

Fano Symmetric Varieties with Low Rank

by

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Abstract

The symmetric projective varieties of rank one are all smooth and Fano by a classical result of Akhiezer. We classify the locally factorial (respectively smooth) projective symmetric G -varieties of rank 2 that are Fano. When G is semisimple we also classify the locally factorial (respectively smooth) projective symmetric G -varieties of rank 2 that are only quasi-Fano. Moreover, we classify the Fano symmetric G -varieties of rank 3 obtainable from a wonderful variety by a sequence of blow-ups along G -stable varieties. Finally, we classify the Fano symmetric varieties of arbitrary rank that are obtainable from a wonderful variety by a sequence of blow-ups along closed orbits.

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A Gorenstein (projective) normal algebraic variety X over \mathbb{C} is called a *Fano variety* if the anticanonical divisor is ample. The Fano surfaces are classically called *Del Pezzo surfaces*. The importance of Fano varieties in the theory of higher dimensional varieties is similar to the importance of Del Pezzo surfaces in the theory of surfaces. Moreover Mori's program predicts that every uniruled variety is birational to a fibration whose general fiber is a Fano variety (with terminal singularities).

Let θ be an involution of a reductive group G (over \mathbb{C}) and let H be a closed subgroup of G such that $G^\theta \subset H \subset N_G(G^\theta)$. A normal G -variety with an open orbit isomorphic to G/H is called a *symmetric variety*. Symmetric varieties are a generalization of toric varieties. The toric smooth Fano varieties with rank at most four have been classified. By Theorem 4.2 of [AIBr04], there are finitely many Fano smooth symmetric varieties with a fixed open orbit. In [Ru07] we have classified the smooth compact symmetric varieties with Picard number one and G semisimple,

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while in [Ru10] we have given an explicit geometrical description of such varieties; they are automatically Fano.

In this work, we want to classify the Fano symmetric varieties with low rank (and G semisimple). First, we consider a special case of arbitrary rank. We say that a variety X is *quasi \mathbb{Q} -Fano* if $-K_X$ is a nef and big \mathbb{Q} -divisor. For any fixed open orbit G/H with G semisimple, there is a unique maximal compactification among those that have only one closed orbit. Such a variety is called the *standard compactification*. Vice versa, if such a compactification exists, then we can suppose that G is semisimple. If the standard compactification is smooth, it is called the *wonderful compactification*; this is the case, for example, if $H = N_G(G^\theta)$ (see [dCoPr83, Theorem 3.1]). We prove that the standard symmetric varieties are all quasi \mathbb{Q} -Fano and we describe when they are Fano. In particular, we prove:

Theorem A. *Let X be a wonderful symmetric variety. Then:*

- *The anticanonical divisor of X is always nef and big.*
- *If X is not a Fano variety and G is simple, then $H = N(G^\theta)$ and there is a maximal torus of G over which θ acts as the inverse map.*

We also determine the Fano symmetric varieties obtainable from a wonderful one by a sequence of blow-ups along closed orbits. In particular, we prove:

Theorem B. *Let X be a Fano symmetric variety obtainable from a wonderful one X_0 by a sequence of blow-ups along closed orbits. Then X is either X_0 or the blow-up of X_0 along the unique closed orbit.*

Next we consider symmetric varieties of rank at most three. The *rank* of a symmetric variety X is the maximal codimension of an orbit under the action of the unipotent radical of any Borel subgroup of G . The symmetric varieties with rank one are all wonderful; moreover one can show that they are isomorphic, under the action of $\text{Aut}^0(X)$, either to a projective homogeneous variety G/P with P maximal, or to $\mathbb{P}^n \times \mathbb{P}^n$ (see [A83]). Thus they are all Fano.

We classify all the locally factorial (resp. smooth) Fano symmetric varieties of rank 2. When G is semisimple, we also classify the locally factorial (resp. smooth) symmetric varieties that are only quasi-Fano. In the proof of this result we obtain a classification of the toroidal Fano varieties of rank 2 with G semisimple (without the regularity assumption). To state the previous results we need the theory of colored fans. However, if we suppose that there is a wonderful compactification X_0 and that X is toroidal (i.e. X dominates X_0), then the previous theorems have a more elementary form. Supposing the existence of a wonderful compactification, the toroidal varieties are classified by the toric varieties that dominate the affine

space. Indeed, X_0 contains a T -subvariety isomorphic to $\mathbb{A}^{\text{rank}(G/H)}$ and there is a morphism $p : X \rightarrow X_0$. Finally, the toric variety associated to X is the inverse image of $\mathbb{A}^{\text{rank}(G/H)}$ in X .

Theorem C. *Let X be a Fano toroidal symmetric variety of rank two with G semisimple and such that the standard compactification is wonderful. Then, up to isomorphism, the fan of the associated toric variety Z has maximal cones*

$$\{\text{cone}(e_1, e_1 + ne_2), \text{cone}(e_2, e_1 + ne_2)\}$$

for an appropriate $n \in \mathbb{Z}^+$. Here $\{e_1, e_2\}$ is the lattice basis associated to $\mathbb{A}^2 \subset X_0$. Moreover X is smooth if and only if $n = 1$.

Theorem D. *Let X be a quasi-Fano smooth toroidal symmetric variety of rank two with G semisimple and such that the standard compactification is wonderful. Then, up to isomorphism, the fan of the associated toric variety Z has maximal cones*

$$\{\text{cone}(e_1, e_1 + e_2), \text{cone}(e_1 + e_2, e_1 + 2e_2), \dots, \text{cone}(e_1 + (n - 1)e_2, e_1 + ne_2), \\ \text{cone}(e_1 + ne_2, e_2)\}$$

for an appropriate $n \in \mathbb{Z}^+$. Here $\{e_1, e_2\}$ is the lattice basis associated to $\mathbb{A}^2 \subset X_0$.

Finally, we classify the smooth Fano symmetric varieties of rank three that are obtainable from a wonderful one by a sequence of blow-ups along G -subvarieties (in particular G is semisimple). This class of varieties is quite large; indeed any compact symmetric variety is dominated by a variety obtained from a wonderful variety by a sequence of blow-ups along G -subvarieties of codimension two (see [dCoPr85, Theorem 2.4]). In particular, we prove the following necessary condition:

Theorem E. *Let X be a Fano symmetric variety of rank three that is obtainable from a wonderful one X_0 by a sequence of blow-ups along G -subvarieties. Then we have the following possibilities:*

- X is the wonderful compactification X_0 ;
- X is a blow-up of X_0 along a G -subvariety;
- X is obtainable from X_0 by a blow-up along a G -subvariety, followed by a blow-up along a G -subvariety of codimension two.

Moreover, there are at most eleven Fano varieties obtainable from a wonderful variety by a sequence of blow-ups.

This last result on rank-3 varieties can be generalized to varieties obtainable from generic wonderful varieties of rank 3 by a sequence of blow-ups along G -subvarieties (without supposing G/H symmetric).

This paper is organized as follows. In §§1.1–1.5 we explain the classification of symmetric varieties. Then, in §1.6 and §1.7 we define the symmetric varieties which we are most interested in: the wonderful varieties and toroidal varieties. In §1.8 we describe the Picard group of a symmetric variety. In §2 we state and explain our results. In §3 we prove Theorem A, while in §4 we prove Theorem B. In §5 we prove the results in rank three, in particular Theorem E. Finally in §6 we classify the 2-rank Fano symmetric varieties. This last section is divided into three parts: 1) In §6.1 we classify the locally factorial (resp. smooth) Fano symmetric varieties with G semisimple; in particular we prove Theorem C. 2) In §6.2 we classify the locally factorial (resp. smooth) quasi-Fano symmetric varieties with G semisimple; in particular we prove Theorem D. 3) In §6.3 we classify the locally factorial (resp. smooth) Fano symmetric varieties with G non-semisimple.

§1. Introduction and notation

In this section we introduce the necessary notation. The reader interested in the embedding theory of spherical varieties can consult [Kn91], [Br97a] or [T06]. In [Vu90] this theory is explained in the particular case of symmetric varieties.

§1.1. First definitions

Let G be a connected reductive algebraic group over \mathbb{C} and let θ be an involution of G . Given a closed subgroup H such that $G^\theta \subset H \subset N_G(G^\theta)$, we say that G/H is a *symmetric space* and that H is a *symmetric subgroup*. A normal G -variety is called a *spherical variety* if it contains a dense B -orbit (B is a chosen Borel subgroup of G). We say that a subtorus of G is *split* if $\theta(t) = t^{-1}$ for all its elements t ; moreover it is a *maximal split torus* if it has maximal dimension. A maximal torus containing a maximal split torus is *maximally split*; any maximally split torus is θ stable (see [T06, Lemma 26.5]). We fix arbitrarily a maximal split torus T^1 and a maximally split torus T containing T^1 . We denote by n the rank of G (i.e. the dimension of T). We can choose a Borel subgroup $B \supset T$ such that, for any positive root α , either $\theta(\alpha) = \alpha$ or $\theta(\alpha)$ is negative. Moreover, BH is dense in G (see [dCoPr83, Lemma 1.2 and Proposition 1.3]). In particular, every normal equivariant open embedding of G/H is spherical; we call it a *symmetric variety*. Let R_G be the root system of G with respect to T and let R_G^+ be the set of positive roots with respect to B .

We can assume that G is the direct product of a simply connected, semisimple group with a central split torus (see [Vu90, §2.1] or [Ru07, Remark 1]).

§1.2. Colored fans

Now, we introduce some details about the classification of the symmetric varieties with fixed G/H by their colored fans (this classification is defined more generally for spherical varieties). Let $D(G/H)$ be the set of B -stable prime divisors of G/H ; its elements are called *colors*. We say that a spherical variety is *simple* if it contains only one closed orbit. Let X be a simple symmetric variety with closed orbit Y . We define the set of *colors of X* as the subset $D(X)$ of $D(G/H)$ consisting of the colors whose closure in X contains Y . To each prime divisor D of X , we can associate the normalized discrete valuation v_D of $\mathbb{C}(G/H)$ whose ring is $\mathcal{O}_{X,D}$; D is G -stable if and only if v_D is G -invariant. Let N be the set of all G -invariant valuations of $\mathbb{C}(G/H)$ taking values in \mathbb{Z} and let $N(X)$ be the set of valuations associated to the G -stable prime divisors of X . Observe that each irreducible component of $X \setminus (G/H)$ has codimension one, because G/H is affine. Let $S := T/T \cap H \simeq T \cdot (eH/H)$. One can show that the group $\mathbb{C}(G/H)^{(B)}/\mathbb{C}^*$ is isomorphic to the character group $\chi(S)$ of S (see [Vu90, §2.3]); in particular, it is a free abelian group. We define the *rank* of G/H as the rank of $\chi(S)$. We can identify the dual group $\text{Hom}_{\mathbb{Z}}(\mathbb{C}(G/H)^{(B)}/\mathbb{C}^*, \mathbb{Z})$ with the group $\chi_*(S)$ of one-parameter subgroups of S ; so we can identify $\chi_*(S)_{\mathbb{R}}$ with $\text{Hom}_{\mathbb{Z}}(\chi(S), \mathbb{R})$. The restriction map to $\mathbb{C}(G/H)^{(B)}/\mathbb{C}^*$ is injective over N (see [Br97a, §3.1, Corollaire 3]), so we can identify N with a subset of $\chi_*(S)_{\mathbb{R}}$. We say that N is the *valuation monoid* of G/H . For each color D , we define $\rho(D)$ as the restriction of v_D to $\chi(S)$. In general, the map $\rho : D(G/H) \rightarrow \chi_*(S)_{\mathbb{R}}$ is not injective. We will describe more explicitly $\rho(D(G/H))$ and N in §1.5. Let $C(X)$ be the cone in $\chi_*(S)_{\mathbb{R}}$ generated by $N(X)$ and $\rho(D(X))$. We say that the pair $(C(X), D(X))$ is the *colored cone* of X ; it determines X uniquely (see [Br97a, §3.3, Théorème]).

Let Y be an orbit of a symmetric variety X . The set $\{x \in X : \overline{G \cdot x} \supset Y\}$ is an open simple G -subvariety of X with closed orbit Y , because any spherical variety contains finitely many G -orbits. Let $\{X_i\}$ be the set of open simple subvarieties of X and define the set of *colors of X* , $D(X)$, as $\bigcup_{i \in I} D(X_i)$. The family $\mathcal{F}(X) := \{(C(X_i), D(X_i))\}_{i \in I}$ is called the *colored fan* of X and determines X completely (see [Br97a, §3.4, Théorème 1]). Moreover X is compact if and only if $\text{cone}(N)$ is contained in the support $|\mathcal{F}(X)| := \bigcup_{i \in I} C(X_i)$ of the colored fan (see [Br97a, §3.4, Théorème 2]).

Let (C, F) be a colored cone of X . We say that the blow-up of X along the subvariety associated to (C, F) is the *blow-up of X along (C, F)* . Moreover, we say that blow-up of a variety along a subvariety of codimension i is an *i -blow-up*.

Given a symmetric variety X we denote by Δ (or Δ_X) the fan associated to the colored fan of X , by $\Delta(i)$ the set of i -dimensional cones in Δ and by

$\Delta[p]$ the set of primitive generators of 1-dimensional cones of Δ . The fan Δ is formed by all the faces of the cones C such that there is a colored cone $(C, F) \in \mathcal{F}(X)$. The toric varieties are a special case of symmetric varieties. If X is a toric variety, then $D(G/H)$ is empty and we need only consider the fan Δ_X (actually the theory of colored fans is a generalization of the classification of toric varieties by fans).

§1.3. Restricted root system

To describe the sets N and $\rho(D(G/H))$, we associate a root system to G/H . We can identify $\chi(T^1)_{\mathbb{R}}$ with $\chi(S)_{\mathbb{R}}$ because $[\chi(S) : \chi(T^1)]$ is finite. We denote again by θ the involution induced on $\chi(T)_{\mathbb{R}}$. The inclusion $T^1 \subset T$ induces an isomorphism of $\chi(T^1)_{\mathbb{R}}$ with the (-1) -eigenspace of $\chi(T)_{\mathbb{R}}$ under the action of θ (see [T06, §26]). Denote by W_G the Weyl group of G (with respect to T). We can identify $\chi(T^1)_{\mathbb{R}}$ with its dual $\chi_*(T^1)_{\mathbb{R}}$ by the restriction (\cdot, \cdot) to $\chi(T^1)_{\mathbb{R}}$ of a fixed W_G -invariant non-degenerate symmetric bilinear form on $\chi(T)_{\mathbb{R}}$. Let R_G^0 be the set of roots fixed by θ and let $R_G^1 = R_G \setminus R_G^0$ and $R_G^{i,+} := R_G^i \cap R_G^+$.

The set $R_{G,\theta} := \{\beta - \theta(\beta) : \beta \in R_G^1\}$ is a (possibly non-reduced) root system in $\chi(S)_{\mathbb{R}}$ (see [Vu90, §2.3 Lemme]), which we call the *restricted root system* of (G, θ) ; we call the non-zero $\beta - \theta(\beta)$ the *restricted roots*. If G is semisimple, then the rank of $R_{G,\theta}$ is equal to the rank of G/H . Usually we denote by β (resp. by α) a root of R_G (resp. of $R_{G,\theta}$); often we denote by ϖ (resp. by ω) a weight of R_G (resp. of $R_{G,\theta}$). We denote by $\bar{R}_G = \{\beta_1, \dots, \beta_n\}$ the basis of R_G associated to B and by $\varpi_1, \dots, \varpi_n$ the fundamental weights of R_G . Let \bar{R}_G^i be $\bar{R}_G \cap R_G^i$. We denote by $\alpha_1, \dots, \alpha_s$ the elements of the basis $\bar{R}_{G,\theta} := \{\beta - \theta(\beta) : \beta \in \bar{R}_G^1\}$ of $R_{G,\theta}$. If $R_{G,\theta}$ is irreducible we order $\bar{R}_{G,\theta}$ as in [Bo68]. Let b_i be equal to $\frac{1}{2}$ if $2\alpha_i$ belongs to $R_{G,\theta}$ and equal to one otherwise; for each i we define α_i^\vee as the coroot $\frac{2b_i}{(\alpha_i, \alpha_i)}\alpha_i$. The set $\{\alpha_1^\vee, \dots, \alpha_s^\vee\}$ is a basis of the dual root system $R_{G,\theta}^\vee$. We call the elements of $R_{G,\theta}^\vee$ the *restricted coroots*. Let $\omega_1, \dots, \omega_s$ be the fundamental weights of $R_{G,\theta}$ with respect to $\{\alpha_1, \dots, \alpha_s\}$ and let $\omega_1^\vee, \dots, \omega_s^\vee$ be the fundamental weights of $R_{G,\theta}^\vee$ with respect to $\{\alpha_1^\vee, \dots, \alpha_s^\vee\}$. Let C^+ be the positive closed Weyl chamber of $\chi(S)_{\mathbb{R}}$ and let $C^- := -C^+$.

There is a permutation $\bar{\theta}$ of \bar{R}_G^1 such that, for all $\beta \in \bar{R}_G^1$, $\theta(\beta) + \bar{\theta}(\beta)$ is a linear combination of roots in \bar{R}_G^0 . We say that a dominant weight $\varpi \in \chi(T)$ is a *spherical weight* if $V(\varpi)$ contains a non-zero vector fixed by G^θ . In this case, $V(\varpi)^{G^\theta}$ is one-dimensional and $\theta(\varpi) = -\varpi$, so ϖ belongs to $\chi(S)_{\mathbb{R}}$. One can show that the set of dominant weights of $R_{G,\theta}$ is the set of spherical weights and that C^+ is the intersection of $\chi(S)_{\mathbb{R}}$ with the positive closed Weyl chamber of R_G . Suppose $\beta_j - \theta(\beta_j) = \alpha_i$. Then ω_i is a positive multiple of $\varpi_j + \varpi_{\bar{\theta}(j)}$.

More precisely, we have the following possibilities: (i) $\omega_i = \varpi_j + \varpi_{\bar{\theta}(j)}$ if $\bar{\theta}(j) \neq j$; (ii) $\omega_i = 2\varpi_j$ if $\bar{\theta}(j) = j$ and β_j is orthogonal to R_G^0 ; (iii) $\omega_i = \varpi_j$ in the other cases (see [ChMa03, Theorem 2.3] or [T06, Proposition 26.4]). We say that a spherical weight is *regular* if it is strictly dominant as weight of the restricted root system.

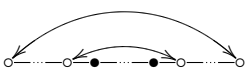

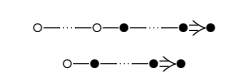
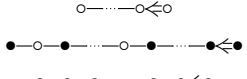
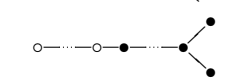
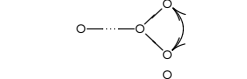
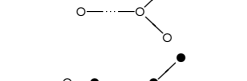
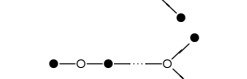
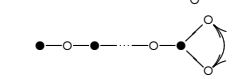
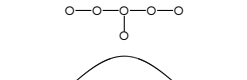

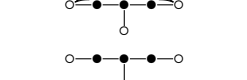
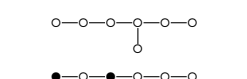
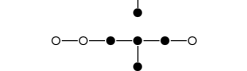




§1.4. Classification of symmetric spaces

We say that (G, θ) is *indecomposable* if the unique normal, connected, θ -stable subgroup of G is the trivial one. Otherwise, we say that (G, θ) is *decomposable*. In this case G/H is, up to a finite cover, a product of smaller symmetric spaces. Indeed $G = G_1 \times G_2$ where G_1 and G_2 are θ -stable, normal subgroups, so $G/G^\theta = G_1/G_1^\theta \times G_2/G_2^\theta$. If θ is indecomposable, there are three possibilities: (i) G is simple; (ii) $G = \dot{G} \times \dot{G}$ with \dot{G} simple and $\theta(x, y) = (y, x)$; (iii) $G = \mathbb{C}^*$ and $\theta(t) = t^{-1}$. The classification, up to conjugation, of the involutions of a simple group is as in Tables 1 and 2 (see also [Wa72, §1.1] and [T06, Table 5.9]). In the fourth column of these tables the Satake diagram of the involution is shown. The Satake diagram of any involution (G, θ) is obtained from the Dynkin diagram of G as follows: 1) the vertices corresponding to elements of \bar{R}_G^0 (resp. of \bar{R}_G^1) are black (resp. white); 2) two simple roots $\beta_1, \beta_2 \in \bar{R}_G^1$ such that $\bar{\theta}(\beta_1) = \beta_2$ are linked by a double-headed arrow. Finally, if $G = \dot{G} \times \dot{G}$ (with \dot{G} simple) and $\theta(x, y) := (y, x)$, then R_G^0 is empty and the Satake diagram consists of two copies of the Dynkin diagram of \dot{G} . Moreover, each vertex of a copy of the Dynkin diagram of \dot{G} is joined to the corresponding vertex of the other copy. For example, if \dot{G} is SL_{l+1} , we have the following diagram:



In §4, we will need to write the simple restricted roots as a linear combination of simple roots of G for the following involutions (see also [T06, Table 5.9]): 1) The involution θ of type *AII*; in this case $G/G^\theta = SL_{2n}/Sp_{2n}$. Let $I_n \in M_n(\mathbb{C})$ be the identity matrix and let Ω_n be $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$; then $\theta(g) = \Omega_n(g^t)^{-1}\Omega_n^{-1}$ for all $g \in SL_{2n}$. 2) The involution θ of type *CII*; in this case $G/G^\theta = Sp_{2n}/(Sp_{2l} \times Sp_{2n-2l})$. Let $K_{n,l}$ be the diagonal matrix with entries $(I_l, -I_{n-l}, I_l, -I_{n-l})$; then $\theta(g) = K_{n,l}gK_{n,l}$ for all $g \in Sp_{2n}$. 3) The involution θ of type *DIII* with $n = 2l + 1$; then $G/G^\theta = SL_{4l+2}/GL_{2l+1}$ and $\theta(g) = \Omega_n g \Omega_n^{-1}$ for all $g \in SL_{4l+2}$. These descriptions allow one to prove that $\alpha_i = \beta_{i-1} + 2\beta_i + \beta_{i+1}$ for each $i < l$.

