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Generalized Jacquet Modules of Parabolically Induced Representations

by

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Abstract

In this paper we study a generalization of the Jacquet module of a parabolically induced representation and construct a filtration on it. The successive quotients of the filtration are written by using the twisting functor.

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§1. Introduction

The Jacquet module of a representation of a semisimple (or reductive) Lie group was introduced by Casselman [Cas80]. One of the motivations of considering the Jacquet module is to investigate homomorphisms to principal series representations. The space of homomorphisms to principal series representations is an important invariant of a representation.

One of the powerful tools to study the Jacquet module of a parabolically induced representation is the Bruhat filtration [CHM00]. This is a filtration on the Jacquet module defined by the Bruhat decomposition. Casselman–Hecht–Miličić [CHM00] used the Bruhat filtration to determine the dimension of the (moderate-growth) Whittaker model of a principal series representation (another proof of Kostant's result [Kos78, Theorem I, Theorem J]). In this paper, we study the Bruhat filtration and show that its successive quotients are described by the twisting functor defined by Arkhipov [Ark04]. The successive quotients become

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"twisted" inductions, which have the same character as that of an induced representation but a different module structure.

Moreover, we investigate its generalization, which is related to the Whittaker model. In [Cas80], Casselman suggested generalizing the notion of the Jacquet module. For this generalized Jacquet module, we can also define a Bruhat filtration and the successive quotients of the filtration are described in terms of the generalized twisting functor.

This result gives a strategy to determine all Whittaker models of a parabolically induced representation. To determine it, it suffices to study the successive quotients and extensions of the filtration. In a special case, we can carry out these steps.

Now we state our results precisely. Let G be a connected semisimple linear Lie group, $G = KA_0N_0$ an Iwasawa decomposition and $P_0 = M_0A_0N_0$ a minimal parabolic subgroup and its Langlands decomposition. As usual, the complexifications of the Lie algebras is denoted by the corresponding German letter (for example, $\mathfrak{g} = \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$). Fix a character η of N_0 and denote its differential also by η. Then for a representation V of G, the *generalized Jacquet modules* $J'_\eta(V)$ and $J^*_\eta(V)$ are defined as follows. Let $V_{K\text{-finite}}$ be the space of K-finite vectors in V.

Definition 1.1. Let V be a finite-length moderate growth Fréchet representation of G (see Casselman [Cas89, p. 391]). We define $\mathfrak{g}$-modules $J'_\eta(V)$ and $J^*_\eta(V)$ by

$$J'_\eta(V) = \left\{ v \in V' \;\middle|\; \begin{array}{l} \text{for some } k \text{ and for all } X \in \mathfrak{n}_0, \\ (X - \eta(X))^k v = 0 \end{array} \right\},$$
$$J^*_\eta(V) = \left\{ v \in (V_{K\text{-finite}})^* \;\middle|\; \begin{array}{l} \text{for some } k \text{ and for all } X \in \mathfrak{n}_0, \\ (X - \eta(X))^k v = 0 \end{array} \right\},$$

where V' is the continuous dual space of V and $(V_{K\text{-finite}})^*$ is the full dual space $\mathrm{Hom}_{\mathbb{C}}(V_{K\text{-finite}}, \mathbb{C})$. If η is the trivial representation, J'_η (resp. J^*_η) is denoted by J' (resp. J^*). The module $J^*(V)$ is called the *Jacquet module* of V.

(We will use the notation $Y^* = \mathrm{Hom}_{\mathbb{C}}(Y, \mathbb{C})$ for any $\mathbb{C}$-vector space Y throughout this paper.)

In this paper, we consider $J'_\eta(V)$ and $J^*_\eta(V)$ when V is a parabolically induced representation. Let P be a parabolic subgroup containing P_0 and take a Langlands decomposition $P = MAN$ such that $A_0 \supset A$. For $\lambda \in \mathfrak{a}^*$ and an irreducible representation σ of M, we define $I(\sigma, \lambda) = \mathrm{Ind}_P^G(\sigma \otimes e^{\lambda+\rho})$ where $\rho \in \mathfrak{a}^*$ is the half sum of positive roots. In this paper, we deal with $J'_\eta(I(\sigma, \lambda))$ and $J^*_\eta(I(\sigma, \lambda))$.

First we discuss $J'_\eta(I(\sigma, \lambda))$. By definition, $I(\sigma, \lambda)$ is realized as the space of C^∞-sections of a certain vector bundle on G/P. Hence an element of its continuous dual space is regarded as a distribution on G/P. Using the Bruhat decomposition on G/P, we can get a filtration $\{I_i\}$ of $J'_\eta(I(\sigma,\lambda))$, which is called the *Bruhat filtration*. The first aim of this paper is to understand the structure of I_i/I_{i-1}.

We give a precise definition of I_i. Let W (resp. W_M) be the little Weyl group of G (resp. M). Then N_0-orbits on G/P are parameterized by W/W_M. Let $W(M)$ be a subset of W consisting of w such that $w(\alpha)$ is positive for any positive restricted root α of M. Then $W(M) \xrightarrow{\sim} W/W_M$. Enumerate $W(M) = \{w_1,\dots,w_r\}$ so that $\bigcup_{j\le i} N_0w_jP/P$ is a closed subset of G/P. Now we define a submodule $I_i \subset J'_\eta(I(\sigma,\lambda))$ by

$$I_i = \left\{x \in J'_\eta(I(\sigma,\lambda)) \;\middle|\; \operatorname{supp} x \subset \bigcup_{j\le i} N_0w_jP\right\}.$$

To describe I_i/I_{i-1}, we need a functor $T_{w_i,\eta}$ which is a generalization of the twisting functor [Ark04]. The generalized twisting functor $T_{w,\eta}$ is defined as follows. Let $\overline{\mathfrak{n}_0}$ be the nilradical of the parabolic subalgebra opposite to $\mathfrak{p}_0$ and $\{e_1,\dots,e_l\}$ a basis of $\operatorname{Ad}(w)\overline{\mathfrak{n}}_0 \cap \mathfrak{n}_0$ such that each e_i is a root vector with respect to $\mathfrak{h}$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ which contains $\mathfrak{a}_0$. Moreover, we choose e_i such that $\bigoplus_{i\le j-1}\mathbb{C}e_i$ is an ideal of $\bigoplus_{i\le j}\mathbb{C}e_i$ for all j. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and $U(\mathfrak{g})_{e_i-\eta(e_i)}$ the localization of $U(\mathfrak{g})$ with respect to the multiplicative set $\{(e_i-\eta(e_i))^n \mid n \in \mathbb{Z}_{>0}\}$. Put

$$S_{w,\eta} = (U(\mathfrak{g})_{e_1-\eta(e_1)}/U(\mathfrak{g})) \otimes_{U(\mathfrak{g})} \cdots \otimes_{U(\mathfrak{g})} (U(\mathfrak{g})_{e_l-\eta(e_l)}/U(\mathfrak{g})).$$

Then $S_{w,\eta}$ is a $U(\mathfrak{g})$-bimodule and its $U(\mathfrak{g})$-bimodule structure is independent of the choice of $\{e_1,\dots,e_l\}$. The twisting functor $T_{w,\eta}$ is an end-functor of the category of $\mathfrak{g}$-modules, defined by $T_{w,\eta}V = S_{w,\eta}\otimes_{U(\mathfrak{g})}(wV)$ for a $\mathfrak{g}$-module V, where wV is the representation twisted by w (i.e., $Xv = \operatorname{Ad}(w)^{-1}(X)\cdot v$ for $X \in \mathfrak{g}$ and $v \in wV$, where the dot means the original action). If η is the trivial representation, then $T_{w,\eta}$ is equal to the twisting functor defined by Arkhipov [Ark04]. In this case, we denote it by T_w.

Now we give the theorem.

Theorem 1.2 (Theorems 4.7 and 6.1). *The filtration $\{I_i\}$ has the following properties.*

(1) *If the character η is not unitary, then $J'_\eta(I_i/I_{i-1}) = 0$ for each $i = 1,\dots,r$. Therefore, $J'_\eta(I(\sigma,\lambda)) = 0$.*

(2) *Assume that η is unitary. The module I_i/I_{i-1} is nonzero if and only if η is trivial on $w_iNw_i^{-1}\cap N_0$ and $J'_{w_i^{-1}\eta}(\sigma\otimes e^{\lambda+\rho}) \ne 0$.*

(3) *If $I_i/I_{i-1} \ne 0$ then $I_i/I_{i-1} \simeq T_{w_i,\eta}(U(\mathfrak{g})\otimes_{U(\mathfrak{p})} J'_{w_i^{-1}\eta}(\sigma\otimes e^{\lambda+\rho}))$ where $\mathfrak{n}$ acts on $J'_{w_i^{-1}\eta}(\sigma\otimes e^{\lambda+\rho})$ trivially.*

Here are some remarks on notation. As $w_i \in W(M)$, we have $\mathrm{Ad}(w_i)(\mathfrak{m}\cap\mathfrak{n}_0) \subset \mathfrak{n}_0$. Hence we can define a character $w_i^{-1}\eta$ of $\mathfrak{m}\cap\mathfrak{n}_0$ by $(w_i^{-1}\eta)(X) = \eta(\mathrm{Ad}(w_i)X)$. Using this character, we can define an $\mathfrak{m}\oplus\mathfrak{a}$-module $J'_{w_i^{-1}\eta}(\sigma\otimes e^{\lambda+\rho})$.

Under the assumptions that P is a minimal parabolic subgroup, and that σ is the trivial representation, $I(\sigma,\lambda)$ has the unique Langlands quotient and η is the trivial representation, this theorem is proved in [Abe08]. The proof we gave in [Abe08] was algebraic, while we give an analytic and geometric proof in this paper.

Next we consider $J^*_\eta(I(\sigma,\lambda))$. For a $U(\mathfrak{g})$-module V, put $\Gamma_\eta(V) = \{v \in V \mid$ for some k and for all $X \in \mathfrak{n}_0$, $(X-\eta(X))^k v = 0\}$ and $C(V) = ((V^*)_{\mathfrak{h}\text{-finite}})^*$. We prove the following theorem.

Theorem 1.3 (Theorem 7.5). *There exists a filtration* $0 = \widetilde{I_0} \subset \widetilde{I_1} \subset \cdots \subset \widetilde{I_r} = J^*_\eta(I(\sigma,\lambda))$ *such that* $\widetilde{I_i}/\widetilde{I_{i-1}} \simeq \Gamma_\eta(C(T_{w_i}(U(\mathfrak{g})\otimes_{U(\mathfrak{p})} J^*(\sigma\otimes e^{\lambda+\rho}))))$ *where* $\mathfrak{n}$ *acts on* $J^*(\sigma\otimes e^{\lambda+\rho})$ *trivially.*

Let us discuss an application. The space $\mathrm{Wh}_\eta(D)$ of *Whittaker vectors* for a $U(\mathfrak{g})$-module D is defined by $\mathrm{Wh}_\eta(D) = \{x \in D \mid (X-\eta(X))x = 0$ for all $X \in \mathfrak{n}_0\}$. If V is a moderate growth Fréchet representation of G, an element of $\mathrm{Wh}_\eta(V')$ corresponds to a moderate growth homomorphism $V \to \mathrm{Ind}_{N_0}^G \eta$ and an element of $\mathrm{Wh}_\eta((V_{K\text{-finite}})^*)$ corresponds to an algebraic homomorphism $V_{K\text{-finite}} \to \mathrm{Ind}_{N_0}^G \eta$. In particular, when η is the trivial representation, these correspond to homomorphisms to principal series representations. Obviously, we have $\mathrm{Wh}_\eta(V') = \mathrm{Wh}_\eta(J'_\eta(V))$ and $\mathrm{Wh}_\eta((V_{K\text{-finite}})^*) = \mathrm{Wh}_\eta(J^*_\eta(V))$. Hence using the above theorems, we can determine the dimension of $\mathrm{Wh}_\eta(I(\sigma,\lambda)')$ and $\mathrm{Wh}_\eta((I(\sigma,\lambda)_{K\text{-finite}})^*)$ if λ satisfies some (generic) condition.

