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Decomposition of Solutions of the Cauchy Problem of a Quasi-Homogeneous Partial Differential Equation

by

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Abstract

We give a decomposition formula for the formal solution of the Cauchy problem for a quasi-homogeneous partial differential equation with constant coefficients in the two-dimensional complex plane. The decomposition formula, Theorem 1.1, is given in a form associated with the factorization of the relevant operator which is similar to the decomposition of the solution of an ordinary differential equation with constant coefficients.

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§1. Introduction

Let p and q be relatively prime positive integers, and $P \equiv P(\partial_t^p, \partial_x^q)$ be a quasi-homogeneous partial differential operator with constant coefficients in the two-dimensional complex plane $\mathbb{C}^2 = \mathbb{C}_t \times \mathbb{C}_x$, where ∂_t and ∂_x denote differentiation in the usual sense. We assume that the following factorization of P is given:

$$(1.1) \qquad P(\partial_t^p, \partial_x^q) = \prod_{j=1}^{\mu} P_j(\partial_t^p, \partial_x^q)^{\ell_j}, \quad P_j(\partial_t^p, \partial_x^q) = \partial_t^p - \alpha_j \partial_x^q,$$

where $\{\alpha_j\}$ are nonzero complex constants which are mutually different.

We put $L := \sum_{j=1}^{\mu} \ell_j$, which represents the total number of factors of P, and assume that $L \geq 2$.

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Let $U(t,x) = \sum_{k\ge 0} U_k(x)t^k$ be the formal solution of the following Cauchy problem:

$$(1.2)\qquad \begin{cases} P(\partial_t^p, \partial_x^q)U(t,x) = 0, \\ \partial_t^k U(0,x) = 0 \quad (0 \le k \le pL-2), \\ \partial_t^{pL-1}U(0,x) = \varphi(x), \end{cases}$$

where the Cauchy data $\varphi(x)$ is holomorphic in a neighbourhood of the origin. When $q > p$, the formal solution $U(t,x)$ diverges in general.

The purpose of this paper is to prove a decomposition formula for $U(t,x)$ in terms of the formal series

$$(1.3)\qquad u_m(t,x) = \sum_{j\ge 0} \alpha_m^j \varphi^{(qj)}(x) \frac{t^{pj+p-1}}{(pj+p-1)!} \qquad (1 \le m \le \mu),$$

which is the formal solution of the Cauchy problem for the factor P_m:

$$(1.4)\qquad \begin{cases} P_m u \equiv (\partial_t^p - \alpha_m \partial_x^q)\, u(t,x) = 0, \\ \partial_t^k u(0,x) = 0 \quad (0 \le k \le p-2), \\ \partial_t^{p-1}u(0,x) = \varphi(x). \end{cases}$$

Our main result is as follows.

Main Theorem 1.1. *Let $U(t,x)$ be the formal solution of the Cauchy problem (1.2). Then there are uniquely determined L constants $\{c_{mn}\,;\, 1 \le m \le \mu,\, 1 \le n \le \ell_m\}$, which satisfy the relation (2.13) below, such that the following decomposition formula holds:*

$$(1.5)\qquad U(t,x) = \sum_{m=1}^{\mu} \sum_{n=1}^{\ell_m} c_{mn} \partial_t^{(1-pL)+p(n-1)} \frac{[(1/p)\delta_t]_{n-1}}{(n-1)!} \partial_t^{p(1-n)+(p-1)} u_m(t,x),$$

where $\delta_t = t\partial_t$ denotes the Euler operator and $[(1/p)\delta_t]_{n-1}$ is given by

$$(1.6)\qquad \left[\frac{1}{p}\delta_t\right]_k := \begin{cases} \dfrac{1}{p}\delta_t\left(\dfrac{1}{p}\delta_t - 1\right)\cdots\left(\dfrac{1}{p}\delta_t - k + 1\right), & k \ge 1, \\ 1, & k = 0. \end{cases}$$

Here ∂_t^k denotes differentiation or integration if $k > 0$ or $k < 0$, respectively, and $\partial_t^{-1} = \int_0^t$.

Remark. The formal solutions in the above decomposition formula are divergent in general when $q > p$, and therefore the formula is only in the formal sense. But, if the divergent solution $U(t,x)$ is Borel summable in some direction in the t-plane, then each $u_m(t,x)$ is also Borel summable in the same direction, and the formula (1.5) holds between the Borel sums associated with the divergent solutions. (The details on Borel summability of divergent solutions of this kind can be found in the papers by M. Miyake, K. Ichinobe and others [2]–[7].) Moreover, by the study of K. Ichinobe [3, 4] on the integral kernel for the Borel sum, the formula (1.5) does hold between the integral kernels for the Borel sums.

The motivation for the study in this paper comes from the decomposition formula for solutions of ordinary differential equations with constant coefficients, which is explained by the following simple equation. Consider the solution $u(t)$ of

$$\left(\frac{d}{dt}-\alpha\right)\left(\frac{d}{dt}-\beta\right)u(t)=0, \quad u(0)=0,\ u'(0)=1. \tag{1.7}$$

First, consider the case $\alpha \neq \beta$. Then the decomposition

$$\mathrm{id}=\frac{1}{\alpha-\beta}\left\{\left(\frac{d}{dt}-\beta\right)-\left(\frac{d}{dt}-\alpha\right)\right\}$$

implies

$$u(t)=\frac{1}{\alpha-\beta}\{u_\alpha(t)-u_\beta(t)\}$$

with $u_\alpha(t)=(d/dt-\beta)u(t)$ and $u_\beta(t)=(d/dt-\alpha)u(t)$. Since $(d/dt-\alpha)u_\alpha(t)=0$ and $(d/dt-\beta)u_\beta(t)=0$, we get $u_\alpha(t)=C_1e^{\alpha t}$ and $u_\beta(t)=C_2e^{\beta t}$ as general solutions, respectively. Now the initial conditions $\{u(0)=0,\ u'(0)=1\}$ show that $C_1=C_2=1$, that is, $u_\alpha(t)=e^{\alpha t}$ and $u_\beta(t)=e^{\beta t}$, which satisfy $u_\alpha(0)=1$ and $u_\beta(0)=1$.