Table 1. Involutions of simple groups

type	G	G^θ	Satake diagram	$R_{G,\theta}$	rank G/G^θ
<i>AI</i>	SL_{l+1}	SO_{l+1}	$\circ - \cdots - \circ$	A_l	$l \geq 1$
<i>AII</i>	SL_{2l+2}	Sp_{2l+2}	$\bullet - \circ - \bullet - \cdots - \bullet - \circ - \bullet$	A_l	$l \geq 1$
<i>AIII</i>	SL_{n+1}	$S(GL_l \times GL_{n+1-l})$	$\circ - \cdots - \bullet - \cdots - \bullet - \cdots - \circ$ 	BC_l	$\frac{n}{2} \geq l \geq 2$
<i>AIII</i>	SL_{2l}	$S(GL_l \times GL_l)$	$\circ - \cdots - \bullet - \cdots - \bullet - \cdots - \circ$ 	C_l	$l \geq 2$
<i>AIV</i>	SL_{n+1}	GL_n	$\circ - \cdots - \bullet - \cdots - \bullet - \cdots - \circ$ 	BC_1	$n > l = 1$
<i>BI</i>	SO_{2n+1}	$S(O_l \times O_{2n+1-l})$	$\circ - \cdots - \bullet - \cdots - \bullet - \cdots - \circ$ 	B_l	$n \geq l \geq 2$
<i>BII</i>	SO_{2n+1}	$S(O_1 \times O_{2n})$	$\circ - \cdots - \bullet - \cdots - \bullet - \cdots - \circ$ 	A_1	$n - 1 \geq l = 1$
<i>CI</i>	Sp_{2l}	GL_l	$\circ - \cdots - \circ \leftarrow \circ$	C_l	$l \geq 3$
<i>CII</i>	Sp_{2n}	$Sp_{2l} \times Sp_{2n-2l}$	$\bullet - \circ - \bullet - \cdots - \bullet - \cdots - \bullet - \circ \leftarrow \bullet$	BC_l	$\frac{n-1}{2} \geq l \geq 1$
<i>CII</i>	Sp_{4l}	$Sp_{2l} \times Sp_{2l}$	$\bullet - \circ - \bullet - \cdots - \bullet - \cdots - \bullet - \circ \leftarrow \bullet$	C_l	$l \geq 2$
<i>DI</i>	SO_{2n}	$S(O_l \times O_{2n-l})$	$\circ - \cdots - \bullet - \cdots - \bullet - \cdots - \circ$ 	B_l	$n - 2 \geq l \geq 2$
<i>DI</i>	SO_{2l+2}	$S(O_l \times O_{l+2})$	$\circ - \cdots - \circ - \circ$ 	B_l	$l \geq 3$
<i>DI</i>	SO_{2l}	$S(O_l \times O_l)$	$\circ - \cdots - \circ - \circ$ 	D_l	$l \geq 4$
<i>DII</i>	SO_{2n}	$S(O_1 \times O_{2n-1})$	$\circ - \cdots - \bullet - \cdots - \bullet - \cdots - \circ$ 	A_1	$l = 1$
<i>DIII</i>	SO_{4l}	GL_{2l}	$\bullet - \circ - \bullet - \cdots - \bullet - \cdots - \bullet - \circ$ 	C_l	$l \geq 2$
<i>DIII</i>	SO_{4l+2}	GL_{2l+1}	$\bullet - \circ - \bullet - \cdots - \bullet - \cdots - \bullet - \circ$ 	BC_l	$l \geq 2$
<i>EI</i>	E_6	C_4	$\circ - \circ - \circ - \circ - \circ - \circ$ 	E_6	$l = 6$
<i>EII</i>	E_6	$A_5 \times A_1$	$\circ - \circ - \circ - \circ - \circ - \circ$ 	F_4	$l = 4$
<i>EIII</i>	E_6	$D_5 \times \mathbb{C}^*$	$\circ - \bullet - \bullet - \bullet - \bullet - \circ$ 	BC_2	$l = 2$
<i>EIV</i>	E_6	F_4	$\circ - \bullet - \bullet - \bullet - \bullet - \circ$ 	A_2	$l = 2$
<i>EV</i>	E_7	A_7	$\circ - \circ - \circ - \circ - \circ - \circ - \circ$ 	E_7	$l = 7$
<i>EVI</i>	E_7	$D_6 \times A_1$	$\bullet - \circ - \bullet - \bullet - \bullet - \bullet - \circ$ 	F_4	$l = 4$
<i>EVII</i>	E_7	$E_6 \times \mathbb{C}^*$	$\circ - \circ - \bullet - \bullet - \bullet - \bullet - \circ$ 	C_3	$l = 3$

<i>EVIII</i>	E_8	D_8		E_8	$l = 8$
<i>EIX</i>	E_8	$E_7 \times A_1$		F_4	$l = 4$
<i>FI</i>	F_4	$C_3 \times A_1$		F_4	$l = 4$
<i>FII</i>	F_4	B_4		BC_1	$l = 1$
<i>G</i>	G_2	$A_1 \times A_1$		G_2	$l = 2$

§1.5. The sets N and $D(G/H)$

The set N equals $C^- \cap \chi_*(S)$; in particular, it consists of the lattice vectors of the rational, polyhedral, convex cone $C^- = \text{cone}(N)$. The set $\rho(D(G/H))$ equals $\overline{R}_{G,\theta}^\vee$ and any fiber $\rho^{-1}(\alpha^\vee)$ contains at most two colors.

Now, we explain the following property: *if ρ is injective, then the symmetric varieties are classified by the fan associated to the colored fan.* For any simple spherical variety X , $N(X)$ is formed by the primitive generators of the 1-faces of $C(X)$ which are contained in $\text{cone}(N)$. When X is symmetric, also $\rho(X)$ can be recovered from $C(X)$: its elements generate the 1-faces of $C(X)$ which are not contained in C^- . Indeed, given $\text{cone}(\alpha_1^\vee, \dots, \alpha_r^\vee, \omega_1^\vee, \dots, \omega_s^\vee)$ with $\omega_1^\vee, \dots, \omega_s^\vee$ antidominant, α_j^\vee is the unique generator of a cone such that $(\alpha_j^\vee, \alpha_j) > 0$.

Finally, we state some technical properties of $D(G/H)$, which we will use to study the local factoriality and the smoothness of a symmetric variety. If (G, θ) is indecomposable, then the number of colors is at most $\text{rank}(G/H) + 1$. If moreover $\sharp D(G/H) > \text{rank}(G/H)$, we say that G/H is *Hermitian*; we have two possibilities (see [Ru07, pp. 7–8] or [T06, pp. 177–178]): 1) $G^\theta = H = N_G(G^\theta)$; 2) $H = G^\theta$ and $[G^\theta : N_G(G^\theta)] = 2$. In the last case any element of $N_G(G^\theta) \setminus G^\theta$ exchanges two colors and $R_{G,\theta}$ has type A_1, B_2 or C_n . We say that a simple restricted root α is *exceptional* if $\sharp \rho^{-1}(\alpha^\vee) = 2$ and 2α is a restricted root. In this case the irreducible factor of $R_{G,\theta}$ containing α is associated to an indecomposable factor of G/G^θ as in 1). We then say that also (G, θ) and any symmetric variety (with open orbit G/H) are exceptional. We denote by D_α the sum of the colors in $\rho^{-1}(\alpha^\vee)$ and by D_ω the G -stable divisor corresponding to $(\mathbb{R}^{\geq 0}\omega, \emptyset) \in \mathbb{F}(X)$.

§1.6. Wonderful symmetric varieties

In this section we define the wonderful compactification of G/H , while in the following section we will use this variety to describe the toroidal symmetric varieties. A wonderful compactification exists only if $N(H)/H$ is finite; in particular the connected center of G acts trivially on G/H . Thus we can suppose that G is semisimple. In this case one can define a canonical compactification which has many properties. This compactification, called the *standard compactification* X_0

(or the standard symmetric variety), is associated to $(\text{cone}(N), \emptyset)$ and it is the maximal simple compactification of G/H in the dominant order. We define e_i as the primitive positive multiple of $-\omega_i^\vee$ (in $\chi_*(S)$), so $\Delta_{X_0}[p] = \{e_1, \dots, e_l\}$.

The standard compactification is called *wonderful* if it is also smooth. A standard symmetric variety is wonderful if and only if $\chi_*(S) = \bigoplus \mathbb{Z}e_i$. De Concini and Procesi have proved that X_0 is wonderful if $H = N_G(G^\theta)$, or equivalently $\chi_*(S) = \bigoplus \mathbb{Z}\omega_i^\vee$ (see [dCoPr83, Theorem 3.1]). Looking at the Cartan matrix of $R_{G,\theta}$ it is not difficult to prove that a standard indecomposable symmetric variety X is wonderful exactly in the following two cases: (i) $H = N_G(G^\theta)$; (ii) $R_{G,\theta}$ has type A_1 , B_2 or C_1 .

A standard (resp. wonderful) symmetric variety is the standard (resp. wonderful) compactification of a symmetric space.

§1.7. Toroidal symmetric varieties

In this section we want to define a special class of varieties, which are closely related to toric varieties. We say that a spherical variety is *toroidal* if $D(X) = \emptyset$. We want to explain how to associate a toric variety Z to a toroidal symmetric variety X . Many local properties of X can be studied on Z . Moreover, in the toroidal case the key step of our classification is the study of the anticanonical bundle of the associated toric varieties. In the rest of this section we suppose that G is semisimple.

The standard compactification X_0 contains an affine toric S -variety Z_0 , which is a quotient of an affine space by a finite group; moreover $\dim Z_0 = \text{rank } X_0$. The toroidal varieties are the symmetric varieties which dominate the standard compactification and are in one-to-one correspondence with the S -toric varieties which dominate Z_0 (see [dCoPr83, §2.3] and [dCoPr85, §5.2]).

Let P be the stabilizer of the B -stable affine open set $U := X_0 \setminus \bigcup_{D \in D(G/H)} \overline{D}$. This open set is P -isomorphic to $R_u P \times Z_0$, where $R_u P = \prod_{\beta \in R_G^{1,+}} U_\beta$ is the unipotent radical of P . To any toroidal variety X we associate the inverse image Z of Z_0 under the projection $X \rightarrow X_0$. Moreover, $X \setminus \bigcup_{D \in D(G/H)} \overline{D}$ is P -isomorphic to $R_u P \times Z$. Thus the restriction of the anticanonical bundle of X to Z is isomorphic to the anticanonical bundle of Z . In particular, if X is Fano then $-K_Z$ is ample. Note that Z_0 is an affine space if and only if X_0 is wonderful.

§1.8. The Picard group

The class group of a symmetric variety is generated by the classes of the B -stable prime divisors modulo the relations $\text{div}(f)$ with $f \in \mathbb{C}(G/H)^{(B)}$. Indeed $\text{Cl}(BH/H) = \text{Pic}(BH/H)$ is trivial. Given $\omega \in \chi(S)$ we denote by f_ω the element of $\mathbb{C}(G/H)^{(B)}$ with weight ω and such that $f_\omega(H/H) = 1$.

A Weyl divisor $\sum_{D \in D(G/H)} a_D D + \sum_{E \in N(X)} b_E E$ is a Cartier divisor if and only if, for any $(C, F) \in \mathcal{F}(X)$, there is $h_C \in \chi(S)$ such that $h_C(E) = a_E$ for all $E \in C$ and $h_C(\rho(D)) = a_D$ for all $D \in F$. Let $PL'(X)$ be the set of functions on the support $|\mathcal{F}(X)|$ that are: 1) linear on each colored cone; 2) integer on $\chi_*(S) \cap |\mathcal{F}(X)|$. Let $L(X) \subset PL'(X)$ be the subset composed of the restrictions of linear functions and let $PL(X) := PL'(X)/L(X)$. Then $\{h_C\}$, corresponding to any Cartier divisor, defines an element of $PL(X)$. If X is compact, there is an exact sequence (see [Br89, Théorème 3.1])

$$0 \rightarrow \bigoplus_{D \in D(G/H) \setminus D(X)} \mathbb{Z}D \rightarrow \text{Pic}(X) \rightarrow PL(X) \rightarrow 0.$$

A Cartier divisor is globally generated (resp. ample) if and only if the associated function is convex (resp. strictly convex) and $h_C(\rho(D)) \leq a_D$ (resp. $h_C(\rho(D)) < a_D$) for all $(C, F) \in \mathcal{F}(X)$ and for all $D \in D(G/H) \setminus F$. Thus, a Cartier divisor on a projective symmetric variety is nef if and only if it is globally generated. Given any linearized line bundle L , the space $H^0(X, L)$ is a multiplicity free G -module and, if L is globally generated, the highest weights of $H^0(X, L)$ are the elements of $\chi(S) \cap \text{hull}(\{h_C\}_{\dim C=l})$, where $l = \text{rank}(G/H)$ and $\text{hull}(\{x_1, \dots, x_m\})$ is the convex hull of x_1, \dots, x_m (see [Br89, §3]). Moreover, a nef G -stable Cartier divisor on a projective symmetric variety is big if and only if the associated piecewise linear function h is such that $(\sum_{C \in \Delta(l)} h_C, R^\vee) \neq 0$ for each irreducible factor R^\vee of $R_{G,\theta}^\vee$ (see [Ru09, Theorem 4.2]). In particular, when θ is indecomposable every non-zero nef G -stable divisor is big.

When X is toroidal we have an exact split sequence

$$0 \rightarrow \text{Pic}(X_0) \rightarrow \text{Pic}(X) \rightarrow \text{Pic}(Z) \rightarrow 0.$$

A normal variety X is *locally factorial* if the Picard group is isomorphic to the class group, while X is \mathbb{Q} -factorial if $\text{Pic}(X)_\mathbb{Q} \cong \text{Cl}(X)_\mathbb{Q}$. A simple symmetric variety associated to a colored cone (C, F) is locally factorial if: (i) C is generated by a subset of a basis of $\chi_*(S)$ and (ii) ρ is injective over F (see [Br97b] for a general statement in the spherical case). When the variety is toroidal, local factoriality is equivalent to smoothness.

We can describe an anticanonical divisor by [Br97b, Theorem 4.2] and by the local description of X_0 ; we also use the fact that, by the cited theorem, the non- G -stable part of such a divisor depends only on the open G -orbit. An anticanonical divisor $-K_X$ of X is $\sum_{\alpha \in \overline{R}_{G,\theta}^\vee} a_\alpha D_\alpha + \sum_{E \in N(X)} E$ with $\sum a_\alpha \omega_\alpha = 2\delta - 2\delta_0$. Here $2\delta := 2\delta_{R_G} = \sum_{\alpha \in R_G^+} \omega_\alpha$ is the sum of all the positive roots of R_G , while $2\delta_0 := 2\delta_{R_G^0}$ is the sum of the positive roots in R_G^0 .

Let k (or k_X) be the piecewise linear function associated to $-K_X$. The anticanonical divisor $-K_X$ is linearly equivalent to a unique G -stable divisor $-\tilde{K}_X$. The piecewise linear function \tilde{k} (or \tilde{k}_X) associated to $-\tilde{K}_X$ equals $k - 2\delta + 2\delta_0$ over $N(X)$ and to 0 over $\rho(D(X))$. Indeed $-\tilde{K}_X$ is $-K_X + \text{div}(\prod_{\alpha \in \bar{R}_{G,\theta}} f_\alpha^{a_\alpha})$, where $f_\alpha \in \mathbb{C}(G/H)^{(B)}$ is an equation of D_α (of weight ω_α). In particular $\tilde{k} = k - 2\delta + 2\delta_0$ if X is toroidal.

§2. Statement of the main theorems

In this section we state the main theorems of this work. *In the following, unless explicitly stated, we always suppose that G is semisimple (we will consider the general reductive case mainly in §2.4.3 and §6.3). Moreover, we often denote the normalizer $N_G(H)$ by $N(H)$.*

§2.1. Fano standard varieties

When the rank of G/H is one, the standard compactification X_0 of G/H is the unique G -equivariant compactification. In that case X_0 is a homogeneous projective variety with respect to $\text{Aut}^0(X)$ by [A83]; moreover it is wonderful and Fano, because either it is $\mathbb{P}^n \times \mathbb{P}^n$ or it has Picard number one. A standard symmetric variety is always \mathbb{Q} -factorial; in particular, K_X is a \mathbb{Q} -Cartier divisor. We prove that in general a standard symmetric variety is a quasi- \mathbb{Q} -Fano variety; moreover we classify the Fano ones.

First of all, we reduce ourselves to the indecomposable case. Write (G, θ) as a product $\prod(G_j, \theta)$ of indecomposable involutions and let X_j be a standard compactification of $G_j/(G_j \cap H)$. If X is wonderful then also the X_j are wonderful and $X = \prod X_j$ (see [Ru09, Corollary 2.1]).

Lemma 2.1. *The variety X is (quasi-) \mathbb{Q} -Fano if and only if all the X_j are (quasi-) \mathbb{Q} -Fano.*

Proof. The weight \tilde{k}_X equals $\sum \tilde{k}_{X_j}$. □

In §6.3 we will prove the following result, which in particular implies Theorem A.

Theorem 2.1. *Let X be a standard indecomposable symmetric variety. Let n be the rank of G and let l be the rank of X . Then:*

- *The anticanonical divisor of X is always a nef and big \mathbb{Q} -divisor.*
- *Suppose X is wonderful. Then it is not a Fano variety if and only if the following three properties hold: (i) if the involution induced on $\chi(S)_\mathbb{R}$ is $-\text{id}$; (ii) $R_{G,\theta}$ is different from A_l and B_l ; (iii) $H = N_G(G^\theta)$.*

Table 2. Non- \mathbb{Q} -Fano standard indecomposable symmetric varieties

G/H	θ	n, l	wonderful
$Spin_{2n+1}/(Spin_1 \times Spin_{2n+1-l})$	BI	$n \geq l \geq 4$	no
$Spin_7/(Spin_3 \times Spin_4)$	BI	$n = l = 3$	no
$Sp_{2n}/N(GL_n)$	CI	$n = l \geq 3$	yes
$Spin_{2l}/N(Spin_l \times Spin_l)$	DI	$n = l \geq 4$	yes
$Spin_{2n}/H$ ($G^\theta \subset H \subsetneq N(G^\theta)$)	DI	$n = l \geq 6$	no
$Spin_{2n}/(Spin_1 \times Spin_{2n-l})$	DI	$n > l \geq 4$	no
$Spin_8/(Spin_3 \times Spin_5)$	DI	$n = 4, l = 3$	no
$E_6/N(C_4)$	EI	$n = l = 6$	yes
$E_7/N(A_7)$	EV	$n = l = 7$	yes
E_7/A_7	EV	$n = l = 7$	no
E_8/D_8	EVIII	$n = l = 8$	yes
$F_4/(C_3 \times A_1)$	FI	$n = l = 4$	yes
$G_2/(A_1 \times A_1)$	G	$n = l = 2$	yes

- The standard indecomposable varieties whose anticanonical divisor is not ample are compactifications of the symmetric spaces in Table 2.

§2.2. Fano blow-ups of a wonderful symmetric variety along closed orbits

In this section we state a partial result in arbitrary rank. We restrict ourselves to the smooth toroidal case. In §1.7 we have associated a toric variety Z to any toroidal symmetric variety X ; moreover $-K_Z$ is ample if X is Fano. For toric varieties of rank 2 one can easily prove the following property:

- (*) Let Z be a smooth toric variety of rank 2 and let Z' be a smooth toric variety birationally proper over Z . If the anticanonical bundle of Z' is ample, then also the anticanonical bundle of Z is ample.

This allows one to prove easily that a smooth toric variety proper over \mathbb{A}^2 with ample anticanonical bundle is either \mathbb{A}^2 or its blow-up at the origin. We would use a property like (*) to classify the smooth toroidal Fano symmetric varieties. More precisely, the idea of the proof is to proceed as follows: 1) prove that for any non-Fano symmetric variety X and any blow-up X' of X , X' is not Fano (generalizing (*)); 2) find an explicit set of non-Fano varieties $\{X_i\}$ such that there are finitely many symmetric varieties that do not dominate any X_i ; 3) study these last varieties.

Unfortunately this strategy is too naive. Actually, property (*) is false already in rank three. Indeed, let \bar{Z} be the 3-dimensional toric variety whose fan $\bar{\Delta}$ has

maximal cones $\text{cone}(e_1, e_2, e_3)$ and $\text{cone}(e_1, e_2, e_1 + e_2 - e_3)$, where $\{e_1, e_2, e_3\}$ is any basis of $\chi_*(S)$. The function associated to its anticanonical bundle is linear, so $-K_{\bar{Z}}$ is nef but non-ample. Furthermore, the blow-up \bar{Z}' of \bar{Z} along $\text{cone}(e_1, e_2)$ is Fano. We can take $\bar{Z} \times \mathbb{A}^m$ as higher dimensional example.

In the previous example, we have considered a blow-up along a subvariety with strictly positive dimension. In Section 4 we will prove a property similar to (*) considering only blow-ups along compact orbits, i.e. S -fixed points. This property will allow us to prove the following classification, which in particular implies Theorem B. We also prove a more precise result when the rank is three (see the next section).

