assumptions of Theorem 3.1, $T(g)$, $g \in \mathrm{Diff}_0(M)$, and $R(\sigma)$, $\sigma \in \mathfrak{S}_\infty$, form a dual pair.*

Proof. Using the same reasonings as in the proof of Theorem 3.1, we can assume that M satisfies condition (cc).

Given $A \in (T(g), g \in \mathrm{Diff}_0(M))'$, relation (3.1) in Theorem 3.1 follows by the same arguments as before. At this stage, the uniform boundedness of the operators $\sum_{\sigma\in\mathfrak{S}_n} \lambda_{n,\sigma} R(\sigma)$ on the whole space $\mathrm{L}^2_{\nu_E}(M^\infty)$ is important. We prove it as follows:

First of all,

$$|\lambda_{n,\sigma}| \le \|A\| \quad \text{for all } n \in \mathbb{N} \text{ and } \sigma \in \mathfrak{S}_n,$$

by (3.3). In addition, the family $\mathcal{O} := \{\sum_{\sigma\in\mathfrak{S}_n} \lambda_\sigma R(\sigma)\}_{|\lambda_\sigma|\le\|A\|}$ of operators on $\mathrm{L}^2_{\mu^n}(M^n)$ is compact in the uniform topology, and given $\epsilon > 0$, there exists a compact set $\Omega := K \times \cdots \times K \subset M^n$ such that

$$\|T\| - \epsilon < \sup_{f\in \mathrm{L}^2_{\mu^n}(M^n)} \frac{\|T(f\cdot\chi_\Omega)\|}{\|f\cdot\chi_\Omega\|} \quad \text{for all } T \in \mathcal{O}. \tag{3.4}$$

Cover K by open coordinate neighbourhoods O'_i $(i = 1, \ldots, s)$ such that

$$K \subset O'_1 \cup \cdots \cup O'_s \quad \text{and} \quad \mu(\overline{O'_i} \setminus O'_i) = 0 \quad (i = 1, \ldots, s).$$

Put

$$O_1 := O'_1, \quad O_2 := O'_2 \setminus \overline{O'_1}, \ \ldots, \ O_s := O'_s \setminus (\overline{O'_1} \cup \cdots \cup \overline{O'_{s-1}}),$$

and

$$\hat{G} := O_1 \cup \cdots \cup O_s, \qquad \Omega_1 := \prod_{k=1}^n \hat{G}.$$

Then $\hat{G}$ is a union of disjoint open sets and Ω_1 has the same property (3.4) as Ω.

Finally, reasoning locally on each O_i, we may find a sequence of disjoint open sets W_i $(i = 1, \ldots, T)$ diffeomorphic to cubes in $\mathbb{R}^d$, and such that their union G approximates $\hat{G}$ from the inside. That is, for any $\delta > 0$, we may find such an open set G with $\mu(\hat{G} \setminus G) < \delta$ and $W_i \cap W_j = \emptyset$ $(i \ne j)$. In addition, after changing Ω_1 to $\Omega_2 := \prod_{k=1}^n G$, we have an equality of type (3.4), if we take δ sufficiently small:

$$\|T\| - 3\epsilon < \sup_{f\in \mathrm{L}^2_{\mu^n}(M^n)} \frac{\|T(f\cdot\chi_{\Omega_2})\|}{\|f\cdot\chi_{\Omega_2}\|} \quad \text{for all } T \in \mathcal{O}. \tag{3.5}$$

Let us denote the centre of each cube W_i in G by x_i $(i = 1, \ldots, T)$, and take other points $\xi_1, \ldots, \xi_T$ in $E_{\infty,n}$, mutually distinct. Removing a small mass of $\overline{E_{\infty,n}}$

from $\hat{G}$ in advance, as necessary, we may assume $\{\xi_1, \dots, \xi_T\} \cap G = \emptyset$. Second, connect x_i and ξ_i with open slim tubes Γ_i that are mutually disjoint, $\Gamma_i \cap W_j = \emptyset$ $(i \neq j)$ and take maps $\psi_i \in \mathrm{Diff}_0(M)$ such that $\operatorname{supp}\psi_i \subset \Gamma_i$ and $\psi_i(x_i) = \xi_i$. Finally, take open sets $V(\xi_i) \subset E_{\infty,n}$, and $U(x_i) \subset W_i$ such that $\psi_i(U(x_i)) \subset V(\xi_i)$, and take a map ϕ_i such that $\phi_i(W_i) \subset U(x_i)$ and $\operatorname{supp}\phi_i \cap (W_j \cup \Gamma_j) = \emptyset$ $(i \neq j)$. Put $\phi := \prod_{i=1}^T \psi_i \circ \phi_i$. Then $\phi(G) \subset E_{\infty,n}$.

Now return to relation (3.1), and fix $n \in \mathbb{N}$ for a while. Take a function $k_\lambda \in \mathrm{L}^2_{\mu^n}(M^n)$ for each $\lambda := \{\lambda_{n,\sigma}\}_{\sigma\in\mathfrak{S}_n}$ bounded by $\|A\|$, satisfying

$$\Big\|\sum_{\sigma\in\mathfrak{S}_n}\lambda_{n,\sigma}R(\sigma)\Big\| - 3\epsilon < \frac{\|(\sum_{\sigma\in\mathfrak{S}_n}\lambda_{n,\sigma}R(\sigma))k_\lambda\chi_{\Omega_2}\|_{\mathrm{L}^2_{\mu^n}(M^n)}}{\|k_\lambda\chi_{\Omega_2}\|_{\mathrm{L}^2_{\mu^n}(M^n)}}.$$

Next, set

$$f_\lambda(x) := \prod_{k=1}^n \sqrt{\frac{d\phi\mu}{d\mu}}(x_k)\frac{(k_\lambda\cdot\chi_{\Omega_2})(\phi^{-1}x_1,\dots,\phi^{-1}x_n)}{\|k_\lambda\chi_{\Omega_2}\|_{\mathrm{L}^2_{\mu^n}(M^n)}}\prod_{k=n+1}^\infty \chi_{E_k}(x_k).$$