Next, in the case $\alpha=\beta$ in (1.7), by letting

$$\lim_{\beta\to\alpha}\frac{u_\alpha(t)-u_\beta(t)}{\alpha-\beta}=te^{\alpha t},$$

we get the solution of (1.7).

This observation on decomposition of solutions can be found in S. Mizohata's book [8].

Next, we illustrate the idea of decomposition by a simple example. Let α and β be different non-zero complex numbers, and consider the solution $U(t,x)$ of the Cauchy problem

$$\begin{cases}(\partial_t-\alpha\partial_x^2)(\partial_t-\beta\partial_x^2)U(t,x)=0,\\ U(0,x)=0, \quad \partial_tU(0,x)=\varphi(x).\end{cases} \tag{1.8}$$

We notice the following decomposition of ∂_t:

$$\partial_t=\frac{1}{\alpha-\beta}\{\alpha(\partial_t-\beta\partial_x^2)-\beta(\partial_t-\alpha\partial_x^2)\}.$$

This implies

$$\mathrm{id} = \frac{1}{\alpha-\beta}\{\alpha\partial_t^{-1}(\partial_t - \beta\partial_x^2) - \beta\partial_t^{-1}(\partial_t - \alpha\partial_x^2)\},$$

since $\partial_t^{-1}\circ\partial_t = \mathrm{id}$ for functions with $U(0,x) = 0$. Then we have the following decomposition of $U(t,x)$:

$$U(t,x) = \frac{\alpha}{\alpha-\beta}\partial_t^{-1}u_\alpha(t,x) + \frac{\beta}{\beta-\alpha}\partial_t^{-1}u_\beta(t,x), \tag{1.9}$$

where $u_\alpha(t,x) := (\partial_t - \beta\partial_x^2)U(t,x)$ is the formal solution of the Cauchy problem

$$(\partial_t - \alpha\partial_x^2)u_\alpha(t,x) = 0, \qquad u_\alpha(0,x)\ (= \partial_t U(0,x)) = \varphi(x),$$

and similarly for $u_\beta(t,x) := (\partial_t - \alpha\partial_x^2)U(t,x)$.

Example 1.2 (cf. Proposition 3.1). 1. The case $P = \prod_{j=1}^{\mu} P_j$, that is, $L = \mu$:

$$U(t,x) = \sum_{m=1}^{\mu} c_m\partial_t^{p-p\mu}u_m(t,x), \quad c_m = \frac{\alpha_m^{\mu-1}}{\prod_{1\le j\le\mu,\, j\ne m}(\alpha_m-\alpha_j)}. \tag{1.10}$$

2. The case $P = P_1^n$ (cf. Lemma 2.1):

$$U(t,x) = \partial_t^{1-p}\frac{[(1/p)\delta_t]_{n-1}}{(n-1)!}\partial_t^{p(1-n)+(p-1)}u_1(t,x). \tag{1.11}$$

3. The case $p = 1$ (cf. Lemma 2.1): Since $[\delta_t]_{n-1} = t^{n-1}\partial_t^{n-1}$, we have

$$U(t,x) = \sum_{m=1}^{\mu}\sum_{n=1}^{\ell_m} c_{mn}\partial_t^{n-L}\frac{t^{n-1}}{(n-1)!}u_m(t,x). \tag{1.12}$$

To end the introduction, we remark that the results of this paper are valid for formal solutions of operators in $\mathbb{C}_t\times\mathbb{C}_x^d$ of the form

$$P(\partial_t,\partial_x) = \prod_{j=1}^{\mu}\left(\partial_t^p - \alpha_j p(\partial_x)\right)^{\ell_j}, \tag{1.13}$$

with mutually different nonzero constants $\{\alpha_j\}$ and $p(\partial_x) = \sum_{|\alpha|\le q} p_\alpha\partial_x^\alpha$, where $x\in\mathbb{C}_x^d$. It is enough to replace ∂_x^q by $p(\partial_x)$ in the following proofs.

§2. Proof of Main Theorem

As a first step, we shall prove the decomposition formula in the case where $\mu = 1$.

Lemma 2.1. *Let $v^{[n]}(t,x)$ be the formal solution of the Cauchy problem*

$$\begin{cases} P_0^n v(t,x) \equiv (\partial_t^p - \alpha\partial_x^q)^n v(t,x) = 0, \\ \partial_t^k v(0,x) = 0 \quad (0 \le k \le pn-2), \quad \partial_t^{pn-1} v(0,x) = \varphi(x). \end{cases} \tag{2.1}$$

Then

$$v^{[n]}(t,x) = \sum_{j\ge 0} \alpha^j \frac{(j+1)_{n-1}}{(n-1)!} \varphi^{(qj)}(x) \frac{t^{pj+pn-1}}{(pj+pn-1)!} \tag{2.2}$$

$$= \partial_t^{1-p} \frac{[(1/p)\delta_t]_{n-1}}{(n-1)!} \partial_t^{p(1-n)+(p-1)} v^{[1]}(t,x), \tag{2.3}$$

where

$$(j+1)_{n-1} = \begin{cases} (j+1)(j+2)\cdots(j+n-1), & n \ge 2, \\ 1 & n = 1, \end{cases} \tag{2.4}$$

and $v^{[1]}(t,x)$ is the formal solution of (2.1) for $n=1$.