Theorem 2.2. *Let G/H be a symmetric space of rank $l (> 1)$ associated to an involution θ and let X be a symmetric variety obtained from the wonderful compactification X_0 of G/H by a sequence of blow-ups along closed orbits.*

Table 3. Fano $X_{1,\dots,l}$

G'/H'	$\theta _{G'}$	$\text{rank}(G/H)$	$\text{rank}(G'/H')$
SL_3	A_2	$l = 2$	2
$Spin_5$	B_2	$l = 2$	2
$SL_6/N(Sp_6)$	AII	$l = 3$	2
$SL_{n+1}/S(GL_2 \times GL_{n-1}), n \geq 4$	$AIII$	$l = 2$	2
$Sp_{2n}/(Sp_4 \times Sp_{2n-4}), n \geq 5$	CII	$l \leq 3$	2
$Sp_8/(Sp_4 \times Sp_4)$	CII	$l = 3$	2
$Sp_8/N(Sp_4 \times Sp_4)$	CII	$l = 2$	2
SO_{10}/GL_5	$DIII$	$l \leq 3$	2
$E_6/N(D_5 \times \mathbb{C}^*)$	$EIII$	$l \leq 4$	2
$E_6/N(F_4)$	EIV	$l \leq 5$	2
PSL_2	A_1	$l = 2$	1
SL_2	A_1	$l \leq 3$	1
$SL_{n+1}/N(SO_n)$	AI	$l = 2$	1
SL_{n+1}/SO_n	AI	$l = 2$	1
$SL_6/N(Sp_6)$	AII	$l \leq 3$	1
SL_6/Sp_6	AII	$l \leq 5$	1
$SL_{n+1}/N(S(GL_1 \times GL_n))$	AIV	$l \leq n + 1$	1
$SO_{2n+1}/N(SO_1 \times SO_{2n})$	BII	$l \leq n + 1$	1
$SO_{2n+1}/(SO_1 \times SO_{2n})$	BII	$l \leq 2n$	1
$Sp_{2n}/(Sp_2 \times Sp_{2n-2})$	CII	$l \leq 2n$	1
$SO_{2n}/N(SO_1 \times SO_{2n-1})$	DII	$l \leq n$	1
$SO_{2n}/(SO_1 \times SO_{2n-1})$	DII	$l \leq 2n - 1$	1
F_4/B_4	FII	$l \leq 12$	1

1. If X is a Fano variety then it is either the wonderful variety X_0 or the blow-up $X_{1,\dots,l}$ of X_0 along the closed orbit.
2. If there is an indecomposable factor of (G, θ) of rank at least 3 then $X_{1,\dots,l}$ is not Fano.
3. If (G, θ) has rank at least 6 and has an indecomposable factor of rank 2, then $X_{1,\dots,l}$ is not Fano.
4. If $X_{1,\dots,l}$ is Fano, the possibilities for the indecomposable factors of G/H are as in Table 3 (we also indicate the conditions on $\text{rank}(G/H)$ so that such a factor can appear).

§2.3. Fano symmetric varieties of rank 3

In this section we suppose X_0 is wonderful of rank three; recall that $\{e_1, e_2, e_3\}$ is the basis of $\chi_*(S)$ which generates C^- . We classify all the Fano symmetric varieties obtainable from X_0 by a succession of blow-ups along G -subvarieties. This class of varieties contains many varieties; indeed each compact symmetric variety is dominated by a smooth toroidal variety obtained by a succession of blow-ups along G -subvarieties of codimension two (see [dCoPr85, Proposition 2.4]).

Given a toric variety Z (above \mathbb{A}^3) that does not contain a subvariety isomorphic to \bar{Z} we will prove a property very similar to $(*)$ (see Lemma 5.1). Instead, if Z contains \bar{Z} we will prove that it is sufficient to study the blow-up of Z along the subvariety corresponding to $\text{cone}(e_1, e_2)$ (see Lemma 5.3). These lemmas allow us to prove the following classification, which implies Theorem E. We state separately the classification depending on whether (G, θ) is indecomposable, is a product of two indecomposable involutions, or a product of three involutions. We introduce the following notation:

- X_{ij} is the blow-up of X_0 along $(\text{cone}(e_i, e_j), \emptyset)$;
- X_{123} is the blow-up of X_0 along the closed G -orbit;
- $X_{123,ij}$ is the blow-up of X_{123} along $(\text{cone}(e_i, e_j), \emptyset)$;
- $X_{123,i}$ is the blow-up of X_{123} along $(\text{cone}(e_i, e_1 + e_2 + e_3), \emptyset)$.

By [Ru09, Corollary 2.1], if $(G, \theta) = (G_1, \theta) \times (G_2, \theta)$ and X_0 is wonderful, then $H = H_1 \times H_2$, where $H_i := H \cap G_i$. Given a 1-rank symmetric space G_i/H_i , let $\psi_i(r) := -2\delta + 2\delta_0 - re_i^*$, $m_i := \max\{r : \psi_i(r)(\alpha_i^\vee) < 0\}$ and $\bar{m}_i := \max\{r : \psi_i(r)(\alpha_i^\vee) \leq 0\}$. Table 4 gives the values of m_i and \bar{m}_i for the various G_i/H_i . Moreover, we indicate by e (resp. h) when G_i/H_i is exceptional (resp. Hermitian non-exceptional).

Table 4. Weights of rank 1 symmetric spaces

G/H	θ	\bar{m}_1	m_1	
PSL_2	A_1	1	0	
SL_2	A_1	2	1	
$SL_2/N(SO_2)$	AI	0	0	h
SL_2/SO_2	AI	1	0	h
$SL_2/N(Sp_2)$	AII	2	1	
SL_2/Sp_2	AII	4	3	
$SL_n/S(GL_1 \times GL_{n-1})$	AIV	n	$n-1$	e
$SO_{2n+1}/N(SO_1 \times SO_{2n})$	BII	$n-1$	$n-1$	
$SO_{2n+1}/(SO_1 \times SO_{2n})$	BII	$2n-1$	$2n-2$	
$Sp_{2n}/(Sp_2 \times Sp_{2n-2})$	CII	$2n-1$	$2n-2$	
$SO_{2n}/S(O_1 \times O_{2n-1})$	DII	$n-1$	$n-2$	
$SO_{2n}/(SO_1 \times SO_{2n-1})$	DII	$2n-2$	$2n-3$	
F_4/B_4	FII	11	10	

Table 5. Fano indecomposable symmetric varieties of rank 3

G/H	θ	X
PSL_4	A_3	X_0, X_{13}
SO_7	B_3	X_0
PSp_6	C_3	X_0, X_{13}
Sp_6	C_3	X_0, X_{13}
$SL_4/N(SO_4)$	AI	X_0
$SL_8/N(Sp_8)$	AII	$X_0, X_{12}, X_{13}, X_{23}, X_{123,13}$
$SL_{n+1}/S(L_3 \times L_{n-2}), n \geq 6$	$AIII$	X_0, X_{13}
$SL_6/N(S(L_3 \times L_3))$	$AIII$	X_0
$SL_6/S(L_3 \times L_3)$	$AIII$	X_0
$SO_{2n+1}/N(SO_3 \times SO_{2n-2})$	BI	X_0
$Sp_6/N(GL_3)$	CI	\emptyset
Sp_6/GL_3	CI	X_0
$Sp_{12}/N(Sp_6 \times Sp_6)$	CII	$X_0, X_{12}, X_{13}, X_{23}$
$Sp_{12}/(Sp_6 \times Sp_6)$	CII	$X_0, X_{12}, X_{13}, X_{23}, X_{123,13}$
$Sp_{2n}/(Sp_6 \times Sp_{2n-6}), n > 6$	CII	$X_0, X_{12}, X_{13}, X_{23}, X_{123,13}$
$SO_{2n}/N(SO_3 \times SO_{2n-3})$	DI	X_0
$SO_{12}/N(GL_6)$	$DIII$	X_0, X_{12}
$SO_{12}/N(GL_6)$	$DIII$	X_0, X_{12}
SO_{14}/GL_7	$DIII$	$X_0, X_{12}, X_{13}, X_{23}, X_{123,13}$
$E_7/N(E_6 \times \mathbb{C}^*)$	$EVII$	X_0, X_{12}
$E_7/(E_6 \times \mathbb{C}^*)$	$EVII$	X_0, X_{12}

Theorem 2.3. *Let G/H be an indecomposable symmetric space of rank 3 such that its standard compactification X_0 is wonderful. If X is a smooth Fano compactification of G/H obtained from X_0 by a sequence of blow-ups along G -subvarieties, then it is $X_0, X_{12}, X_{13}, X_{23}$ or $X_{123,13}$. More precisely, the Fano ones are those appearing in Table 5.*

Theorem 2.4. *Let G/H be a symmetric space such that X_0 is wonderful and $(G, \theta) = (G_1, \theta) \times (G_2, \theta)$ with $\text{rank}(G_i/G_i^\theta) = i$. If X is a smooth Fano compactification of G/H obtained from X_0 by a sequence of blow-ups along G -subvarieties, then it is $X_0, X_{12}, X_{13}, X_{23}, X_{123}, X_{123,12}, X_{123,13}, X_{123,23}, X_{123,1}, X_{123,12}$ or $X_{123,3}$. More precisely, the classification of such varieties is as in Table 6. In the second column, we indicate the conditions on m_1 so that X is Fano.*

Theorem 2.5. *Let G/H be a symmetric space such that X_0 is wonderful. Suppose that $(G, \theta) = (G_1, \theta) \times (G_2, \theta) \times (G_3, \theta)$ with $\text{rank}(G_i/G_i^\theta) = 1$ and let x_r be the number of factors G_i such that $\psi_i(r)$ is antidominant and regular. If X is a smooth Fano compactification of G/H obtained from X_0 by a sequence of blow-ups along G -subvarieties, then it is $X_0, X_{12}, X_{13}, X_{23}, X_{123}, X_{123,12}, X_{123,13}, X_{123,23}, X_{123,1}, X_{123,2}$ or $X_{123,3}$. More precisely, we have the following classification (depending on G/H):*

- *If $x_1 \leq 1$, then the smooth Fano compactifications of G/H are X_0, X_{12}, X_{13} and X_{23} . In particular, there are four of them.*
- *If $x_1 = 2$, then there are five Fano varieties. Let $i < j$ be the indices such that $\psi_i(1)$ and $\psi_j(1)$ are anti-dominant and regular. The smooth Fano compactifications of G/H are $X_0, X_{12}, X_{13}, X_{23}$ and $X_{123,ij}$.*
- *If (x_1, x_2) is to $(3, 0)$ or $(3, 1)$, then there are eight Fano varieties: $X_0, X_{12}, X_{13}, X_{23}, X_{123}, X_{123,12}, X_{123,13}$ and $X_{123,23}$.*
- *If $(x_1, x_2) = (3, 2)$, then there are nine Fano varieties. Suppose that, for all $j \neq i$, $\psi_j(2)$ is anti-dominant and regular. Then the Fano varieties are $X_0, X_{12}, X_{13}, X_{23}, X_{123}, X_{123,12}, X_{123,13}, X_{123,23}$ and $X_{123,i}$.*
- *If $(x_1, x_2) = (3, 3)$, then there are eleven Fano varieties: $X_0, X_{12}, X_{13}, X_{23}, X_{123}, X_{123,12}, X_{123,13}, X_{123,23}, X_{123,1}, X_{123,2}$ and $X_{123,3}$.*

We will see that a weaker statement holds in a more general context.

Corollary 2.1. *Given any wonderful G -variety X_0 of rank 3 (even non-symmetric) and any Fano variety X obtained from X_0 by a succession of blow-ups along G -stable subvarieties, then X is $X_0, X_{12}, X_{13}, X_{23}, X_{123}, X_{123,12}, X_{123,13}, X_{123,23}, X_{123,1}, X_{123,2}$ or $X_{123,3}$.*

Table 6. Fano decomposable symmetric varieties of rank 3

G_2/H_2	m_1	X
PSL_3	–	$X_0, X_{12}, X_{13}, X_{23}$
SO_5	–	X_0, X_{12}
$Spin_5$	–	$X_0, X_{12}, X_{13}, X_{23}$
G_2	–	X_0, X_{13}
$SL_3/N(SO_3)$	–	X_0
$SL_6/N(Sp_6)$	–	$X_0, X_{12}, X_{13}, X_{23}, X_{123,23}$
	$m_1 \geq 1$	$X_{123}, X_{123,12}, X_{123,13}$
$SL_{n+1}/S(GL_2 \times GL_{n-1})$	–	$X_0, X_{12}, X_{13}, X_{23}$
$(n \geq 5)$	$m_1 \geq 1$	$X_{123,13}$
$SL_5/S(GL_2 \times GL_3)$	–	$X_0, X_{12}, X_{13}, X_{23}$
$SL_4/N(S(GL_2 \times GL_2))$	–	X_0
$SL_4/S(GL_2 \times GL_2)$	–	X_0, X_{12}
$SO_5/S(O_2 \times O_3)$	–	X_0
$SO_5/(SO_2 \times SO_3)$	–	X_0
$SO_{2n+1}/S(O_2 \times O_{2n-1})$	–	X_0, X_{13}
$SO_{2n+1}/(SO_2 \times SO_{2n-1})$	–	X_0, X_{13}
$Sp_{2n}/(Sp_4 \times Sp_{2n-4})$	–	$X_0, X_{12}, X_{13}, X_{23}, X_{123,23}$
$(n \geq 5)$	$m_1 \geq 1$	$X_{123}, X_{123,12}, X_{123,13}$
	$m_1 \geq 2$	$X_{123,2}$
$Sp_8/N(Sp_4 \times Sp_4)$	–	$X_0, X_{12}, X_{13}, X_{23}$
$Sp_8/(Sp_4 \times Sp_4)$	–	$X_0, X_{12}, X_{13}, X_{23}, X_{123,23}$
	$m_1 \geq 1$	$X_{123}, X_{123,12}, X_{123,13}$
$SO_{2n}/S(O_2 \times O_{2n-2})$	–	X_0, X_{13}
$SO_{2n}/SO_2 \times SO_{2n-2}$	–	X_0, X_{13}
$SO_8/N(GL_4)$	–	X_0, X_{12}
SO_8/GL_4	–	X_0, X_{12}
SO_{10}/GL_5	–	$X_0, X_{12}, X_{13}, X_{23}, X_{123,23}$
	$m_1 \geq 1$	$X_{123}, X_{123,12}, X_{123,13}$
$E_6/(D_5 \times \mathbb{C}^*)$	–	$X_0, X_{12}, X_{13}, X_{23}, X_{123,23}$
	$m_1 \geq 1$	$X_{123}, X_{123,12}, X_{123,13}, X_{123,1}$
	$m_1 \geq 2$	$X_{123,2}, X_{123,3}$
$E_6/N(F_4)$	–	$X_0, X_{12}, X_{13}, X_{23}, X_{123,23}$
	$m_1 \geq 1$	$X_{123}, X_{123,12}, X_{123,13}, X_{123,1}$
	$m_1 \geq 2$	$X_{123,2}, X_{123,3}$
$G_2/(A_1 \times A_1)$	–	\emptyset

§2.4. (Quasi-) Fano symmetric varieties of rank 2

Finally we state the results about symmetric varieties of rank two. We define $\psi_i(r)$, m_i and \bar{m}_i as in the previous section. Note that, supposing $R_{G,\theta} = A_1 \times A_1$, the m_i have the same value if $\chi_*(S) = \mathbb{Z}(\omega_1^\vee + \omega_2^\vee) \oplus \mathbb{Z}2\omega_2^\vee$ or if $H = G^\theta$. Moreover, $G/H = \prod G_i/(H \cap G_i)$ if H is either G^θ or $N_G(G^\theta)$. So to find the values of the m_i it is sufficient to check Table 4.

Notice that to unequivocally specify a projective 2-rank symmetric variety with ρ injective over $\varrho^{-1}(\varrho(D(X)))$, it is sufficient to give $\Delta[p]$. Indeed, we have seen in §1.5 that if ρ is injective over $\varrho^{-1}(\varrho(D(X)))$ then Δ determines $\mathcal{F}(X)$. Moreover, Δ is specified by the maximal cones, which have maximal dimension if X is projective. When the rank is 2, the maximal cones are $\text{cone}(v, w)$ with $v, w \in \Delta[p]$ and such that: (i) $v \neq w$; (ii) $\text{cone}(v, w) \cap \Delta[p] = \{v, w\}$; (iii) the interior of $\text{cone}(v, w)$ intersects C^- .

2.4.1. Fano symmetric varieties. First we consider the locally factorial (resp. smooth) Fano symmetric varieties with G semisimple.

Theorem 2.6. *Let G/H be a symmetric space of rank 2 (with G semisimple).*

- If a (projective) symmetric variety X is Fano then $\rho^{-1}(\rho(D(X))) = D(X)$. If moreover X is locally factorial, then $\sharp\rho(D(X)) = \sharp\rho^{-1}(\rho(D(X)))$.
- If $R_{G,\theta}$ is irreducible and X is a Fano locally factorial equivariant compactification of G/H , we have exactly the following possibilities for $\Delta[p]$:
 1. $\Delta[p] = \{e_1, \alpha_1^\vee\}$ if $\sharp\rho^{-1}(\alpha_1^\vee) = 1$ and $R_{G,\theta}$ is not G_2 ;
 2. $\Delta[p] = \{e_2, \alpha_2^\vee\}$ if $\sharp\rho^{-1}(\alpha_2^\vee) = 1$ and $R_{G,\theta}$ is not B_2 ;
 3. $\Delta[p] = \{e_2, \alpha_2^\vee\}$ if $R_{G,\theta} = B_2$ and $H = G^\theta$;
 4. $\Delta[p] = \{-\omega_1^\vee, -\omega_2^\vee\}$ if $G/H \neq G_2/(SL_2 \times SL_2)$ and $H = N(G^\theta)$;
 5. $\Delta[p] = \{-2\omega_1^\vee, -\omega_2^\vee\}$ if $R_{G,\theta} = B_2$ and $H = G^\theta$;
 6. $\Delta[p] = \{\alpha_1^\vee, \alpha_2^\vee, -\alpha_1^\vee - \alpha_2^\vee\}$ if $R_{G,\theta} = A_2$ and $H = G^\theta$;
 7. $\Delta[p] = \{\alpha_1^\vee, -\omega_1^\vee - \omega_2^\vee, -3\omega_1^\vee\}$ (and $\Delta[p] = \{\alpha_2^\vee, -\omega_1^\vee - \omega_2^\vee, -3\omega_2^\vee\}$) if $H = G^\theta$ and $R_{G,\theta} = A_2$;
 8. $\Delta[p] = \{\alpha_1^\vee, -\omega_1^\vee - \omega_2^\vee, -4\omega_1^\vee - \omega_2^\vee, -3\omega_1^\vee\}$ (and $\Delta[p] = \{\alpha_2^\vee, -\omega_1^\vee - \omega_2^\vee, -\omega_1^\vee - 4\omega_2^\vee, -3\omega_2^\vee\}$) if $H = G^\theta$, $R_{G,\theta} = A_2$ and $\theta \neq -\text{id}$ over $\chi_*(T)$;
 9. $\Delta[p] = \{e_1, e_2, e_1 + e_2\}$ if $\chi_*(S) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ and $-2\delta + 2\delta_0 + e_i^* \in \text{int}(C^-)$ for each i .

The previous varieties are singular in the following cases:

- $\Delta[p] = \{e_i, \alpha_i^\vee\}$ and $i = 1, 2$ if $R_{G,\theta} = A_2$;

- $\Delta[p] = \{e_1, \alpha_1^\vee\}$ if $R_{G,\theta} = B_2$, $H = G^\theta$ (and $\sharp D(G/H) = 2$);
- $\Delta[p] = \{e_2, \alpha_2^\vee\}$ if $R_{G,\theta} = BC_2$ (and $\sharp D(G/H) = 2$).
- Suppose $(G, \theta) = (G_1, \theta) \times (G_2, \theta)$ and $\chi_*(S) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$; we have the following locally factorial Fano varieties:
 1. $\Delta[p] = \{e_1, e_2\}$;
 2. $\Delta[p] = \{e_1, e_2, e_1 + e_2\}$;
 3. $\Delta[p] = \{\alpha_1^\vee, -\alpha_1^\vee, -r\alpha_1^\vee + e_2\}$ if $r \leq m_2 + 1$ and $\sharp \rho^{-1}(\alpha_1^\vee) = 1$;
 4. $\Delta[p] = \{\alpha_2^\vee, -\alpha_2^\vee, e_1 - r\alpha_2^\vee\}$ if $r \leq m_1 + 1$ and $\sharp \rho^{-1}(\alpha_2^\vee) = 1$;
 5. $\Delta[p] = \{\alpha_1^\vee, -\alpha_1^\vee, -r\alpha_1^\vee + e_2, -(r+1)\alpha_1^\vee + e_2\}$ if $r \leq m_2$ and $\sharp \rho^{-1}(\alpha_1^\vee) = 1$;
 6. $\Delta[p] = \{\alpha_2^\vee, -\alpha_2^\vee, e_1 - r\alpha_2^\vee, e_1 - (r+1)\alpha_2^\vee\}$ if $r \leq m_1$ and $\sharp \rho^{-1}(\alpha_2^\vee) = 1$;
 7. $\Delta[p] = \{\alpha_1^\vee, \alpha_2^\vee, -\alpha_1^\vee - \alpha_2^\vee\}$ if $\sharp D(G/H) = 2$.

Only the first two are smooth.