Then

$$T(\phi)^{-1}f_\lambda(x) = \frac{(k_\lambda\chi_{\Omega_2})(x_1,\dots,x_n)}{\|k_\lambda\chi_{\Omega_2}\|_{\mathrm{L}^2_{\mu^n}(M^n)}}\prod_{k=n+1}^\infty\sqrt{\frac{d\phi^{-1}\mu}{d\mu}}(x_k)\chi_{E_k}(\phi(x_k)),$$

and $f_\lambda \in P_n(\mathrm{L}^2_{\nu_E}(M^\infty))$. It follows from (3.1) that

$$\begin{aligned}T(\phi)^{-1}P_nAP_nf_\lambda(x) &= \Big(\sum_{\sigma\in\mathfrak{S}_n}\lambda_{n,\sigma}R(\sigma)\Big)T(\phi)^{-1}f_\lambda(x)\\ &= \|k_\lambda\chi_{\Omega_2}\|^{-1}_{\mathrm{L}^2_{\mu^n}(M^n)}\Big(\sum_{\sigma\in\mathfrak{S}_n}\lambda_{n,\sigma}R(\sigma)\Big)(k_\lambda\chi_{\Omega_2})(x_1,\dots,x_n)\\ &\quad\cdot\prod_{k=n+1}^\infty\sqrt{\frac{d\phi^{-1}\mu}{d\mu}}(x_k)\chi_{E_k}(\phi(x_k)).\end{aligned}$$

Hence,

$$\begin{aligned}\|T(\phi)^{-1}P_nAP_nf_\lambda\|^2_{\mathrm{L}^2_{\nu_E}(M^\infty)} &> \Big(\Big\|\sum_{\sigma\in\mathfrak{S}_n}\lambda_{n,\sigma}R(\sigma)\Big\| - 3\epsilon\Big)^2\prod_{k=n+1}^\infty\mu(E_k)\\ &= \Big(\Big\|\sum_{\sigma\in\mathfrak{S}_n}\lambda_{n,\sigma}R(\sigma)\Big\| - 3\epsilon\Big)^2\|f_\lambda\|^2_{\mathrm{L}^2_{\nu_E}(M^\infty)}.\end{aligned}$$

This indicates that

$$\Big\|\sum_{\sigma\in\mathfrak{S}_n}\lambda_{n,\sigma}R(\sigma)\Big\| < \|A\| + 3\epsilon.$$

In addition, the operator norm of $\sum_{\sigma\in\mathfrak{S}_n}\lambda_{n,\sigma}R(\sigma)$ is left invariant under a change of the basic space from $\mathrm{L}^2_{\mu^n}(M^n)$ to $\mathrm{L}^2_{\nu_E}(M^\infty)$. This gives the uniform boundedness. As a result, we easily see that $\sum_{\sigma\in\mathfrak{S}_n}\lambda_{n,\sigma}R(\sigma)$ converges strongly to A, and we have $A\in(R(\sigma),\,\sigma\in\mathfrak{S}_\infty)''$. In other words,

$$(T(g), g\in \mathrm{Diff}_0(M))' \subseteq (R(\sigma), \sigma\in\mathfrak{S}_\infty)''.$$

The reverse inclusion is obvious. □

§4. Irreducible decompositions of $(T(g), \mathrm{L}^2_{\nu_E}(M^\infty))$, $g\in\mathrm{Diff}_0(M)$

As mentioned in the Introduction, an irreducible decomposition of T is provided in this section.

Theorem 4.1. *Take the Hilbert space* $\mathrm{L}^2_{\nu_E}(M^\infty)$ *considered in Section* 2*; in addition, let* M *satisfy condition* (mcc) *and suppose* $\dim(M)\geq 3$*. Then the natural representation* $(T(g), \mathrm{L}^2_{\nu_E}(M^\infty))$, $g\in\mathrm{Diff}_0(M)$*, has an irreducible decomposition, and the irreducible components are the spaces* $(T(g),\mathcal{H}(\Sigma_\lambda))$*, where* $\mathcal{H}(\Sigma_\lambda)$ *are the Hilbert spaces defined in the Introduction,* $\Sigma_\lambda := (\mu, E, H_\lambda)$*, and* (Π_λ, H_λ) *are irreducible components of the left regular representation of* $\mathfrak{S}_\infty$*.*

Proof. The proof is divided into several steps.

Step 1. First, we will give a unitary operator U_1 from $\mathrm{L}^2_{\nu_E}(M^\infty)$ onto $\ell^2(\mathfrak{S}_\infty)\otimes \mathrm{L}^2_{\nu_E}(D_E)$.

Fix a c.o.n.s. $\{h_n(x)\}_n$ in $\mathrm{L}^2_{\nu_E}(D_E)$. Of course, $\{h_n(x\sigma^{-1})\}_n$ forms a c.o.n.s. in $\mathrm{L}^2_{\nu_E}(D_E\sigma)$ for any $\sigma\in\mathfrak{S}_\infty$. Given $n\in\mathbb{N}$, $f\in \mathrm{L}^2_{\nu_E}(M^\infty)$ and $\sigma\in\mathfrak{S}_\infty$, put

$$a_n(\sigma) := \int_{D_E\sigma} f(x)\overline{h_n(x\sigma^{-1})}\,\nu_E(dx) = \int_{D_E} f(x\sigma)\overline{h_n(x)}\,\nu_E(dx).$$

It follows directly that

$$\sum_{\sigma\in\mathfrak{S}_\infty}\sum_{n=1}^\infty |a_n(\sigma)|^2 < \infty.$$

Define a map U_1 by

$$U_1(f) := \sum_{n=1}^\infty a_n\otimes h_n.$$

One may easily observe that U_1 is the desired unitary operator.

Step 2. Let $(L(\sigma), \ell^2(\mathfrak{S}_\infty))$, $\sigma\in\mathfrak{S}_\infty$, be the left regular representation. Decompose it, applying the reduction theory of the von Neumann algebra generated by

$L(\sigma), \sigma \in \mathfrak{S}_\infty$, and by the maximal abelian ring of its commutant (cf. [8, 9, 19]):

$$\ell^2(\mathfrak{S}_\infty) \sim \int^\oplus H_\lambda \sqrt{d\sigma(\lambda)}, \qquad L(\sigma) \sim \sum \Pi_\lambda(\sigma).$$

It then follows that there exists a realization of the orthogonal generalized direct sum of the $H_\lambda \otimes \mathrm{L}^2_{\nu_E}(D_E)$ to $\ell^2(\mathfrak{S}_\infty) \otimes \mathrm{L}^2_{\nu_E}(D_E)$ with the weight function $\sigma(\lambda)$:

$$\ell^2(\mathfrak{S}_\infty) \otimes \mathrm{L}^2_{\nu_E}(D_E) \sim \int^\oplus H_\lambda \otimes \mathrm{L}^2_{\nu_E}(D_E)\sqrt{d\sigma(\lambda)} \sim \int^\oplus \mathrm{L}^2_{\nu_E}(D_E, H_\lambda)\sqrt{d\sigma(\lambda)}.$$