Let us give such a formula. Let Σ (resp. Σ_M) be the restricted root system for (G,A_0) (resp. $(M, M\cap A_0)$), Σ^+ the positive system of Σ corresponding to N_0, and $\Pi \subset \Sigma$ the set of simple roots determined by Σ^+. For $\alpha \in \Sigma$, the coroot of α is denoted by $\check{\alpha}$. Put $\Sigma_M^+ = \Sigma_M \cap \Sigma^+$. Let $\widetilde{W}$ (resp. $\widetilde{W_M}$) be the (complex) Weyl group of $\mathfrak{g}$ (resp. $\mathfrak{m}$). Let $\widetilde{\mu} \in (\mathfrak{m}\cap\mathfrak{h})^*$ be the infinitesimal character of σ. Using the decomposition $\mathfrak{h} = \mathfrak{a}\oplus(\mathfrak{m}\cap\mathfrak{h})$, we regard $(\mathfrak{m}\cap\mathfrak{h})^* \subset \mathfrak{h}^*$. Let Δ be the root system for $(\mathfrak{g},\mathfrak{h})$. Put $\Sigma_\eta^+ = (\sum_{\eta|_{\mathfrak{g}_\beta}\neq 0,\,\beta\in\Pi} \mathbb{Z}\beta) \cap \Sigma^+$. Let $\rho_0 \in \mathfrak{a}_0^*$ be the half sum of positive roots counted with multiplicities. Recall that $\nu \in (\mathfrak{m}\cap\mathfrak{a}_0)^*$ is called an *exponent* of σ if $\nu + \rho_0|_{\mathfrak{m}\cap\mathfrak{a}_0}$ is an $(\mathfrak{m}\cap\mathfrak{a}_0)$-weight of $\sigma/(\mathfrak{m}\cap\mathfrak{n}_0)\sigma$. Using $\mathfrak{a}_0 = (\mathfrak{m}\cap\mathfrak{a}_0)\oplus\mathfrak{a}$, we regard ν as an element of $\mathfrak{a}_0^*$. We also have $\mathfrak{a}^* \subset \mathfrak{a}_0^*$.

Theorem 1.4 (Theorems 8.8 and 8.16). *For* $\lambda \in \mathfrak{a}^*$ *and an irreducible representation* σ *of* M*, the following formulas hold.*

(1) *Assume that for any* $w \in W$ *such that* $\eta|_{wNw^{-1}\cap N_0} = 1$, *the following two conditions hold:*

(a) $\langle\check{\alpha}, \lambda + \nu\rangle \notin \mathbb{Z}_{\leq 0}$ *for each exponent* ν *of* σ *and* $\alpha \in \Sigma^+ \setminus w^{-1}(\Sigma_M^+ \cup \Sigma_\eta^+)$.

(b) $\lambda - \widetilde{w}(\lambda + \widetilde{\mu})|_{\mathfrak{a}} \notin \mathbb{Z}_{\leq 0}((\Sigma^+ \setminus \Sigma_M^+) \cap w^{-1}\Sigma^+)|_{\mathfrak{a}} \setminus \{0\}$ *for all* $\widetilde{w} \in \widetilde{W}$.

Then

$$\dim \mathrm{Wh}_\eta(I(\sigma, \lambda)') = \sum_{w \in W(M),\, \eta|_{wNw^{-1}\cap N_0} = 1} \dim \mathrm{Wh}_{w^{-1}\eta}(\sigma').$$

(2) *Assume that* $(\lambda + \widetilde{\mu}) - \widetilde{w}(\lambda + \widetilde{\mu}) \notin \mathbb{Z}\Delta$ *for all* $\widetilde{w} \in \widetilde{W} \setminus \widetilde{W_M}$. *Then*

$$\dim \mathrm{Wh}_\eta((I(\sigma, \lambda)_{K\text{-finite}})^*) = \sum_{w \in W(M)} \dim \mathrm{Wh}_{w^{-1}\eta}((\sigma_{M\,\cap\, K\text{-finite}})^*).$$

For σ finite-dimensional, we have the following theorem announced by T. Oshima (a talk at National University of Singapore, January 11, 2006). Let Δ_M be the root system for $(\mathfrak{m} \oplus \mathfrak{a}, \mathfrak{h})$ and take a positive system Δ_M^+ compatible with Σ_M^+. Put $\widetilde{\rho_M} = (1/2)\sum_{\alpha \in \Delta_M^+} \alpha$. For subsets Θ_1, Θ_2 of Π, put $\Sigma_{\Theta_i} = \mathbb{Z}\Theta_i \cap \Sigma$, $W(\Theta_i) = \{w \in W \mid w(\Theta_i) \subset \Sigma^+\}$, W_{Θ_i} the Weyl group of Σ_{Θ_i} and $W(\Theta_1, \Theta_2) = \{w \in W(\Theta_1) \cap W(\Theta_2)^{-1} \mid w(\Sigma_{\Theta_1}) \cap \Sigma_{\Theta_2} = \emptyset\}$. The parabolic subgroup P defines a subset of Π, denoted by Θ. Let $w_0 \in W$ be the longest element.

Theorem 1.5. *Assume that* σ *is an irreducible finite-dimensional representation of* M *with highest weight* $\widetilde{\nu}$. *Let* $\dim_{M_0}(\lambda + \widetilde{\nu})$ *be the dimension of a finite-dimensional irreducible representation of* M_0A_0 *with highest weight* $\lambda + \widetilde{\nu}$.

(1) *Assume that for all* $w \in W$ *such that* $\eta|_{wN_0w^{-1}\cap N_0} = 1$ *the following two conditions hold:*

(a) $\langle\check{\alpha}, \lambda + w_0\widetilde{\nu}\rangle \notin \mathbb{Z}_{\leq 0}$ *for all* $\alpha \in \Sigma^+ \setminus w^{-1}(\Sigma_M^+ \cup \Sigma_\eta^+)$.

(b) $\lambda - \widetilde{w}(\lambda + \widetilde{\nu} + \widetilde{\rho_M})|_{\mathfrak{a}} \notin \mathbb{Z}_{\leq 0}((\Sigma^+ \setminus \Sigma_M^+) \cap w^{-1}\Sigma^+)|_{\mathfrak{a}} \setminus \{0\}$ *for all* $\widetilde{w} \in \widetilde{W}$.

Then

$$\dim \mathrm{Wh}_\eta(I(\sigma, \lambda)') = \#W(\operatorname{supp}\eta, \Theta) \times \dim_{M_0}(\lambda + \widetilde{\nu}).$$

(2) *Assume that* $(\lambda + \widetilde{\nu}) - \widetilde{w}(\lambda + \widetilde{\nu}) \notin \Delta$ *for all* $\widetilde{w} \in \widetilde{W} \setminus \widetilde{W_M}$. *Then*

$$\dim \mathrm{Wh}_\eta((I(\sigma, \lambda)_{K\text{-finite}})^*) = \#W(\operatorname{supp}\eta, \Theta) \times \#W_{\operatorname{supp}\eta} \times \dim_{M_0}(\lambda + \widetilde{\nu}).$$

We summarize the content of this paper. In §2, we introduce the Bruhat filtration. From §2 to §6 we study the module $J'_\eta(I(\sigma, \lambda))$. In §3 we prove that successive quotients of the Bruhat filtration are zero under some conditions. The structure of the successive quotients is investigated in §4. We give the definition and properties of the generalized twisting functor in §5, and in §6 we reveal the relation between the twisting functor and the successive quotients. We complete the proof of Theorem 1.2 in that section. Theorem 1.3 is proved in §7. In §8, the dimension of the space of Whittaker vectors is determined, and Theorems 1.4 and 1.5 are proved.

List of symbols

Notation

Throughout this paper we use the following notation. As usual we denote the ring of integers, the set of non-negative integers, the set of positive integers, the real number field and the complex number field by $\mathbb{Z}, \mathbb{Z}_{\geq 0}, \mathbb{Z}_{>0}, \mathbb{R}$ and $\mathbb{C}$, respectively. Let G be a connected semisimple linear Lie group and $\mathfrak{g}$ the complexification of its Lie algebra. Fix a Cartan involution θ of G and denote its derivation by the same letter θ. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be the decomposition of $\mathfrak{g}$ into the $+1$ and -1 eigenspaces for θ. Set $K = \{g \in G \mid \theta(g) = g\}$. Let $P_0 = M_0 A_0 N_0$ be a minimal parabolic subgroup and its Langlands decomposition such that $M_0 \subset K$ and $\mathrm{Lie}(A_0) \subset \mathfrak{s}$. Denote the complexifications of the Lie algebras of P_0, M_0, A_0, N_0 by $\mathfrak{p}_0, \mathfrak{m}_0, \mathfrak{a}_0, \mathfrak{n}_0$, respectively. Take a parabolic subgroup P which contains P_0 and denote its Langlands decomposition by $P = MAN$. Here we assume $A \subset A_0$. Let $\mathfrak{p}, \mathfrak{m}, \mathfrak{a}, \mathfrak{n}$ be the complexifications of the Lie algebras of P, M, A, N. Put $\overline{P_0} = \theta(P_0)$, $\overline{N_0} = \theta(N_0)$, $\overline{P} = \theta(P)$, $\overline{N} = \theta(N)$, $\overline{\mathfrak{p}_0} = \theta(\mathfrak{p}_0)$, $\overline{\mathfrak{n}_0} = \theta(\mathfrak{n}_0)$, $\overline{\mathfrak{p}} = \theta(\mathfrak{p})$ and $\overline{\mathfrak{n}} = \theta(\mathfrak{n})$.

In general, we denote the dual space $\mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ of a $\mathbb{C}$-vector space V by V^*. Let $\Sigma \subset \mathfrak{a}_0^*$ be the restricted root system for $(\mathfrak{g}, \mathfrak{a}_0)$ and $\mathfrak{g}_\alpha$ the root space for $\alpha \in \Sigma$. Then $\sum_{\alpha \in \Sigma} \mathbb{R}\alpha$ is a real form of $\mathfrak{a}_0^*$. We denote the real part of $\lambda \in \mathfrak{a}_0^*$ with respect to this real form by $\mathrm{Re}\,\lambda$ and the imaginary part by $\mathrm{Im}\,\lambda$. Let Σ^+ be the positive system determined by $\mathfrak{n}_0$. Put $\rho_0 = \sum_{\alpha \in \Sigma^+} (\dim \mathfrak{g}_\alpha/2)\alpha$ and $\rho = \rho_0|_{\mathfrak{a}}$. The positive system Σ^+ determines the set Π of simple roots. Fix a total order on $\sum_{\alpha \in \Sigma} \mathbb{R}\alpha$ such that the following conditions hold: (1) If $\alpha > \beta$ and $\gamma \in \sum_{\alpha \in \Sigma} \mathbb{R}\alpha$ then $\alpha + \gamma > \beta + \gamma$. (2) If $\alpha > 0$ and c is a positive real number then $c\alpha > 0$. (3) For all $\alpha \in \Sigma^+$ we have $\alpha > 0$. Write W for the little Weyl group for $(\mathfrak{g}, \mathfrak{a}_0)$, e for the unit element of W and w_0 for the longest element of W. For $w \in W$, we fix a representative in $N_K(\mathfrak{a})$ and denote it also by w. For $\alpha \in \Sigma$, let $\check{\alpha}$ be its coroot.

Let $\mathfrak{t}_0$ be a Cartan subalgebra of $\mathfrak{m}_0$ and T_0 the corresponding Cartan subgroup of M_0. Then $\mathfrak{h} = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ is a Cartan subalgebra of $\mathfrak{g}$. Let Δ be the root system for $(\mathfrak{g}, \mathfrak{h})$ and take a positive system Δ^+ compatible with Σ^+, i.e., if $\alpha \in \Delta^+$ is such that $\alpha|_{\mathfrak{a}_0} \neq 0$ then $\alpha|_{\mathfrak{a}_0} \in \Sigma^+$. Let $\mathfrak{g}_\alpha^{\mathfrak{h}}$ be the root space of $\alpha \in \Delta$ and $\widetilde{W}$ the Weyl group of Δ. Put $\widetilde{\rho} = (1/2)\sum_{\alpha \in \Delta^+} \alpha$. By the decompositions $(\mathfrak{m} \cap \mathfrak{a}_0)^* \oplus \mathfrak{a}^* = \mathfrak{a}_0^*$ and $\mathfrak{t}_0^* \oplus \mathfrak{a}_0^* = \mathfrak{h}^*$, we always regard $\mathfrak{a}^* \subset \mathfrak{a}_0^* \subset \mathfrak{h}^*$.

We use the same notation for M, i.e., Σ_M is the restricted root system of M, $\Sigma_M^+ = \Sigma_M \cap \Sigma^+$, W_M is the little Weyl group of M, Δ_M is the root system of M, $\Delta_M^+ = \Delta_M \cap \Delta^+$, $\widetilde{W_M}$ is the Weyl group of M and $w_{M,0}$ is the longest element of W_M.

We can define an anti-isomorphism of $U(\mathfrak{g})$ by $X \mapsto -X$ for $X \in \mathfrak{g}$. We denote this anti-isomorphism by $u \mapsto \check{u}$.

For a $\mathfrak{g}$-module V and $g \in G$, we define a $\mathfrak{g}$-module gV as follows: The representation space is V and the action of $X \in \mathfrak{g}$ is $X \cdot v = (\mathrm{Ad}(g)^{-1}X)v$ for $v \in gV$.