- If G/H is decomposable and $\chi_*(S) = \mathbb{Z}(\frac{1}{2}\alpha_1^\vee + \frac{1}{2}\alpha_2^\vee) \oplus \mathbb{Z}\alpha_2^\vee$, we have the following locally factorial Fano compactifications of G/H :
 1. $\Delta[p] = \{\alpha_1^\vee, \alpha_2^\vee, -\frac{1}{2}\alpha_1^\vee - \frac{1}{2}\alpha_2^\vee\}$;
 2. $\Delta[p] = \{\alpha_1^\vee, -\alpha_1^\vee, -\frac{2r+1}{2}\alpha_1^\vee - \frac{1}{2}\alpha_2^\vee\}$ if $0 \leq r \leq \frac{m_2+1}{2}$;
 3. $\Delta[p] = \{\alpha_2^\vee, -\alpha_2^\vee, -\frac{1}{2}\alpha_1^\vee - \frac{2r+1}{2}\alpha_2^\vee\}$ if $0 \leq r \leq \frac{m_1+1}{2}$;
 4. $\Delta[p] = \{\alpha_1^\vee, -\alpha_1^\vee, -\frac{2r+1}{2}\alpha_1^\vee - \frac{1}{2}\alpha_2^\vee, -\frac{2r+3}{2}\alpha_1^\vee - \frac{1}{2}\alpha_2^\vee\}$ if $0 \leq r \leq \frac{m_2-1}{2}$;
 5. $\Delta[p] = \{\alpha_2^\vee, -\alpha_2^\vee, -\frac{1}{2}\alpha_1^\vee - \frac{2r+1}{2}\alpha_2^\vee, -\frac{1}{2}\alpha_1^\vee - \frac{2r+3}{2}\alpha_2^\vee\}$ if $0 \leq r \leq \frac{m_1-1}{2}$.

The first one is smooth, while the other ones are smooth if and only if $r = 0$.

Proposition 2.1. *Let G/H be a symmetric space of rank 2 with X_0 smooth. If θ is indecomposable, the Fano toroidal compactifications of G/H are as in Table 7. If θ is decomposable, the Fano toroidal compactifications of G/H are the following:*

- $\Delta[p] = \{e_1, e_2\}$;
- $\Delta[p] = \{e_1, e_2, re_1 + e_2\}$ with $r \leq m_2 + 1$;
- $\Delta[p] = \{e_1, e_2, e_1 + re_2\}$ with $r \leq m_1 + 1$.

2.4.2. Smooth quasi-Fano varieties. Now, we consider the smooth (resp. locally factorial) quasi-Fano symmetric varieties with rank two and G semisimple. A Gorenstein (projective) variety is called *quasi-Fano* if its anticanonical divisor is big and nef.

Table 7. Fano toroidal indecomposable symmetric varieties with rank 2

G/H	θ	$\Delta[p]$	$\Delta[p]$
PSL_3	A_2	$\{-\omega_1^\vee, -\omega_2^\vee\}$	$\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee\}$
SO_3	B_2	$\{-\omega_1^\vee, -\omega_2^\vee\}$	
$Spin_3$	B_2	$\{-2\omega_1^\vee, -\omega_2^\vee\}$	$\{-2\omega_1^\vee, -\omega_2^\vee, -2\omega_1^\vee - \omega_2^\vee\}$
G_2	G_2	$\{-\omega_1^\vee, -\omega_2^\vee\}$	
$SL_3/N(SO_3)$	AI	$\{-\omega_1^\vee, -\omega_2^\vee\}$	
$SL_6/N(Sp_6)$	AII	$\{-\omega_1^\vee, -\omega_2^\vee\}$ $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee\}$	$\{-\omega_1^\vee, -\omega_2^\vee, -2\omega_1^\vee - \omega_2^\vee\}$ $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - 2\omega_2^\vee\}$
$SL_{n+1}/(GL_2 \times GL_{n-1})$	$AIII$	$\{-\omega_1^\vee, -\omega_2^\vee\}$	$\{-\omega_1^\vee, -\omega_2^\vee, -r\omega_1^\vee - \omega_2^\vee\}$, $r \leq n - 3$
$SL_4/N(S(GL_2 \times GL_2))$	$AIII$	$\{-\omega_1^\vee, -\omega_2^\vee\}$	
$SL_4/S(GL_2 \times GL_2)$	$AIII$	$\{-\omega_1^\vee, -2\omega_2^\vee\}$	
$SO_{2n+1}/(O_2 \times O_{2n-1})$	BI	$\{-\omega_1^\vee, -\omega_2^\vee\}$	
$SO_{2n+1}/SO_2 \times SO_{2n-1}$	BI	$\{-2\omega_1^\vee, -\omega_2^\vee\}$	
$Sp_{2n}/(Sp_4 \times Sp_{2n-4})$	CII	$\{-\omega_1^\vee, -\omega_2^\vee\}$	$\{-\omega_1^\vee, -\omega_2^\vee, -r\omega_1^\vee - \omega_2^\vee\}$, $r \leq 2n - 6$
$n \geq 5$		$\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - 2\omega_2^\vee\}$	
$Sp_8/N(Sp_4 \times Sp_4)$	CII	$\{-\omega_1^\vee, -\omega_2^\vee\}$	$\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee\}$
$Sp_8/(Sp_4 \times Sp_4)$	CII	$\{-\omega_1^\vee, -2\omega_2^\vee\}$ $\{-\omega_1^\vee, -2\omega_2^\vee, -\omega_1^\vee - 2\omega_2^\vee\}$	$\{-\omega_1^\vee, -2\omega_2^\vee, -2\omega_1^\vee - 2\omega_2^\vee\}$ $\{-\omega_1^\vee, -2\omega_2^\vee, -\omega_1^\vee - 4\omega_2^\vee\}$
$SO_{2n}/(O_2 \times O_{2n-2})$	DI	$\{-\omega_1^\vee, -\omega_2^\vee\}$	
$SO_{2n}/(SO_2 \times SO_{2n-2})$	DI	$\{-2\omega_1^\vee, -\omega_2^\vee\}$	
$SO_8/N(SO_4)$	$DIII$	$\{-\omega_1^\vee, -\omega_2^\vee\}$	
SO_8/SO_4	$DIII$	$\{-\omega_1^\vee, -2\omega_2^\vee\}$	
SO_{10}/SO_5	$DIII$	$\{-\omega_1^\vee, -\omega_2^\vee\}$ $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - 2\omega_2^\vee\}$	$\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee\}$ $\{-\omega_1^\vee, -\omega_2^\vee, -2\omega_1^\vee - \omega_2^\vee\}$
$E_6/(D_5 \times \mathbb{C}^*)$	$EIII$	$\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - r\omega_2^\vee\}$, $r \leq 3$	$\{-\omega_1^\vee, -\omega_2^\vee, -r\omega_1^\vee - \omega_2^\vee\}$, $r \leq 4$
		$\{-\omega_1^\vee, -\omega_2^\vee\}$	
$E_6/N(F_4)$	EIV	$\{-\omega_1^\vee, -\omega_2^\vee\}$ $\{-\omega_1^\vee, -\omega_2^\vee, -r\omega_1^\vee - \omega_2^\vee\}$, $r \leq 4$	$\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - r\omega_2^\vee\}$, $r \leq 4$
$G_2/(A_1 \times A_1)$	G	$\#$	

Theorem 2.7. *Let G/H be a symmetric space of rank 2 (with G semisimple).*

- *The nefness of the anticanonical bundle of a compactification of G/H depends only on the fan associated to the colored fan (and not on the whole colored fan).*
- *The fans of the locally factorial quasi-Fano (but non-Fano) compactifications of an indecomposable symmetric space of rank 2 (with G semisimple) are those in Table 8 (we also have to require that ρ is injective over $D(X)$). Such a variety is singular if and only if $\rho(D(X))$ contains an exceptional root.*

Table 8. Quasi-Fano indecomposable symmetric varieties with rank 2

G/H	θ	$\Delta[p]$
PSL_3	(A_2)	$\#$
SL_3	(A_2)	$\{\alpha_i^\vee, -\omega_i^\vee - \omega_{i^c}^\vee, -4\omega_i^\vee - \omega_{i^c}^\vee, -5\omega_i^\vee - 2\omega_{i^c}^\vee, -3\omega_i^\vee\}, i = 1, 2$
SO_5	(B_2)	$\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee\}$
$Spin_5$	(B_2)	$\{-2\omega_1^\vee, -\omega_2^\vee, -2\omega_1^\vee - \omega_2^\vee, -2\omega_1^\vee - 2\omega_2^\vee\}$
G_2	(G_2)	$\#$
$SL_3/N(SO_3)$	AI	$\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee\}$
SL_3/SO_3	AI	$\{\alpha_i^\vee, -3\omega_i^\vee, -\omega_i^\vee - \omega_{i^c}^\vee, -4\omega_i^\vee - \omega_{i^c}^\vee\}, i = 1, 2$
$SL_6/N(Sp_6)$	AII	$\{-\omega_1^\vee, -\omega_2^\vee, -\omega_i^\vee - \omega_{i^c}^\vee, -2\omega_i^\vee - \omega_{i^c}^\vee\}, i = 1, 2$
SL_6/Sp_6	AII	$\{\alpha_i^\vee, -\omega_i^\vee - \omega_{i^c}^\vee, \dots, -(r+3)\omega_i^\vee - r\omega_{i^c}^\vee, -3\omega_i^\vee\},$ $i = 1, 2; r = 2, 3, 4$
$SL_{n+1}/S(GL_2 \times GL_{n-1})$ $n \geq 4$	$AIII$	$\{\alpha_i^\vee, -\omega_i^\vee - \omega_{i^c}^\vee, -4\omega_i^\vee - \omega_{i^c}^\vee, -7\omega_i^\vee - \omega_{i^c}^\vee, -3\omega_i^\vee\}, i = 1, 2$ $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, \dots, -r\omega_1^\vee - \omega_2^\vee\}, 2 \leq r \leq n - 2$ $\{\alpha_2^\vee, -\omega_2^\vee\}$
$SL_4/N(S(GL_2 \times GL_2))$	$AIII$	$\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee\}$
$SL_4/S(GL_2 \times GL_2)$	$AIII$	$\{-\omega_1^\vee, -2\omega_2^\vee, -\omega_1^\vee - 2\omega_2^\vee\}$ $\{\alpha_2^\vee, -2\omega_2^\vee\}$
$SO_{2n+1}/S(O_2 \times O_{2n-1})$	BI	$\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, \dots, -r\omega_1^\vee - \omega_2^\vee\}, r \leq n - 2; n \geq 3$
$SO_5/S(O_2 \times O_3)$	BI	$\#$
$SO_{2n+1}/(SO_2 \times SO_{2n-1})$	BI	$\{-2\omega_1^\vee, -\omega_2^\vee, -2\omega_1^\vee - \omega_2^\vee, \dots, -2r\omega_1^\vee - \omega_2^\vee\}, r \leq n - 1$ $\{\alpha_1^\vee, -\omega_1^\vee\}$
$Sp_{2n}/(Sp_4 \times Sp_{2n-4}),$ $n > 4$	CII	$\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, \dots, -r\omega_1^\vee - \omega_2^\vee\}, 2 \leq r \leq n - 5$ $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, -\omega_1^\vee - 2\omega_2^\vee\}$
$Sp_8/N(Sp_4 \times Sp_4)$	CII	$\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, -2\omega_1^\vee - \omega_2^\vee\}$ $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, -\omega_1^\vee - 2\omega_2^\vee\}$
$Sp_8/(Sp_4 \times Sp_4)$	CII	$\{-\omega_1^\vee, -2\omega_2^\vee, -\omega_1^\vee - 2\omega_2^\vee, \dots, -r\omega_1^\vee - 2\omega_2^\vee\}, 2 \leq r \leq 3$ $\{-\omega_1^\vee, -2\omega_2^\vee, -\omega_1^\vee - 2\omega_2^\vee, -\omega_1^\vee - 4\omega_2^\vee\}$
$SO_{2n}/S(O_2 \times O_{2n-2})$	DI	$\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, \dots, -r\omega_1^\vee - \omega_2^\vee\}, r \leq n - 2$
$SO_{2n}/(SO_2 \times SO_{2n-2})$	DI	$\{-2\omega_1^\vee, -\omega_2^\vee, -2\omega_1^\vee - \omega_2^\vee, \dots, -2r\omega_1^\vee - \omega_2^\vee\}, r \leq n - 2$ $\{\alpha_1^\vee, -2\omega_1^\vee\}$

$SO_8/N(SO_4)$	<i>DIII</i>	$\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee\}$ $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, -\omega_1^\vee - 2\omega_2^\vee\}$
SO_8/SO_4	<i>DIII</i>	$\{-\omega_1^\vee, -2\omega_2^\vee, \dots, -\omega_1^\vee - 2r\omega_2^\vee\}, r = 1, 2$ $\{\alpha_2^\vee, -2\omega_2^\vee\}$
SO_{10}/SO_5	<i>DIII</i>	$\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, \dots, -r\omega_1^\vee - \omega_2^\vee\}, r = 2, 3$ $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, -\omega_1^\vee - 2\omega_2^\vee\}$ $\{\alpha_2^\vee, -\omega_2^\vee\}$
$E_6/(D_5 \times \mathbb{C}^*)$	<i>EIII</i>	$\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, \dots, -r\omega_1^\vee - \omega_2^\vee\}, 2 \leq r \leq 5$ $\{-\omega_1^\vee, -\omega_2^\vee, -\omega_1^\vee - \omega_2^\vee, \dots, -\omega_1^\vee - r\omega_2^\vee\}, 2 \leq r \leq 3$ $\{\alpha_2^\vee, -\omega_2^\vee\}$
$E_6/N(F_4)$	<i>EIV</i>	$\{-\omega_1^\vee, -\omega_2^\vee, -\omega_i^\vee - \omega_{i^c}^\vee, \dots, -r\omega_i^\vee - \omega_{i^c}^\vee\},$ $i = 1, 2; r = 2, 3, 4$
E_6/F_4	<i>EIV</i>	$\{\alpha_i^\vee, -\omega_i^\vee - \omega_{i^c}^\vee, \dots, -(r+3)\omega_i^\vee - r\omega_{i^c}^\vee, -3\omega_i^\vee\},$ $i = 1, 2; r = 2, \dots, 8$ $\{\alpha_i^\vee, -\omega_i^\vee - \omega_{i^c}^\vee, \dots, -(3r+1)\omega_i^\vee - \omega_{i^c}^\vee, -3\omega_i^\vee\},$ $i = 1, 2; r = 2, 3, 4$
$G_2/(A_1 \times A_1)$	<i>G</i>	$\{-\omega_1^\vee, -\omega_2^\vee\}$

• If $(G, \theta) = (G_1, \theta) \times (G_2, \theta)$, and $\chi_*(S) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, let $v_j(i) := -i\alpha_j^\vee + e_{j^c}$ and $w_j(x, y) := -(xy + 1)\alpha_j^\vee + ye_{j^c}$. We have the following locally factorial quasi-Fano compactifications of G/H , which are not Fano (we always suppose ρ injective over $D(X)$):

1. $\Delta[p] = \{\alpha_1^\vee, -\alpha_1^\vee, v_1(r), v_1(r+1), \dots, v_1(r+s)\}$ if (i) $s = 0, 1$, (ii) $r + s \leq \bar{m}_2 + 1$, and (iii) either $r + s > \bar{m}_2 + 1$ or $\sharp\rho^{-1}(\alpha_1^\vee) = 2$;
2. $\Delta[p] = \{\alpha_2^\vee, -\alpha_2^\vee, v_2(r), v_2(r+1), \dots, v_2(r+s)\}$ if (i) $s = 0, 1$, (ii) $r + s \leq \bar{m}_1 + 1$, and (iii) either $r + s > \bar{m}_1 + 1$ or $\sharp\rho^{-1}(\alpha_2^\vee) = 2$;
3. $\Delta[p] = \{\alpha_1^\vee, -\alpha_1^\vee, v_1(r), v_1(r+1), \dots, v_1(r+s)\}$ if (i) $s \geq 2$ and (ii) $r + s \leq \bar{m}_2 + 1$;
4. $\Delta[p] = \{\alpha_2^\vee, -\alpha_2^\vee, v_2(r), v_2(r+1), \dots, v_2(r+s)\}$ if (i) $s \geq 2$ and (ii) $r + s \leq \bar{m}_1 + 1$;
5. $\Delta[p] = \{\alpha_1^\vee, -\alpha_1^\vee, -r\alpha_1^\vee + e_2, w_1(r, 1), \dots, w_1(r, s)\}$ if $r \leq \bar{m}_2$ and $2 \leq s \leq \bar{m}_1 + 1$;
6. $\Delta[p] = \{\alpha_2^\vee, -\alpha_2^\vee, -r\alpha_2^\vee + e_1, w_2(r, 1), \dots, w_2(r, s)\}$ if $r \leq \bar{m}_1$ and $2 \leq s \leq \bar{m}_2 + 1$;
7. $\Delta[p] = \{\alpha_1^\vee, \alpha_2^\vee, -\alpha_1^\vee - \alpha_2^\vee\}$ if $\sharp D(G/H) > 2$ and $H = G^\theta$;
8. $\Delta[p] = \{e_1, e_1 + e_2, e_1 + 2e_2, \dots, e_1 + (s-1)e_2, e_1 + se_2, e_2\}$ if $2 \leq s \leq \bar{m}_1 + 1$;
9. $\Delta[p] = \{e_2, e_1 + e_2, 2e_1 + e_2, \dots, (s-1)e_1 + e_2, se_1 + e_2, e_1\}$ if $2 \leq s \leq \bar{m}_2 + 1$.

These varieties are smooth if they are toroidal or if, for all $\alpha^\vee \in \rho(D(X))$, $\sharp\rho^{-1}(\alpha^\vee) = 2$ and $2\alpha \notin R_{G,\theta}$.

- If G/H is decomposable and $\chi_*(S) = \mathbb{Z}(\frac{1}{2}\alpha_1^\vee + \frac{1}{2}\alpha_2^\vee) \oplus \mathbb{Z}\alpha_2^\vee$, let $v_j(i) := -\frac{2i+1}{2}\alpha_j^\vee - \frac{1}{2}\alpha_{j^c}^\vee$ and $w_j(x, y) := -\frac{2xy+y+2}{2}\alpha_j^\vee - \frac{y}{2}\alpha_{j^c}^\vee$. We have the following locally factorial quasi-Fano compactifications of G/H , which are not Fano:

1. $\Delta[p] = \{\alpha_1^\vee, -\alpha_1^\vee, v_1(r), v_1(r+1), \dots, v_1(r+s)\}$ if (i) $r \geq 0$, (ii) $s \geq 2$ and (iii) $r+s \leq \frac{\bar{m}_2+1}{2}$;
2. $\Delta[p] = \{\alpha_2^\vee, -\alpha_2^\vee, v_2(r), v_2(r+1), \dots, v_2(r+s)\}$ if (i) $r \geq 0$, (ii) $s \geq 2$ and (iii) $r+s \leq \frac{\bar{m}_1+1}{2}$;
3. $\Delta[p] = \{\alpha_1^\vee, -\alpha_1^\vee, v_1(r), v_1(r+1), \dots, v_1(r+s)\}$ if (i) $r \geq 0$, (ii) $s = 0, 1$ and (iii) $\frac{\bar{m}_2+1}{2} < r+s \leq \frac{\bar{m}_2+1}{2}$;
4. $\Delta[p] = \{\alpha_2^\vee, -\alpha_2^\vee, v_2(r), v_2(r+1), \dots, v_2(r+s)\}$ if (i) $r \geq 0$, (ii) $s = 0, 1$ and (iii) $\frac{\bar{m}_1+1}{2} < r+s \leq \frac{\bar{m}_1+1}{2}$;
5. $\Delta[p] = \{\alpha_1^\vee, -\alpha_1^\vee, -\frac{2r+1}{2}\alpha_1^\vee - \frac{1}{2}\alpha_2^\vee, w_1(r, 1), \dots, w_1(r, s)\}$ if (i) $r \leq \frac{\bar{m}_2-1}{2}$ and (ii) $2 \leq s \leq \bar{m}_1 + 1$;
6. $\Delta[p] = \{\alpha_2^\vee, -\alpha_2^\vee, -\frac{1}{2}\alpha_1^\vee - \frac{2r+1}{2}\alpha_2^\vee, w_2(r, 1), \dots, w_2(r, s)\}$ if (i) $r \leq \frac{\bar{m}_1-1}{2}$ and (ii) $2 \leq s \leq \bar{m}_2 + 1$;
7. $\Delta[p] = \{-\alpha_1^\vee, -\alpha_2^\vee, -\frac{1}{2}\alpha_1^\vee - \frac{1}{2}\alpha_2^\vee\}$.

The last variety is smooth, while the other ones are smooth if and only if $r = 0$.

2.4.3. Symmetric Fano varieties with G reductive. In this section we consider the 2-rank locally factorial Fano symmetric varieties over which a group G acts that is only reductive. If G is a torus, then X is the projective space. So, we can suppose $G = G' \times \mathbb{C}^*$ with $H \cap \mathbb{C}^* = (\mathbb{C}^*)^\theta = \{\pm \text{id}\}$. Write $\chi_*(\mathbb{C}^*/\{\pm \text{id}\}) = \mathbb{Z}f$.

If $R_{G,\theta} = BC_1$, then $H = G^\theta$ and $\chi_*(S) = \mathbb{Z}f \oplus \mathbb{Z}\alpha^\vee$. Instead, if $R_{G,\theta} = A_1$ there are three possibilities: 1) $\chi_*(S) = \mathbb{Z}f \oplus \mathbb{Z}\alpha^\vee$ and $H = G^\theta$; 2) $\chi_*(S) = \mathbb{Z}f \oplus \mathbb{Z}\frac{1}{2}\alpha^\vee$ and $H = N_{G'}(G^\theta) \times \{\pm 1\}$; 3) $\chi_*(S) = \mathbb{Z}f \oplus \mathbb{Z}\frac{\alpha^\vee+f}{2}$. In the last case H is generated by G^θ and by n_1n_2 , where $n_1 \in N_{G'}((G')^\theta) \setminus (G')^\theta$ and $n_2 \in \mathbb{C}^*$ has order four; in particular $[G^\theta : H] = 2$. Let e be the primitive positive multiple of $-\alpha^\vee$ and let $\{e^*, f^*\}$ be the dual basis of $\{e, f\}$.