Thus, we have a unitary map

$$U_2 : \ell^2(\mathfrak{S}_\infty) \otimes \mathrm{L}^2_{\nu_E}(D_E) \to \int^\oplus \mathrm{L}^2_{\nu_E}(D_E, H_\lambda)\sqrt{d\sigma(\lambda)}$$

such that

$$U_2\Big(\sum_n a_n \otimes h_n\Big) := \int_- \underline{f}_\lambda \sqrt{d\sigma(\lambda)},$$

where

$$\underline{f}_\lambda(x) = \sum_{n=1}^\infty \underline{a_{n,\lambda}} h_n(x) \qquad \text{for all } x \in D_E;$$

we follow the notation of von Neumann [13]. It is evident that U_2 is actually a unitary operator.

Step 3. Finally, extend the domain D_E of each component function $\underline{f}_\lambda$ in order to get another function $\underline{F}_\lambda$ in $\mathcal{H}(\Sigma_\lambda)$ defined by;

$$\underline{F}_\lambda(x\sigma) := \Pi_\lambda^{-1}(\sigma)\underline{f}_\lambda(x) \qquad \text{for all } x \in D_E \text{ and } \sigma \in \mathfrak{S}_\infty.$$

In addition, define

$$U_3 := \int_- \underline{f}_\lambda \sqrt{d\sigma(\lambda)} \in \int^\oplus \mathrm{L}^2_{\nu_E}(D_E, H_\lambda)\sqrt{d\sigma(\lambda)} \mapsto \int_- \underline{F}_\lambda \sqrt{d\sigma(\lambda)} \in \int^\oplus \mathcal{H}(\Sigma_\lambda)\sqrt{d\sigma(\lambda)}.$$

Step 4. Now let us compose these three maps:

$$U := U_3 \circ U_2 \circ U_1.$$

It turns out that U is independent of the set D_E and of the c.o.n.s. in $\mathrm{L}^2_{\nu_E}(D_E)$. As the latter is easily checked, we now verify the independence from D_E.

Take another D_F with the same properties as D_E. For each $\rho \in \mathfrak{S}_\infty$, we take a c.o.n.s. $\{h_n^\rho(x)\}_{n<N_\rho}$ in $\mathrm{L}^2_{\nu_E}(D_E \cap D_F\rho)$, where N_ρ is an integer or ∞. Of course $\{h_n^\rho(x\rho)\}_{n<N_\rho}$ is a c.o.n.s. in $\mathrm{L}^2_{\nu_E}(D_E\rho^{-1} \cap D_F)$, and $\{h_n^\rho(x\rho)\}_{\rho,n<N_\rho}$ is a c.o.n.s.

in $\mathrm{L}^2_{\nu_E}(D_F)$. For any $\sigma \in \mathfrak{S}_\infty$, set

$$a_{\rho,n}(\sigma) := \int_{D_E \cap D_F\rho} f(x\sigma)\overline{h_n^\rho(x)}\,\nu_E(dx),$$
$$b_{\rho,n}(\sigma) := \int_{D_F \cap D_E\rho^{-1}} f(x\sigma)\overline{h_n^\rho(x\rho)}\,\nu_E(dx).$$

Then it follows that

$$b_{\rho,n}(\sigma) = a_{\rho,n}(\rho^{-1}\sigma) \quad \text{for all } \sigma, \rho \in \mathfrak{S}_\infty,$$

which leads to

$$\Pi_\lambda(\rho)\underline{a_{\rho,n,\lambda}} = \underline{b_{\rho,n,\lambda}} \quad \text{for a.e. } \lambda.$$

For any $f \in \mathrm{L}^2_{\nu_E}(M^\infty)$, denote the corresponding element in the cases of D_E and D_F by

$$\int_- \underline{F}_{\lambda,E}\sqrt{d\sigma(\lambda)} \quad \text{and} \quad \int_- \underline{F}_{\lambda,F}\sqrt{d\sigma(\lambda)},$$

respectively. Then, for any $x \in D_E$ and $\sigma \in \mathfrak{S}_\infty$,

$$\underline{F}_{\lambda,E}(x\sigma) = \sum_{\rho,n<N_\rho} h_n^\rho(x)\Pi_\lambda^{-1}(\sigma)(\underline{a_{\rho,n,\lambda}}),$$

and for any $y \in D_F$ and $\tau \in \mathfrak{S}_\infty$,

$$\underline{F}_{\lambda,F}(y\tau) = \sum_{\rho,n<N_\rho} h_n^\rho(y\rho)\Pi_\lambda^{-1}(\tau)(\underline{b_{\rho,n,\lambda}}).$$

If $x\sigma = y\tau$, then $x = y\tau\sigma^{-1} \in D_F\tau\sigma^{-1}$, and $h_n^\rho(x) = 0$ except for $\rho = \tau\sigma^{-1}$. Hence,

$$\underline{F}_{\lambda,E}(x\sigma) = \sum_{n<N_{\tau\sigma^{-1}}} h_n^{\tau\sigma^{-1}}(x)\Pi_\lambda^{-1}(\sigma)(\underline{a_{\tau\sigma^{-1},n,\lambda}}).$$

On the other hand, $y = x\sigma\tau^{-1} \in D_E\sigma\tau^{-1}$, and $h_n^\rho(y\rho) = 0$ except for $\rho = \tau\sigma^{-1}$. Thus,

$$\begin{aligned}\underline{F}_{\lambda,F}(y\tau) &= \sum_{n<N_{\tau\sigma^{-1}}} h_n^{\tau\sigma^{-1}}(y\tau\sigma^{-1})\Pi_\lambda^{-1}(\tau)(\underline{b_{\tau\sigma^{-1},n,\lambda}}) \\ &= \sum_{n<N_{\tau\sigma^{-1}}} h_n^{\tau\sigma^{-1}}(x)\Pi_\lambda^{-1}(\tau)\Pi_\lambda(\tau\sigma^{-1})(\underline{a_{\tau\sigma^{-1},n,\lambda}}) = \underline{F}_{\lambda,E}(x\sigma).\end{aligned}$$

This demonstrates the independence.