For $\xi = (\xi_1, \dots, \xi_l) \in \mathbb{Z}^l$, put $|\xi| = \xi_1 + \dots + \xi_l$.

§2. Parabolic induction and the Bruhat filtration

Fix a character η of $\mathfrak{n}_0$ and put $\mathrm{supp}_G\,\eta = \mathrm{supp}\,\eta = \{\alpha \in \Pi \mid \eta|_{\mathfrak{g}_\alpha} \neq 0\}$. The character η is called *non-degenerate* if $\mathrm{supp}\,\eta = \Pi$. We denote the character of N_0 whose differential is η by the same letter η.

Definition 2.1. Let V be a finite-length moderate growth Fréchet representation of G (see Casselman [Cas89, p. 391]). We define $\mathfrak{g}$-modules $J'_\eta(V)$ and $J^*_\eta(V)$ by

$$J'_\eta(V) = \left\{ v \in V' \;\middle|\; \begin{array}{l} \text{for some } k \text{ and for all } X \in \mathfrak{n}_0, \\ (X - \eta(X))^k v = 0 \end{array} \right\},$$

$$J^*_\eta(V) = \left\{ v \in (V_{K\text{-finite}})^* \;\middle|\; \begin{array}{l} \text{for some } k \text{ and for all } X \in \mathfrak{n}_0, \\ (X - \eta(X))^k v = 0 \end{array} \right\},$$

where V' is the continuous dual space of V.

Put $J'(V) = J'_0(V)$ and $J^*(V) = J^*_0(V)$ where 0 is the trivial representation of $\mathfrak{n}_0$. The module $J^*(V)$ is the (dual of the) *Jacquet module* defined by Casselman [Cas80]. By the automatic continuity theorem [Wal83, Theorem 4.8], we have $J'(V) = J^*(V)$. The correspondences $V \mapsto J'_\eta(V)$ and $V \mapsto J^*_\eta(V)$ are functors from the category of G-modules to the category of $\mathfrak{g}$-modules.

Remark 2.2. The character $\eta\colon \mathfrak{n}_0 \to \mathbb{C}$ gives a $\mathbb{C}$-algebra homomorphism $U(\mathfrak{n}_0) \to \mathbb{C}$. We denote this homomorphism again by η and let $\operatorname{Ker}\eta$ be its kernel. Then the following conditions are equivalent:

(1) For some k and for all $X \in \mathfrak{n}_0$, $(X - \eta(X))^k v = 0$.

(2) For all $X \in \mathfrak{n}_0$ there exists k such that $(X - \eta(X))^k v = 0$.

(3) For some k, $(\operatorname{Ker}\eta)^k v = 0$.

In fact, this holds for any nilpotent Lie algebra. Obviously, (3) implies (1) and (1) implies (2). We prove that (2) implies (3) by induction on $\dim \mathfrak{n}_0$. Replacing V with $V \otimes (-\eta)$, we may assume $\eta = 0$. Take a codimension 1 ideal $\mathfrak{c} \subset \mathfrak{n}_0$ and $X \in \mathfrak{n}_0 \setminus \mathfrak{c}$. Then $\mathfrak{c}^k v = 0$ for some k by inductive hypothesis. Put $V' = U(\mathfrak{n}_0)v$. Then $V' = U(\mathfrak{c})U(\mathbb{C}X)v$. By (2), $U(\mathbb{C}X)v$ is finite-dimensional. Since $\mathfrak{c}$ is an ideal, $\mathfrak{c}^k U(\mathbb{C}X)v \subset U(\mathfrak{n}_0)\mathfrak{c}^k v = 0$. Hence V' is finite-dimensional. Since each finite-dimensional irreducible representation of a nilpotent algebra is a character, V' is given by an extension of characters. By the assumption (2), each irreducible subquotient of V' is trivial. Hence $\mathfrak{n}_0^{k'} v = 0$ for some k'.

In this paper, we study the module $J'_\eta(V)$ for a parabolically induced representation V. An element of $\mathfrak{a}^*$ is identified with a character of A. We denote the character of A corresponding to $\lambda + \rho$ by $e^{\lambda+\rho}$ where $\lambda \in \mathfrak{a}^*$. For an irreducible moderate growth Fréchet representation σ of M and $\lambda \in \mathfrak{a}^*$, put

$$I(\sigma, \lambda) = C^\infty\text{-}\operatorname{Ind}_P^G(\sigma \otimes e^{\lambda+\rho}).$$

(For moderate growth Fréchet representations, see Casselman [Cas89].) The representation $I(\sigma, \lambda)$ has a natural structure of a moderate growth Fréchet representation. Denote its continuous dual space by $I(\sigma,\lambda)'$. Let $\mathcal{L}$ be a vector bundle on G/P attached to the representation $\sigma\otimes e^{\lambda+\rho}$. Then $I(\sigma,\lambda)$ is the space of C^∞-sections of $\mathcal{L}$.

Remark 2.3. A C^∞-section of $\mathcal{L}$ corresponds to a σ-valued C^∞-function f on G such that $f(gman)=\sigma(m)^{-1}e^{-(\lambda+\rho)(\log a)}f(g)$ for $g\in G$, $m\in M$, $a\in A$, $n\in N$. In particular a C^∞-function on G/P corresponds to a right P-invariant C^∞-function on G. We use this identification throughout this paper.

We use the notation of Appendix A. We can regard $J'_\eta(I(\sigma,\lambda))$ as a subspace of $\mathcal{D}'(G/P,\mathcal{L})$ as follows. Let $G/P=\bigcup_\gamma U_\gamma$ be an open covering such that $\mathcal{L}$ is trivial on U_γ. For each γ, $C_c^\infty(U_\gamma,\mathcal{L})$ is identified with a subspace $\{\varphi\in C^\infty(G/P,\mathcal{L})\mid \varphi|_{(G/P)\setminus U_\gamma}=0\}$ of $C^\infty(G/P,\mathcal{L})=I(\sigma,\lambda)$. Hence an element of $I(\sigma,\lambda)'$ gives an element of $(C_c^\infty(U_\gamma,\mathcal{L}))'$. By the definition of $\mathcal{D}'(G/P,\mathcal{L})$, the collection of these elements in $(C_c^\infty(U_\gamma,\mathcal{L}))'$ over γ's patches together to give an element of $\mathcal{D}'(G/P,\mathcal{L})$. Hence we get $I(\sigma,\lambda)'\to\mathcal{D}'(G/P,\mathcal{L})$. It is easy to see that this is an injective $\mathfrak{g}$-module homomorphism.

Set $W(M)=\{w\in W\mid w(\Sigma_M^+)\subset\Sigma^+\}$. Then it is known that the multiplication map $W(M)\times W_M\to W$ is bijective [Kos61, Proposition 5.13]. By the Bruhat decomposition, we have

$$G/P=\bigsqcup_{w\in W(M)} N_0wP/P.$$

(Recall that we fix a representative of $w\in W$, see Notation.) Enumerate $W(M)=\{w_1,\dots,w_r\}$ so that $\bigcup_{j\le i}N_0w_jP/P$ is a closed subset of G/P for each i. (For example, choose w_i such that $\dim(N_0w_1P/P)\le\dots\le\dim(N_0w_rP/P)$.) Then we can define a submodule I_i of $J'_\eta(I(\sigma,\lambda))$ by

$$I_i=\Bigl\{x\in J'_\eta(I(\sigma,\lambda))\Bigm| \operatorname{supp}x\subset\bigcup_{j\le i}N_0w_jP/P\Bigr\}.$$

The filtration $\{I_i\}$ is called a *Bruhat filtration* [CHM00]. In the rest of this section, we study the modules I_i/I_{i-1}. Put $U_i=w_i\overline{N}P/P$ and $O_i=N_0w_iP/P$. By the lemma below, U_i is an open subset of G/P containing O_i, and $U_i\cap O_j=\emptyset$ if $j<i$.

Lemma 2.4. *Let $w,w'\in W$ and assume that $w\overline{N_0}P\cap N_0w'P\neq\emptyset$. Then $w'\ge w$ with respect to the Bruhat order.*

Proof. Take $H\in\operatorname{Lie}(A_0)$ such that $\alpha(H)<0$ for all $\alpha\in\Sigma^+$ and put $a_t=\exp(tH)$ for $t\in\mathbb{R}_{>0}$. Then $\lim_{t\to\infty}a_t\overline{n}a_t^{-1}=1$ for all $\overline{n}\in\overline{N_0}$. By assumption, there exists $\overline{n}\in\overline{N_0}$ such that $w\overline{n}P/P\in N_0w'P/P\subset G/P$. Since $N_0w'P/P\subset G/P$ is stable under the action of A_0, we have $(wa_tw^{-1})w\overline{n}P/P \in N_0w'P/P$. Since $a_t \in A_0 \subset P$, we have $wa_t\overline{n}a_t^{-1}P/P \in N_0w'P/P$. Hence $wP/P = \lim_{t\to\infty} wa_t\overline{n}a_t^{-1}P/P \in \overline{N_0w'P/P}$ where $\overline{N_0w'P/P}$ is the closure of $N_0w'P/P$ in G/P, proving the lemma. □

Hence, the restriction map $\mathrm{Res}_i\colon I_i \to \mathcal{D}'(U_i,\mathcal{L})$ induces an injective map

$$\mathrm{Res}_i\colon I_i/I_{i-1} \to \mathcal{D}'(U_i,\mathcal{L}).$$

Moreover, $\mathrm{Im}\,\mathrm{Res}_i \subset \mathcal{T}_{O_i}(U_i,\mathcal{L})$ where $\mathcal{T}_{O_i}(U_i,\mathcal{L})$ is the space of tempered $\mathcal{L}$-distributions on U_i with respect to G/P whose supports are contained in O_i.

The map $n \mapsto nw_iP/P$ yields isomorphisms $w_i\overline{N}w_i^{-1} \simeq U_i$ and $w_i\overline{N}w_i^{-1} \cap N_0 \simeq O_i$. Since the exponential map $\exp\colon \mathrm{Ad}(w_i)\overline{\mathfrak{n}} \to w_i\overline{N}w_i^{-1}$ is a diffeomorphism, the space U_i is diffeomorphic to a Euclidean space and O_i is a subspace of U_i. Moreover, since $w_i\overline{N}w_i^{-1} \simeq (w_i\overline{N}w_i^{-1}\cap\overline{N_0})(w_i\overline{N}w_i^{-1}\cap N_0)$, a basis of $\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\overline{\mathfrak{n}_0}$ satisfies the conditions of Appendix A.2. Hence $\mathcal{T}_{O_i}(U_i,\mathcal{L}) \hookrightarrow U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\overline{\mathfrak{n}_0})\mathcal{D}'(O_i,\mathcal{L}) \simeq U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\overline{\mathfrak{n}_0}) \otimes_{\mathbb{C}} \mathcal{D}'(O_i,\mathcal{L})$ by Proposition A.3.

Fix a Haar measure on $w_i\overline{N}w_i^{-1}\cap N_0$. We define $\delta_i \in \mathcal{D}'(O_i,\mathcal{L})$ by

$$\langle \delta_i,\varphi\rangle = \int_{w_i\overline{N}w_i^{-1}\cap N_0} \varphi(nw_i)\,dn$$

for $\varphi \in C_c^\infty(O_i,\mathcal{L})$. Recall that U_i has the structure of a vector space and O_i is a subspace. Let $\mathcal{P}(O_i)$ be the ring of polynomials on O_i (cf. [CG90] or Appendix A.3). Define a C^∞-function η_i on O_i by $\eta_i(nw_iP/P) = \eta(n)$ for $n \in w_i\overline{N}w_i^{-1}\cap N_0$. For a C^∞-function f on O_i and $u' \in \sigma'$, we define $f\otimes u' \in C^\infty(O_i,\sigma')$ by $(f\otimes u')(x) = f(x)u'$. Since $w_i \in W(M)$, $\mathrm{Ad}(w_i)(\mathfrak{m}\cap\mathfrak{n}_0) \subset \mathfrak{n}_0$. Hence we can define a character $w_i^{-1}\eta$ of $\mathfrak{m}\cap\mathfrak{n}_0$ by $(w_i^{-1}\eta)(X) = \eta(\mathrm{Ad}(w_i)X)$. Using this character, we can define the Jacquet module $J'_{w_i^{-1}\eta}(\sigma\otimes e^{\lambda+\rho})$ of the MA-representation $\sigma\otimes e^{\lambda+\rho}$. It is an $\mathfrak{m}\oplus\mathfrak{a}$-module. Put

$$I_i' = \left\{ \sum_{k=1}^{l} E_k(((f_k\eta_i^{-1})\otimes u_k')\delta_i) \;\middle|\; \begin{array}{l} E_k \in U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\overline{\mathfrak{n}_0}),\ f_k \in \mathcal{P}(O_i), \\ u_k' \in J'_{w_i^{-1}\eta}(\sigma\otimes e^{\lambda+\rho}) \end{array} \right\}.$$

(Recall that $\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\overline{\mathfrak{n}_0}$ is a normal direction to O_i in U_i.) The space I_i' is a $U(\mathfrak{g})$-submodule of $\mathcal{D}'(U_i,\mathcal{L})$. Our aim is to prove that if $I_i/I_{i-1} \neq 0$ then Res_i gives an isomorphism $I_i/I_{i-1} \simeq I_i'$.