Theorem 2.8. *Let G/H be a symmetric space of rank two such that G is neither semisimple nor abelian. As before, write $\psi(r) = -2\delta + 2\delta_0 - re^*$ and $m_1 := \max\{r : \psi(r) < 0\}$. The Fano locally factorial compactifications of G/H are the following:*

1. $\Delta[p] = \{f, -f, e+rf\}$ if $\chi_*(S) = \mathbb{Z}e \oplus \mathbb{Z}f$, $r \in \mathbb{Z}$, $r \leq m_1+1$ and $-r \leq m_1+1$;
2. $\Delta[p] = \{f, -f, e+rf, e+(r+1)f\}$ if $\chi_*(S) = \mathbb{Z}e \oplus \mathbb{Z}f$, $r \in \mathbb{Z}$, $r \leq m_1$ and $-r \leq m_1+1$;

3. $\Delta[p] = \{\alpha^\vee, -f, -\alpha^\vee + f\}$ (and $\Delta[p] = \{\alpha^\vee, f, -\alpha^\vee - f\}$) if $\chi_*(S) = \mathbb{Z}\alpha^\vee \oplus \mathbb{Z}f$ and $\sharp D(G/H) = 1$;
4. $\Delta[p] = \{\alpha^\vee, -f, -\alpha^\vee, -\alpha^\vee + f\}$ (and $\Delta[p] = \{\alpha^\vee, f, -\alpha^\vee, -\alpha^\vee - f\}$) if $\chi_*(S) = \mathbb{Z}\alpha^\vee \oplus \mathbb{Z}f$ and $\sharp D(G/H) = 1$;
5. $\Delta[p] = \{f, -f, -\frac{1}{2}\alpha^\vee + \frac{2r+1}{2}f\}$ if $\chi_*(S) = \mathbb{Z}f \oplus \mathbb{Z}(\frac{\alpha^\vee+f}{2})$, $r \in \mathbb{Z}$, $r \leq \frac{m_1+1}{2}$ and $-r \leq \frac{m_1+3}{2}$;
6. $\Delta[p] = \{f, -f, -\frac{1}{2}\alpha^\vee + \frac{2r+1}{2}f, -\frac{1}{2}\alpha^\vee + \frac{2r+3}{2}f\}$ if $\chi_*(S) = \mathbb{Z}f \oplus \mathbb{Z}(\frac{\alpha^\vee+f}{2})$, $r \in \mathbb{Z}$, $r \leq \frac{m_1-1}{2}$ and $-r \leq \frac{m_1+3}{2}$;
7. $\Delta[p] = \{\alpha^\vee, -\frac{1}{2}\alpha^\vee + \frac{1}{2}f, -\frac{1}{2}\alpha^\vee - \frac{1}{2}f, -\alpha^\vee\}$ if $\chi_*(S) = \mathbb{Z}f \oplus \mathbb{Z}(\frac{\alpha^\vee+f}{2})$;
8. $\Delta[p] = \{\alpha^\vee, -\frac{1}{2}\alpha^\vee + \frac{1}{2}f, -\frac{1}{2}\alpha^\vee - \frac{1}{2}f\}$ if $\chi_*(S) = \mathbb{Z}f \oplus \mathbb{Z}(\frac{\alpha^\vee+f}{2})$;
9. $\Delta[p] = \{\alpha^\vee, -f, -\frac{1}{2}\alpha^\vee + \frac{1}{2}f\}$ (and $\Delta[p] = \{\alpha^\vee, f, -\frac{1}{2}\alpha^\vee - \frac{1}{2}f\}$) if $\chi_*(S) = \mathbb{Z}f \oplus \mathbb{Z}(\frac{\alpha^\vee+f}{2})$;
10. $\Delta[p] = \{\alpha^\vee, -f, -\frac{1}{2}\alpha^\vee + \frac{1}{2}f, -\frac{1}{2}\alpha^\vee - \frac{1}{2}f\}$ (and $\Delta[p] = \{\alpha^\vee, f, -\frac{1}{2}\alpha^\vee + \frac{1}{2}f, -\frac{1}{2}\alpha^\vee - \frac{1}{2}f\}$) if $\chi_*(S) = \mathbb{Z}f \oplus \mathbb{Z}(\frac{\alpha^\vee+f}{2})$.

The only singular varieties are the ones in cases 3 and 4.

§3. Standard symmetric varieties

In this section we prove Theorem 2.1 (and Theorem A). Recall that the standard compactification X_0 of G/H is the simple symmetric variety associated to the colored cone (C^-, \emptyset) , so \tilde{k}_{X_0} is always strictly convex.

Proof of Theorem 2.1. We can suppose θ is indecomposable (see Lemma 2.1). We have to determine when $(\tilde{k}_{C^-}, \alpha_i) \leq 0$ for each $i \in \{1, \dots, l\}$. We can write $-2\delta + 2\delta_0$ as the sum of the spherical weights $-2\tilde{\delta} = -2\sum_{\beta_j \in \overline{R}_G^1} \varpi_j$ and $2\tilde{\delta}_0 = -2\sum_{\beta_j \in \overline{R}_G^0} \varpi_j + 2\delta_0$. Write $\beta_j - \theta(\beta_j) = \alpha_i$, so that $(\tilde{k}_{C^-}, \alpha_i) = 2(\tilde{k}_{C^-}, \beta_j)$ and $(2\tilde{\delta}_0, \alpha_i) = 4(\tilde{\delta}_0, \beta_j) \leq 0$. Thus $(-2\delta + 2\delta_0)(\alpha_i^\vee) = -1$ if $\omega_i = 2\varpi_j$ and $(-2\delta + 2\delta_0)(\alpha_i^\vee) \leq -2$ otherwise. Suppose now H is autonormalizing, i.e. $H = N(G^\theta)$; the case where $H \subsetneq N(H)$ is very similar. We want to study $k_{C^-} = -\sum_{i=1}^l \alpha_i$. By the expression of the Cartan matrix of $R_{G,\theta}$, $k_{C^-}(\alpha_i^\vee) \leq 1$ for each i . Therefore \tilde{k}_{C^-} is always antidominant. If \tilde{k}_{C^-} is not regular, then there is a (unique) $\alpha_i = 2\beta_j \in \overline{R}_{G,\theta}$ such that $k_X(\alpha_i^\vee) = 1$; in particular G is simple. By the classification of the involutions by their Satake diagrams, \tilde{k}_{C^-} is not regular if and only if $\theta = -\text{id}$ over $\chi(S)$ and $R_{G,\theta}$ is different from A_l and B_l . \square

§4. Blows-ups along closed orbits

In this section we prove a partial result in arbitrary rank: Theorem 2.2 (and Theorem B). First, we prove a property similar to (*) (see §2.2). Recall that we have associated a toric variety Z to any toroidal symmetric variety X ; moreover $-K_Z$ is ample if X is Fano.

Lemma 4.1. *Let Z be a smooth l -dimensional toric variety whose fan contains two l -dimensional cones σ_+ and σ_- such that: (i) $\sigma_+ \cap \sigma_-$ has dimension $l-1$ and (ii) $\sigma_+ \cup \sigma_-$ is strictly convex. Assume moreover that the piecewise linear function associated to the anticanonical bundle of Z is not strictly convex on $\sigma_+ \cup \sigma_-$. Then the anticanonical bundle of any toric variety obtained from Z by a sequence of blow-ups centered at S -fixed points is not ample.*

We can reformulate the first hypothesis in a more combinatorial way. Indeed, we can write $\sigma_+ = \text{cone}(v_1, \dots, v_{l-1}, v_+)$ and $\sigma_- = \text{cone}(v_1, \dots, v_{l-1}, v_-)$ with $v_1, \dots, v_{l-1}, v_+, v_-$ primitive and $v_+ + v_- = \sum a_i v_i$, where the a_i are positive integers, not all zero.

Proof of Lemma 4.1. The anticanonical bundle of a variety with satisfies the hypotheses of the lemma is not ample. Thus, it is sufficient to show that the blow-up Z' of Z centered in any $\sigma \in \Delta(l)$ satisfies again the hypotheses of the lemma. We can suppose $\sigma = \sigma_+$ by symmetry. Then the fan of Z' contains σ_- and $\sigma' := \text{cone}(v_1, \dots, v_{l-1}, v_+ + \sum v_i)$. We have $(v_+ + \sum v_i) + v_- = \sum (a_i + 1)v_i$, so Z' satisfies the hypotheses with respect to σ_- and σ' . \square

Now we can classify the toric varieties with ample anticanonical bundle which are obtained from \mathbb{A}^l by a sequence of blow-ups centered at S -fixed points.

Proposition 4.1. *Let Z be a smooth toric variety with ample anticanonical bundle which is obtained from \mathbb{A}^l by a sequence of blow-ups centered at S -fixed points. Then it is either \mathbb{A}^l or the blow-up of \mathbb{A}^l at the S -fixed point.*

Proof. It is not difficult to see that the blow-up Z_1 of \mathbb{A}^l at the S -stable point has ample anticanonical bundle. The blow-up of Z_1 at the S -fixed point corresponding to $\text{cone}(e_1, \dots, \hat{e}_j, \dots, e_l, \sum e_i)$ satisfies the hypotheses of the previous lemma with respect to $\text{cone}(e_1, \dots, \hat{e}_h, \dots, e_l, \sum_{i=1}^l e_i)$ and $\text{cone}(e_1, \dots, \hat{e}_h, \dots, \hat{e}_j, \dots, e_l, \sum_{i=1}^l e_i, 2 \sum_{i=1}^l e_i - e_j)$, where $h \neq j$. \square

Thus, a symmetric variety obtained from a wonderful one by a sequence of blow-ups along closed orbits can be Fano only if it is either the wonderful variety or its blow-up along the closed orbit (the restriction of the anticanonical bundle of the

symmetric variety to the associated toric variety is the anticanonical bundle of this last variety by §1.7). We have already considered the wonderful case. Now, we prove in particular that, when such a blow-up is Fano, the rank of every indecomposable factor of $R_{G,\theta}$ is at most 3.

Lemma 4.2. *Let $X_{1,\dots,l}$ be the blow-up of the wonderful compactification of G/H along the closed orbit and suppose that $R_{G,\theta}$ contains an irreducible factor of rank at least three. If $X_{1,\dots,l}$ is Fano then it is indecomposable, has rank 3 and $H \subsetneq N(G^\theta)$.*

Proof. The weights $\{\tilde{k}_C\}$ associated to $-\tilde{K}_{X_{1,\dots,l}}$ are $\lambda_i = -2\delta + 2\delta_0 - (l-2)e_i^* + \sum_{j \neq i} e_j^*$ with $i = 1, \dots, l$. First, suppose G/H is indecomposable and write $e_i^* = -x_i\alpha_i$. If $H \subsetneq N(G^\theta)$, then $R_{G,\theta}$ has type A_1 , B_2 or C_l because X_0 is wonderful (see the remark at the end of §3).

We consider two cases. First, suppose that there is $\beta_h \in \overline{R}_G^1$ orthogonal to R_G^0 . Write $\beta_h - \theta(\beta_h) = \alpha_j$, so $0 > (\lambda_j, \alpha_j^\vee) = (-2\delta + 2\delta_0, \alpha_j^\vee) + (l-2)x_j\alpha_j^\vee, \alpha_j^\vee) - (\sum_{i \neq j} x_i\alpha_i^\vee, \alpha_j^\vee) \geq -2 + 2(l-2)x_j + 0$. Notice that we have used that $(\delta_0, \alpha_j^\vee) = 0$. By the previous remark on $R_{G,\theta}$ we have $x_j^{-1} \leq 2$, so $l = 3$ and $H \subsetneq N(G^\theta)$. Note that the x_i are all 1 if $H = N(G^\theta)$.

Suppose now that there is not such a root. By the classification of indecomposable symmetric spaces, we have the following possibilities: 1) θ has type A_{II} and G/G^θ is SL_{2l+2}/Sp_{2l+2} ; 2) θ has type C_{II} and G/G^θ is $Sp_{2n}/(Sp_{2l} \times Sp_{2n-2l})$; 3) θ has type D_{III} and G/G^θ is SO_{4l+2}/GL_{2l+1} . Then there are $\beta_3, \beta_5 \in R_G^0$ orthogonal to $\overline{R}_G^0 \setminus \{\beta_3, \beta_5\}$ and $\beta_4 \in R_G^1$ such that $\alpha_2 = \beta_3 + 2\beta_4 + \beta_5$ (see §1.4). Moreover, $(\beta_3, \beta_3) = (\beta_4, \beta_4) = (\beta_5, \beta_5)$, $\alpha_2^\vee = \frac{1}{(\beta_2, \beta_2)}\alpha_2$ and $x_i = 1$ if $i < l$. Thus $0 > (\lambda_2, \alpha_2^\vee) = (-2\delta + 2\delta_0, \alpha_2^\vee) + (l-2)(\alpha_2, \alpha_2^\vee) - (\alpha_1, \alpha_2^\vee) - x_3(\alpha_3, \alpha_2^\vee) \geq -4 + 2(l-2) + 1 + x_3$, so we again have $l = 3$ and $H \subsetneq N(G^\theta)$.

Finally, suppose θ is decomposable. Let $(G, \theta) = (G_1, \theta_1) \times (G_2, \theta_2)$ with $l' := \text{rank}(G_1/G_1^\theta) \geq 3$ and define the weight λ'_i for G_1 analogously to λ_i . We have $\lambda_i = \lambda'_i - (l-l')e_i^* + \omega$ where ω is orthogonal to $R_{G_1,\theta}$. By the previous part of the proof there is always an i with $\lambda'_i(\alpha_i^\vee) \geq -1$, so $\lambda_i(\alpha_i^\vee) \geq \lambda'_i(\alpha_i^\vee) + \frac{1}{2}(\alpha_i, \alpha_i^\vee) \geq 0$, a contradiction. \square

One can prove Theorem 2.2 by an explicit analysis of the indecomposable involutions of rank at most three.

§5. Regular Fano varieties of rank 3

In this section we prove Theorems 2.3–2.5. We begin by proving a result similar to Lemma 4.1. Let \tilde{Z} be the toric variety whose fan $\tilde{\Delta}$ has maximal cones

$\text{cone}(v_1, v_2, v_+)$ and $\text{cone}(v_1, v_2, v_- = x_1v_1 + x_2v_2 - v_+)$, where $\{v_1, v_2, v_+\}$ is a basis of $\chi_*(S)$, $x_1 + x_2 > 0$ and $x_1 \geq x_2 \geq 0$. The anticanonical bundle of \tilde{Z} is ample if and only if $x_1 = x_1 + x_2 = 1$. In this case, \tilde{Z} is the blow-up of \mathbb{A}^3 along a stable subvariety of codimension 2. Moreover, the anticanonical bundle of \tilde{Z} is nef but non-ample if and only if $x_1 + x_2 = 2$. We have two possibilities: either $v_+ + v_- = v_1 + v_2$ or $v_+ + v_- = 2v_1$. In the first case we have a variety isomorphic to the variety \bar{Z} of §2.2. This is the more problematic case, so we will study it later.

Lemma 5.1. *Let Z be a smooth 3-dimensional toric variety whose fan contains two maximal cones $\text{cone}(v_1, v_2, v_+)$ and $\text{cone}(v_1, v_2, v_-)$ such that $v_+ + v_- = x_1v_1 + x_2v_2$, where x_1 and x_2 are integers with $x_1 \geq x_2 \geq 0$. Suppose moreover that Z satisfies the following condition:*

$$(H) \quad x_1 \geq 2.$$

Then the anticanonical bundle of any toric variety Z' obtained from Z by a sequence of blow-ups along S -subvarieties is not ample.

Proof. Note that the anticanonical bundle of Z is not ample. We say that a variety satisfies weakly the hypotheses of the lemma if it satisfies the weaker condition

$$(WH) \quad x_1 + x_2 \geq 2.$$

We use the following trivial observation: $x_1 + x_2 > 2$ implies $x_1 \geq 2$. One can try to prove this lemma by induction as Lemma 4.1. Unfortunately we can only prove the following weaker statement.

Lemma 5.2. *Let Z be a toric variety which satisfies (WH) and let Z' be the blow-up of Z along a cone τ .*

1. *If $\tau \neq \text{cone}(v_1, v_2)$, then Z' satisfies (H).*
2. *If Z satisfies (H), then Z' satisfies (WH).*

Proof. We can suppose $\tau \subset \text{cone}(v_1, v_2, v_-)$ by symmetry. If $\tau \neq \text{cone}(v_1, v_2)$, we have three possibilities: $\tau = \text{cone}(v_1, v_-)$, $\tau = \text{cone}(v_2, v_-)$ or $\tau = \text{cone}(v_1, v_2, v_-)$. We always have $\Delta_{Z'}[p] = \Delta_Z[p] \cup \{v' := v_- + b_1v_1 + b_2v_2\}$ with $b_1, b_2 \in \{0, 1\}$. Moreover, $\Delta_{Z'}$ contains the cones $\text{cone}(v_1, v_2, v_+)$ and $\text{cone}(v_1, v_2, v')$ and we have $v' + v_+ = (x_1 + b_1)v_1 + (x_2 + b_2)v_2$ with $(x_1 + b_1) + (x_2 + b_2) > 2$, so Z' satisfies (H).

Finally let $\tau = \text{cone}(v_1, v_2)$. The fan of Z' contains $\text{cone}(v_1, v_1 + v_2, v_+)$ and $\text{cone}(v_1, v_1 + v_2, v_-)$. We have $v_+ + v_- = (x_1 - x_2)v_1 + x_2(v_1 + v_2)$ with $(x_1 - x_2) + x_2 = x_1 \geq 2$. □

Now, we consider the general case. We have a sequence $Z = Z_0 \leftarrow Z_1 \leftarrow \dots \leftarrow Z_r = Z'$ where Z_{i+1} is the blow-up of Z_i along the cone τ_i . Let $\Delta_i = \Delta_{Z_i}$ and let j be maximal such that Z_j satisfies (H) (with respect to $\text{cone}(w_1, w_2, w_+)$ and $\text{cone}(w_1, w_2, w_-)$). By the previous lemma Z_{j+1} satisfies (WH) , in particular its anticanonical bundle is not ample. By the maximality of j , Z_{j+1} does not satisfy (H) , thus, by the previous lemma, $\tau_j = \text{cone}(w_1, w_2)$ and Z_{j+1} contains a variety isomorphic to \bar{Z} . By the previous proof, we can suppose that the maximal cones in the fan $\bar{\Delta}$ of that variety are $\sigma_+ = \text{cone}(w_1, w_1 + w_2, w_+)$ and $\sigma_- = \text{cone}(w_1, w_1 + w_2, w_-)$; in particular Z_{j+1} satisfies (WH) with respect to σ_+ and σ_- . Moreover $w_+ + w_- = 2w_1 + w_2$.

If Δ_r contains $\bar{\Delta}$ then $-K_{Z'}$ is not ample. Indeed, otherwise there is a minimal h such that $\bar{\Delta}$ is not contained in Δ_{h+1} ; in particular τ_h is contained in $\sigma^+ \cup \sigma^-$. We claim that Z_{h+1} satisfies (H) , which gives a contradiction.

By the first part of Lemma 5.2 we can suppose $\tau_h = \text{cone}(w_1, w_1 + w_2)$; otherwise Z_{h+1} satisfies (H) . The fan of Z_{h+1} contains two cones $\text{cone}(w_1 + w_2, w_+, 2w_1 + w_2)$ and $\text{cone}(w_1 + w_2, w_+, w')$, with $w' = w_2 + x_1(w_1 + w_2) + x_2w_+$ and $x_1, x_2 \geq 0$. Therefore Z_{h+1} satisfies (H) with respect to these cones. Indeed $(2w_1 + w_2) + w' = (2 + x_1)(w_1 + w_2) + x_2w_+$. □

Now we want to study the varieties that contain an open subvariety isomorphic to \bar{Z} . Observe that these varieties are never Fano varieties. Let Z be such a variety and let Z' be the blow-up of Z along the subvariety of \bar{Z} associated to $\text{cone}(e_1, e_2)$. Such a subvariety is the unique S -stable projective curve contained in \bar{Z} . We prove that, if Z' satisfies (H) , then there does not exist a Fano variety obtainable from Z by a sequence of blow-ups.

Lemma 5.3. *Let Z be a smooth 3-dimensional toric variety whose fan contains $\text{cone}(v_1, v_2, v_3)$ and $\text{cone}(v_1, v_2, v_1 + v_2 - v_3)$ for suitable v_1, v_2, v_3 . Let Z' be the blow-up of Z along the stable subvariety corresponding to $\text{cone}(v_1, v_2)$ and let Z'' be a toric variety obtained from Z by a sequence of blow-ups along S -subvarieties. If the anticanonical bundle of Z'' is ample, then Z'' is obtainable from Z' by a sequence of blow-ups along S -subvarieties.*

Proof. We cannot proceed as in the previous lemma, because we do not know the other cones of Δ_Z . We again have a sequence $Z = Z_0 \leftarrow Z_1 \leftarrow \dots \leftarrow Z_h = Z''$ where $\pi_{i+1} : Z_{i+1} \rightarrow Z_i$ is the blow-up along τ_i . First of all, there is a cone τ_j contained in $\text{cone}(v_1, v_2, v_3, v_1 + v_2 - v_3)$, because otherwise the anticanonical bundle of Z'' is not ample. We take j minimal. By Lemma 5.1 and by the minimality of j , τ_j is $\text{cone}(v_1, v_2)$. We want to reorder the cones associated to the subvarieties along which we are blowing-up. Clearly this operation is not well defined in general.