Step 5. Finally, we will observe that the map U preserves every $T(g)$, $g \in \mathrm{Diff}_0(M)$. Take any $f \in \mathrm{L}^2_{\nu_E}(M^\infty)$ and let

$$U(T(g)f) := \int_- \underline{K}_\lambda\sqrt{d\sigma(\lambda)}.$$

Then, take gD_E as D_F and take a c.o.n.s. $\{T(g)h_n\}_n$ of $\mathrm{L}^2_{\nu_E}(gD_E)$, where $\{h_n\}_n$ is a c.o.n.s. of $\mathrm{L}^2_{\nu_E}(D_E)$. Then, for all $\sigma \in \mathfrak{S}_\infty$,

$$\int_{gD_E} (T(g)f)(x\sigma)\overline{(T(g)h_n)(x)}\,\nu_E(dx) = \int_{D_E} f(x\sigma)\overline{h_n(x)}\,\nu_E(dx) =: a_n(\sigma).$$

Thus, for any $y \in gD_E$ and $\sigma \in \mathfrak{S}_\infty$,

$$\begin{aligned}\underline{K}_\lambda(y\sigma) &= \sum_n (T(g)h_n)(y)\Pi_\lambda^{-1}(\sigma)(\underline{a_{n,\lambda}})\\ &= \sqrt{\frac{dg\nu_E}{d\nu_E}}(y)\sum_n h_n(g^{-1}y)\Pi_\lambda^{-1}(\sigma)(\underline{a_{n,\lambda}})\\ &= \sqrt{\frac{dg\nu_E}{d\nu_E}}(y\sigma)\sum_n h_n(g^{-1}y)\Pi_\lambda^{-1}(\sigma)(\underline{a_{n,\lambda}})\\ &= \sqrt{\frac{dg\nu_E}{d\nu_E}}(y\sigma)\underline{F}_\lambda(g^{-1}y\sigma) = (T(g)\underline{F}_\lambda)(y\sigma).\end{aligned}$$

This shows that

$$U(T(g)f) = \int_- T(g)\underline{F}_\lambda\sqrt{d\sigma(\lambda)}.$$

Since $(T(g), \mathcal{H}(\Sigma_\lambda))$, $g \in \mathrm{Diff}_0(M)$, is irreducible, an irreducible decomposition of the natural representation $(T(g), \mathrm{L}^2_{\nu_E}(M^\infty))$, $g \in \mathrm{Diff}_0(M)$, is obtained. □

Appendix

§A.1. Irreducibility

We begin this section with the restatement and proof of Theorem 2.3.

Theorem 2.3 (irreducibility). *Let E be a disjoint μ-unital sequence, and (Π, H) an irreducible unitary representation of $\mathfrak{S}_\infty$. Form a triplet $\Sigma = (\mu, E, \Pi)$, as before. If $\dim(M) \geq 3$ and M satisfies condition* (mcc)*, then the unitary representation $(T, \mathcal{H}(\Sigma))$ is irreducible.*

Proof. The proof consists of several steps. First we need the following lemma, which was already used in Section 3, and whose proof runs parallel to arguments in the finite-dimensional case.

Lemma A. *Let $G = \{G_n\}_n$ be a disjoint μ-unital sequence, and assume that each G_n is connected and open. Then the natural representation T of the restricted product group $\prod_n^* \mathrm{Diff}_0(G_n)$ on $\mathrm{L}^2_{\nu_G}(G)$, defined below, is irreducible:*

$$T(g) : f(x) \mapsto \sqrt{\frac{dg\nu_G}{d\nu_G}}(x)f(g^{-1}x).$$

Next using the same reasonings as above, assume that M satisfies condition (cc) and $E = \{E_n\}_n$ has the properties in Lemma 2.1. Put

$$E_{\infty,n} := \Big(\overline{\sum_{k=n+1}^{\infty} E_k}\Big)^c \quad \text{(a connected open set in } M\text{)},$$

$$\tilde{E}_n := \bigcup_{\sigma\in\mathfrak{S}_\infty} \Big((E_{\infty,n})^n \times \prod_{n+1}^{\infty} E_k\Big)\sigma \quad \text{(a symmetric set in } M^\infty\text{)},$$

$$P_n : f \in \mathcal{H}(\Sigma) \mapsto \chi_{\tilde{E}_n} \cdot f \in \mathcal{H}(\Sigma) \quad \text{(an increasing projection tending to id)}.$$

In addition, take any non-empty disjoint connected open sets $G_1, \ldots, G_n$ in $E_{\infty,n}$ such that for all i, $0 < \mu(G_i) < \infty$, form a unital sequence

$$GE := \{G_1, \ldots, G_n, E_{n+1}, \ldots, E_k, \ldots\}$$

and define a map Q^{Π}_{GE} by

$$Q^{\Pi}_{GE} : f(x) \in \mathrm{L}^2_{\nu_E}\Big(\prod_{k=1}^{n} G_k \times \prod_{k=n+1}^{\infty} E_k, H\Big) \mapsto \sum_{\sigma\in\mathfrak{S}_\infty} \Pi(\sigma) f(x\sigma) \in \mathcal{H}(\Sigma).$$

Now let A be an intertwining operator of $(T, \mathcal{H}(\Sigma))$. Then, after defining maps similar to Q^{Π}_{GE}, for example

$$f(x) \in \mathrm{L}^2_{\nu_E}\Big(\prod_{k=1}^{n} G_{i_k} \times \prod_{k=n+1}^{\infty} E_k, H\Big) \mapsto \sum_{\sigma\in\mathfrak{S}_\infty} \Pi(\sigma) f(x\sigma) \in \mathcal{H}(\Sigma),$$

where $i_k = 1, \ldots, n$ or $= \infty$ with $G_\infty := E_{\infty,n} \setminus \bigcup_{i=1}^{n} G_i$, and after some additional arguments, we get

$$\mathrm{Image}(P_n A Q^{\Pi}_{GE}) \subset \mathrm{Image}(Q^{\Pi}_{GE}),$$

in view of Lemma A. It follows that

$$P_n A Q^{\Pi}_{GE} = Q^{\Pi}_{GE}(\mathrm{id}_{\mathrm{L}^2_{\nu_{GE}}} \otimes A_{n,G}),$$

where $A_{n,G}$ is a bounded operator on H, due to the same reason and the irreducibility assumption on (Π, H).

As $A_{n,G}$ does not depend on the choice of $(G_1, \ldots, G_n)$ in view of the connectedness of $E_{\infty,n}$, we may simply write A_n instead of $A_{n,G}$.