Remark 2.5. Since $w_i \in W(M)$, it follows that $\mathrm{Ad}(w_i)(\mathfrak{m}\cap\mathfrak{n}_0) \subset \mathfrak{n}_0$ and $\mathrm{Ad}(w_i)(\mathfrak{m}\cap\overline{\mathfrak{n}_0}) \subset \overline{\mathfrak{n}_0}$. Hence

$$\mathrm{Ad}(w_i)\mathfrak{m}\cap\mathfrak{n}_0 = (\mathrm{Ad}(w_i)(\mathfrak{m}\cap\mathfrak{n}_0)\oplus\mathrm{Ad}(w_i)(\mathfrak{m}\cap\overline{\mathfrak{n}_0}))\cap\mathfrak{n}_0 = \mathrm{Ad}(w_i)(\mathfrak{m}\cap\mathfrak{n}_0).$$

By the same argument,

$$\mathrm{Ad}(w_i)\mathfrak{m} \cap \mathfrak{p}_0 = \mathrm{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{p}_0).$$

We use these formulas frequently.

Lemma 2.6. (1) $\mathrm{Ad}(w_i)(\overline{\mathfrak{n}} \oplus (\mathfrak{m} \cap \mathfrak{n}_0))$ *is a subalgebra.*

(2) $\mathrm{Ad}(w_i)(\overline{\mathfrak{n}} \oplus (\mathfrak{m} \cap \mathfrak{n}_0)) = (\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0}) \oplus (\mathrm{Ad}(w_i)(\overline{\mathfrak{n}} \oplus \mathfrak{m}) \cap \mathfrak{n}_0)$ *and both direct summands are subalgebras.*

(3) $\mathrm{Ad}(w_i)(\mathfrak{m} \oplus \overline{\mathfrak{n}}) \cap \mathfrak{n}_0 = (\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0) \oplus (\mathrm{Ad}(w_i)\mathfrak{m} \cap \mathfrak{n}_0)$ *and both direct summands are subalgebras.*

Proof. (1) This subspace is the nilpotent radical of a minimal parabolic subalgebra.

(2) We have

$$\mathrm{Ad}(w_i)(\overline{\mathfrak{n}} \oplus (\mathfrak{m} \cap \mathfrak{n}_0)) = (\mathrm{Ad}(w_i)(\overline{\mathfrak{n}} \oplus (\mathfrak{m} \cap \mathfrak{n}_0)) \cap \overline{\mathfrak{n}_0}) \oplus (\mathrm{Ad}(w_i)(\overline{\mathfrak{n}} \oplus (\mathfrak{m} \cap \mathfrak{n}_0)) \cap \mathfrak{n}_0).$$

By Remark 2.5, $\mathrm{Ad}(w_i)(\overline{\mathfrak{n}} \oplus (\mathfrak{m} \cap \mathfrak{n}_0)) \cap \overline{\mathfrak{n}_0} = (\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0}) \oplus (\mathrm{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{n}_0) \cap \overline{\mathfrak{n}_0}) = \mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0}$. We also have $\mathrm{Ad}(w_i)(\mathfrak{m} \oplus \overline{\mathfrak{n}}) \cap \mathfrak{n}_0 = (\mathrm{Ad}(w_i)(\overline{\mathfrak{n}} \oplus (\mathfrak{m} \cap \mathfrak{n}_0)) \cap \mathfrak{n}_0) \oplus (\mathrm{Ad}(w_i)(\mathfrak{m} \cap \overline{\mathfrak{n}_0}) \cap \mathfrak{n}_0) = \mathrm{Ad}(w_i)(\overline{\mathfrak{n}} \oplus (\mathfrak{m} \cap \mathfrak{n}_0)) \cap \mathfrak{n}_0$ since $\mathrm{Ad}(w_i)(\mathfrak{m} \cap \overline{\mathfrak{n}_0}) \subset \overline{\mathfrak{n}_0}$.

(3) This is obvious. □

Lemma 2.7. *Let* $E_1, \ldots, E_n$ *be a basis of* $\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0}$ *such that each* E_s *is a restricted root vector for some root (say* α_s*) and* $F \in \mathrm{Ad}(w_i)(\overline{\mathfrak{n}} \oplus \mathfrak{m}) \cap \mathfrak{n}_0$*. For* $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{Z}^n_{\geq 0}$*, set* $E^\xi = E_1^{\xi_1} \cdots E_n^{\xi_n}$*. Then for all* $c \in \mathbb{C}$ *we have*

$$[(F-c)^k, E^\xi] \in \Big(\sum_{\xi' \in A(\xi)} \mathbb{C}E^{\xi'}\Big) U(\mathrm{Ad}(w_i)(\overline{\mathfrak{n}} \oplus \mathfrak{m}) \cap \mathfrak{n}_0) \subset U(\mathrm{Ad}(w_i)(\overline{\mathfrak{n}} \oplus (\mathfrak{m} \cap \mathfrak{n}_0)))$$

where $A(\xi) = \{\xi' \in \mathbb{Z}^n_{\geq 0} \mid |\xi'| < |\xi|,\ or\ (|\xi'| = |\xi|\ and\ \sum \xi'_i\alpha_i > \sum \xi_i\alpha_i)\}$*.*

Proof. Notice that α_s is negative.

We may assume $k = 1$. We argue by induction on $|\xi|$. We have

$$[F-c, E^\xi] = [F, E^\xi] = \sum_{s=1}^{n} \sum_{l=0}^{\xi_s - 1} E_1^{\xi_1} \cdots E_{s-1}^{\xi_{s-1}} E_s^l [F, E_s] E_s^{\xi_s - l - 1} E_{s+1}^{\xi_{s+1}} \cdots E_n^{\xi_n}.$$

Hence, it is sufficient to prove

$$E_1^{\xi_1} \cdots E_{s-1}^{\xi_{s-1}} E_s^l [F, E_s] E_s^{\xi_s - l - 1} E_{s+1}^{\xi_{s+1}} \cdots E_n^{\xi_n} \in \Big(\sum_{\xi' \in A(\xi)} \mathbb{C}E^{\xi'}\Big) U(\mathrm{Ad}(w_i)(\overline{\mathfrak{n}} \oplus \mathfrak{m}) \cap \mathfrak{n}_0).$$

We may assume that F is a restricted root vector. If $[F, E_s] \in \mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0}$ then the left hand side is in $U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0})$ and its $\mathfrak{a}_0$-weight is greater than that of E^ξ. Hence it belongs to $\sum_{\xi' \in A(\xi)} \mathbb{C}E^{\xi'}$.

Assume that $[F, E_s] \in \mathrm{Ad}(w_i)(\overline{\mathfrak{n}} \oplus \mathfrak{m}) \cap \mathfrak{n}_0$. Define $\xi^{(1)}, \xi^{(2)} \in \mathbb{Z}^n$ by $\xi^{(1)} = (\xi_1, \dots, \xi_{s-1}, l, 0, \dots, 0)$ and $\xi^{(2)} = (0, \dots, 0, \xi_s - l - 1, \xi_{s+1}, \dots, \xi_n)$. Using inductive hypothesis, we have

$$\begin{aligned} E^{\xi^{(1)}}\big[[F, E_s], E^{\xi^{(2)}}\big] &\in E^{\xi^{(1)}}\Big(\sum_{\xi' \in A(\xi^{(2)})} \mathbb{C}E^{\xi'}\Big) U(\mathrm{Ad}(w_i)(\overline{\mathfrak{n}} \oplus \mathfrak{m}) \cap \mathfrak{n}_0) \\ &\subset \Big(\sum_{\xi' \in A(\xi^{(1)} + \xi^{(2)})} \mathbb{C}E^{\xi'}\Big) U(\mathrm{Ad}(w_i)(\overline{\mathfrak{n}} \oplus \mathfrak{m}) \cap \mathfrak{n}_0) \\ &\subset \Big(\sum_{\xi' \in A(\xi)} \mathbb{C}E^{\xi'}\Big) U(\mathrm{Ad}(w_i)(\overline{\mathfrak{n}} \oplus \mathfrak{m}) \cap \mathfrak{n}_0). \end{aligned}$$

On the other hand,

$$\begin{aligned} E^{\xi^{(1)}} E^{\xi^{(2)}}[F, E_s] &\in \Big(\sum_{|\xi'| \le |\xi^{(1)} + \xi^{(2)}|} \mathbb{C}E^{\xi'}\Big)[F, E_s] \\ &\subset \Big(\sum_{|\xi'| \le |\xi^{(1)} + \xi^{(2)}|} \mathbb{C}E^{\xi'}\Big)(\mathrm{Ad}(w_i)(\overline{\mathfrak{n}} \oplus \mathfrak{m}) \cap \mathfrak{n}_0). \end{aligned}$$

Since $|\xi^{(1)} + \xi^{(2)}| = |\xi| - 1 < |\xi|$, we get the assertion. $\square$

Let X be an element of the normalizer of $\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0$ in $\mathfrak{g}$. For $f \in C^\infty(O_i)$ we define $D_i(X)f \in C^\infty(O_i)$ by

$$(D_i(X)f)(nw_i) = \frac{d}{dt} f(\exp(-tX) n \exp(tX) w_i)\Big|_{t=0}$$

where $n \in w_i \overline{N} w_i^{-1} \cap N_0$.

Lemma 2.8. *Fix $f \in C^\infty(O_i)$, $u' \in (\sigma \otimes e^{\lambda+\rho})'$ and $X \in \mathfrak{g}$.*

(1) *If $X \in \mathfrak{a}_0$, then X normalizes $\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0$ and*

$$\begin{aligned} X((f \otimes u')\delta_i) = ((D_i(X)f) \otimes u')\delta_i + (f \otimes ((\mathrm{Ad}(w_i)^{-1}X)u'))\delta_i \\ + (w_i\rho_0 - \rho_0)(X)(f \otimes u')\delta_i. \end{aligned}$$

(2) *If $X \in \mathrm{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{n}_0)$ or $X \in \mathfrak{m}_0$, then X normalizes $\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0$ and*

$$X((f \otimes u')\delta_i) = ((D_i(X)f) \otimes u')\delta_i + (f \otimes ((\mathrm{Ad}(w_i)^{-1}X)u'))\delta_i.$$

Proof. First we prove that X normalizes $\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0$. If $X \in \mathfrak{m}_0 + \mathfrak{a}_0$, then X normalizes each restricted root space. Hence, X normalizes $\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0$. If $X \in \mathrm{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{n}_0)$, then $X \in \mathfrak{n}_0$ by Remark 2.5. Hence, X normalizes $\mathfrak{n}_0$. Since $\mathfrak{m}$ normalizes $\overline{\mathfrak{n}}$, X normalizes $\mathrm{Ad}(w_i)\overline{\mathfrak{n}}$.