We consider the following sequence of blow-ups: $Z = Z'_0 \leftarrow Z'_1 \leftarrow \cdots \leftarrow Z'_{j+1}$, where $\pi'_0 : Z'_1 \rightarrow Z'_0$ is the blow-up along τ_j and $\pi'_{i+1} : Z'_{i+1} \rightarrow Z'_i$ is the blow-up along τ_{i-1} for each $i \geq 1$. Let $\Delta'_i = \Delta_{Z'_i}$. We show that these blow-ups are well defined and $Z'_{j+1} = Z_{j+1}$.

The cone τ_{i-1} belongs to Δ'_i for each $1 \leq i \leq j$ because τ_i is contained in $|\Delta| \setminus \text{cone}(v_1, v_2, v_3, v_1 + v_2 - v_3)$ for each $i \leq j$. Moreover, the elements of $\Delta'_1(3)$ not contained in $\text{cone}(v_1, v_2, v_3, v_1 + v_2 - v_3)$ are exactly the elements of $\Delta_0(3) \setminus \{\text{cone}(v_1, v_2, v_3), \text{cone}(v_1, v_2, v_1 + v_2 - v_3)\}$.

Z is the union of the following two open S -subvarieties: U_1 whose fan has maximal cones $\text{cone}(v_1, v_2, v_3)$ and $\text{cone}(v_1, v_2, v_1 + v_2 - v_3)$; and U_2 whose fan has maximal cones $\Delta(3) \setminus \{\text{cone}(v_1, v_2, v_3), \text{cone}(v_1, v_2, v_1 + v_2 - v_3)\}$.

The blow-up π'_0 induces an isomorphism between U_2 and its inverse image, because $\text{cone}(v_1, v_2)$ is not contained in any maximal cone of U_2 . In the same way π_j induces an isomorphism between the inverse image of U_2 in Z_j and its inverse image in Z_{j+1} . So the inverse image of U_2 in Z_{j+1} is isomorphic to the inverse image of U_2 in Z'_{j+1} . Moreover $\pi'_j \circ \cdots \circ \pi'_2$ induces an isomorphism between $(\pi'_1)^{-1}(U_1)$ and its inverse image. In the same way $\pi_{j-1} \circ \cdots \circ \pi_1$ induces an isomorphism between U_1 and its inverse image. So the inverse image of U_1 in Z_{j+1} is isomorphic to the inverse image of U_1 in Z'_{j+1} . The lemma follows because there is at most one morphism between two toric S -varieties extending the identity automorphism of S . □

We now restrict the possible Fano symmetric varieties with rank three (and fixed G/H) that are obtainable as before to a finite explicit list.

Proposition 5.1. *The toric varieties obtainable from \mathbb{A}^3 by a sequence of blow-ups and with ample anticanonical bundle are, up to isomorphisms:*

1. \mathbb{A}^3 ;
2. a 2-blow-up of \mathbb{A}^3 ;
3. a 3-blow-up of \mathbb{A}^3 ;
4. the variety whose fan has maximal cones: $\text{cone}(e_1, e_1 + e_2, e_1 + e_2 + e_3)$, $\text{cone}(e_1, e_3, e_1 + e_2 + e_3)$, $\text{cone}(e_2, e_3, e_1 + e_2 + e_3)$ and $\text{cone}(e_2, e_1 + e_2, e_1 + e_2 + e_3)$; this variety is obtainable from \mathbb{A}^3 by two consecutive blow-ups along subvarieties of codimension two;
5. the variety whose fan has maximal cones: $\text{cone}(e_1, e_3, e_1 + e_2 + 2e_3)$, $\text{cone}(e_1, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3)$, $\text{cone}(e_1, e_2, e_1 + e_2 + e_3)$, $\text{cone}(e_2, e_1 + e_2 + e_3, e_1 + e_2 + 2e_3)$ and $\text{cone}(e_2, e_3, e_1 + e_2 + 2e_3)$; this variety is obtainable from \mathbb{A}^3 by a 3-blow up followed by a 2-blow up.

Proof. Similarly to §4, we want to find an explicit set $\{X_i\}$ of symmetric varieties such that: 1) they satisfy (H) , in particular they are not Fano; 2) there are finitely many symmetric varieties $\{Y_j\}$ that do not dominate any X_i , but are obtainable from X_0 through a sequence of blow-ups. In such a case it is sufficient to study the Y_j ; indeed we will study them to find the X_i .

We proceed as follows: the anticanonical bundle of \mathbb{A}^3 is ample, so we consider all the possible blow-ups of \mathbb{A}^3 . Let Z be a blow-up of \mathbb{A}^3 : 1) if Z satisfies (H) we know that there is no toric variety with ample anticanonical bundle and obtainable from Z by a sequence of blow-ups; 2) if Z satisfies (WH) , we study the variety Z' of Lemma 5.3; 3) finally, if the anticanonical bundle Z is ample, we reiterate the procedure. Observe that a priori it is possible that Z belongs to none of the previous cases. If two blow-ups of a given variety are isomorphic, we can examine only one of them.

We consider explicitly only some examples to illustrate the strategy. We draw the fans of the varieties we wish to study. In the figure we indicate by (A) that the anticanonical bundle of the variety is ample, and by (H) , respectively (WH) , that the variety satisfies the hypotheses of Lemma 5.1, resp. 5.3, with respect to σ and σ' . In the last two cases we indicate $\sigma \cap \sigma'$ with a wavy line. We will use the following notation: Z_0 is the variety \mathbb{A}^3 associated with the wonderful compactification; Z_1, Z_2, \dots are the blow-ups of Z_0 . Given a variety $Z_{i_1 \dots i_k}$ we denote its blow-ups by $Z_{i_1 \dots i_k 1}, \dots, Z_{i_1 \dots i_k n}$. Moreover, given a variety Z_{i_1, \dots, i_k} which satisfies (WH) we denote by $Z_{i_1, \dots, i_k b}$ the variety defined in Lemma 5.3; if $Z_{i_1, \dots, i_k b}$ again satisfies (WH) we denote by $Z_{i_1, \dots, i_k c}$ the variety defined in Lemma 5.3 and so on. Finally, we denote a vector $ae_1 + be_2 + ce_3$ by (a, b, c) .

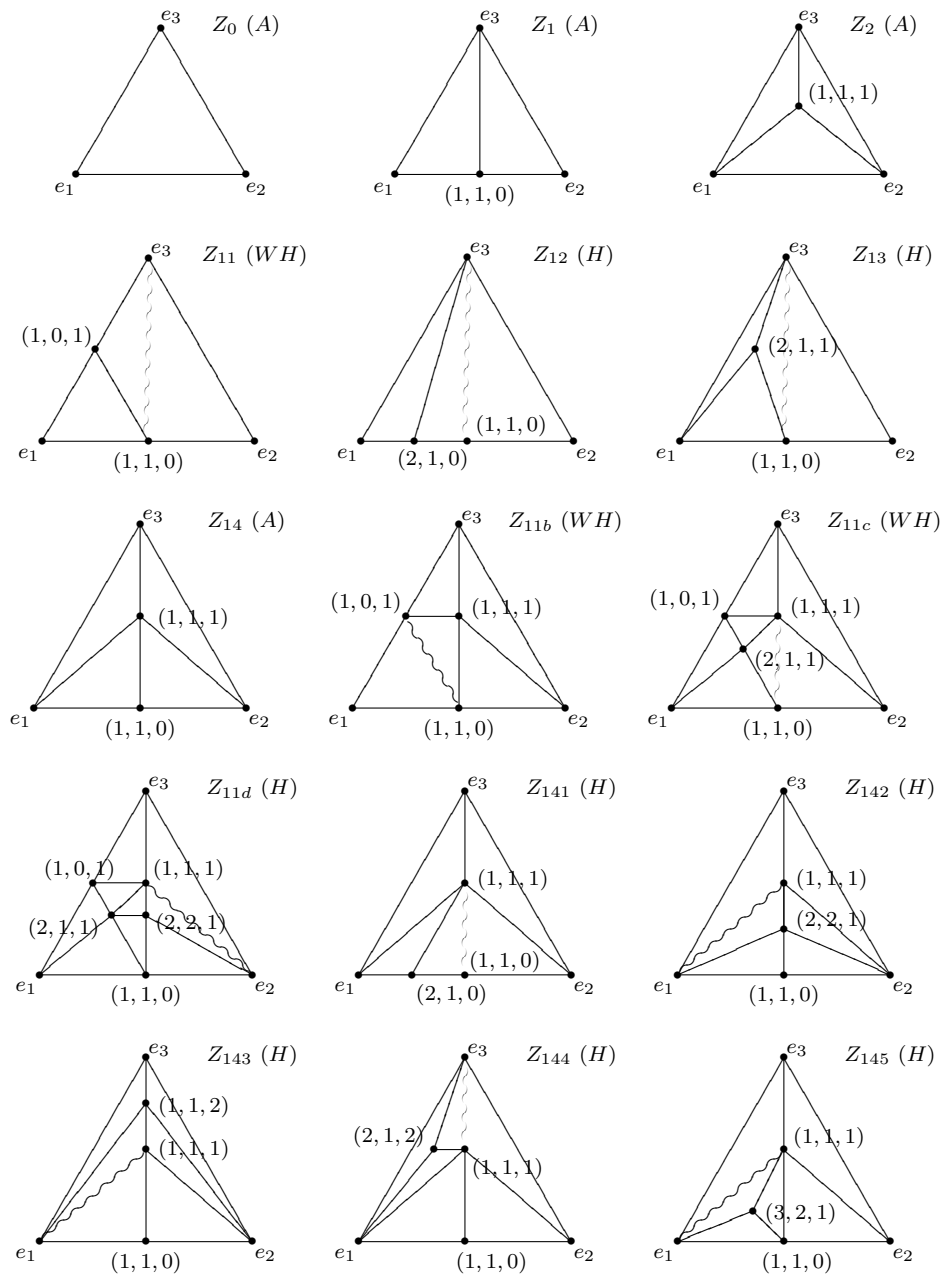
Let $Z_0 = \mathbb{A}^3$; it has ample canonical bundle. Indeed it has trivial Picard group. There are three 2-blow-ups of Z_0 , but they are all isomorphic to the blow-up Z_1 along $\text{cone}(e_1, e_2)$. There is a unique 3-blow-up Z_2 of Z_0 (along $\text{cone}(e_1, e_2, e_3)$).

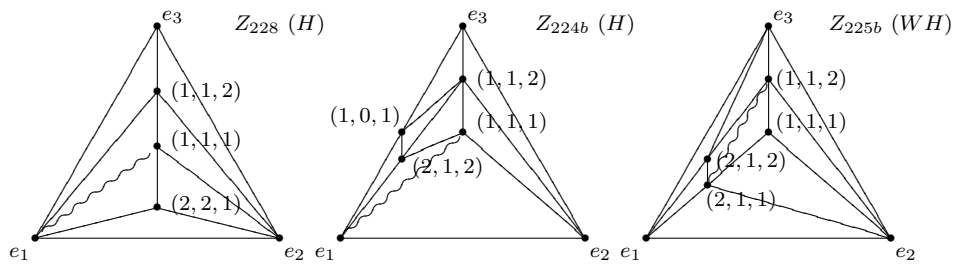
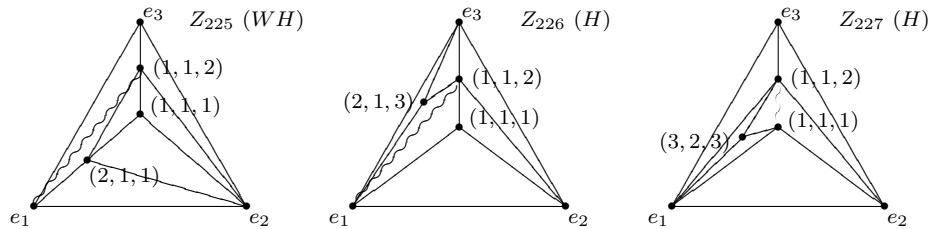
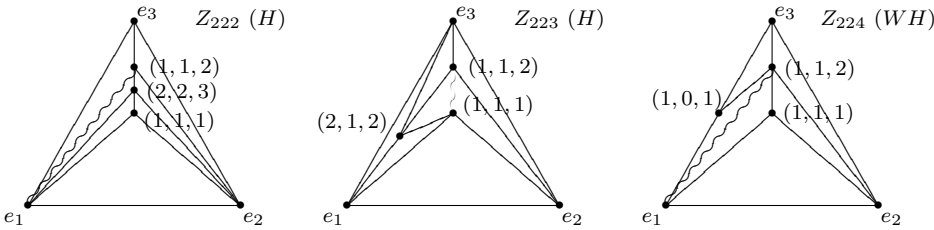
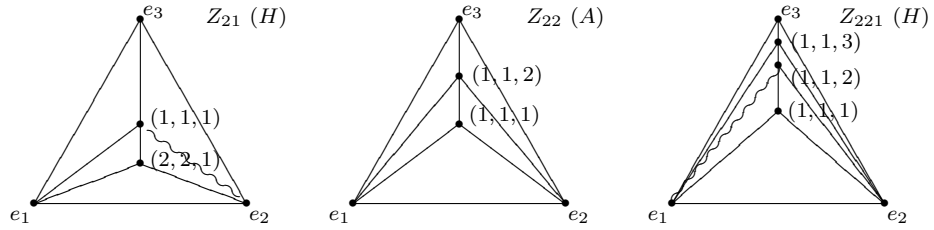
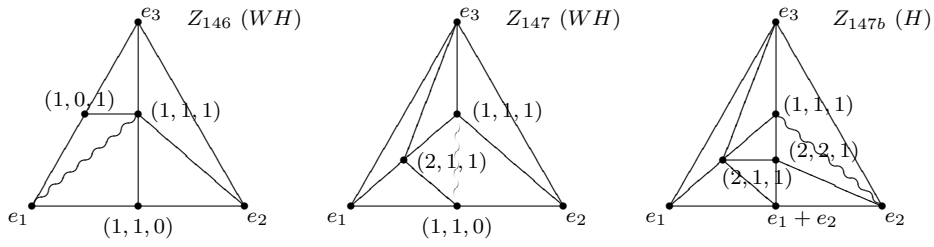
Let Z_{11} be the blow-up of Z_1 along $\text{cone}(e_1, e_3)$; it satisfies (WH) with respect to $\text{cone}(e_3, e_1 + e_2, e_1 + e_3)$ and $\text{cone}(e_3, e_1 + e_2, e_2)$; indeed $(e_1 + e_3) + e_2 = e_3 + (e_1 + e_2)$ and $\text{cone}(e_3, e_1 + e_2, e_1 + e_3) \cap \text{cone}(e_3, e_1 + e_2, e_2) = \text{cone}(e_3, e_1 + e_2)$. Hence we have to study the blow-up Z_{11b} of Z_{11} along $\text{cone}(e_3, e_1 + e_2)$.

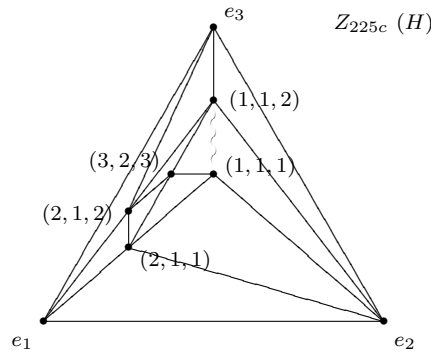
Let Z_{12} be the blow-up Z_1 along $\text{cone}(e_1, e_1 + e_2)$; it satisfies (H) with respect to $\text{cone}(e_3, e_1 + e_2, 2e_1 + e_2)$ and $\text{cone}(e_3, e_1 + e_2, e_2)$. Indeed, $(2e_1 + e_2) + e_2 = 2(e_1 + e_2)$ and $\text{cone}(e_3, e_1 + e_2, 2e_1 + e_2) \cap \text{cone}(e_3, e_1 + e_2, e_2) = \text{cone}(e_3, e_1 + e_2)$.

The following varieties have not been drawn: 1) the blow-up Z_{146b} of Z_{146} along $\text{cone}(e_1, e_1 + e_2 + e_3)$, which is Z_{14c} ; 2) the blow-up Z_{23} of Z_2 along $\text{cone}(e_1, e_2)$, which is Z_{14} ; 3) the blow-up Z_{229} of Z_{22} along $\text{cone}(e_1, e_2)$, which is Z_{143} . \square

Now, one can easily prove Theorems 2.3, 2.4 and 2.5 by verifying when the weights associated to the anticanonical bundle of a variety as in the previous lemma are strictly antidominant.







§6. (Quasi-) Fano symmetric varieties of rank 2

In this section we consider the quasi-Fano locally factorial symmetric varieties with rank 2 (and G only reductive). We have seen in §2.4 that if ρ is injective then X is determined by $\Delta_X[p]$. To prove the classification of 2-rank Fano symmetric varieties we will consider three cases separately: 1) X is simple; 2) X is toroidal; 3) X is neither simple nor toroidal. In the first case, one can study very easily the ampleness of $-K_X$, but to study the smoothness of X is slightly more complicated. In the second case, first we will prove an upper bound on the number of 1-cones of Δ_X . In the third case we use the local factoriality of X to prove that $R_{G,\theta}$ is A_2, A_1, BC_1 or decomposable. The case of quasi-Fano varieties is roughly reduced to the Fano case by Lemma 6.4, which allows one to associate to any quasi-Fano symmetric variety a new quasi-Fano symmetric variety that is “almost always” Fano.

§6.1. Fano symmetric varieties

We begin by classifying the Fano varieties with G semisimple. Recall that we have to verify that, given any $(C, F) \in \mathcal{F}(X)$, $k_C(v) < 1$ for each $v \in N(X) \setminus C$ and $k_C(\rho(D)) < (2\delta - 2\delta_0)(\rho(D))$ for each $D \in D(G/H) \setminus F$. One can show that if X is simple then it is quasi- \mathbb{Q} -Fano. For example, if ρ is injective, then $\text{Pic}(X) = \mathbb{Z}$ and K_X is not big because X is rational. Moreover, every simple and normal G -variety is projective by a classical result of Sumihiro. But in general a simple symmetric variety is not smooth or even locally factorial. Now, we want to find some necessary conditions for the ampleness of $-K_X$ when X is non-simple. We consider two cases: (i) G/H is indecomposable, while X is neither simple nor toroidal; (ii) X is toroidal.

Lemma 6.1. *Let X be a locally factorial projective symmetric variety. Suppose that G/H is indecomposable and X is neither simple nor toroidal. Then $R_{G,\theta} = A_2, H = G^\theta$ and $\mathcal{F}(X)$ contains $(\text{cone}(-\alpha_1^\vee - \alpha_2^\vee), \emptyset)$.*

There is a simple restricted coroot α_i^\vee in $\Delta[p]$ because X is not toroidal. Thus there is a colored cone $(\text{cone}(\alpha_i^\vee, v), F)$ in $\mathcal{F}(X)$, where v is an antidominant and regular weight. Indeed $\text{int}(\text{cone}(\alpha_i^\vee, v)) \cap C^- \neq \emptyset$ and X is not simple (so $v \neq -\omega_{i^c}^\vee$).

Proof. We do a case-by-case analysis.

1) Suppose $R_{G,\theta} = A_2$ and $H = N(G^\theta)$. Then, up to reindexing $R_{G,\theta}$, we have $\text{cone}(\alpha_1^\vee, v) \in \Delta$ with $v := -\frac{x}{3}\alpha_1^\vee - \frac{y}{3}\alpha_2^\vee \in \text{int}(C^-)$ primitive. We can write $-\omega_2^\vee$ as a positive integral combination of α_1^\vee and v ; thus $y \in \{1, 2\}$. Moreover, $0 < (-v, \alpha_2) = -x/3 + 2y/3$, so $0 < x < 2y$. If $y = 1$, then $v = -\frac{1}{3}(\omega_1^\vee + \omega_2^\vee) \notin \chi_*(S)$. If $y = 2$, we have three possibilities: (i) $v = -\omega_2^\vee$ which is not regular; (ii) $v = -\frac{2}{3}(\omega_1^\vee + \omega_2^\vee) \notin \chi_*(S)$; (iii) $v = -\alpha_1^\vee - \frac{2}{3}\alpha_2^\vee \notin \chi_*(S)$.

2) Suppose $R_{G,\theta} = A_2$ and $H = G^\theta$. Then, up to reindexing $R_{G,\theta}$, we have $\text{cone}(\alpha_1^\vee, v) \in \Delta$ with $v := -x\alpha_1^\vee - y\alpha_2^\vee$ primitive. Hence $y = 1$ because $\{\alpha_1^\vee, v\}$ is a basis of $\chi_*(S)$. Moreover, $0 < (-v, \alpha_2) = -x + 2$, so $v = -\alpha_1^\vee - \alpha_2^\vee$ as in the statement.

3) Suppose $R_{G,\theta} = B_2$ and $\text{cone}(\alpha_1^\vee, v) \in \Delta$, with $v := -x\alpha_1^\vee - \frac{y}{2}\alpha_2^\vee$ primitive. We can write $-\omega_2^\vee = -\alpha_1^\vee - \alpha_2^\vee$ as a positive integral combination of α_1^\vee and v ; thus $y \in \{1, 2\}$. Moreover, $0 < (-v, \alpha_2) = -x + y$, so $0 < x < y$. Therefore $v = -\omega_2^\vee$ is not regular.

4) Suppose $R_{G,\theta} = B_2$ and $\text{cone}(\alpha_2^\vee, v) \in \Delta$. If $H = N(G^\theta)$, then $\{\alpha_2^\vee, v\}$ cannot be a basis of $\chi_*(S) = \mathbb{Z}\alpha_1^\vee \oplus \mathbb{Z}\frac{\alpha_2^\vee}{2}$.

5) Suppose $R_{G,\theta} = B_2$, $H = G^\theta$, and $\text{cone}(\alpha_2^\vee, v) \in \Delta$ with $v := -x\alpha_1^\vee - y\alpha_2^\vee$ primitive. Then $x = 1$ because $\{\alpha_2^\vee, v\}$ is a basis of $\chi_*(S)$. Moreover, $0 < (-v, \alpha_1) = 2 - 2y$, so $0 < y < 1$.