Now after some calculations using standard arguments of representation theory, we have

$$\forall\sigma \in \mathfrak{S}_n, \quad \Pi(\sigma)A_n = A_n\Pi(\sigma), \quad \text{and} \quad \forall n, \quad A_n = A_{n+1} \ (=: A_\infty).$$

Thus, we see that there exists $c \in \mathbb{C}$ such that $A_\infty = c\,\mathrm{Id}$, because of the irreducibility of (Π, H).

In other words,

$$P_n A Q^{\Pi}_{GE} = c Q^{\Pi}_{GE} \quad \text{and} \quad P_n A P_n = c P_n, \tag{5.1}$$

where the first equality follows from the fact that the family of subspaces $\{\mathrm{Image}(Q^{\Pi}_{GE})\}_G$, $G := (G_1, \ldots, G_n)$, generates the space $P_n(\mathcal{H}(\Sigma))$, whenever G runs through all possible pairs of sets in $(E_{\infty,n})^n$.

Finally, letting $n \to \infty$ in (5.1), we conclude that A is a scalar operator $c\,\mathrm{Id}$. This gives also another proof of Theorem 2.3, though we have omitted the details. □

§A.2. Equivalence

The rest of this section is devoted to a study of the mutual equivalence of $(T, \mathcal{H}(\Sigma))$, when E runs through μ-unital sets.

Theorem A. *Assume that* $\dim(M) \geq 3$ *and let* M *satisfy condition* (mcc). *Given* $\Sigma_1 = (\mu, E, \Pi_1)$ *and* $\Sigma_2 = (\mu, F, \Pi_2)$, $(T, \mathcal{H}(\Sigma_1))$ *and* $(T, \mathcal{H}(\Sigma_2))$ *are unitarily equivalent if and only if*

(1) *there exists a permutation* a *on* $\mathbb{N}$ (*maybe infinite*) *such that* $E \sim F a^{-1}$,

(2) Π_1 *is equivalent to* ${}^a\Pi_2$ *defined by*

$${}^a\Pi_2(\sigma) := \Pi_2(a^{-1}\sigma a) \quad \text{for } \sigma \in \mathfrak{S}_\infty.$$

Proof. The sufficiency is obvious. The difficulty lies in showing the necessity, in particular, that there exists an infinite permutation σ on $\mathbb{N}$ such that $\sum_{n=1}^{\infty} \mu(E_n \ominus F_{\sigma(n)}) < \infty$ (Lemma G). To do this, we first prepare a useful lemma (Lemma B), which has many applications as an ordinary check of the irreducibility of natural representations of the normal type. Applying the lemma, we find that E_n overlaps widely with a unique $F_{\sigma(n)}$ for sufficiently large n. That is, $\lim_{n\to\infty} \mu(E_n \ominus F_{\sigma(n)}) = 0$ (Lemma C). To complete the proof, we need to analyze M further, and this requires another lemma (Lemma D) that states the possibility of incompressive transportation of mass from one part to another of an open, connected set in M. The rest of the proof discusses how to use this transportation lemma, and requires a lengthy technical argument. Finally, we claim that σ is actually a permutation of $\mathbb{N}$. This is an outline of our original proof. Note that we may assume that M satisfies (cc) as before.

Lemma B. *Let* M *satisfy* (cc), *suppose that* $(T, \mathcal{H}(\Sigma_1))$ *and* $(T, \mathcal{H}(\Sigma_2))$ *are unitarily equivalent and let* A *be an intertwining unitary operator,* $A : \mathcal{H}(\Sigma_1) \to \mathcal{H}(\Sigma_2)$. *Given a Borel set* $B \in \mathfrak{B}(M)$, *we introduce a projection* P_B *on* $\mathcal{H}(\Sigma_i)$

$(i = 1, 2)$ *by*

$$(P_B f)(x) := \prod_{n=1}^{\infty} \chi_B(x_n) f(x).$$

Then

$$AP_B = P_B A.$$

Proof. Without loss of generality we may, of course, assume that $E = \{E_n\}_n$ and $F = \{F_n\}_n$ have the properties in Lemma 2.1.

Step 0 (Preparation). Suppose that D a given relatively compact, open set. Given η_1, take a compact subset K of D such that $\mu(D \setminus K) < \eta_1$ and cover K with a finite collection $\{W_t\}_{t=1}^{T}$ of relatively compact, open sets diffeomorphic to disks in $\mathbb{R}^d$:

$$K \subset \bigcup_{t=1}^{T} W_t \subseteq D.$$

Without loss of generality, we may assume that the image of $\mu|W_t$ under the coordinate map $\phi_t : W_t \to \mathbb{R}^d$ is the restriction of the Lebesgue measure to $\phi_t(W_t)$ and that $\mu(\overline{W}_t \setminus W_t) = 0$. Put

$$V_1 := W_1, \quad V_t := W_t \setminus (\overline{W}_1 \cup \cdots \cup \overline{W}_{t-1}) \quad (t = 2, \ldots, T).$$

Then V_t $(t = 1, \ldots, T)$ are mutually disjoint, open sets,

$$\mu\Big(\bigcup_{t=1}^{T} W_t \setminus \bigcup_{t=1}^{T} V_t\Big) = 0, \quad \text{and hence} \quad \mu\Big(K \setminus \bigcup_{t=1}^{T} V_t\Big) = 0.$$

Given $\eta_2 > 0$, take an open set U_t such that

$$\overline{U}_t \subset V_t \quad \text{and} \quad \mu(V_t \setminus U_t) < \eta_2 \quad (t = 1, \ldots, T).$$

Moreover, for $0 < a < 1$ and each t, take $\tilde{g}_t^{a,\eta} \in \mathrm{Diff}_0(\phi_t(W_t))$ such that

$$(\tilde{g}_t^{a,\eta})^{-1}(x) = ax \quad \text{on } \phi_t(\overline{U}_t).$$

Finally, put

$$g_t^{a,\eta} := \phi_t^{-1} \circ \tilde{g}_t^{a,\eta} \circ \phi_t \quad \text{and} \quad g^{a,\eta} := \prod_{t=1}^{T} g_t^{a,\eta}.$$

Then $g^{a,\eta} \in \mathrm{Diff}_0(D)$ and

$$\int_D \sqrt{\frac{dg^{a,\eta}\mu}{d\mu}(P)}\, \mu(dP) \le a^{d/2} \mu\Big(\bigcup_{t=1}^{T} U_t\Big) + \sqrt{\mu(D)} \sqrt{\mu\Big(D \setminus \bigcup_{t=1}^{T} U_t\Big)}.$$

Hence, letting first $a \to +0$ and then $\eta_1, \eta_2 \to +0$, we get

$$\lim_{a,\eta \to +0} \int_D \sqrt{\frac{dg^{a,\eta}\mu}{d\mu}}(P)\, \mu(dP) = 0. \tag{5.2}$$

Step 1. In this step, we will prove the assertion for the case when B is the complement of a relatively compact, open set D.