In particular, when $p=1$ the formula (2.3) reduces to

$$v^{[n]}(t,x) = \frac{t^{n-1}}{(n-1)!} v^{[1]}(t,x). \tag{2.5}$$

Proof. The formula (2.5) is an immediate consequence of (2.3) in view of $[\delta_t]_{n-1} = t^{n-1}\partial_t^{n-1}$. Hence we shall prove (2.2) and (2.3). We put

$$P_0(\partial_t^p, \partial_x^q)^n \equiv (\partial_t^p - \alpha\partial_x^q)^n = \partial_t^{pn} - \sum_{k=1}^n a_k \partial_t^{p(n-k)} \partial_x^{qk},$$

where ${}_nC_k \alpha^k = (-1)^{k-1} a_k$ $(1 \le k \le n)$. For the operator P_0^n, the following polynomial is called the *characteristic polynomial*:

$$P_0(\lambda, 1)^n \equiv (\lambda - \alpha)^n = \lambda^n - \sum_{k=1}^n a_k \lambda^{n-k}.$$

By a careful calculation, we see that

$$v^{[n]}(t,x) = \sum_{j=0}^{\infty} A(j) \varphi^{(qj)}(x) \frac{t^{pj+pn-1}}{(pj+pn-1)!},$$

where the coefficients $\{A(j)\}_{j=0}^{\infty}$ satisfy the recurrence formula

$$A(j+n) = \sum_{k=1}^n a_k A(j+n-k), \quad j = -n+1, -n+2, \dots, \tag{2.6}$$

with the initial conditions

$$A(0) = 1, \qquad A(j) = 0 \qquad (j < 0). \tag{2.7}$$

The formula (2.2) will be proved once we show

$$A(j) = \frac{(n)_j}{j!}\alpha^j = \frac{(j+1)_{n-1}}{(n-1)!}\alpha^j \qquad (j \geq 0), \tag{2.8}$$

where $(n)_j = n(n+1)\cdots(n+j-1) = \Gamma(n+j)/\Gamma(n)$.

For the proof, let $f(x)$ be the generating function of the sequence $\{A(j)\}$ defined by

$$f(x) := \sum_{j=0}^{\infty} A(j)x^j.$$

From the recurrence formula (2.6), we have

$$\sum_{j=0}^{\infty} A(j+n)x^{j+n} = \sum_{k=1}^{n}\Big\{a_k x^k \sum_{j=0}^{\infty} A(j+n-k)x^{j+n-k}\Big\}.$$

Hence,

$$f(x) - \sum_{j=0}^{n-1} A(j)x^j = \sum_{k=1}^{n}\Big\{a_k x^k \Big(f(x) - \sum_{j=0}^{n-k-1} A(j)x^j\Big)\Big\}, \tag{2.9}$$

which implies

$$f(x) = \frac{A(0) + \sum_{j=1}^{n-1} x^j\{A(j) - \sum_{k=1}^{j} a_k A(j-k)\}}{1 - \sum_{k=1}^{n} a_k x^k}. \tag{2.10}$$

By the recurrence formula (2.6) with the initial conditions (2.7) we have

$$A(j) - \sum_{k=1}^{j} a_k A(j-k) = 0, \qquad j = 1, \ldots, n-1.$$

Thus

$$f(x) = \frac{1}{1 - \sum_{k=1}^{n} a_k x^k} = \frac{1}{(1-\alpha x)^n}.$$

Taking $0 < \varepsilon < |\alpha|^{-1}$, we have

$$\begin{aligned} A(j) &= \frac{1}{2\pi i}\oint_{|x|=\varepsilon} \frac{f(x)}{x^{j+1}}\,dx = \lim_{x\to 0}\frac{1}{j!}f^{(j)}(x) \\ &= \frac{n(n+1)\cdots(n+j-1)}{j!}\alpha^j = \frac{(j+1)_{n-1}}{(n-1)!}\alpha^j. \end{aligned} \tag{2.11}$$

This proves (2.2).

For the proof of (2.3), we recall that

$$v^{[1]}(t,x)=\sum_{j=0}^{\infty}\alpha^j\varphi^{(qj)}(x)\,\frac{t^{pj+p-1}}{(pj+p-1)!}.$$

We notice the following relations:

$$\frac{t^{pj+p-1}}{(pj+p-1)!}\xrightarrow{\partial_t^{p(1-n)+(p-1)}}\frac{t^{pj+pn-p}}{(pj+pn-p)!}\xrightarrow{[(1/p)\delta_t]_{n-1}}(j+1)_{n-1}\frac{t^{pj+pn-p}}{(pj+pn-p)!}$$
$$\xrightarrow{\partial_t^{1-p}}(j+1)_{n-1}\frac{t^{pj+pn-1}}{(pj+pn-1)!}.$$

By applying these relations to (2.2), we obtain (2.3). □

Proof of Main Theorem 1.1. We recall that

$$P=\prod_{j=1}^{\mu}P_j^{\ell_j},\quad P_j=\partial_t^p-\alpha_j\partial_x^q. \tag{2.12}$$

First of all, we assume that we can choose L constants $\{c_{mn}\,;\,1\le m\le\mu,\,1\le n\le\ell_m\}$ so that the following operator equality holds:

$$\partial_t^{p(L-1)}=\sum_{m=1}^{\mu}\sum_{n=1}^{\ell_m}c_{mn}\partial_t^{p(n-1)}\prod_{j=1,\,j\neq m}^{\mu}P_j^{\ell_j}P_m^{\ell_m-n}. \tag{2.13}$$

Indeed, the linear equations for $\{c_{mn}\}$ are obtained by comparing the coefficients of $\partial_t^{p(L-j)}\partial_x^{qj}$ on both sides of the above equality. The details will be studied in the following sections.

Let $U(t,x)$ be the formal solution of the Cauchy problem (1.2). Then

$$\partial_t^{p(L-1)}U(t,x)=\sum_{m=1}^{\mu}\sum_{n=1}^{\ell_m}c_{mn}\partial_t^{p(n-1)}\prod_{j=1,\,j\neq m}^{\mu}P_j^{\ell_j}P_m^{\ell_m-n}U(t,x). \tag{2.14}$$

Now we put

$$U_m^{[n]}(t,x):=\prod_{j=1,\,j\neq m}^{\mu}P_j^{\ell_j}P_m^{\ell_m-n}U(t,x). \tag{2.15}$$

Then $U_m^{[n]}(t,x)$ is the formal solution of the Cauchy problem

$$\begin{cases}P_m^nU_m^{[n]}(t,x)=0,\\ \partial_t^kU_m^{[n]}(0,x)=0\quad(0\le k\le pn-2),\\ \partial_t^{pn-1}U_m^{[n]}(0,x)=\varphi(x).\end{cases} \tag{2.16}$$

Indeed, $P_m^n U_m^{[n]} = PU$ and $U(t,x)$ is the formal solution of (1.2). Hence by Lemma 2.1 and since $U_m^{[1]}(t,x) = u_m(t,x)$, which is given by (1.3), we have

$$\partial_t^{p(L-1)} U(t,x) = \sum_{m=1}^{\mu} \sum_{n=1}^{\ell_m} c_{mn} \partial_t^{p(n-1)} \partial_t^{1-p} \frac{[(1/p)\delta_t]_{n-1}}{(n-1)!} \partial_t^{p(1-n)+(p-1)} u_m(t,x).$$

By applying $\partial_t^{p(1-L)}$ to both sides, we obtain the desired formula (1.5). □

Hence, the proof of the main theorem is reduced to proving the formula (2.13).