6) Suppose $R_{G,\theta} = BC_2$ and $\text{cone}(\alpha_1^\vee, v) \in \Delta$ with $v := -x\alpha_1^\vee - y\alpha_2^\vee$ primitive. As before $y = 1$. Moreover, $0 < (-v, \alpha_2) = -x + 1$, so $0 < x < 1$.

7) Suppose $R_{G,\theta} = BC_2$ and $\text{cone}(\alpha_2^\vee, v) \in \Delta$ with $v := -x\alpha_1^\vee - y\alpha_2^\vee$ primitive. As before $x = 1$ and $0 < (-v, \alpha_1) = 2 - y$, so $v = -\omega_1^\vee$ is not regular.

8) Suppose $R_{G,\theta} = G_2$ and $\text{cone}(\alpha_1^\vee, v) \in \Delta$ with $v := -x\alpha_1^\vee - y\alpha_2^\vee$ primitive. As before $y = 1$ and $0 < (-v, \alpha_2) \leq -3x + 2$, so $0 < x < 2/3$.

9) Suppose $R_{G,\theta} = G_2$ and $\text{cone}(\alpha_2^\vee, v) \in \Delta$ with $v := -x\alpha_1^\vee - y\alpha_2^\vee$ primitive. As before $x = 1$. Moreover, $0 < (-v, \alpha_1) = 2 - y$ and $0 < (-v, \alpha_2) = -3 + 2y$; thus $3 < 2y < 4$. □

In the next two lemmas, we do not make any hypothesis on the regularity of X .

Lemma 6.2. *Let X be a Fano symmetric variety of rank 2 with G semisimple. Then $\mathcal{F}(X)$ contains at most three colored 1-cones.*

Proof. Suppose that $\mathcal{F}(X)$ contains $(\sigma_i := \text{cone}(v_i, v_{i+1}), \emptyset)$ with $i = 1, 2, 3$ and v_1, v_2, v_3, v_4 primitive. Recall that all the 1-dimensional cones are contained in C^- , so they do not have colors. We can write v_2 as a positive linear combination $xv_1 + yv_4$ of v_1 and v_4 . Then $k_{\sigma_2}(v_1) < k(v_1) = 1$, $k_{\sigma_2}(v_4) < k(v_4) = 1$ and $1 = k_{\sigma_2}(v_2) = xk_{\sigma_2}(v_1) + yk_{\sigma_2}(v_4) \leq 0$, which gives a contradiction. \square

Lemma 6.3. *Let X be a Fano non-simple toroidal symmetric variety. Then X_0 is smooth and $\Delta[p]$ is either $\{e_1, e_1 + re_2, e_2\}$ or $\{e_1, re_1 + e_2, e_2\}$. These varieties are smooth if and only if $r = 1$.*

Proof. By the previous lemma, we have $\Delta[p] = \{e_1, e_2, v\}$ for an appropriate v . First suppose that X_0 is smooth, i.e. $\chi_*(S) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, and write $v = x_1e_1 + x_2e_2$. For each i , let $\sigma_i = \text{cone}(e_i, v)$ and $\{i, i^c\} = \{1, 2\}$, so k_{σ_i} is $e_i^* + \frac{1-x_i}{x_{i^c}}e_{i^c}^*$. If $x_1 \geq 2$ and $x_2 \geq 2$, then $\frac{1-x_1}{x_2}$ and $\frac{1-x_2}{x_1}$ are strictly negative integers, so $x_1 \geq x_2 + 1 \geq x_1 + 2$, a contradiction.

Suppose now that X_0 is singular. Then $R_{G,\theta}$ is either A_2 or $A_1 \times A_1$. In the first case, $e_i = -3\omega_i^\vee$ and the strictly positive integer $k(-\omega_1^\vee - \omega_2^\vee)$ is smaller than $\frac{1}{3}k(-3\omega_1^\vee) + \frac{1}{3}k(-3\omega_2^\vee) = \frac{2}{3}$, a contradiction. Finally suppose $R_{G,\theta} = A_1 \times A_1$ and $\chi_*(S) = \mathbb{Z}2\omega_2^\vee \oplus \mathbb{Z}(\omega_1^\vee + \omega_2^\vee)$. Then there is i such that $-\omega_1^\vee - \omega_2^\vee = av + b(-2\omega_i^\vee)$ with $a, b \geq 0$. The integer $a + b = k(-\omega_1 - \omega_2)$ is strictly less than $\frac{1}{2}k(-2\omega_1) + \frac{1}{2}k(-2\omega_2) = 1$, a contradiction. \square

Note that the previous two lemmas apply also to a toroidal symmetric variety with $-K_X$ ample, $|\mathcal{F}(X)|$ convex and generated by a basis of $\chi_*(S)$ (without supposing X is compact). Using Lemma 6.3, one can easily prove Proposition 2.1. Now, we prove Theorem 2.6 (and Theorem C).

Proof of Theorem 2.6. Recall that an element $v \in \Delta[p]$ is a simple coroot if it is the image of a color by ρ , while it is an antidominant weight if it is the image of a G -stable divisor (in this case $v \in N(X)$). The first item of the theorem holds because $-K_X$ is linearly equivalent to a G -stable divisor. Indeed suppose that there are $D_1 \in D(X)$ and $D_2 \in D(G/H)$ such that $\rho(D_1) = \rho(D_2)$; then the coefficients a_{D_1} and a_{D_2} of $-K_X$ with respect to D_1 and D_2 are equal. Let h_X be the piecewise linear function associated to $-K_X$. If $-K_X$ is ample and $D_2 \notin D(X)$, then $a_{D_1} = h_X(\rho(D_1)) < h_X(\rho(D_2)) = a_{D_2}$, which gives a contradiction. Finally, observe that if X is locally factorial, then ρ is injective over $D(X)$.

Let $a_i := (2\delta - 2\delta_0)(\alpha_i^\vee)$ for each i ; recall that $a_1, a_2 \geq 1$ (see §3). In the following we always suppose $\sharp\rho(D(X)) = \sharp\rho^{-1}(\rho(D(X)))$.

(I) Suppose X simple. If X is toroidal (i.e. $X = X_0$) we have to exclude the following cases by §3: (i) $G/H = G_2/(SL_2 \times SL_2)$; (ii) $R_{G,\theta} = A_2$ and $H = G^\theta$;

(iii) $R_{G,\theta} = A_1 \times A_1$ and $\chi_*(S) = \mathbb{Z}2\omega_1^\vee \oplus \mathbb{Z}(\omega_1^\vee + \omega_2^\vee)$. In the first case X_0 is not Fano, while in the last two cases X_0 is not smooth. If $\rho(D(X)) = \{\alpha_1^\vee\}$, then θ is indecomposable because $C(X)$ is strictly convex. Moreover, $\Delta[p] = \{\alpha_1^\vee, e_1 = -x_1\alpha_1^\vee - x_2\alpha_2^\vee\}$ and $k_X = a_1\omega_1 - \frac{x_1a_1+1}{x_2}\omega_2$; so $k_X(\alpha_2^\vee) \leq 0 < a_2$. Thus we have only to verify that X is locally factorial. We have to exclude two cases: 1) $\Delta(2) = \{\text{cone}(\alpha_1^\vee, -\omega_1^\vee)\}$ and $R_{G,\theta} = G_2$; 2) $\Delta(2) = \{\text{cone}(\alpha_2^\vee, -\omega_2^\vee)\}$, $R_{G,\theta} = B_2$ and $H = N(G^\theta)$.

(II) Suppose now that X is not simple. In the toroidal case we have $\Delta[p] = \{e_1, e_2, e_1 + e_2\}$ and X_0 is smooth by Lemma 6.3. This variety is Fano if and only if $-2\delta + 2\delta_0 + e_i^*$ is antidominant and regular for each i .

(III) Assume moreover that X is not toroidal. Suppose first θ is indecomposable. Then, by Lemma 6.1, $R_{G,\theta} = A_2$, $H = G^\theta$ and $\Delta(2)$ contains $\text{cone}(\alpha_1^\vee, -\alpha_1^\vee - \alpha_2^\vee)$, up to reindexing. If $\rho(D(X))$ also contains α_2^\vee then $\Delta(2) = \{\text{cone}(\alpha_1^\vee, -\alpha_1^\vee - \alpha_2^\vee), \text{cone}(\alpha_2^\vee, -\alpha_1^\vee - \alpha_2^\vee)\}$. Moreover, $\tilde{k}_X(\alpha_1^\vee) = \tilde{k}_X(\alpha_2^\vee) = 0$ and $\tilde{k}_X(-\alpha_1^\vee - \alpha_2^\vee) \geq 1$, so \tilde{k}_X is strictly convex.

If $\rho(D(X)) = \{\alpha_1^\vee\}$, then $\Delta[p]$ contains $-3\omega_1^\vee$. Note that $\{-\alpha_1^\vee - \alpha_2^\vee, -3\omega_1^\vee\}$ is a basis of $\chi_*(S)$. By Lemma 6.3 we have two possibilities for $\Delta[p]$: $\{\alpha_1^\vee, -\alpha_1^\vee - \alpha_2^\vee, -3\omega_1^\vee\}$ or $\{\alpha_1^\vee, -\alpha_1^\vee - \alpha_2^\vee, -4\omega_1^\vee - \omega_2^\vee, -3\omega_1^\vee\}$. In the first case there is no condition because the weights of k are $a_1\omega_1 - (a_1 + 1)\omega_2$ and $-\omega_2$. In the last case we have to impose that $a_1, a_2 > 1$. Indeed the weights of k are $a_1\omega_1 - (a_1 + 1)\omega_2, -\alpha_2$ and $-\omega_1 + \omega_2$. Moreover $(-\alpha_2)(\alpha_1^\vee) < a_1$ and $(-\omega_1 + \omega_2)(\alpha_2^\vee) < a_2$. In §3, we have seen that $a_1 \leq 1, a_2 \leq 1$ if and only if $\theta = -\text{id}$.

(IV) Suppose now θ is decomposable. Note that $\sharp D(G/H) = 2$ if $\chi_*(S) = \mathbb{Z}(\omega_1^\vee + \omega_2^\vee) \oplus \mathbb{Z}2\omega_2^\vee$, because any $n \in H \setminus G^\theta$ exchanges the colors of G/G^θ associated to the same coroot. Thus in this case any projective symmetric variety is \mathbb{Q} -factorial.

First suppose $D(X) = D(G/H)$. Then $\Delta(2)$ contains $\text{cone}(\alpha_1^\vee, x(-m\alpha_1^\vee - \alpha_2^\vee))$ and $\text{cone}(\alpha_2^\vee, x(-\alpha_1^\vee - r\alpha_2^\vee))$ with $x \in \{1, 1/2\}$. Note that $\mathbb{R}^+(-m\alpha_1^\vee - \alpha_2^\vee) = \mathbb{R}^+(-\alpha_1^\vee - \frac{1}{m}\alpha_2^\vee)$ and $x(-\alpha_1^\vee - r\alpha_2^\vee) \notin \text{cone}(\alpha_1^\vee, x(-m\alpha_1^\vee - \alpha_2^\vee))$ so $r \leq 1/m$. Therefore $\Delta(2) = \{\text{cone}(\alpha_1^\vee, x(-\alpha_1^\vee - \alpha_2^\vee)), \text{cone}(\alpha_2^\vee, x(-\alpha_1^\vee - \alpha_2^\vee))\}$.

(V) Suppose now that $\rho(D(X))$ contains exactly one coroot, say α_1^\vee . Suppose X_0 is smooth and let $\sigma_0 := \text{cone}(\alpha_1^\vee, -m\alpha_1^\vee + e_2)$ be in $\Delta(2)$. The case of X_0 singular is very similar. We can apply Lemma 6.3 to the maximal open toroidal subvariety X' of X (whose colored fan has support $\text{cone}(-\alpha_1^\vee, -m\alpha_1^\vee + e_2)$). There are two possibilities for $\Delta_{X'}$: its maximal cones are either $\{\sigma_1 := \text{cone}(-\alpha_1^\vee, -m\alpha_1^\vee + e_2)\}$ or $\{\sigma_1 := \text{cone}(-m\alpha_1^\vee + e_2, -(m+1)\alpha_1^\vee + e_2), \sigma_2 := \text{cone}(-\alpha_1^\vee, -(m+1)\alpha_1^\vee + e_2)\}$. In the first case the unique non-trivial condition is $k_{\sigma_1}(\alpha_2^\vee) < a_2$, which is equivalent to $\tilde{k}_{\sigma_1}(\alpha_2^\vee) = (\psi_2(m-1))(\alpha_2^\vee) < 0$. In the second case the unique non-trivial condition is $\tilde{k}_{\sigma_2}(\alpha_2^\vee) = (\psi_2(m))(\alpha_2^\vee) < 0$.

Now, we are interested in which of these varieties are smooth. First, we explain the conditions for a projective locally factorial symmetric variety X with rank two to be smooth (see [Ru07, Theorems 2.1 and 2.2]). Let Y be an open simple G -subvariety of X whose closed orbit is compact; then the associated colored cone (C, F) is 2-dimensional. Write $C = \text{cone}(v_1, v_2)$ and let $C^\vee = \text{cone}(v_1^*, v_2^*)$ be the dual cone; we take v_1 and v_2 primitive. If X is smooth then ρ is injective over F and $\rho(F)$ does not contain any exceptional root. Suppose the latter conditions are satisfied and let R' be the root subsystem of $R_{G,\theta}$ generated by the simple roots α such that $\alpha^\vee \in \rho(F)$ and $\sharp\rho^{-1}(\alpha^\vee) = 1$. If there is no such root, then Y is smooth. Otherwise it is smooth if and only if: (i) R' has type A_1 ; (ii) up to reindexing, $\frac{1}{2}(2v_1^* - v_2^*)$ is the fundamental weight of R' .

Suppose now that $C = \text{cone}(\alpha_1^\vee, -r\alpha_1^\vee - \alpha_2^\vee)$ with $-r\alpha_1^\vee - \alpha_2^\vee$ primitive. If X is Fano, then $\sharp\rho^{-1}(\alpha_1^\vee) = 1$. If moreover θ is indecomposable, then $R_{G,\theta} = A_2$, $H = G^\theta$ and $r = 1$. Furthermore R' is A_1 and $C^\vee = \text{cone}(v_1^* = \omega_1 - \omega_2, v_2^* = -\omega_2)$. Hence $\frac{1}{2}(2v_1^* - v_2^*) = \frac{1}{2}\alpha_1$; thus Y is smooth. If $G/G^\theta = G_1/G_1^\theta \times G_2/G_2^\theta$, then $R' = R_{G_1,\theta}$, $H = G_1^\theta \times (H \cap G_2)$ and $C^\vee = \text{cone}(v_1^* = \omega_1 - r\omega_2, v_2^* = -\omega_2)$. Hence $R_{G_1,\theta}$ has to be A_1 and $\frac{1}{2}(2v_1^* - v_2^*) = \omega_1 + (\frac{1}{2} - r)\omega_2$ has to be $\frac{1}{2}\alpha_1$. Thus Y is not smooth. Suppose now that $R_{G,\theta} = A_1 \times A_1$, $\chi_*(S) = \mathbb{Z}2\omega_1^\vee \oplus \mathbb{Z}(\omega_1^\vee + \omega_2^\vee)$ and $C = \text{cone}(\alpha_1^\vee, -\frac{2r+1}{2}\alpha_1^\vee - \frac{1}{2}\alpha_2^\vee)$. Then R' is A_1 and we have $C^\vee = \text{cone}(v_1^* = \omega_1 - (2r+1)\omega_2, v_2^* = -2\omega_2)$; moreover, $\frac{1}{2}(2v_1^* - v_2^*) = \omega_1 - 2r\omega_2$; thus Y is smooth if and only if $r = 0$. The other cases are similar. \square

§6.2. Smooth quasi-Fano varieties

Now, we prove Theorem 2.7 (and Theorem D) about the quasi-Fano symmetric varieties. Recall that we have to verify that, given any $(C, F) \in \mathcal{F}(X)$, $k_C(v) \leq 1$ for each $v \in N(X)$ and $k_C(\alpha^\vee) \leq (2\delta - 2\delta_0)(\alpha^\vee)$ for each simple restricted co-root α^\vee . The idea is to use Theorem 2.6 and the following lemma. In the case where θ is indecomposable, if $-K_X$ is nef then it is also big (see §1.8).

Lemma 6.4. *Let X be a projective symmetric variety with $-K_X$ nef. Then there is a symmetric variety X' below X such that the piecewise linear function associated to $-K_{X'}$ is strictly convex (over the colored fan of X') and coincides with the function associated to $-K_X$. If X is toroidal, we can choose X' toroidal.*

Proof. Let Δ' be the fan whose maximal cones are the maximal cones over which k_X is linear. For any cone $C \in \Delta'$, define F_C as $\{D \in D(G/H) : \rho(D) \in C\}$. We claim that $\{(C, F_C)\}_{C \in \Delta', \text{int } C \cap C' = \emptyset}$ is a colored fan associated to a symmetric variety that satisfies the conditions of the lemma. Note that $|\Delta'| = |\mathcal{F}(X)|$. We have only to prove that the maximal cones of Δ' are strictly convex. Let $C = \text{cone}(\alpha_1^\vee, \dots, \alpha_r^\vee, -\varpi_1^\vee, \dots, -\varpi_s^\vee) \in \Delta'(l)$ (with $\varpi_1^\vee, \dots, \varpi_s^\vee$

dominant) and suppose for contradiction that it contains the line generated by $v = \sum_{i=1}^r a_i \alpha_i^\vee + \sum_{j=1}^s b_j (-\varpi_j^\vee)$, where $a_i, b_j \geq 0$. Then C also contains $\mathbb{R}v'$, where $v' = \sum b_j (-\varpi_j^\vee)$. Indeed $\alpha_1^\vee, \dots, \alpha_r^\vee, -\varpi_1^\vee, \dots, -\varpi_s^\vee, -v$ belong to C and $-v' = -v + \sum_{i=1}^r a_i \alpha_i^\vee$; thus v' and $-v'$ belong to C . Write $-v' = \sum_{\alpha \in \bar{R}_{G,\theta}} c_\alpha \alpha^\vee$; then C contains all the α^\vee such that $c_\alpha \neq 0$, because $-v' \in C \cap C^+$ and any spherical weight is a positive rational combination of simple restricted roots. Thus, if $v' \neq 0$, then $k_C(-v') = \sum c_\alpha k_C(\alpha^\vee) \geq 0$, while $k_C(v') = \sum b_j k_C(-\varpi_j^\vee) > 0$. Suppose now that $v = \sum_{i=1}^r a_i \alpha_i^\vee$ with $a_j \neq 0$. Then C contains $\mathbb{R}\alpha_j^\vee$. Write $-\alpha_j^\vee = \sum_{i=1}^r a'_i \alpha_i^\vee + \sum_{i=1}^s b'_i (-\varpi_i^\vee)$ with positive coefficients. Then there is j_0 such that $b_{j_0} \neq 0$. So $k_C(-\alpha_j^\vee) \geq b_{j_0} > 0$ and $k_C(\alpha_j^\vee) > 0$, a contradiction. \square

Proof of Theorem 2.7. The first property holds because the inequalities in the conditions for the nefness of a Cartier divisor are not strict; moreover given two colors in the inverse image of a simple coroot, the coefficients of $-K_X$ with respect to these divisors are equal.

(I) First suppose ρ non-injective. We have to consider the varieties whose fan is as in Theorem 2.6, but that are not Fano because $\sharp\rho^{-1}(\rho(D(X))) \neq \sharp\rho(D(X))$. In such cases ρ has to be injective over $D(X)$, so that X is locally factorial. If θ is indecomposable we have the following possibilities: (i) $R_{G,\theta} = BC_2$ and $\Delta(2) = \{\text{cone}(\alpha_2^\vee, -\omega_2)\}$; (ii) $R_{G,\theta} = B_2$, $H = G^\theta$ and $\Delta(2) = \{\text{cone}(\alpha_1^\vee, -2\omega_1)\}$. If θ is decomposable, we have to consider all the possibilities listed in Theorem 2.6 that correspond to non-toroidal varieties with $\chi_*(S) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$. We also have to replace the conditions of type $\psi_i(m) < 0$ by the corresponding conditions $\psi_i(m) \leq 0$.

(II) Suppose X toroidal and let X' be as in Lemma 6.4. If $X' = X$ and is simple, then $G/H = G_2/(SL_2 \times SL_2)$. If $X' \neq X$ and is simple, then it must be singular. Otherwise, given any $w = x_1e_1 + x_2e_2 \in \Delta_X[p] \setminus \Delta_{X'}[p]$, we have $x_i \in \mathbb{Z}^{>0}$ and $x_1 + x_2 = k(w) = 1$. If $R_{G,\theta} = A_2$ and $H = G^\theta$, then $k_X = -\frac{1}{3}(\omega_1 + \omega_2) \notin \chi(S)$, so X is not locally factorial. Thus $R_{G,\theta} = A_1 \times A_1$ and $\chi_*(S) = \mathbb{Z}2\omega_1^\vee \oplus \mathbb{Z}(\omega_1^\vee + \omega_2^\vee)$. Let $w = -x_1\alpha_1^\vee - x_2\alpha_2^\vee$ be in $\Delta_X[p] \setminus \Delta_{X'}[p]$; then $2x_1, 2x_2 \in \mathbb{Z}^{>0}$ and $x_1 + x_2 = k(w) = 1$. Thus $\Delta_X[p] = \{-\alpha_1^\vee, -\alpha_2^\vee, -\frac{1}{2}\alpha_1^\vee - \frac{1}{2}\alpha_2^\vee\}$. In this case $-K_X$ is nef but non-ample.

If X' is not simple, then the standard compactification of G/H has to be smooth because of Lemma 6.3. So it is sufficient to prove the following lemma.