To this end, we need to check that

$$\langle P_{D^c} A\phi, A\phi\rangle_{\mathcal{H}(\Sigma_2)} = \langle P_{D^c}\phi, \phi\rangle_{\mathcal{H}(\Sigma_1)} \quad \text{for all } \phi \in \mathcal{H}(\Sigma_1), \tag{5.3}$$

and this is ensured if we show that

$$\langle T(g^{a,\eta})\phi, \phi\rangle_{\mathcal{H}(\Sigma_i)} \to \langle P_{D^c}\phi, \phi\rangle_{\mathcal{H}(\Sigma_i)} \tag{5.4}$$

as $a, \eta \to +0$. Since the same proof works for $i = 1, 2$, we have only to check it for $i = 1$. Moreover, as can be easily seen, we may assume that $\|\phi\|_{H_1}$ is bounded ($\|\phi\|_\infty < \infty$), as we prove (5.3).

Let p_k denote the natural kth projection from M^∞ to M, and $M_E^{(n)} := M^n \times \prod_{k=n+1}^\infty E_k$, and set

$$\begin{aligned} L_k^{a,\eta} &:= \int_{D_E \cap p_k^{-1}(D)} \sqrt{\frac{dg^{a,\eta}\nu_E}{d\nu_E}}(x) |\langle \phi((g^{a,\eta})^{-1}(x)), \phi(x)\rangle_{H_1}|\, \nu_E(dx) \\ &= L_{k,n}^{a,\eta,1} + L_{k,n}^{a,\eta,2}, \end{aligned}$$

where

$$L_{k,n}^{a,\eta,1} := \int_{D_E \cap p_k^{-1}(D) \cap M_E^{(n)}} (\dots), \qquad L_{k,n}^{a,\eta,2} := \int_{D_E \cap p_k^{-1}(D) \cap (M_E^{(n)})^c} (\dots).$$

It can be easily checked that

$$L_{k,n}^{a,\eta,2} \le \|\phi\|_{\mathcal{H}(\Sigma_1)} \left\{ \int_{D_E \cap (M_E^{(n)})^c} \|\phi(x)\|_{H_1}^2\, \nu_E(dx) \right\}^{1/2}, \tag{5.5}$$

hence it converges to 0 uniformly in a, η as $n \to \infty$.

Take a sequence $\{K_s\}_{s\in\mathbb{N}}$ of compact sets such that $K_s \uparrow M$ $(s \to \infty)$, and put

$$M_E^{(n,s)} := (K_s)^n \times \prod_{k=n+1}^\infty E_k,$$

and

$$L_{k,n,s}^{a,\eta,1} := \int_{D_E \cap p_k^{-1}(D) \cap M_E^{(n,s)}} \sqrt{\frac{dg^{a,\eta}\nu_E}{d\nu_E}}(x) |\langle \phi((g^{a,\eta})^{-1}(x)), \phi(x)\rangle_{H_1}|\, \nu_E(dx).$$

Then we get

$$(5.6)\qquad |L^{a,\eta,1}_{k,n} - L^{a,\eta,1}_{k,n,s}| \leq \|\phi\|_{\mathcal{H}(\Sigma_1)} \left\{ \int_{D_E \cap (M_E^{(n)} \setminus M_E^{(n,s)})} \|\phi(x)\|^2_{H_1} \, \nu_E(dx) \right\}^{1/2},$$

which converges to 0 uniformly in (a, η) for each fixed n as $s \to \infty$. Moreover, after some calculations, we get

$$\begin{aligned} L^{a,\eta,1}_{k,n,s} &\leq \|\phi\|^2_\infty \int_{M_E^{(n,s)} \cap p_k^{-1}(D)} \sqrt{\frac{dg^{a,\eta}\nu_E}{d\nu_E}}(x) \, \nu_E(dx) \\ &\leq \|\phi\|^2_\infty (\mu(K_s))^{n-1} \prod_{k=n+1}^{\infty} \mu(E_k) \int_D \sqrt{\frac{dg^{a,\eta}\mu}{d\mu}}(P) \, \mu(dP), \end{aligned}$$

provided that $k < n$ and s and n are so large that

$$D \subset K_s \quad \text{and} \quad E_k \cap D = \emptyset \qquad \text{for all } k \geq n+1.$$

It follows from (5.2), (5.5) and (5.6) that

$$(5.7)\qquad \lim_{a,\eta \to +0} L^{a,\eta}_k = 0 \qquad \text{for all } k \in \mathbb{N}.$$

Obviously, we have

$$\begin{aligned} (5.8)\qquad & \langle T(g^{a,\eta})\phi, \phi \rangle_{\mathcal{H}(\Sigma_1)} \\ &= \int_{D_E \cap \bigcup_k p_k^{-1}(D)} \sqrt{\frac{dg^{a,\eta}\nu_E}{d\nu_E}}(x) \langle \phi((g^{a,\eta})^{-1}(x)), \phi(x) \rangle_{H_1} \, \nu_E(dx) \\ &\quad + \int_{D_E \cap \bigcap_k p_k^{-1}(D^c)} \|\phi(x)\|^2_{H_1} \, \nu_E(dx), \end{aligned}$$

and the second term on the right-hand side is equal to $\langle P_{D^c}\phi, \phi \rangle_{\mathcal{H}(\Sigma_1)}$.

We take n so large that $D \cap E_k = \emptyset$ for all $k \geq n+1$ and split the first term on the right-hand side in (5.8) as follows:

$$\int_{D_E \cap \bigcup_k p_k^{-1}(D) \cap M_E^{(n)}} (\dots) + \int_{D_E \cap \bigcup_k p_k^{-1}(D) \cap (M_E^{(n)})^c} (\dots).$$

The absolute value of the first integral is smaller than

$$\sum_{k=1}^{n} \int_{D_E \cap p_k^{-1}(D)} |(\dots)|,$$

and it converges to 0 as $a, \eta \to +0$ for fixed n.