Put $g_t = \exp(tX)$ for $t \in \mathbb{R}$. Set $D(t) = |\det(\mathrm{Ad}(g_t)^{-1}|_{\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\mathfrak{n}_0})|$. Take $\varphi \in C_c^\infty(U_i,\mathcal{L})$ and regard φ as a σ-valued C^∞-function on $w_i\overline{N}P$ (Remark 2.3). In each case, $w_i^{-1}g_tw_i \in P$. Hence $\varphi(xw_i^{-1}g_tw_i) = (\sigma\otimes e^{\lambda+\rho})(w_i^{-1}g_tw_i)^{-1}\varphi(x)$. Then

$$\begin{aligned}
&\langle X((f\otimes u')\delta_i),\varphi\rangle = \langle (f\otimes u')\delta_i, -X\varphi\rangle \\
&\quad= \frac{d}{dt}\int_{w_i\overline{N}w_i^{-1}\cap N_0} u'(\varphi(g_tnw_i))f(nw_i)\,dn\Big|_{t=0} \\
&\quad= \frac{d}{dt}\int_{w_i\overline{N}w_i^{-1}\cap N_0} u'(\varphi((g_tng_t^{-1})w_i(w_i^{-1}g_tw_i)))f(nw_i)\,dn\Big|_{t=0} \\
&\quad= \frac{d}{dt}\int_{w_i\overline{N}w_i^{-1}\cap N_0} u'((\sigma\otimes e^{\lambda+\rho})(w_i^{-1}g_tw_i)^{-1}\varphi((g_tng_t^{-1})w_i))f(nw_i)\,dn\Big|_{t=0} \\
&\quad= \frac{d}{dt}\int_{w_i\overline{N}w_i^{-1}\cap N_0} u'((\sigma\otimes e^{\lambda+\rho})(w_i^{-1}g_tw_i)^{-1}\varphi(nw_i))f(g_t^{-1}ng_tw_i)D(t)\,dn\Big|_{t=0} \\
&\quad= \frac{d}{dt}\int_{w_i\overline{N}w_i^{-1}\cap N_0} ((w_i^{-1}g_tw_i)u')(\varphi(nw_i))f(g_t^{-1}ng_tw_i)D(t)\,dn\Big|_{t=0}.
\end{aligned}$$

This implies

$$\begin{aligned}
X((f\otimes u')\delta_i) &= ((D_i(X)f)\otimes u')\delta_i + (f\otimes((\mathrm{Ad}(w_i)^{-1}X)u'))\delta_i \\
&\quad + \frac{d}{dt}|\det(\mathrm{Ad}(g_t)^{-1}|_{\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\mathfrak{n}_0})|\Big|_{t=0}((f\otimes u')\delta_i).
\end{aligned}$$

(1) Assume that $X \in \mathfrak{a}_0$. Since $w_i \in W(M)$, we have $w_i\overline{N}w_i^{-1}\cap N_0 = w_i\overline{N}_0w_i^{-1}\cap N_0$. This implies that $\det(\mathrm{Ad}(g_t)^{-1}|_{\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\mathfrak{n}_0}) = e^{t(w_i\rho_0-\rho_0)(X)}$.

(2) First assume that $X \in \mathfrak{m}_0$. Since $M_0 \ni g \mapsto \det(\mathrm{Ad}(g)^{-1}|_{\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\mathfrak{n}_0})$ is a 1-dimensional representation, it is unitary since M_0 is compact. Hence we have $|\det(\mathrm{Ad}(g_t)^{-1}|_{\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\mathfrak{n}_0})| = 1$. Next assume $X \in \mathrm{Ad}(w_i)\mathfrak{m}\cap\mathfrak{n}_0$. Then $\mathrm{ad}(X)$ is nilpotent. Hence, $\mathrm{Ad}(g_t)-1$ is nilpotent. This implies $\det(\mathrm{Ad}(g_t)^{-1}|_{\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\mathfrak{n}_0}) = 1$. □

Lemma 2.9. *Let $x \in \mathcal{T}_{O_i}(U_i,\mathcal{L})$. Assume that there exists a positive integer k such that $(X-\eta(X))^kx = 0$ for all $X \in \mathrm{Ad}(w_i)\overline{\mathfrak{p}}\cap\mathfrak{n}_0$. Then $x \in I_i'$. In particular* $\operatorname{Im}\operatorname{Res}_i \subset I_i'$.

Proof. Let E_s and α_s be as in Lemma 2.7. For $\xi = (\xi_1,\dots,\xi_n) \in \mathbb{Z}_{\geq 0}^n$, set $E^\xi = E_1^{\xi_1}\cdots E_n^{\xi_n}$. Since $x \in \mathcal{T}_{O_i}(U_i,\mathcal{L}) \hookrightarrow U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\overline{\mathfrak{n}_0})\mathcal{D}'(O_i,\mathcal{L})$, there exist $x_\xi \in \mathcal{D}'(O_i,\mathcal{L})$ such that $x = \sum_\xi E^\xi x_\xi$ (finite sum).

First we prove $x_\xi \in (\mathcal{P}(O_i)\eta_i^{-1}\otimes(\sigma\otimes e^{\lambda+\rho})')\delta_i$ by backward induction on the lexicographic order of $(|\xi|, -\sum_s\xi_s\alpha_s)$. Fix a nonzero element $F \in \mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\mathfrak{n}_0$. Then $(F-\eta(F))^kx = \sum_\xi[(F-\eta(F))^k, E^\xi](x_\xi) + \sum_\xi E^\xi((F-\eta(F))^kx_\xi)$. Assume that $(F-\eta(F))^k x=0$. Define the set $A(\xi)$ as in Lemma 2.7. By that lemma,

$$\sum_{\xi} E^{\xi}((F-\eta(F))^k x_{\xi}) = -\sum_{\xi}[(F-\eta(F))^k, E^{\xi}](x_{\xi}) \in \sum_{\xi}\Big(\sum_{\xi'\in A(\xi)} \mathbb{C}E^{\xi'}\Big) U(\mathrm{Ad}(w_i)(\overline{\mathfrak{n}}\oplus\mathfrak{m})\cap\mathfrak{n}_0)(x_{\xi}).$$

Put $B(\xi)=\{\xi' \mid |\xi'|>|\xi|, \text{ or } (|\xi'|=|\xi| \text{ and } \sum \xi'_s\alpha_s < \sum \xi_s\alpha_s)\}$. Notice that $U(\mathrm{Ad}(w_i)(\overline{\mathfrak{n}}\oplus\mathfrak{m})\cap\mathfrak{n}_0)(x_{\xi}) \subset \mathcal{D}'(O_i,\mathcal{L})$. Since we have $U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\overline{\mathfrak{n}_0})\mathcal{D}'(O_i,\mathcal{L}) \simeq U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\overline{\mathfrak{n}_0})\otimes\mathcal{D}'(O_i,\mathcal{L})$, it follows that

$$(F-\eta(F))^k x_{\xi} \in \sum_{\xi'\in B(\xi)} U(\mathrm{Ad}(w_i)(\overline{\mathfrak{n}}\oplus\mathfrak{m})\cap\mathfrak{n}_0)(x_{\xi'}).$$

By inductive hypothesis, $x_{\xi'} \in (\mathcal{P}(O_i)\eta_i^{-1}\otimes(\sigma\otimes e^{\lambda+\rho})')\delta_i$ for all $\xi'\in B(\xi)$. Hence we have $(F-\eta(F))^k x_{\xi} \in (\mathcal{P}(O_i)\eta_i^{-1}\otimes(\sigma\otimes e^{\lambda+\rho})')\delta_i$. Therefore $x_{\xi} \in (\mathcal{P}(O_i)\eta_i^{-1}\otimes(\sigma\otimes e^{\lambda+\rho})')\delta_i$ by Corollary A.5.

Hence, we can write $x=\sum_{\xi} E^{\xi}\sum_l (f_{\xi,l}\eta_i^{-1}\otimes u'_{\xi,l})\delta_i$ (finite sum), where $f_{\xi,l}\in\mathcal{P}(O_i)$ and $u'_{\xi,l}\in(\sigma\otimes e^{\lambda+\rho})'$. Moreover, we may assume that $f_{\xi,l}$ is an $\mathfrak{a}_0$-weight vector with respect to D_i and $\{f_{\xi,l}\}_l$ is linearly independent for each ξ. We prove $u'_{\xi,l}\in J'_{w_i^{-1}\eta}(\sigma\otimes e^{\lambda+\rho})$. Take $F\in\mathfrak{m}\cap\mathfrak{n}_0$. By Lemma 2.8,

$$\begin{aligned}
&(\mathrm{Ad}(w_i)F-\eta(\mathrm{Ad}(w_i)F))^k x\\
&\quad=\sum_{\xi,l}[(\mathrm{Ad}(w_i)F-\eta(\mathrm{Ad}(w_i)F))^k, E^{\xi}]((f_{\xi,l}\eta_i^{-1}\otimes u'_{\xi,l})\delta_i)\\
&\qquad+\sum_{\xi,l}E^{\xi}\sum_{p=1}^{k}\binom{k}{p}(((D_i(\mathrm{Ad}(w_i)F))^p(f_{\xi,l})\eta_i^{-1})\otimes(F-\eta(\mathrm{Ad}(w_i)F))^{k-p}(u'_{\xi,l}))\delta_i\\
&\qquad+\sum_{\xi,l}E^{\xi}(f_{\xi,l}\eta_i^{-1}\otimes(F-\eta(\mathrm{Ad}(w_i)F))^k u'_{\xi,l})\delta_i.
\end{aligned}$$

Now we prove $u'_{\xi,l}\in J'_{w_i^{-1}\eta}(\sigma\otimes e^{\lambda+\rho})$ by backward induction on the lexicographic order of $(|\xi|,-\sum\xi_s\alpha_s,-\mathrm{wt}\, f_{\xi,l})$ where $\mathrm{wt}\, f_{\xi,l}$ is the $\mathfrak{a}_0$-weight of $f_{\xi,l}$ with respect to D_i. Take k such that $(\mathrm{Ad}(w_i)F-\eta(\mathrm{Ad}(w_i)F))^k x=0$. Then

$$\begin{aligned}
&\sum_{\xi,l}E^{\xi}(f_{\xi,l}\eta_i^{-1}\otimes(F-\eta(\mathrm{Ad}(w_i)F))^k u'_{\xi,l})\delta_i\\
&\quad=-\sum_{\xi,l}[(\mathrm{Ad}(w_i)F-\eta(\mathrm{Ad}(w_i)F))^k, E^{\xi}]((f_{\xi,l}\eta_i^{-1}\otimes u'_{\xi,l})\delta_i)\\
&\qquad-\sum_{\xi,l}E^{\xi}\sum_{p=1}^{k}\binom{k}{p}(((D_i(\mathrm{Ad}(w_i)F))^p(f_{\xi,l})\eta_i^{-1})(F-\eta(\mathrm{Ad}(w_i)F))^{k-p}(u'_{\xi,l}))\delta_i.
\end{aligned}$$

Consequently,

$$\begin{aligned}&(f_{\xi,l}\otimes(F-\eta(\mathrm{Ad}(w_i)F))^k(u'_{\xi,l}))\delta_i\\ &\quad\in\sum_{\xi'\in B(\xi),\,l'}U(\mathrm{Ad}(w_i)(\overline{\mathfrak{n}}\oplus\mathfrak{m})\cap\mathfrak{n}_0)((f_{\xi',l'}\eta_i^{-1}\otimes u'_{\xi',l'})\delta_i)\\ &\qquad+\sum_{\mathrm{wt}\,f_{\xi',l'}<\mathrm{wt}\,f_{\xi,l}}\sum_p(((D_i(\mathrm{Ad}(w_i)F))^p f_{\xi',l'}\eta_i^{-1})\otimes(U(\mathbb{C}F)u'_{\xi',l'}))\delta_i.\end{aligned}$$

By inductive hypothesis, $(F-\eta(\mathrm{Ad}(w_i)F))^k u'_{\xi,l}\in J'_{w_i^{-1}\eta}(\sigma\otimes e^{\lambda+\rho})$. This implies that $u'_{\xi,l}\in J'_{w_i^{-1}\eta}(\sigma\otimes e^{\lambda+\rho})$. □

In fact, $\operatorname{Im}\operatorname{Res}_i=I'_i$ if $\operatorname{Im}\operatorname{Res}_i\neq 0$. This is proved in Section 4.

§3. Vanishing lemma

In this section, we fix $i\in\{1,\dots,r\}$ and a basis $\{e_1,\dots,e_l\}$ of $\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\mathfrak{n}_0$. Here we assume that each e_s is a restricted root vector and denote its root by α_s. Moreover, we assume that $\bigoplus_{s\leq t-1}\mathbb{C}e_s$ is an ideal of $\bigoplus_{s\leq t}\mathbb{C}e_s$ for all $t=1,\dots,l$.

By the decomposition (as groups)

$$\begin{aligned}N_0/[N_0,N_0]&\simeq((w_i\overline{P}w_i^{-1}\cap N_0)/(w_i\overline{P}w_i^{-1}\cap[N_0,N_0]))\\ &\quad\times((w_iNw_i^{-1}\cap N_0)/(w_iNw_i^{-1}\cap[N_0,N_0]))\end{aligned}$$

where $[\cdot,\cdot]$ is the commutator group, we can define a character η' of N_0 by $\eta'(n)=\eta(n)$ for $n\in w_i\overline{P}w_i^{-1}\cap N_0$ and $\eta'(n)=1$ for $n\in w_iNw_i^{-1}\cap N_0$. First, we prove the following lemma. This gives a necessary condition for $I_i/I_{i-1}\neq 0$.