§3. Proof of (2.13)

We prepare some notation. Let $\alpha = (\alpha_1, \dots, \alpha_\mu) \in (\mathbb{C} \setminus \{0\})^\mu$ and define

$$f[\alpha] := \prod_{j=1}^{\mu} \alpha_j^{\ell_j}, \qquad f_m^{[n]} \equiv f_m^{[n]}[\alpha] := f[\alpha]/\alpha_m^n \qquad (1 \le m \le \mu,\ 1 \le n \le \ell_m).$$

Moreover, write $\partial_\alpha := \sum_{j=1}^{\mu} \partial_{\alpha_j}$ and

$$\Delta_\alpha^k := \frac{\partial_\alpha^k}{k!} = \sum_{\substack{0 \le k_1, \dots, k_\mu \le k \\ k_1 + \cdots + k_\mu = k}} \frac{1}{k_1! \cdots k_\mu!} \partial_{\alpha_1}^{k_1} \cdots \partial_{\alpha_\mu}^{k_\mu}.$$

Then we have the following proposition.

Proposition 3.1. *The L constants $\{c_{mn}\}$ in (2.13) are determined as the solution of the system of linear equations*

$$\mathcal{A}\vec{c} = \vec{e}, \tag{3.1}$$

where $\mathcal{A}$ is an $L \times L$ matrix and $\vec{c}$, $\vec{e}$ are column L vectors, defined by

$$\mathcal{A} = \begin{pmatrix} \Delta_\alpha^{L-1} f_1^{[1]} & \Delta_\alpha^{L-2} f_1^{[2]} & \cdots & \Delta_\alpha^{L-\ell_1} f_1^{[\ell_1]} & \cdots & \Delta_\alpha^{L-1} f_\mu^{[1]} & \cdots & \Delta_\alpha^{L-\ell_\mu} f_\mu^{[\ell_\mu]} \\ \Delta_\alpha^{L-2} f_1^{[1]} & \Delta_\alpha^{L-3} f_1^{[2]} & & \vdots & \cdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & & \Delta_\alpha^0 f_1^{[\ell_1]} & \cdots & \vdots & & \Delta_\alpha^0 f_\mu^{[\ell_\mu]} \\ \vdots & \vdots & \iddots & 0 & & \vdots & \iddots & \\ \Delta_\alpha^1 f_1^{[1]} & \Delta_\alpha^0 f_1^{[2]} & \iddots & \vdots & \cdots & \vdots & & \\ \Delta_\alpha^0 f_1^{[1]} & 0 & \cdots & 0 & & \Delta_\alpha^0 f_\mu^{[1]} & & O \end{pmatrix}, \tag{3.2}$$

and

$$\vec{c} = {}^t(c_{11}, \dots, c_{1\ell_1}, c_{21}, \cdots, c_{2\ell_2}, \cdots, c_{\mu 1}, \dots, c_{\mu \ell_\mu}), \qquad \vec{e} = {}^t(1, 0, \dots, 0).$$

The unique existence of $\{c_{mn}\}$ is ensured by showing that $\det \mathcal{A} \neq 0$, which will be proved in the next section (cf. Lemma 3.3).

Example 3.2. 1. The case $P = (\partial_t^p - \alpha\partial_x^q)(\partial_t^p - \beta\partial_x^q)$: Then $f[\alpha,\beta] = \alpha\beta$ and

$$\mathcal{A}\vec{c} = \begin{pmatrix} 1 & 1 \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix} = \frac{1}{\alpha - \beta} \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}.$$

2. The case $P = (\partial_t^p - \alpha\partial_x^q)^2(\partial_t^p - \beta\partial_x^q)$: Then $f[\alpha,\beta] = \alpha^2\beta$ and

$$\mathcal{A}\vec{c} = \begin{pmatrix} 1 & 1 & 1 \\ \alpha+\beta & \beta & 2\alpha \\ \alpha\beta & 0 & \alpha^2 \end{pmatrix} \begin{pmatrix} c_{11} \\ c_{12} \\ c_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} c_{11} \\ c_{12} \\ c_{21} \end{pmatrix} = \frac{1}{(\alpha-\beta)^2} \begin{pmatrix} -\alpha\beta \\ \alpha(\alpha-\beta) \\ \beta^2 \end{pmatrix}.$$

3. The case $P = (\partial_t^q - \alpha\partial_x^q)^L$: Then $f[\alpha] = \alpha^L$ and

$$\mathcal{A}\vec{c} = \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ \vdots & \vdots & & \iddots & \\ \frac{(L-1)(L-2)}{2!}\alpha^{L-3} & (L-2)\alpha^{L-3} & \iddots & & \\ (L-1)\alpha^{L-2} & \alpha^{L-2} & & & \\ \alpha^{L-1} & 0 & & & O \end{pmatrix} \begin{pmatrix} c_{11} \\ c_{12} \\ \vdots \\ c_{1L} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$c_{1n} = 0$ $(1 \leq n \leq L-1)$, $c_{1L} = 1$.