On the other hand, the absolute value of the second integral is smaller than

$$\|\phi\|_{\mathcal{H}(\Sigma_1)}\left\{\int_{D_E\cap(M_E^{(n)})^c}\|\phi(x)\|_{H_1}^2\,\nu_E(dx)\right\}^{1/2},$$

which also converges to 0 as $n\to\infty$. It follows that the proof of (5.4) is now complete.

Step 2. Suppose that B is a closed set in M. Take a sequence $\{D_n\}_{n\in\mathbb{N}}$ of relatively compact, open sets such that $D_n\uparrow M$ and put $B_n^c:=D_n\cap B^c$. Then $B_n\downarrow B$, so that $P_{B_n}\downarrow P_B$. It follows from Step 1 that

$$AP_{B_n}=P_{B_n}A,\quad\text{and hence}\quad AP_B=P_BA.$$

Step 3. In particular, when $\mu(\overline{B}\setminus B)=0$, the above conclusion follows, because $P_{\overline{B}}=P_B$.

Step 4. Let us consider a general Borel set B. Put $B_n:=B\cup\bigcup_{k=n+1}^{\infty}(E_k\cup F_k)$. Then $B_n\downarrow B$, and hence

$$P_{B_n}\downarrow P_B. \tag{5.9}$$

Further, take an increasing sequence $\{T_m\}_{m\in\mathbb{N}}$ of closed subsets of B such that $\mu(B\setminus\bigcup_{m=1}^{\infty}T_m)=0$, and put $B_{n,m}:=T_m\cup\bigcup_{k=n+1}^{\infty}(E_k\cup F_k)$. Since $\mu(\overline{B_{n,m}}\setminus B_{n,m})=0$, due to Lemma 2.1, we get

$$AP_{B_{n,m}}=P_{B_{n,m}}A \tag{5.10}$$

by the step above. Now, put

$$M_{E\cup F}^{(l)}:=M^l\times\prod_{l+1}^{\infty}(E_k\cup F_k)\quad\text{and}\quad B_n':=\bigcup_{m=1}^{\infty}T_m\cup\bigcup_{k=n+1}^{\infty}(E_k\cup F_k).$$

It can be easily seen that, for $n<l$,

$$(B_{n,m}\times B_{n,m}\times\cdots)\cap M_{E\cup F}^{(l)}\uparrow(B_n'\times B_n'\times\cdots)\cap M_{E\cup F}^{(l)}\quad(m\to\infty).$$

Thus for fixed n,

$$P_{B_{n,m}}\uparrow P_{B_n'}=P_{B_n}\qquad\text{on }\mathcal{H}(\Sigma_i)\ (i=1,2).$$

Hence, the conclusion follows from (5.9) and (5.10). □

Lemma C. *Under the same assumptions and the same notation as in Theorem A and assumption* (cc) *on* M*, the following holds: for sufficiently large* $n\in\mathbb{N}$*, there exists* $\sigma(n)\in\mathbb{N}$ *such that*

$$\lim_{n\to\infty}\mu(E_n\ominus F_{\sigma(n)})=0.$$

Proof. As before, we may assume that $E = \{E_n\}_n$ and $F = \{F_n\}_n$ have the properties as in Lemma 2.1. Take $h \in H_1$ with $\|h\|_{H_1} = 1$ and put

$$g(x) := AQ_E^{\Pi_1}\Big(\prod_{n=1}^{\infty} \chi_{E_n} \otimes h\Big)(x).$$

Applying the above lemma to $B := E_k^c$ for each k, we have

$$P_{E_k^c} g = 0.$$

Next, approximating g with a sum of $Q_F^{\Pi_2}$-images of tame functions

$$Q_F^{\Pi_2}\Big(\rho(x_1, \dots, x_l) \prod_{n=l+1}^{\infty} \chi_{F_n}(x_n) \cdot h'\Big),$$

where ρ is a square summable function and $h' \in H_2$, we find that

$$\forall \epsilon > 0,\ \exists N_\epsilon \in \mathbb{N},\ \forall k \geq N_\epsilon, \qquad \prod_{n=1}^{\infty} \mu(E_k^c \cap F_n) < \epsilon.$$

It follows that

$$\exists \sigma(k) \in \mathbb{N}, \qquad \mu(E_k^c \cap F_{\sigma(k)}) < \epsilon K_U,$$

with a universal constant K_U. By proceeding in a similar fashion, but changing E to F, we can show that $\mu(E_k \cap F_{\sigma(k)}^c)$ is equally small if k is large. □

For the proof of Theorem A, we need more analysis on M. The following lemma is useful; it shows the possibility of incompressive transportation of mass from one part to another with slim tubes in connected open sets in M.

Lemma D. *Assume that* $d := \dim(M) \geq 2$. *Let* F *be a connected, open subset of* M *and* U_i $(i = 1, 2)$ *be open subsets of* F *such that* $U_1 \cap U_2 = \emptyset$ *and* $\mu(U_1) < \mu(U_2) < \infty$. *Then, given* $\epsilon > 0$, *we have a* μ*-preserving diffeomorphism* $g_\epsilon \in \mathrm{Diff}_0(F)$ *and a Borel subset* $B_\epsilon \subset U_1$ *such that*

$$\mu(U_1 \setminus B_\epsilon) < \epsilon \quad \textit{and} \quad g_\epsilon(B_\epsilon) \subset U_2.$$

Proof. In this proof, a λ-*neighbourhood* $V(P)$ of $P \in M$ is an open set including x with the following properties:

(1) $V(P)$ is diffeomorphic to a disk in $\mathbb{R}^d$ under a coordinate map ϕ,

(2) the image measure of $\mu|V(P)$ by ϕ is the restriction of the Lebesgue measure to $\phi(V(P))$.