Lemma 3.1. *Let $X\in\mathfrak{n}_0$. Then for all $x\in I'_i$ there exists a positive integer k such that $(X-\eta'(X))^k x=0$.*

For the proof, we need some notation and lemmas. For $X\in\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\mathfrak{n}_0$, we define a differential operator $R'_i(X)$ on O_i by

$$(R'_i(X)\varphi)(nw_iP/P)=\frac{d}{dt}\varphi(n\exp(tX)w_iP/P)\bigg|_{t=0}$$

where $n\in w_i\overline{N}w_i^{-1}\cap N_0$. (Recall that $w_i\overline{N}w_i^{-1}\cap N_0\simeq O_i$ via the map $n\mapsto nw_iP/P$.)

For $X\in\mathfrak{g}$, we define a differential operator $R(X)$ on G by

$$(R(X)\varphi)(g)=\frac{d}{dt}\varphi(g\exp(tX))\bigg|_{t=0}$$

for a C^∞-function φ on G. We define $R'_i(E)$ ($E \in U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0)$) and $R(E)$ ($E \in U(\mathfrak{g})$) in the usual way. For $E \in U(\mathfrak{g})$, $f \in C^\infty(O_i)$ and $u' \in (\sigma \otimes e^{\lambda+\rho})'$, we define $\delta_i(E, f, u') \in \mathcal{D}'_{O_i}(U_i, \mathcal{L})$ by

$$\langle \delta_i(E, f, u'), \varphi \rangle = \int_{w_i \overline{N} w_i^{-1} \cap N_0} f(nw_i) u'((R(\mathrm{Ad}(w_i)^{-1}E)\varphi)(nw_i))\, dn$$

where $\varphi \in C_c^\infty(U_i, \mathcal{L})$ and we regard φ as a function on $w_i \overline{N} P$ (Remark 2.3).

Lemma 3.2. *We have the following properties:*

(1) $\delta_i(XE, f, u') = \delta_i(E, R'_i(-X)(f), u')$ *for* $X \in \mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0$.
(2) $\delta_i(EX, f, u') = \delta_i(E, f, \mathrm{Ad}(w_i)^{-1}Xu')$ *for* $X \in \mathrm{Ad}(w_i)\mathfrak{p}$.
(3) *The map* $C^\infty(O_i) \otimes_{U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0)} U(\mathfrak{g}) \otimes_{U(\mathrm{Ad}(w_i)\mathfrak{p})} w_i(\sigma \otimes e^{\lambda+\rho})' \to \mathcal{D}'_{O_i}(U_i, \mathcal{L})$ *defined by* $f \otimes E \otimes u' \mapsto \delta_i(E, f, u')$ *is injective.*

Proof. (1) and (2) are obvious. To prove (3), the same argument in the proof of Proposition A.3 can be applied. □

Lemma 3.3. *Let* $E \in \mathfrak{g}$, $E' \in U(\mathfrak{g})$, $f \in C^\infty(O_i)$ *and* $u' \in (\sigma \otimes e^{\lambda+\rho})'$. *Then*

$$E\delta_i(E', f, u') = \sum_{(k_1, \ldots, k_l) \in \mathbb{Z}^l_{\geq 0}} \delta_i\left((\mathrm{ad}(e_l)^{k_l} \cdots \mathrm{ad}(e_1)^{k_1} E)E', f \prod_{s=1}^{l} \frac{(-x_s)^{k_s}}{k_s!}, u' \right),$$

where x_i *is a polynomial on* O_i *given by* $\exp(a_1 e_1) \cdots \exp(a_l e_l) w_i P/P \mapsto a_i$. (*Notice that the right hand side is a finite sum since* $\mathrm{ad}(e_i)$ *is nilpotent.*)

Proof. We remark that $(a_1, \ldots, a_l) \mapsto \exp(a_1 e_1) \cdots \exp(a_l e_l)$ yields a diffeomorphism $\mathbb{R}^l \simeq w_i \overline{N} w_i^{-1} \cap N_0$, and a Haar measure of $w_i \overline{N} w_i^{-1} \cap N_0$ corresponds to the Euclidean measure of $\mathbb{R}^l$. Take $\varphi \in C_c^\infty(w_i \overline{N} P, \sigma \otimes e^{\lambda+\rho})$. Put $n(a) = \exp(a_1 e_1) \cdots \exp(a_l e_l)$ for $a = (a_1, \ldots, a_l)$. By the definition, the action of $E \in \mathfrak{g}$ and $R_i(E')$ ($E' \in \mathfrak{g}$) commute with each other. For $E \in \mathfrak{g}$, we have

$$\begin{aligned}
&\langle E\delta_i(E', f, u'), \varphi \rangle \\
&\quad = \int_{\mathbb{R}^l} u'(((-E)R(\mathrm{Ad}(w_i)^{-1}E')\varphi)(n(a)w_i)) f(n(a)w_i)\, da \\
&\quad = \frac{d}{dt} \int_{\mathbb{R}^l} u'(R(\mathrm{Ad}(w_i)^{-1}E')\varphi)(\exp(tE)n(a)w_i)) f(n(a)w_i)\, da \Big|_{t=0} \\
&\quad = \frac{d}{dt} \int_{\mathbb{R}^l} u'(R(\mathrm{Ad}(w_i)^{-1}E')\varphi)(n(a)\exp(t\,\mathrm{Ad}(n(a))^{-1}E)w_i)) f(n(a)w_i)\, da \Big|_{t=0}.
\end{aligned}$$

The formula

$$\begin{aligned}\mathrm{Ad}(n(a))^{-1}E &= e^{-\,\mathrm{ad}(a_l e_l)}\cdots e^{-\,\mathrm{ad}(a_1 e_1)}E\\ &= \sum_{(k_1,\ldots,k_l)\in\mathbb{Z}^l_{\geq 0}} \frac{(-a_1)^{k_1}}{k_1!}\cdots\frac{(-a_l)^{k_l}}{k_l!}\,\mathrm{ad}(e_l)^{k_l}\cdots\mathrm{ad}(e_1)^{k_1}E\end{aligned}$$

gives the lemma. □

For $\mathbf{k}=(k_1,\ldots,k_l)$, we denote the operator $\mathrm{ad}(e_l)^{k_l}\cdots\mathrm{ad}(e_1)^{k_1}$ on $\mathfrak{g}$ by $\mathrm{ad}(e)^{\mathbf{k}}$ and the polynomial $((-x_1)^{k_1}/k_1!)\cdots((-x_l)^{k_l}/k_l!)\in\mathcal{P}(O_i)$ by $f_{\mathbf{k}}$; here the polynomial x_i is defined in Lemma 3.3.

Lemma 3.4. *Let* $\mathbf{k}=(k_1,\ldots,k_l)\in\mathbb{Z}^l_{\geq 0}$ *and* $X\in\mathfrak{n}_0$. *Assume that* $\mathrm{ad}(e)^{\mathbf{k}}X\in\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\mathfrak{n}_0$. *Then* $R'_i(\mathrm{ad}(e)^{\mathbf{k}}X)f_{\mathbf{k}}=0$.

Proof. We may assume that X is a restricted root vector and denote its restricted root by α. We consider the $\mathfrak{a}_0$-weight with respect to D_i. The polynomial $f_{\mathbf{k}}$ is an $\mathfrak{a}_0$-weight vector of weight $-\sum_s k_s\alpha_s$. This implies that $R'_i(\mathrm{ad}(e)^{\mathbf{k}}X)f_{\mathbf{k}}$ is an $\mathfrak{a}_0$-weight vector of weight α. However, $\mathcal{P}(O_i)$ has a decomposition into the direct sum of $\mathfrak{a}_0$-weight spaces and its weight belongs to $\{\sum_{\beta\in\Sigma^+}b_\beta\beta\mid b_\beta\in\mathbb{Z}_{\leq 0}\}\not\ni\alpha$. Hence, $R'_i(\mathrm{ad}(e)^{\mathbf{k}}X)f_{\mathbf{k}}=0$. □

For $f\in\mathcal{P}(O_i)$ and $X\in\mathfrak{n}_0$ we define $L(X)(f)$ by

$$L(X)(f)(nw_i)=\frac{d}{dt}f(\exp(-tX)nw_i)\bigg|_{t=0}.$$

Recall that the C^∞-function η_i on O_i is defined by $\eta_i(nw_iP/P)=\eta(n)$ for $n\in w_i\overline{N}w_i^{-1}\cap N_0$, and the character η' of N_0 is defined by $\eta'(n)=\eta(n)$ for $n\in w_i\overline{P}w_i^{-1}\cap N_0$ and $\eta'(n)=1$ for $n\in w_iNw_i^{-1}\cap N_0$.

Lemma 3.5. *Let* $X\in\mathfrak{n}_0$ *be a restricted root vector. For* $f\in\mathcal{P}(O_i)$ *and* $u'\in J'_{w_i^{-1}\eta}(\sigma\otimes e^{\lambda+\rho})$, *we have*

$$\begin{aligned}(X-\eta'(X))\delta_i(1,f\eta_i^{-1},u') &= \delta_i(1,L(X)(f)\eta_i^{-1},u')\\ &\quad+\sum_{\mathrm{ad}(e)^{\mathbf{k}}X\in\mathrm{Ad}(w_i)\mathfrak{n}_0\cap\mathfrak{n}_0}\delta_i(1,ff_{\mathbf{k}}\eta_i^{-1},(\mathrm{Ad}(w_i)^{-1}(\mathrm{ad}(e)^{\mathbf{k}}X)-\eta'(\mathrm{ad}(e)^{\mathbf{k}}X))u').\end{aligned}$$

(*Again the sum on the right hand side is finite.*)

In particular, if $X\in\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\mathfrak{n}_0$, *then*

$$(X-\eta'(X))\delta_i(1,f\eta_i^{-1},u')=\delta_i(1,L(X)(f)\eta_i^{-1},u').$$

Proof. We have

$$X\delta_i(1, f\eta_i^{-1}, u') = \sum_{\mathbf{k}\in\mathbb{Z}^l_{\geq 0}} \delta_i(\operatorname{ad}(e)^{\mathbf{k}}X, ff_{\mathbf{k}}\eta_i^{-1}, u').$$

by Lemma 3.3. Since $\operatorname{ad}(e)^{\mathbf{k}}X$ belongs to $\mathfrak{n}_0$ and is a restricted root vector, we have either $\operatorname{ad}(e)^{\mathbf{k}}X \in \operatorname{Ad}(w_i)\overline{\mathfrak{n}_0} \cap \mathfrak{n}_0$ or $\operatorname{ad}(e)^{\mathbf{k}}X \in \operatorname{Ad}(w_i)\mathfrak{n}_0 \cap \mathfrak{n}_0$. Recall that $\operatorname{Ad}(w_i)\overline{\mathfrak{n}_0} \cap \mathfrak{n}_0 = \operatorname{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0$ since $w_i \in W(M)$. Assume that $\operatorname{ad}(e)^{\mathbf{k}}X \in \operatorname{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0$. By the definition of η_i and η', we have $R'_i(-\operatorname{ad}(e)^{\mathbf{k}}X)(\eta_i^{-1}) = \eta(\operatorname{ad}(e)^{\mathbf{k}}X)\eta_i^{-1} = \eta'(\operatorname{ad}(e)^{\mathbf{k}}X)\eta_i^{-1}$. Hence, using Lemma 3.4,

$$\begin{aligned}\delta_i(\operatorname{ad}(e)^{\mathbf{k}}X, ff_{\mathbf{k}}\eta_i^{-1}, u') &= \delta_i(1, R'_i(-\operatorname{ad}(e)^{\mathbf{k}}X)(ff_{\mathbf{k}}\eta_i^{-1}), u')\\ &= \delta_i(1, R'_i(-\operatorname{ad}(e)^{\mathbf{k}}X)(f)f_{\mathbf{k}}\eta_i^{-1}, u')\\ &\quad + \eta'(\operatorname{ad}(e)^{\mathbf{k}}X)\delta_i(1, ff_{\mathbf{k}}\eta_i^{-1}, u').\end{aligned}$$