Proof of Proposition 3.1. First, we prove that $\{c_{mn}\}$ are solutions of the linear equations (3.1). Since the operators ∂_t^p and ∂_x^q commute, by substituting $(\tau, 1, -\alpha_j)$ into $(\partial_t^p, \partial_x^q, \alpha_j)$ in (2.13), we have

$$\tau^{L-1} = \sum_{m=1}^{\mu}\sum_{n=1}^{\ell_m} c_{mn}\tau^{n-1}\prod_{j=1}^{\mu}(\tau+\alpha_j)^{\ell_j}/(\tau+\alpha_m)^n. \tag{3.3}$$

For any polynomial $g[\alpha] = g[\alpha_1,\ldots,\alpha_\mu]$ we have

$$g[\tau+\alpha_1,\ldots,\tau+\alpha_\mu] = \sum_{p\geq 0}\tau^p\Delta_\alpha^p g[\alpha].$$

Therefore

$$\prod_{j=1}^{\mu}(\tau+\alpha_j)^{\ell_j}/(\tau+\alpha_m)^n = \sum_{p=0}^{L-n}\tau^p\Delta_\alpha^p f_m^{[n]}[\alpha], \tag{3.4}$$

and by employing inner product, we have

$$\tau^{n-1}\sum_{p=0}^{L-n}\tau^p\Delta_\alpha^p f_m^{[n]}[\alpha] = \tau^{n-1}(\tau^{L-n},\dots,\tau^0)\begin{pmatrix}\Delta_\alpha^{L-n}f_m^{[n]}\\ \vdots\\ \Delta_\alpha^0 f_m^{[n]}\end{pmatrix}$$
$$= (\tau^{L-1},\dots,\tau^{n-1})\begin{pmatrix}\Delta_\alpha^{L-n}f_m^{[n]}\\ \vdots\\ \Delta_\alpha^0 f_m^{[n]}\end{pmatrix}.$$

Hence

$$\tau^{L-1} = \sum_{m=1}^{\mu}(\tau^{L-1},\dots,\tau^0)\begin{pmatrix}\Delta_\alpha^{L-1}f_m^{[1]} & \Delta_\alpha^{L-2}f_m^{[2]} & \cdots & \Delta_\alpha^{L-\ell_m}f_m^{[\ell_m]}\\ \Delta_\alpha^{L-1}f_m^{[1]} & \Delta_\alpha^{L-2}f_m^{[2]} & & \vdots\\ \vdots & \vdots & & \Delta_\alpha^0 f_m^{[\ell_m]}\\ \vdots & \vdots & \iddots & \\ \Delta_\alpha^1 f_m^{[1]} & \Delta_\alpha^0 f_m^{[2]} & & \\ \Delta_\alpha^0 f_m^{[1]} & & & O\end{pmatrix}\begin{pmatrix}c_{m1}\\ \vdots\\ c_{m\ell_m}\end{pmatrix}$$
$$=: \sum_{m=1}^{\mu}(\tau^{L-1},\dots,\tau^0)\mathcal{A}_m[L,\ell_m]\ \vec{c}_m[\ell_m].$$

Hence we can rewrite (3.3) as

$$\tau^{L-1} = (\tau^{L-1},\dots,\tau^0)(\mathcal{A}_1[L,\ell_1],\dots,\mathcal{A}_\mu[L,\ell_\mu])\begin{pmatrix}\vec{c}_1[\ell_1]\\ \vdots\\ \vec{c}_\mu[\ell_\mu]\end{pmatrix}$$
$$= (\tau^{L-1},\dots,\tau^0)\mathcal{A}\vec{c},$$

which implies the system of linear equations (3.1) immediately. □

The following lemma proves the unique solvability of (3.1).

Lemma 3.3. *For the matrix $\mathcal{A}$ in Proposition 3.1, we have*

$$\det\mathcal{A} = \begin{cases}\displaystyle\prod_{j=1}^{\mu}(-\alpha_j)^{\ell_j(\ell_j-1)/2}\prod_{1\le i<j\le\mu}(\alpha_i-\alpha_j)^{\ell_i\ell_j}, & \mu\neq 1,\\ (-\alpha_1)^{\ell_1(\ell_1-1)/2}, & \mu = 1.\end{cases} \tag{3.5}$$

§4. Proof of Lemma 3.3

§4.1. Elementary transformations of the matrix $\mathcal{A}$

We prepare some notation. We put

$$ {}^t((\tau^{L-1},\dots,\tau^0)\mathcal{A}_m[L,\ell_m]) = \begin{pmatrix} \sum_{p=0}^{L-1}\tau^p\Delta_\alpha^p f_m^{[1]} \\ \tau\sum_{p=0}^{L-2}\tau^p\Delta_\alpha^p f_m^{[2]} \\ \vdots \\ \tau^{\ell_m-1}\sum_{p=0}^{L-\ell_m}\tau^p\Delta_\alpha^p f_m^{[\ell_m]} \end{pmatrix} =: \vec{F}_m[\ell_m]. $$

With this notation, we have

$$ \begin{aligned} (\tau^{L-1},\dots,\tau^0)\mathcal{A} &= (\tau^{L-1},\dots,\tau^0)(\mathcal{A}_1[L,\ell_1],\dots,\mathcal{A}_\mu[L,\ell_\mu]) \\ &= ({}^t\vec{F}_1[\ell_1],\dots,{}^t\vec{F}_\mu[\ell_\mu]). \end{aligned} $$

Now we define the notation $\|\cdot\|$ so that the following equality holds:

$$ \mathcal{A} = \|{}^t\vec{F}_1[\ell_1],\dots,{}^t\vec{F}_\mu[\ell_\mu]\| = {}^t\left\| \begin{matrix} \vec{F}_1[\ell_1] \\ \vdots \\ \vec{F}_\mu[\ell_\mu] \end{matrix} \right\|. \tag{4.1} $$

The definition should be understood from the following examples.