Now, take a λ-neighbourhood $V(P)$ $(\subset U_1)$ for each $P \in U_1$ and cover U_1 by a countable collection of $\{V(P_n)\}_n$. Put

$$W(P_1) := V(P_1) \quad \text{and} \quad W(P_n) := V(P_n) \setminus (\overline{V(P_1)} \cup \cdots \cup \overline{V(P_{n-1})}).$$

Then $V(P_n) \cap W(P_m) = \emptyset$ $(n \neq m)$ and

$$\mu\Big(U_1 \setminus \sum_{n=1}^{N} W(P_n)\Big) < \frac{1}{3}\epsilon \quad \text{for sufficiently large } N.$$

Since $W(P_n)$ can be approximated as closely as we wish by finite unions of λ-neighbourhoods, in place of $W(P_n)$ we can take an open subset $\hat{W}(P_n)$ such that $\overline{\hat{W}(P_n)} \subset W(P_n)$, $\hat{W}(P_n)$ is a finite union of inverse images of rectangles in $\mathbb{R}^d$ and

$$\mu\Big(\sum_{n=1}^{N} W(P_n) \setminus \sum_{n=1}^{N} \hat{W}(P_n)\Big) < \frac{\epsilon}{3}.$$

In exactly the same manner as for U_2, we have $V(Q_m)$, $W(Q_m)$ and $\hat{W}(Q_m)$ $(m = 1, \ldots, M)$ such that

$$\hat{W}(Q_m) \subset \overline{\hat{W}(Q_m)} \subset W(Q_m) \subset V(Q_m) \subset U_2,$$
$$\mu\Big(U_2 \setminus \sum_{m=1}^{M} \hat{W}(Q_m)\Big) < \mu(U_2) - \mu(U_1).$$

Therefore,

$$\mu\Big(\sum_{n=1}^{N} \hat{W}(P_n)\Big) < \mu(U_1) < \mu\Big(\sum_{m=1}^{M} \hat{W}(Q_m)\Big).$$

Now, we connect $V(P_i)$ and $V(P_{i+1})$ $(i = 1, \ldots, N-1)$ by a slim open tube included in F which intersects neither $\hat{W}(P_n)$ $(n = 1, \ldots, N)$ nor $\hat{W}(Q_m)$ $(m = 1, \ldots, M)$. Similarly we proceed for $V(Q_j)$ and $V(Q_{j+1})$ $(j = 1, \ldots M)$, and we finally connect $V(P_1)$ and $V(Q_1)$ by a slim open tube S included in F which intersects neither $\hat{W}(P_n)$ $(n = 1, \ldots, N)$ nor $\hat{W}(Q_m)$ $(m = 1, \ldots, M)$.

Now, we transport each divided small mass of $\hat{W}(P_1)$ to $\hat{W}(Q_M), \hat{W}(Q_{M-1}), \ldots, \hat{W}(Q_1)$ through S and through the slim tubes for U_2 (a larger subscript of Q has priority in the order of distribution). In fact, this is possible by the following lemma.

Lemma E. *There exists a Lebesgue measure preserving diffeomorphism with compact support that realizes local displacement in* $\mathbb{R}^d$ $(d \geq 2)$.

Proof. Put $n := d - 2$ and take compact intervals

$$[\alpha', \beta'] \subset [\alpha, \beta], \qquad [\gamma', \delta'] \subset [\gamma, \delta], \qquad [\alpha'_i, \beta'_i] \subset [\alpha_i, \beta_i] \quad (i = 1, \ldots, n).$$

We take a vector field $v \equiv (v_1, v_2, w_1, \dots, w_n)$ on $\mathbb{R}^d$ defined by

$$\begin{aligned} v_1(x, y, z_1, \dots, z_n) &:= f_1(x) f_2'(y) g_1(z_1) \cdots g_n(z_n), \\ v_2(x, y, z_1, \dots, z_n) &:= -f_1'(x) f_2(y) g_1(z_1) \cdots g_n(z_n), \\ w_i &\equiv 0 \quad (i = 1, \dots, n), \end{aligned}$$

where f_1, f_2 and g_i $(i = 1, \dots, n)$ are C^∞ functions on $\mathbb{R}^1$ with compact support such that

$$f_1(x) = \begin{cases} 1 & \text{on } [\alpha', \beta'], \\ 0 & \text{on } [\alpha, \beta]^c, \end{cases} \qquad f_2(y) = \begin{cases} y & \text{on } [\gamma', \delta'], \\ 0 & \text{on } [\gamma, \delta]^c, \end{cases} \qquad g_i(x) = \begin{cases} 1 & \text{on } [\alpha_i', \beta_i'] \\ 0, & \text{on } [\alpha_i, \beta_i]^c. \end{cases}$$

It is clear that

$$\operatorname{supp} v \subseteq [\alpha, \beta] \times [\gamma, \delta] \times \prod_{i=1}^n [\alpha_i, \beta_i], \qquad \operatorname{div}(v) = 0,$$

$$v = (1, 0, \dots, 0) \qquad \text{on } T := [\alpha', \beta'] \times [\gamma', \delta'] \times \prod_{i=1}^n [\alpha_i', \beta_i'].$$

Therefore,

$$\exp(tv)(x_0) = x_0 + t(1, 0, \dots, 0) \qquad \text{for all } x_0 \in T,$$

provided that $x_0 + t(1, 0, \dots, 0) \in T$. This is the desired diffeomorphism. □

The rest of the proof of Theorem A relies on the above lemma. We first give the following two lemmas; we omit their proofs, because they are quite technical and complicated (for details, see [18]).

Lemma F. *Under the same assumptions and notation of Lemma C, we have*

$$\sum \max_{k \neq n} \mu(E_k \cap F_{\sigma(n)}) < \infty,$$

where the summation is over all n except a finite number.

Lemma G. *Under the same assumptions and notation of Lemma C, we have*

$$\sum \mu(E_n \ominus F_{\sigma(n)}) < \infty,$$

where the summation is over all n except a finite number.

Now we can complete the proof of Theorem A through standard arguments of representation theory in order to see that σ actually extends to a permutation on $\mathbb{N}$ and to show the equivalence of the irreducible unitary representations (Π, H). □

In a similar fashion, we get an interesting version of Theorem A.

Theorem B. *Assume that* $\dim(M) \geq 3$, *and let* M *satisfy* (mcc). *Given* μ, E, F, H_1, H_2, *we have a nonzero intertwining operator* $A : (T, \mathrm{L}^2_{\nu_E}(M^\infty, H_1)) \to (T, \mathrm{L}^2_{\nu_F}(M^\infty, H_2))$ *if and only if there exists a permutation* a *on* $\mathbb{N}$ (*maybe infinite*) *such that* $E \sim Fa^{-1}$.

Acknowledgements

I wish to express my deep gratitude to Honorary Professor T. Hirai of Kyoto University for introducing me to this subject. In fact, the underlying concepts are due to him.

This research was partially supported by a Grant-in-Aid for Scientific Research (No. 16540162) from the Japan Society of the Promotion of Science.

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