Next assume that $\operatorname{ad}(e)^{\mathbf{k}}X \in \operatorname{Ad}(w_i)\mathfrak{n}_0 \cap \mathfrak{n}_0$. For $h \in \mathcal{P}(O_i)$, define $\widetilde{h} \in \mathcal{P}(U_i)$ by $\widetilde{h}(nn_0w_iP) = h(nw_iP)$ for $n \in w_i\overline{N}w_i^{-1} \cap N_0$ and $n_0 \in w_i\overline{N}w_i^{-1} \cap \overline{N_0}$. Then $(R'_i(Y)h)^{\sim} = R(\operatorname{Ad}(w_i)^{-1}Y)\widetilde{h}$ for all $Y \in \operatorname{Ad}(w_i)\overline{\mathfrak{n}_0} \cap \mathfrak{n}_0$. Since $\widetilde{f}(pnw_i) = \widetilde{f}(pw_i)$ for $p \in w_i\overline{N}Pw_i^{-1}$ and $n \in w_iN_0w_i^{-1} \cap N_0$, we have $R(\operatorname{Ad}(w_i)^{-1}(-\operatorname{ad}(e)^{\mathbf{k}}X))(\widetilde{f}) = 0$. By Lemma 3.2(2),

$$\begin{aligned}\delta_i(\operatorname{ad}(e)^{\mathbf{k}}X, ff_{\mathbf{k}}\eta_i^{-1}, u') &= \delta_i(1, ff_{\mathbf{k}}\eta_i^{-1}, \operatorname{Ad}(w_i)^{-1}(\operatorname{ad}(e)^{\mathbf{k}}X)u')\\ &= \delta_i(1, R(\operatorname{Ad}(w_i)^{-1}(-\operatorname{ad}(e)^{\mathbf{k}}X))(\widetilde{f})|_{O_i}f_{\mathbf{k}}\eta_i^{-1}, u')\\ &\quad + \delta_i(1, ff_{\mathbf{k}}\eta_i^{-1}, \operatorname{Ad}(w_i)^{-1}(\operatorname{ad}(e)^{\mathbf{k}}X)u').\end{aligned}$$

By the same calculation as in the proof of Lemma 3.3,

$$L(X)(f)^{\sim} = L(X)(\widetilde{f}) = \sum_{\mathbf{k}\in\mathbb{Z}^l_{\geq 0}} R(\operatorname{Ad}(w_i)^{-1}(-\operatorname{ad}(e)^{\mathbf{k}}X))(\widetilde{f})\widetilde{f_{\mathbf{k}}}.$$

Hence

$$\begin{aligned}\sum_{\mathbf{k}\in\mathbb{Z}^l_{\geq 0}} &\delta_i(1, R(\operatorname{Ad}(w_i)^{-1}(-\operatorname{ad}(e)^{\mathbf{k}}X))(\widetilde{f})|_{O_i}f_{\mathbf{k}}\eta_i^{-1}, u')\\ &= \delta_i(1, (L(X)(f))^{\sim}|_{O_i}\eta_i^{-1}, u') = \delta_i(1, L(X)(f)\eta_i^{-1}, u').\end{aligned}$$

These imply that

$$\begin{aligned}(X - \eta'(X))\delta_i(1, f\eta_i^{-1}, u') &= \delta_i(1, L(X)(f)\eta_i^{-1}, u')\\ &+ \sum_{\operatorname{ad}(e)^{\mathbf{k}}X\in\operatorname{Ad}(w_i)\mathfrak{n}_0\cap\mathfrak{n}_0} \delta_i(1, ff_{\mathbf{k}}\eta_i^{-1}, \operatorname{Ad}(w_i)^{-1}(\operatorname{ad}(e)^{\mathbf{k}}X)u')\\ &+ \sum_{\operatorname{ad}(e)^{\mathbf{k}}X\in\operatorname{Ad}(w_i)\overline{\mathfrak{n}}\cap\mathfrak{n}_0} \eta'(\operatorname{ad}(e)^{\mathbf{k}}X)\delta_i(1, ff_{\mathbf{k}}\eta_i^{-1}, u') - \eta'(X)\delta_i(1, f\eta_i^{-1}, u').\end{aligned}$$

Since η' is a character, if $\mathbf{k} \neq (0,\dots,0)$ then $\eta'(\mathrm{ad}(e)^{\mathbf{k}}X) = 0$. Hence

$$\sum_{\mathbf{k}\in\mathbb{Z}^n_{\geq 0}} \eta'(\mathrm{ad}(e)^{\mathbf{k}}X)\delta_i(1, ff_{\mathbf{k}}\eta_i^{-1}, u') = \eta'(X)\delta_i(1, f\eta_i^{-1}, u').$$

This implies

$$\Big(\sum_{\mathrm{ad}(e)^{\mathbf{k}}X\in \mathrm{Ad}(w_i)\bar{\mathfrak{n}}\cap\mathfrak{n}_0} \eta'(\mathrm{ad}(e)^{\mathbf{k}}X)\delta_i(1, ff_{\mathbf{k}}\eta_i^{-1}, u')\Big) - \eta'(X)\delta_i(1, f\eta_i^{-1}, u') = -\sum_{\mathrm{ad}(e)^{\mathbf{k}}X\in \mathrm{Ad}(w_i)\mathfrak{n}_0\cap\mathfrak{n}_0} \eta'(\mathrm{ad}(e)^{\mathbf{k}}X)\delta_i(1, ff_{\mathbf{k}}\eta_i^{-1}, u'),$$

proving the lemma. □

Proof of Lemma 3.1. Since $\mathrm{ad}(\mathfrak{n}_0)$ is nilpotent, the subspace

$$\{x \in I_i' \mid \text{for some } k \text{ and for all } X \in \mathfrak{n}_0,\ (X - \eta'(X))^k x = 0\}$$

is $\mathfrak{g}$-stable. Hence we may assume that $x = ((f\eta_i^{-1}) \otimes u')\delta_i = \delta_i(1, f\eta_i^{-1}, u')$ for some $f \in \mathcal{P}(O_i)$ and $u' \in J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$.

Now define $V = U(\mathrm{Ad}(w_i)^{-1}\mathfrak{n}_0 \cap \mathfrak{n}_0)u' \subset J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$ where $\mathfrak{n}$ acts on $J'_{w_i^{-1}\eta}(\sigma\otimes e^{\lambda+\rho})$ trivially. Then V is finite-dimensional. Since $\mathfrak{n}$ acts on V trivially, $X - (w_i^{-1}\eta')(X)$ acts on V as a nilpotent operator for $X \in \mathrm{Ad}(w_i)^{-1}\mathfrak{n}_0\cap\mathfrak{n}_0$ by the definition of η'. By applying Engel's theorem for $V\otimes(-w_i^{-1}\eta')$, there exists a filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_p = V$ such that $(V_s/V_{s-1}) \otimes (-w_i^{-1}\eta'|_{\mathrm{Ad}(w_i)^{-1}\mathfrak{n}_0\cap\mathfrak{n}_0})$ is the trivial representation of $\mathrm{Ad}(w_i)^{-1}\mathfrak{n}_0\cap\mathfrak{n}_0$. Then $V_s/V_{s-1} \simeq w_i^{-1}\eta'|_{\mathrm{Ad}(w_i)^{-1}\mathfrak{n}_0\cap\mathfrak{n}_0}$ for all $s = 1,\dots,p$. We prove the lemma by induction on $p = \dim V$.

We may assume that X is a restricted root vector. By Lemma 3.5,

$$(X - \eta'(X))\delta_i(1, f\eta_i^{-1}, u') \in \delta_i(1, L(X)(f)\eta_i^{-1}, u') + \sum_{h\in\mathcal{P}(O_i),\, v'\in V_{p-1}} \mathbb{C}\delta_i(1, h\eta_i^{-1}, v').$$

Since f is a polynomial, $(L(X))^c(f) = 0$ for some positive integer c. Then we have $(X - \eta'(X))^c\delta_i(1, f\eta_i^{-1}, u') \in \sum_{h\in\mathcal{P}(O_i),\, v'\in V_{p-1}} \mathbb{C}\delta_i(1, h\eta_i^{-1}, v')$. By inductive hypothesis, the lemma follows. □

From the lemma, we get the following vanishing lemma. Recall that we define the character $w_i^{-1}\eta$ of $\mathfrak{m}\cap\mathfrak{n}_0$ by $(w_i^{-1}\eta)(X) = \eta(\mathrm{Ad}(w_i)X)$ and we have the injective homomorphism $\mathrm{Res}_i\colon I_i/I_{i-1} \to I_i'$.

Lemma 3.6. *Assume that $I_i/I_{i-1} \neq 0$. Then:*

(1) *The character η is unitary.*

(2) *The character η is zero on $\mathrm{Ad}(w_i)\mathfrak{n} \cap \mathfrak{n}_0$. (This is equivalent to $\eta = \eta'$.)*

(3) *The module $J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$ is not zero.*

Proof. (2) By Lemma 3.1 and the definition of J'_η, if $I_i/I_{i-1} \neq 0$ then $\eta = \eta'$.

(3) This is clear from Lemma 2.9.

(1) It is sufficient to prove that if η is not unitary then $J'_\eta(V) = 0$ for all irreducible representations V of G. By Casselman's subrepresentation theorem, V is a subrepresentation of a principal series representation. Since J'_η is an exact functor, we may assume V is a principal series representation $\mathrm{Ind}_{P_0}^G(\sigma_0 \otimes e^{\lambda_0+\rho_0})$.

Take the Bruhat filtration $\{I_i\}$ of $J'_\eta(V)$. We will prove $I_i/I_{i-1} = 0$ for all i. By (2), if η is non-trivial on $w_i N_0 w_i^{-1} \cap N_0$ then $I_i/I_{i-1} = 0$. Hence we may assume that η is not unitary on $w_i \overline{N_0} w_i^{-1} \cap N_0$. In this case, by the same argument as in the classical case (for example, see Schwartz's book [Sch66, Ch. VII, §4]), a nonzero element of I'_i is not tempered. Hence $I_i/I_{i-1} = 0$. □

Remark 3.7. In the next section it is proved that the conditions of Lemma 3.6 are also sufficient (Theorem 4.7).

Remark 3.8. If $\Pi = \operatorname{supp}\eta$, Lemma 3.6 follows from [CHM00, Theorem 5.12].

Definition 3.9 (Whittaker vectors). Let V be a $U(\mathfrak{n}_0)$-module. We define a vector space $\mathrm{Wh}_\eta(V)$ by

$$\mathrm{Wh}_\eta(V) = \{v \in V \mid Xv = \eta(X)v \text{ for all } X \in \mathfrak{n}_0\}.$$

An element of $\mathrm{Wh}_\eta(V)$ is called a *Whittaker vector.*

Lemma 3.10. *Assume that $\eta|_{\mathrm{Ad}(w_i)\mathfrak{n}\cap\mathfrak{n}_0} = 0$. Then*

$$\mathrm{Wh}_\eta\Bigl(\Bigl\{\sum_s (f_s\eta_i^{-1} \otimes u'_s)\delta_i \mid f_s \in \mathcal{P}(O_i),\ u'_s \in J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})\Bigr\}\Bigr) = \{(\eta_i^{-1} \otimes u')\delta_i \mid u' \in \mathrm{Wh}_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})\}.$$

Proof. By assumption, we have $\eta = \eta'$. Hence the right hand side is a subspace of the left hand side by Lemma 3.5.

Take $x = \sum_s (f_s\eta_i^{-1} \otimes u'_s) = \sum_s \delta_i(1, f_s\eta_i^{-1}, u'_s) \in \mathrm{Wh}_\eta(I'_i)$. We assume that $\{u'_s\}$ is linearly independent. Take $X \in \mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0$. It then follows that $\sum_s \delta_i(1, L(X)(f_s)\eta_i^{-1}, u'_s) = 0$ by Lemma 3.5. Hence $L(X)(f_s) = 0$. This implies $f_s \in \mathbb{C}$.

From the above argument, $x = \delta_i(1, \eta_i^{-1}, u')$ for some $u' \in J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$. Take $X \in \mathrm{Ad}(w_i)(\mathfrak{m} \cap \mathfrak{n}_0)$. By Lemma 3.5, we have

$$\delta_i(1, \eta_i^{-1}, (\mathrm{Ad}(w_i)^{-1}X - \eta(X))u') \in \sum_{\mathbf{k} \neq 0,\, u_{\mathbf{k}} \in J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})} \mathbb{C}\delta_i(1, f_{\mathbf{k}}\eta_i^{-1}, u_{\mathbf{k}}).$$

If $\mathbf{k} \neq 0$ then the degree of $f_{\mathbf{k}}$ is greater than 0. So the left hand side must be 0. Hence $(\mathrm{Ad}(w_i)^{-1}X - \eta(X))u' = 0$, proving the lemma. □

The following lemma is well-known, but we give a proof for the reader's convenience (cf. Casselman–Hecht–Miličić [CHM00], Yamashita [Yam86]).

Lemma 3.11. *Assume that* $\operatorname{supp}\eta = \Pi$.

(1) $\mathrm{Wh}_\eta(I(\sigma, \lambda)') \hookrightarrow \mathrm{Wh}_\eta(I'_r)$, *where the homomorphism is induced by* Res_r.
(2) *For all* $x \in \mathrm{Wh}_\eta(I'_r)$, *there exists* $u' \in \mathrm{Wh}_{w_r^{-1}\eta}((\sigma \otimes e^{\lambda+\rho})')$ *such that* $x = (\eta_r^{-1} \otimes u')\delta_r$.

Recall that $r = \#W(M) = \#(W/W_M)$ and $w_{M,0}$ is the longest element of the little Weyl group of M.