Lemma 6.5. *The smooth toric varieties birationally proper over \mathbb{A}^2 with nef anticanonical divisor are, up to isomorphisms, \mathbb{A}^2 and Z_m , where Z_m is the variety whose fan has maximal cones $\{\text{cone}(e_1, e_1 + e_2), \text{cone}(e_1 + e_2, e_1 + 2e_2), \dots, \text{cone}(e_1 + (m - 1)e_2, e_1 + me_2), \text{cone}(e_1 + me_2, e_2)\}$.*

Proof of Lemma 6.5. The piecewise linear function k_m associated to the anticanonical bundle of Z_m is linear on $\text{cone}(e_1, e_1 + me_2)$. It is easy to see that this function is convex.

Now we show that, given any Z as in the hypotheses, it is isomorphic to Z_m for an appropriate m . Notice that any smooth toric variety birationally proper over \mathbb{A}^2 is obtained by a sequence of blow-ups. Thus there is nothing to prove if $\#\Delta[p] < 4$. Suppose now that $\#\Delta[p] \geq 4$; we claim that up to isomorphisms, Δ contains $\text{cone}(e_1, e_1 + e_2)$ and $\text{cone}(e_1 + e_2, e_1 + 2e_2)$. We know that $\tau = \mathbb{R}^{\geq 0}(e_1 + e_2)$ is contained in Δ .

First of all, we determine the restrictions of k to the cones containing τ , and afterwards we will determine the cones themselves. Let $\sigma = \text{cone}(e_1 + e_2, b_1e_1 + b_2e_2) \in \Delta(2)$ be a maximal cone containing τ and write $k_\sigma = x_1e_1^* + x_2e_2^*$, so $k_\sigma(e_1 + e_2) = x_1 + x_2 = 1$ and $x_i = k_\sigma(e_i) \leq 1$ for each i . Hence k_σ is e_i^* for an appropriate i and $b_i = k_\sigma(b_1e_1 + b_2e_2) = 1$. Because of the non-singularity of Z the only possibilities for σ are $\text{cone}(e_1 + e_2, e_1)$, $\text{cone}(e_1 + e_2, e_2)$, $\text{cone}(e_1 + e_2, e_1 + 2e_2)$ and $\text{cone}(e_1 + e_2, 2e_1 + e_2)$. The fan Δ does not contain both $\text{cone}(e_1 + e_2, e_1 + 2e_2)$ and $\text{cone}(e_1 + e_2, 2e_1 + e_2)$; otherwise $k_{\text{cone}(e_1 + e_2, e_1 + 2e_2)}(2e_1 + e_2) = 2 > 1 = k(2e_1 + e_2)$. Observe that if Δ contains $\text{cone}(e_1 + e_2, 2e_1 + e_2)$, then Z is isomorphic to a variety whose fan contains $\text{cone}(e_1 + e_2, e_1 + 2e_2)$ by the isomorphism induced by the automorphism of $\chi_*(S)$ that exchanges e_1 and e_2 . So the claim is proved.

Because of the non-singularity of Z , Δ contains a cone $\sigma = \text{cone}(e_1 + me_2, e_2)$ for a suitable integer m ; we want to show that Z is Z_m . Let Z' be the open toric subvariety of Z whose fan Δ' is $\Delta \setminus \{\text{cone}(e_1 + me_2, e_2), \text{cone}(e_2)\}$.

We claim that, for each integer $r > 1$, there is a unique variety \tilde{Z}'_r with the following two properties: 1) the fan $\tilde{\Delta}'_r$ of \tilde{Z}'_r has support $\text{cone}(e_1, e_1 + re_2)$; 2) \tilde{Z}'_r is an open subvariety of a toric variety \tilde{Z}_r with nef anticanonical bundle and birationally proper over \mathbb{A}^2 . In particular, the anticanonical divisor of \tilde{Z}'_r is nef. The open subvariety Z'_r of Z_r whose fan is $\Delta_r \setminus \{\text{cone}(e_1 + re_2, e_2), \text{cone}(e_2)\}$ satisfies these properties. So it is sufficient to prove the claim.

We show the claim by induction on r . We have already verified the case $r = 2$. Let \tilde{Z}'_r be a variety that satisfies the hypotheses of the claim and let σ' be the unique cone in $\tilde{\Delta}'_r(2)$ which contains $e_1 + re_2$. Because of the inductive hypothesis it is sufficient to show that $\sigma' = \text{cone}(e_1 + re_2, e_1 + (r - 1)e_2)$.

Let k be the function associated to the anticanonical bundle of a fixed \tilde{Z}_r . Let $k_{\sigma'} = x_1e_1^* + x_2e_2^*$. Then $1 = k_{\sigma'}(e_1 + re_2) = x_1 + rx_2$ and $x_i = k_{\sigma'}(e_i) \leq 1$ for each i , so the unique possibilities for $k_{\sigma'}$ are e_1^* and $-(r - 1)e_1^* + e_2^*$. Write $\sigma' = \text{cone}(e_1 + re_2, v)$ with $v = c_1e_1 + c_2e_2$ primitive. If $k_{\sigma'} = -(r - 1)e_1^* + e_2^*$, then $c_2 = (r - 1)c_1 + 1$ because $(-(r - 1)e_1^* + e_2^*)(v) = 1$. Because of the non-singularity of Z we have $c_1 - 1 = \pm 1$, so there are two possibilities: either $\sigma' =$

$\text{cone}(e_1 + re_2, e_2)$ or $\sigma' = \text{cone}(e_1 + re_2, 2e_1 + (2r - 1)e_2)$. We exclude the first one because e_2 does not belong to $|\tilde{\Delta}'_r|$. We also exclude the second one because $k_{\text{cone}(e_1, e_1 + e_2)}(2e_1 + (2r - 1)e_2) = 2 > k(v)$. If $k_{\sigma'} = e_1^*$, then $c_1 = k_{\sigma'}(v) = 1$. Because of the smoothness of Z we have $c_2 - r = \pm 1$. Again, we exclude $e_1 + (r + 1)e_2$ because it does not belong to $|\tilde{\Delta}'_r|$. Thus $\sigma' = \text{cone}(e_1 + re_2, e_1 + (r - 1)e_2)$. \square

Suppose $\Delta_X[p] = \{e_1, e_1 + e_2, e_1 + 2e_2, \dots, e_1 + re_2, e_2\}$. Then $-K_X$ is nef if and only if $-2\delta + 2\delta_0 + e_1^*$ and $-2\delta + 2\delta_0 - (r - 1)e_1^* + e_2^*$ are antidominant. In that case, if θ is decomposable, then the sum of these weights is regular, so $-K_X$ is big. If $r = 1$ and the previous weights are regular, then $-K_X$ is ample. Note that if $r = 1$ and θ is decomposable, such weights are always regular.

(III) If X is not toroidal and θ is indecomposable, let X' be as in Lemma 6.4. By Lemma 6.1 and Theorem 2.6, $R_{G,\theta} = A_2$, $H = G^\theta$ and $\Delta_{X'}[p]$ contains properly $\{\alpha_1^\vee, -\alpha_1^\vee - \alpha_2^\vee, -3\omega_1^\vee\}$, up to reindexing. By Lemma 6.5 we have two possibilities for $\Delta_{X'}[p]$: either (i) $\{\alpha_1^\vee, -\omega_1^\vee - \omega_2^\vee, -4\omega_1^\vee - \omega_2^\vee, -5\omega_1^\vee - 2\omega_2^\vee, \dots, -(r + 3)\omega_1^\vee - r\omega_2^\vee, -3\omega_1^\vee\}$ or (ii) $\{\alpha_1^\vee, -\omega_1^\vee - \omega_2^\vee, -4\omega_1^\vee - \omega_2^\vee, -7\omega_1^\vee - \omega_2^\vee, \dots, -(3r + 1)\omega_1^\vee - \omega_2^\vee, -3\omega_1^\vee\}$.

Recall $a_i = (2\delta - 2\delta_0)(\alpha_i^\vee)$. In the first case k is linear over $\text{cone}(\alpha_1^\vee, -\omega_1^\vee - \omega_2^\vee)$, $\sigma := \text{cone}(-\omega_1^\vee - \omega_2^\vee, -(r + 3)\omega_1^\vee - r\omega_2^\vee)$ and $\text{cone}(-(r + 3)\omega_1^\vee - r\omega_2^\vee, -3\omega_1^\vee)$. The unique non-trivial condition is $k_\sigma(\alpha_1^\vee) = r \leq a_1$. In the second case k is linear over $\text{cone}(\omega_1^\vee, -\omega_1^\vee - \omega_2^\vee)$, $\text{cone}(-\omega_1^\vee - \omega_2^\vee, -(3r + 1)\omega_1^\vee - \omega_2^\vee)$ and $\sigma := \text{cone}(-(3r + 1)\omega_1^\vee - \omega_2^\vee, -3\omega_1^\vee)$. The unique non-trivial condition is $k_\sigma(\alpha_2^\vee) = 2r - 1 \leq a_2$ (or equivalently $k_\sigma(\alpha_2^\vee) \leq 0$). If $r = 1$, then $\theta = -\text{id}$ over $\chi_*(T)$ so that $-K_X$ is not ample.

(IV) Finally, suppose X non-toroidal and θ decomposable. Suppose also X_0 is smooth; the case of X_0 singular is very similar. By the proof of Theorem 2.6, $\rho(D(X))$ cannot be $\rho(D(G/H))$. If $\rho(D(X)) = \alpha_1^\vee$, then, by the local factoriality of X , $e_1 = -\alpha_1^\vee$ and there is $r \in \mathbb{Z}^{>0}$ with $\sigma := \text{cone}(\alpha_1^\vee, v' = -r\alpha_1^\vee + e_2) \in \Delta$. We apply Lemma 6.4 to the maximal open toroidal subvariety of X (whose colored fan has support $\text{cone}(-\alpha_1^\vee, v')$). Then, by Lemma 6.5, $\Delta[p]$ has to be $\{-\alpha_1^\vee, v', -\alpha_1^\vee + v', -\alpha_1^\vee + 2v', \dots, -\alpha_1^\vee + sv', \alpha_1^\vee\}$, $\{-\alpha_1^\vee, v', -\alpha_1^\vee + v', -2\alpha_1^\vee + v', \dots, -s\alpha_1^\vee + v', \alpha_1^\vee\}$ or $\{-\alpha_1^\vee, v', \alpha_1^\vee\}$.

In the first case, k is linear over $\text{cone}(\alpha_1^\vee, v')$, $\sigma_1 := \text{cone}(v', -\alpha_1^\vee + sv')$ and $\sigma_2 := \text{cone}(-\alpha_1^\vee + sv', -\alpha_1^\vee)$. The non-trivial conditions are $\tilde{k}_{\sigma_1}(\alpha_1^\vee) \leq 0$ and $\tilde{k}_{\sigma_2}(\alpha_2^\vee) \leq 0$ (or equivalently $s \leq \bar{m}_1 + 1$ and $r \leq \bar{m}_2$). If $s = 1$, then $r > m_2$ because $-K_X$ is not ample. In the second case we can suppose $s \geq 2$; k is linear over $\text{cone}(\alpha_1^\vee, v')$, $\sigma_1 := \text{cone}(v', -s\alpha_1^\vee + v')$ and $\sigma_2 := \text{cone}(-s\alpha_1^\vee + v', -\alpha_1^\vee)$. The unique non-trivial condition is $\tilde{k}_{\sigma_2}(\alpha_2^\vee) \leq 0$, or, equivalently, $r + s - 1 \leq \bar{m}_2$. In the first two cases $\tilde{k}_{\sigma_1}(\alpha_2^\vee) < 0$ and $\tilde{k}_{\sigma_2}(\alpha_1^\vee) < 0$, so $-\tilde{K}_X$ is big. In the last case, we

proceed as in the proof of Theorem 2.6, obtaining $m_2 + 1 < r \leq \bar{m}_2 + 1$. Moreover, $\tilde{k}_{\text{cone}(\alpha_1^\vee, v')}(\alpha_2^\vee) < 0$ and $\tilde{k}_{\text{cone}(-\alpha_1^\vee, v')}(\alpha_1^\vee) < 0$, so $-K_X$ is big.

We can study the smoothness of all the previous varieties as in the proof of Theorem 2.6. Note that if $H = G^\theta$, θ is decomposable, $\sharp\rho^{-1}(\alpha_1^\vee) = 2$ and $(C(Y), D(Y)) = (\text{cone}(\alpha_1^\vee, -r\alpha_1^\vee - \alpha_2^\vee), F)$, then Y is smooth if and only if $\sharp F = 1$ and α_1 is not exceptional (but Y cannot be an open subvariety of a Fano variety). The symmetric varieties in the statement are all projective. Indeed, if $\Delta_X[p] = \{e_1, e_1 + e_2, e_1 + 2e_2, \dots, e_1 + (s-1)e_2, e_1 + se_2, e_2\}$, then the following divisor is ample: $-pK_X + \sum_{i=1}^s i^2 D_{e_1+ie_2} + m \sum_{D \in D(G/H) \setminus D(X)} D$ with $p, m \gg 0$. Indeed the piecewise linear function associated to $\sum_{i=1}^s i^2 D_{e_1+ie_2}$ is strictly convex on $\{C \in \Delta_X : C \subset \text{cone}(e_1, e_1 + s_2)\}$. Moreover, k_X is strictly convex on the fan with maximal cones $\text{cone}(e_1, e_1 + s_2)$ and $\text{cone}(e_1 + s_2, e_2)$. The other cases are similar.

§6.3. Symmetric Fano varieties with G reductive

Finally we classify the 2-rank locally factorial Fano symmetric varieties acted upon by a group G that is only reductive. Recall that $G = G' \times \mathbb{C}^*$, $\chi_*(\mathbb{C}^*/(\mathbb{C}^*)^\theta) = \mathbb{Z}f$ and e is the primitive multiple of $-\alpha^\vee$. Observe that X cannot be simple because the valuation cone is not strictly convex (and $N(H)/H$ is not finite). We begin with a lemma similar to Lemma 6.2.

Lemma 6.6. *Let X be a Fano locally factorial symmetric variety with G as before. If X is toroidal, there are at most four colored 1-cones. Otherwise, there are at most three colored 1-cones.*

Proof. First suppose X is toroidal. Then $k(\pm f) = 1$. Let $\sigma \in \Delta(2)$ be a cone that contains neither f nor $-f$. Then $k_\sigma(\pm f) \leq 0$, so k_σ is a multiple of e^* . If there is another cone $\sigma' \in \Delta(2)$ with the same properties (and such that $\dim(\sigma \cap \sigma') = 1$), then $k_\sigma = k_{\sigma'}$. If X is not toroidal, we can study the maximal open toroidal subvariety X' of X as in Lemma 6.2 (because $|\mathcal{F}(X')|$ is strictly convex). \square

Proof of Theorem 2.8. We have to require, as in Theorem 2.6, that $\sharp\rho(D(X)) = \sharp\rho^{-1}(\rho(D(X)))$, but in this case $\sharp\rho(D(X)) \leq 1$. If X is not toroidal, then $e = -\alpha^\vee$, because there is an appropriate cone (α^\vee, v) in $\Delta(2)$. Suppose first that $\chi_*(S) = \mathbb{Z}e \oplus \mathbb{Z}f$. We have to consider the following cases:

1) If $\Delta[p] = \{f, -f, v\}$, then $v = e + rf$ because $\{f, v\}$ is a basis of $\chi_*(S)$. We have to impose $\tilde{k}_{\text{cone}(f, e+rf)}(\alpha^\vee) = \psi(r-1)(\alpha^\vee) < 0$ and $\tilde{k}_{\text{cone}(-f, e+rf)}(\alpha^\vee) = \psi(-r-1)(\alpha^\vee) < 0$.

2) Suppose $\Delta[p] = \{f, -f, v_1, v_2\}$. As before $v_i = e + x_i f$; moreover $x_1 = x_2 \pm 1$ because $\{v_1, v_2\}$ is a basis of $\chi_*(S)$. Suppose $x_1 = x_2 + 1$. We have to

impose that $\tilde{k}_{\text{cone}(f, e+(x_2+1)f)}(\alpha^\vee) = \psi(x_2)(\alpha^\vee) < 0$ and $\tilde{k}_{\text{cone}(-f, e+x_2f)}(\alpha^\vee) = \psi(-x_2 - 1)(\alpha^\vee) < 0$.

3) Suppose $\Delta[p] = \{\alpha^\vee, -f, v\}$. Then $v = -\alpha^\vee + rf = -m\alpha^\vee \pm f$ because X is locally factorial. But $-f$ is not contained in $\text{cone}(\alpha^\vee, v)$, thus $v = -\alpha^\vee + f$.

4) Suppose $\Delta[p] = \{\alpha^\vee, -f, v_1, v_2\}$ and $\Delta(2) = \{\text{cone}(\alpha^\vee, v_1), \sigma := \text{cone}(v_1, v_2), \text{cone}(-f, v_2)\}$. Then $v_1 = -m\alpha^\vee + f$ and $v_2 = -\alpha^\vee + rf$. Observe that $\mathbb{R}^{\geq 0}v_1 = \mathbb{R}^{\geq 0}(-\alpha^\vee + \frac{1}{m}f)$, so $r \leq 0$. Furthermore $v_2 - rv_1 = -(1 - rm)\alpha^\vee = \pm\alpha^\vee$, so mr is 0 or 2. Thus $v_2 = -\alpha^\vee$. Moreover $h_\sigma(-f) = m - 1 < 1$, so $v_1 = -\alpha^\vee + f$.

5) Suppose $\Delta[p] = \{\alpha^\vee, v_1, v_2\}$ and $\Delta(2) = \{\text{cone}(\alpha^\vee, v_1), \text{cone}(v_1, v_2), \text{cone}(\alpha^\vee, v_2)\}$. Thus $v_1 = -r\alpha^\vee + f$ and $v_2 = -m\alpha^\vee - f$ with $r, m > 0$. Moreover, $v_1 + v_2 = -(r + m)\alpha^\vee = \pm\alpha^\vee$, a contradiction.

6) Suppose $\Delta[p] = \{\alpha^\vee, v_1, v_2, v_3\}$ and $\Delta(2) = \{\text{cone}(\alpha^\vee, v_1), \sigma := \text{cone}(v_1, v_2), \text{cone}(v_2, v_3), \text{cone}(\alpha^\vee, v_3)\}$. Write $v_2 = -x\alpha^\vee + yf$. By the local factoriality of X , $v_1 = -r\alpha^\vee + f$, $v_3 = -m\alpha^\vee - f$, $x = -my \pm 1$ and $x = +ry \pm 1$. The last two conditions plus $x > 0$ imply that $v_2 = -\alpha^\vee$. Moreover $k_\sigma(v_3) = m + r - 1 < 1$, a contradiction.

Now, suppose $\chi_*(S) = \mathbb{Z}^{\frac{f+\alpha^\vee}{2}} \oplus \mathbb{Z}f$.

1) The toroidal case can be studied as before.

2) Suppose that $\Delta(2)$ contains two cones $\sigma_\pm := \text{cone}(v_\pm, \alpha^\vee)$. Let $u = \frac{1}{2}\alpha^\vee + \frac{1}{2}f$. We have $v_+ = -\frac{2m+1}{2}\alpha^\vee + \frac{1}{2}f = -(2m+1)u + (m+1)f$ and $v_- = -\frac{2r+1}{2}\alpha^\vee - \frac{1}{2}f = -(2r+1)u + rf$ with $m, r \geq 0$. First, suppose that there is another $v = -xu + yf$ in $\Delta[p]$; here $x > 0$. By the local factoriality we have $x(m+1) = y(2m+1) \pm 1$ and $xr = y(2r+1) \pm 1$. Thus $x[(m+1)(2r+1) - r(2m+1)] = \pm 1 \mp 1$, so $x(m+r+1) = 2$, $x(m+1) = y(2m+1) + 1$ and $xr = y(2r+1) - 1$. If $x = 1$, then the previous three equations are not compatible. Instead, if $x = 2$ then $\Delta[p] = \{\alpha^\vee, -\frac{1}{2}\alpha^\vee + \frac{1}{2}f, -\alpha^\vee, -\frac{1}{2}\alpha^\vee - \frac{1}{2}f\}$. Next, suppose that $\Delta[p] = \{\alpha^\vee, v_+, v_-\}$; in particular $\{v_+, v_-\}$ is a basis. Thus $v_+ + v_- = -(m+r+1)\alpha^\vee = \pm\alpha^\vee$, so $\Delta[p] = \{\alpha^\vee, -\frac{1}{2}\alpha^\vee + \frac{1}{2}f, -\frac{1}{2}\alpha^\vee - \frac{1}{2}f\}$.

3) Finally, suppose that $\Delta[p]$ contains α^\vee and $-f$. Then Δ contains $\sigma = \text{cone}(\alpha^\vee, v_1)$ and $\text{cone}(-f, v_2)$, with $v_1 = -\frac{2m+1}{2}\alpha^\vee + \frac{1}{2}f$, $m \geq 0$ and $v_2 = -\frac{1}{2}\alpha^\vee - \frac{2r+1}{2}f$. If $v_1 = v_2$, then $\Delta[p] = \{\alpha^\vee, -f, -\frac{1}{2}\alpha^\vee + \frac{1}{2}f\}$. Otherwise, by Lemma 6.6, Δ contains $\text{cone}(v_1, v_2)$. Furthermore, $(2r+1)v_1 + v_2 = \pm\alpha^\vee$, hence $(2r+1)(2m+1)$ is 1 or -3 . But $r \geq 0$ because $v_2 \notin \sigma$, so $v_1 = -\frac{1}{2}\alpha^\vee + f$ and $v_2 = -\frac{1}{2}\alpha^\vee - f$. The smoothness of these varieties can be studied as in Theorems 2.6 and 2.7. \square

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