Example 4.1. 1. $\mathcal{A} = \begin{pmatrix} 1 & 1 \\ \beta & \alpha \end{pmatrix} = {}^t\left\| \begin{matrix} 1\cdot\tau+\beta \\ 1\cdot\tau+\alpha \end{matrix} \right\|.$

2. $\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 \\ \alpha+\beta & \beta & 2\alpha \\ \alpha\beta & 0 & \alpha^2 \end{pmatrix} = {}^t\left\| \begin{matrix} 1\cdot\tau^2+(\alpha+\beta)\tau+\alpha\beta \\ \tau(1\cdot\tau+\beta) \\ 1\cdot\tau^2+2\alpha\tau+\alpha^2 \end{matrix} \right\| = {}^t\left\| \begin{matrix} (\tau+\alpha)(\tau+\beta) \\ \tau(\tau+\beta) \\ (\tau+\alpha)^2 \end{matrix} \right\|.$

3. The matrix

$$ \mathcal{A} = \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ \vdots & \vdots & & \iddots & \\ \frac{(L-1)(L-2)}{2!}\alpha^{L-3} & (L-2)\alpha^{L-3} & \iddots & & \\ (L-1)\alpha^{L-2} & \alpha^{L-2} & & & \\ \alpha^{L-1} & 0 & & & O \end{pmatrix} $$

is represented as

$$
{}^t\left\|\begin{matrix} 1\cdot\tau^{L-1}+(L-1)\alpha\tau^{L-2}+\cdots+\alpha^{L-1} \\ \tau(1\cdot\tau^{L-2}+(L-2)\alpha\tau^{L-3}+\cdots+\alpha^{L-2}) \\ \vdots \\ \tau^{L-2}(1\cdot\tau+\alpha) \\ \tau^{L-1}\cdot 1 \end{matrix}\right\| = {}^t\left\|\begin{matrix} (\tau+\alpha)^{L-1} \\ \tau(\tau+\alpha)^{L-2} \\ \vdots \\ \tau^{L-2}(\tau+\alpha) \\ \tau^{L-1} \end{matrix}\right\|.
$$

Further, we set

$$
f[\tau+\alpha] = \prod_{j=1}^{\mu}(\tau+\alpha_j)^{\ell_j} =: \prod_{j=1}^{\mu}\hat{f}_j(\tau), \qquad \hat{f}_j(\tau) := (\tau+\alpha_j)^{\ell_j},
$$

$$
\check{f}_m(\tau) := f[\tau+\alpha]/(\tau+\alpha_m)^{\ell_m} \qquad (\Leftrightarrow \check{f}_m(\tau)\hat{f}_m(\tau) = f[\tau+\alpha]).
$$

With this notation, we have

$$
\sum_{p=0}^{L-n}\tau^p\Delta_\alpha^p f_m^{[n]}[\alpha] = f[\tau+\alpha]/(\tau+\alpha_m)^n = (\tau+\alpha_m)^{\ell_m-n}\check{f}_m(\tau),
$$

and

$$
\vec{F}_m[\ell_m] = \begin{pmatrix} (\tau+\alpha_m)^{\ell_m-1}\check{f}_m(\tau) \\ \tau(\tau+\alpha_m)^{\ell_m-2}\check{f}_m(\tau) \\ \tau^2(\tau+\alpha_m)^{\ell_m-3}\check{f}_m(\tau) \\ \vdots \\ \tau^{\ell_m-2}(\tau+\alpha_m)\check{f}_m(\tau) \\ \tau^{\ell_m-1}\check{f}_m(\tau) \end{pmatrix}.
$$

Now, by elementary transformations of matrices, we can prove the following lemma.

Lemma 4.2.

$$
\det\mathcal{A} = \det {}^t\left\|\begin{matrix} \vec{F}_1[\ell_1] \\ \vdots \\ \vec{F}_\mu[\ell_\mu] \end{matrix}\right\| = \prod_{j=1}^{\mu}(-\alpha_j)^{\frac{1}{2}\ell_j(\ell_j-1)} \times \det {}^t\left\|\begin{matrix} \check{\mathbf{F}}_1[\ell_1] \\ \vdots \\ \check{\mathbf{F}}_\mu[\ell_\mu] \end{matrix}\right\|, \tag{4.2}
$$

where

$$
\check{\mathbf{F}}_m[\ell_m] = \begin{pmatrix} \tau^{\ell_m-1}\,\check{f}_m(\tau) \\ \tau^{\ell_m-2}\,\check{f}_m(\tau) \\ \vdots \\ \tau\,\check{f}_m(\tau) \\ \check{f}_m(\tau) \end{pmatrix}. \tag{4.3}
$$

Proof. For each m we make the elementary transformations

$$\vec{F}_m[\ell_m] = \begin{pmatrix} (\tau+\alpha_m)^{\ell_m-1}\check{f}_m(\tau) \\ \tau(\tau+\alpha_m)^{\ell_m-2}\check{f}_m(\tau) \\ \vdots \\ \tau^{\ell_m-3}(\tau+\alpha_m)^2\check{f}_m(\tau) \\ \begin{cases}\tau^{\ell_m-2}(\tau+\alpha_m)\check{f}_m(\tau) \\ \tau^{\ell_m-1}\check{f}_m(\tau)\end{cases} \end{pmatrix} \to \begin{pmatrix} (\tau+\alpha_m)^{\ell_m-1}\check{f}_m(\tau) \\ \tau(\tau+\alpha_m)^{\ell_m-2}\check{f}_m(\tau) \\ \vdots \\ \tau^{\ell_m-3}(\tau+\alpha_m)^2\check{f}_m(\tau) \\ \begin{cases}\alpha_m\tau^{\ell_m-2}\check{f}_m(\tau) \\ \tau^{\ell_m-1}\check{f}_m(\tau)\end{cases} \end{pmatrix}$$

$$\to \begin{pmatrix} (\tau+\alpha_m)^{\ell_m-1}\check{f}_m(\tau) \\ \tau(\tau+\alpha_m)^{\ell_m-2}\check{f}_m(\tau) \\ \vdots \\ \tau^{\ell_m-4}(\tau+\alpha_m)^3\check{f}_m(\tau) \\ \begin{cases}\alpha_m^2\tau^{\ell_m-3}\check{f}_m(\tau) \\ \alpha_m\tau^{\ell_m-2}\check{f}_m(\tau) \\ \tau^{\ell_m-1}\check{f}_m(\tau)\end{cases} \end{pmatrix} \to \cdots \to \begin{pmatrix} \alpha_m^{\ell_m-1}\check{f}_m(\tau) \\ \alpha_m^{\ell_m-2}\tau\check{f}_m(\tau) \\ \vdots \\ \alpha_m^3\tau^{\ell_m-4}\check{f}_m(\tau) \\ \alpha_m^2\tau^{\ell_m-3}\check{f}_m(\tau) \\ \alpha_m\tau^{\ell_m-2}\check{f}_m(\tau) \\ \tau^{\ell_m-1}\check{f}_m(\tau) \end{pmatrix}.$$

Since the determinant is invariant under these transformations, we get the formula (4.2), where we have changed the order of rows. □

Thus the proof of Lemma 3.3 is reduced to showing

Lemma 4.3.

$$\det \check{\mathcal{A}} := \det {}^t\begin{Vmatrix} \check{\mathbf{F}}_1[\ell_1] \\ \vdots \\ \check{\mathbf{F}}_\mu[\ell_\mu] \end{Vmatrix} = \prod_{1\le i<j\le\mu} (\alpha_i-\alpha_j)^{\ell_i\ell_j} \ (\neq 0). \tag{4.4}$$

§4.2. Notation and lemmas

In this subsection, we prepare additional notation and three lemmas about matrices. Let $p(\tau) = \sum_{j=0}^{\ell} p_j\tau^j$ and $k \ge 1$. We define a matrix ${}_{\ell+k}M_k[p(\tau)]$ of size $(\ell+k)\times k$ by

$$ {}_{\ell+k}M_k[p(\tau)] := \begin{pmatrix} p_\ell & & O \\ p_{\ell-1} & \ddots & \\ \vdots & & p_\ell \\ \vdots & & p_{\ell-1} \\ p_0 & & \vdots \\ & \ddots & \vdots \\ O & & p_0 \end{pmatrix} = {}^t\begin{Vmatrix} \tau^{k-1}p(\tau) \\ \tau^{k-2}p(\tau) \\ \vdots \\ \tau p(\tau) \\ p(\tau) \end{Vmatrix}. \tag{4.5}$$

With this notation, we have

$$\check{\mathcal{A}} = ({}_L M_{\ell_1}[\check{f}_1(\tau)], \dots, {}_L M_{\ell_\mu}[\check{f}_\mu(\tau)]), \tag{4.6}$$

where $\check{f}_m(\tau) = f[\tau+\alpha]/(\tau+\alpha_m)^{\ell_m} = \prod_{j=1,\, j\neq m}^{\mu}(\tau+\alpha_j)^{\ell_j}$.

In the following, $p(\tau), q(\tau)$ and $r(\tau)$ are polynomials of degree ℓ, m and n, and their coefficients of τ^j are denoted by p_j, q_j and r_j, respectively.

Lemma 4.4. *Let $P(\tau) = p(\tau)r(\tau) := \sum_{k=0}^{\ell+n} P_k \tau^k$. Then*

$$ {}_{(\ell+n)+1}M_1[P(\tau)] = {}_{(\ell+n)+1}M_1[p(\tau)r(\tau)] = {}_{n+(\ell+1)}M_{\ell+1}[r(\tau)]\ {}_{\ell+1}M_1[p(\tau)], \tag{4.7}$$

that is,

$$\begin{pmatrix} P_{\ell+n} \\ \vdots \\ P_0 \end{pmatrix} = \begin{pmatrix} r_n & & O \\ r_{n-1} & \ddots & \\ \vdots & & r_n \\ \vdots & & r_{n-1} \\ r_0 & & \vdots \\ & \ddots & \vdots \\ O & & r_0 \end{pmatrix} \begin{pmatrix} p_\ell \\ \vdots \\ p_0 \end{pmatrix}.$$

Proof. This is an immediate consequence of $P_k = \sum_{i+j=k} p_i r_j$. □

Lemma 4.5. *We have*

$$\begin{aligned} &\left({}_{(\ell+n)+m}M_m[p(\tau)r(\tau)] \,\vdots\, {}_{(m+n)+\ell}M_\ell[q(\tau)r(\tau)]\right) \\ &\qquad = {}_{n+(\ell+m)}M_{\ell+m}[r(\tau)]\ \left({}_{\ell+m}M_m[p(\tau)] \,\vdots\, {}_{m+\ell}M_\ell[q(\tau)]\right). \end{aligned} \tag{4.8}$$

Moreover, $N := \left({}_{\ell+m}M_m[p(\tau)] \,\vdots\, {}_{m+\ell}M_\ell[q(\tau)]\right)$ is a square matrix of size $\ell+m$, and its determinant is the resultant of the polynomials $p(\tau)$ and $q(\tau)$. Hence

$$\det N = p_\ell^m q_m^\ell \prod_{1\le i\le \ell,\, 1\le j\le m} (x_i - y_j), \tag{4.9}$$

where $\{x_i\}$ and $\{y_j\}$ are the roots of $p(\tau)$ and $q(\tau)$, respectively.

Proof. The formula (4.8) follows by applying Lemma 4.4. For the theory of resultants, we refer to [9]. □

The next lemma follows from Lemma 4.5.

Lemma 4.6. *Let* $s(\tau)=\sum_{j=0}^{\ell+m}s_j\tau^j$. *Then*

$$
\begin{aligned}
(4.10)\qquad & \big({}_{(\ell+m)+n}M_n[s(\tau)] \,\vdots\, {}_{(\ell+n)+m}M_m[p(\tau)r(\tau)] \,\vdots\, {}_{(m+n)+\ell}M_\ell[q(\tau)r(\tau)]\big)\\
&= \big({}_{(\ell+m)+n}M_n[s(\tau)] \,\vdots\, {}_{n+(\ell+m)}M_{\ell+m}[r(\tau)]\big)\\
&\quad\times \big(E_n\oplus({}_{\ell+m}M_m[p(\tau)] \,\vdots\, {}_{m+\ell}M_\ell[q(\tau)])\big),
\end{aligned}
$$

where E_n *denotes the unit matrix of size* n.

§4.3. Proof of Lemma 4.3

Lemma 4.3 is obtained by $(\mu-2)$-fold application of Lemma 4.6. Firstly, we recall

$$
(4.11)\qquad \check{\mathcal{A}}=({}_LM_{\ell_1}[\check f_1(\tau)],\dots,{}_LM_{\ell_\mu}[\check f_\mu(\tau)]),
$$

where $\check f_m(\tau)=f[\tau+\alpha]/(\tau+\alpha_m)^{\ell_m}=\prod_{j=1,\,j\neq m}^{\mu}(\tau+\alpha_j)^{\ell_j}$.

For brevity, set

$$
\begin{aligned}
&L[\le i]:=\sum_{j=1}^{i}\ell_j,\quad L[\ge i]:=\sum_{j=i}^{\mu}\ell_j\quad \Big(L[\le i]+L[\ge i+1]=L=\sum_{j=1}^{\mu}\ell_j\Big),\\
&f_{\le m}(\tau):=\prod_{j=1}^{m}(\tau+\alpha_j)^{\ell_j},\quad f_{\ge m}(\tau):=\prod_{j=m}^{\mu}(\tau+\alpha_j)^{\ell_j}\\
&\qquad\qquad (f_{\le m}(\tau)f_{\ge m+1}(\tau)=f[\tau+\alpha]).
\end{aligned}
$$

We denote $\check{\mathcal{A}}=\mathcal{A}_\mu$.

Since the polynomials $\check f_{\mu-1}(\tau)$ and $\check f_\mu(\tau)$ have a common factor $f_{\le\mu-2}(\tau)=\prod_{j=1}^{\mu-2}(\tau+\alpha_j)^{\ell_j}$, the formula (4.10) implies that

$$
\begin{aligned}
\mathcal{A}_\mu &= \big({}_LM_{\ell_1}[\check f_1(\tau)],\dots,{}_LM_{\ell_{\mu-2}}[\check f_{\mu-2}(\tau)] \,\vdots\, {}_LM_{\ell_{\mu-1}}[\check f_{\mu-1}(\tau)] \,\vdots\, {}_LM_{\ell_\mu}[\check f_\mu(\tau)]\big)\\
&= \big({}_LM_{\ell_1}[\check f_1(\tau)],\dots,{}_LM_{\ell_{\mu-2}}[\check f_{\mu-2}(\tau)] \,\vdots\, {}_LM_{L[\ge\mu-1]}[f_{\le\mu-2}(\tau)]\big)\\
&\quad\times\big(E_{L[\le\mu-2]}\oplus({}_{L[\ge\mu-1]}M_{\ell_{\mu-1}}[\hat f_\mu(\tau)] \,\vdots\, {}_{L[\ge\mu-1]}M_{\ell_\mu}[\hat f_{\mu-1}(\tau)])\big)\\
&=: \mathcal{A}_{\mu-1}\cdot\big(E_{L[\le\mu-2]}\oplus\widehat{\mathcal{A}}(\mu,\mu-1)\big).
\end{aligned}
$$

Here $\det\widehat{\mathcal{A}}(\mu,\mu-1)$ is the resultant of $\hat f_\mu(\tau)=(\tau+\alpha_\mu)^{\ell_\mu}$ and $\hat f_{\mu-1}(\tau)=(\tau+\alpha_{\mu-1})^{\ell_{\mu-1}}$, and its value is $(\alpha_{\mu-1}-\alpha_\mu)^{\ell_{\mu-1}\ell_\mu}$.

Secondly, since the polynomials $\check f_{\mu-2}(\tau)$ and $f_{\le\mu-2}(\tau)$ have a common factor $f_{\le\mu-3}(\tau)=\prod_{j=1}^{\mu-3}(\tau+\alpha_j)^{\ell_j}$, we have

$$\begin{aligned}
&\mathcal{A}_{\mu-1}\\
&= \big({}_L M_{\ell_1}[\check{f}_1(\tau)], \dots, {}_L M_{\ell_{\mu-3}}[\check{f}_{\mu-3}(\tau)] \vdots {}_L M_{\ell_{\mu-2}}[\check{f}_{\mu-2}(\tau)] \vdots {}_L M_{L[\geq\mu-1]}[f_{\leq\mu-2}(\tau)]\big)\\
&= \big({}_L M_{\ell_1}[\check{f}_1(\tau)], \dots, {}_L M_{\ell_{\mu-3}}[\check{f}_{\mu-3}(\tau)] \vdots {}_L M_{L[\geq\mu-2]}[f_{\leq\mu-3}(\tau)]\big)\\
&\quad \times \big(E_{L[\leq\mu-3]} \oplus ({}_{L[\geq\mu-2]}M_{\ell_{\mu-2}}[f_{\geq\mu-1}(\tau)] \vdots {}_{L[\geq\mu-2]}M_{L[\geq\mu-1]}[\hat{f}_{\mu-2}(\tau)])\big)\\
&=: \mathcal{A}_{\mu-2} \cdot \big(E_{L[\leq\mu-3]} \oplus \widehat{\mathcal{A}}(\geq\mu-1, \mu-2)\big).
\end{aligned}$$

Here, $\det \widehat{\mathcal{A}}(\geq\mu-1, \mu-2)$ is the resultant of $f_{\geq\mu-1}(\tau) = (\tau+\alpha_{\mu-1})^{\ell_{\mu-1}}(\tau+\alpha_\mu)^{\ell_\mu}$ and $\hat{f}_{\mu-2}(\tau) = (\tau+\alpha_{\mu-2})^{\ell_{\mu-2}}$, and its value is $(\alpha_{\mu-2}-\alpha_{\mu-1})^{\ell_{\mu-2}\ell_{\mu-1}} \cdot (\alpha_{\mu-2}-\alpha_\mu)^{\ell_{\mu-2}\ell_\mu}$.

By continuing these arguments, we finally get

$$\text{(4.12)} \quad \det \check{\mathcal{A}} = \det \mathcal{A}_\mu = \prod_{k=1}^{\mu-1} \det \widehat{\mathcal{A}}(\geq\mu-k+1, \mu-k) = \prod_{1\leq i<j\leq\mu} (\alpha_i-\alpha_j)^{\ell_i\ell_j},$$

where $\widehat{\mathcal{A}}(\geq\mu, \mu-1) = \widehat{\mathcal{A}}(\mu, \mu-1)$ and $\mathcal{A}_2 = \widehat{\mathcal{A}}(\geq 2, 1)$. □

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