Proof. Assume that $i < r$. Then $w_i w_{M,0}$ is not the longest element of W. There exists a simple root $\alpha \in \Pi$ such that $s_\alpha w_i w_{M,0} > w_i w_{M,0}$. This means that $w_i w_{M,0}\Sigma^+ \cap \Sigma^+ = s_\alpha(s_\alpha w_i w_{M,0}\Sigma^+ \cap \Sigma^+) \cup \{\alpha\}$. The left hand side is $w_i(\Sigma^+ \setminus \Sigma^+_M) \cap \Sigma^+$. Hence, η is not trivial on $\mathrm{Ad}(w_i)\mathfrak{n} \cap \mathfrak{n}_0$. By Lemma 3.6, $I_i/I_{i-1} = 0$. This implies that $J'_\eta(I(\sigma, \lambda)) \subset I'_r$. Since $\mathrm{Ad}(w_r)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0} = 0$, there exists a polynomial $f_s \in \mathcal{P}(O_r)$ and $u'_s \in J'_{w_r^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$ such that $x = \sum_s((f_s\eta_r^{-1}) \otimes u'_s)\delta_r$. Now Lemma 3.10 yields the assertion. □

§4. Analytic continuation

The aim of this section is to prove that $\operatorname{Im}\mathrm{Res}_i = I'_i$ if $I_i/I_{i-1} \neq 0$. Namely, we extend an element of I'_i (which is a distribution on U_i) to G/P. An element of I'_i and $\varphi \in C_c^\infty(U_i)$ is given by an integral. Formally, this integral is valid for any $\varphi \in I(\sigma, \lambda)$. We prove the integral converges if λ is sufficiently dominant. Moreover, as a function of λ, we prove this integral has a meromorphic continuation to $\mathfrak{a}^*$. (These are essentially known, but we give a proof for the sake of completeness.) The resulting distribution is a distribution on G/P with a parameter λ. If it has no pole, this is an extension we need. In general, we can modify the distribution and remove the pole. This is the outline of the proof.

For $w \in W$, there is an open dense subset $w\overline{N}P/P$ of G/P and it is diffeomorphic to $\overline{N}$. Then for $w, w' \in W$, there exists a map $\Phi_{w,w'}$ from some open dense subset $U \subset \overline{N}$ to $\overline{N}$ such that $w\overline{n}P/P = w'\Phi_{w,w'}(\overline{n})P/P$ for $\overline{n} \in U$. The map $\Phi_{w,w'}$ is a rational function.

Define $H\colon G \to \mathrm{Lie}(A)$ by $g \in KM\exp(H(g))N$ via the Iwasawa decomposition.

Lemma 4.1. (1) *The map $\overline{N} \to \mathbb{R}$ defined by $\overline{n} \mapsto e^{8\rho(H(\overline{n}))}$ is a polynomial.*

(2) *For all $\overline{n} \in \overline{N}$ we have $e^{8\rho(H(\overline{n}))} \geq 1$.*

(3) *Take $H_0 \in \mathrm{Lie}(A)$ such that $\alpha(H_0) = -1$ for all $\alpha \in \Pi \setminus \Sigma_M$. There exists a continuous function $Q(\overline{n}) \geq 0$ on $\overline{N}$ such that the following conditions hold:*

- (a) *The function Q vanishes only at the unit element.*
- (b) $e^{8\rho(H(\overline{n}))} \geq Q(\overline{n})$.
- (c) $Q(\exp(tH_0)\overline{n}\exp(-tH_0)) \geq e^{8t}Q(\overline{n})$ *for $t \in \mathbb{R}_{>0}$ and $\overline{n} \in \overline{N}$.*

Proof. By Knapp [Kna01, Proposition 7.19], there exists an irreducible finite-dimensional representation $V_{4\rho}$ of G with highest weight $4\rho \in \mathfrak{a}_0^* \subset \mathfrak{h}^*$. Let $v_{4\rho} \in V_{4\rho}$ be a highest weight vector and $v_{-4\rho}^* \in V_{4\rho}^*$ a lowest weight vector of $V_{4\rho}^*$. Take $\overline{n} \in \overline{N}$ and decompose $\overline{n} = kan$ where $k \in K$, $a \in A_0$ and $n \in N_0$. Then $\log(a) \in (\mathfrak{m} \cap \mathfrak{a}_0) + H(\overline{n})$. Hence $\rho(\log(a)) = \rho(H(\overline{n}))$.

First we prove (1). We have $\theta(\overline{n})^{-1}\overline{n} = \theta(n)^{-1}a^2n$. Hence

$$\begin{aligned}\langle \theta(\overline{n})^{-1}\overline{n}v_{4\rho}, v_{-4\rho}^*\rangle &= \langle \theta(n)^{-1}a^2nv_{4\rho}, v_{-4\rho}^*\rangle = \langle a^2nv_{4\rho}, \theta(n)v_{-4\rho}^*\rangle \\ &= e^{8\rho(H(\overline{n}))}\langle v_{4\rho}, v_{-4\rho}^*\rangle.\end{aligned}$$

The left hand side is a polynomial.

Next we prove (2) and (3). Fix a compact real form of $\mathfrak{g}$ containing $\mathrm{Lie}(K)$ and take an inner product on $V_{4\rho}$ which is invariant under the action of this compact real form. We normalize an inner product $\|\cdot\|$ so that $\|v_{4\rho}\| = 1$. Then $\|\overline{n}v_{4\rho}\| = \|kanv_{4\rho}\| = \|av_{4\rho}\| = e^{4\rho(H(\overline{n}))}\|v_{4\rho}\| = e^{4\rho(H(\overline{n}))}$. For $\nu \in \mathfrak{h}^*$ let $Q_\nu(\overline{n}) \in V_{4\rho}$ be a vector of weight ν such that $\overline{n}v_{4\rho} = \sum_\nu Q_\nu(\overline{n})$. Then $e^{8\rho(H(\overline{n}))} = \sum_\nu \|Q_\nu(\overline{n})\|^2$. Since $Q_{4\rho}(\overline{n}) = v_{4\rho}$, we have $e^{8\rho(H(\overline{n}))} \geq 1$.

Put $Q(\overline{n}) = \sum_{w \in W(M)\setminus\{e\}} \|Q_{4w\rho}(\overline{n})\|^2$. Assume that $\overline{n} \neq e$. Then there exist $w \in W(M) \setminus \{e\}$, $m' \in M$, $a' \in A$, $n' \in N$ and $\overline{n}' \in \overline{N}$ such that $\overline{n} = w\overline{n}'m'a'n'$. Let $v_{-4w\rho}^* \in V_{4\rho}^*$ be a weight vector with $\mathfrak{h}$-weight $-4w\rho$ such that for all $v \in V_{4w\rho}$, $|\langle v, v_{-4w\rho}^*\rangle| = \|v\|$. Then

$$\begin{aligned}\|Q_{4w\rho}(\overline{n})\| &= |\langle \overline{n}v_{4\rho}, v_{-4w\rho}^*\rangle| = |\langle w\overline{n}'m'a'n'v_{4\rho}, v_{-4w\rho}^*\rangle| \\ &= |\langle a'v_{4\rho}, w^{-1}v_{-4w\rho}^*\rangle| = e^{4\rho(\log a')}|\langle v_{4\rho}, w^{-1}v_{-4w\rho}^*\rangle| \neq 0.\end{aligned}$$

Hence, if $\overline{n} \in \overline{N} \setminus \{e\}$ then $Q(\overline{n}) \neq 0$.

Let $t > 0$. Using $Q_\nu(\exp(tH_0)\overline{n}\exp(-tH_0)) = e^{t(\nu-4\rho)(H_0)}Q_\nu(\overline{n})$, we have

$$Q(\exp(tH_0)\overline{n}\exp(-tH_0)) = \sum_{w\in W(M)\setminus\{e\}} e^{8t(w\rho-\rho)(H_0)}\|Q_{4w\rho}(\overline{n})\|^2.$$

Since $(w\rho-\rho)(H_0) \geq 1$ for $w \in W(M)\setminus\{e\}$, we get the lemma. □

Remark 4.2. The conditions in Lemma 4.1(3) imply that $\lim_{\overline{n}\to\infty} Q(\overline{n}) = \infty$. Indeed, take H_0 as in Lemma 4.1. Let $\{e_1,\dots,e_l\}$ be a basis of $\overline{\mathfrak{n}}$. We assume that each e_s is a restricted root vector and denote its root by α_s. Any $\overline{n} \in \overline{N}$ can be written as $\overline{n} = \exp(\sum_{s=1}^l a_s e_s)$ where $a_s \in \mathbb{R}$. We have $\alpha_s(H_0) > 0$ for all $s = 1,\dots,l$. Put $r(\overline{n}) = \sum_{s=1}^l |a_s|^{1/\alpha_s(H_0)}$. Set $C = \min_{r(\overline{n})=1} Q(\overline{n})$. Since $Q(\overline{n}) > 0$ if $\overline{n}$ is not the unit element, $C > 0$. Put $t = \log r(\overline{n})$ and set $\overline{n}' = \exp(-tH_0)\overline{n}\exp(tH_0)$. Then $\overline{n}' = \exp(\sum_{s=1}^l a_s e^{-t\alpha_s(H_0)}e_s)$. Therefore, $r(\overline{n}') = \sum_{s=1}^l |a_s|^{1/\alpha_s(H_0)}e^{-t} = 1$. Hence, if $r(\overline{n}) > 1$, then $Q(\overline{n}) = Q(\exp(tH_0)\overline{n}'\exp(-tH_0)) \geq Ce^{8t} = Cr(\overline{n})^8$ by Lemma 4.1(3). If $\overline{n}\to\infty$ then $r(\overline{n})\to\infty$. Hence, $Q(\overline{n})\to\infty$.

Lemma 4.3. *Let f be a polynomial on $\overline{N}$. There exists a positive integer k such that a C^∞-function h on $w_i\overline{N}P/P$ defined by $h(w_i\overline{n}P/P) = e^{-k\rho(H(\overline{n}))}f(\overline{n})$ can be extended to a C^∞-function on G/P.*

Proof. By Lemma 4.1 and Remark 4.2, we can choose a positive integer k such that $\lim_{\overline{n}\to\infty} e^{-8k\rho(H(\overline{n}))}f(\overline{n}) = 0$. Let h be a function on U_i defined by $h(w_i\overline{n}P/P) = e^{-8k\rho(H(\overline{n}))}f(\overline{n})$ for $\overline{n}\in\overline{N}$. We prove that h can be extended to G/P as a C^∞-function. Take $w \in W(M)$. Then h is defined on a subset of $w\overline{N}P/P$. Using a diffeomorphism $\overline{N} \simeq w\overline{N}P/P$, h defines a rational function $h\circ\Phi_{w_i,w}$ defined on an open dense subset of $\overline{N}$. By the condition on k, the function $h\circ\Phi_{w_i,w}$ has no pole. Hence, h defines a C^∞-function on $w\overline{N}P/P$. Since $\bigcup_{w\in W(M)} w\overline{N}P/P = G/P$, the lemma follows. □

Recall that for a representation V of $\mathfrak{m}$, $\nu \in (\mathfrak{m}\cap\mathfrak{a}_0)^* \subset \mathfrak{a}_0^*$ is called an *exponent* of V if $\nu + \rho_0|_{\mathfrak{m}\cap\mathfrak{a}_0}$ is an $\mathfrak{a}_0$-weight of $V/(\mathfrak{m}\cap\mathfrak{n}_0)V$.

Proposition 4.4. *Let φ be a σ-valued function on K which satisfies $\varphi(km) = \sigma(m)^{-1}\varphi(k)$ for all $k\in K$ and $m \in M\cap K$. Define $\varphi_\lambda \in I(\sigma,\lambda)$ by $\varphi_\lambda(kman) = e^{-(\lambda+\rho)(\log a)}\sigma(m)^{-1}\varphi(k)$ for $k\in K$, $m\in M$, $a\in A$ and $n \in N$. For $u' \in J'_{w_i^{-1}\eta}(\sigma\otimes e^{\lambda+\rho})$ and $f\in\mathcal{P}(O_i)$, put*

$$I_{f,u'}(\varphi_\lambda) = \int_{w_i\overline{N}w_i^{-1}\cap N_0} u'(\varphi_\lambda(nw_i))\eta(n)^{-1}f(nw_i)\,dn.$$

(1) *If $\langle\check{\alpha},\operatorname{Re}\lambda\rangle$ is sufficiently large for each $\alpha\in\Sigma^+\setminus\Sigma_M^+$ then the integral $I_{f,u'}(\varphi_\lambda)$ absolutely converges.*

(2) *As a function of* λ*, the integral* $I_{f,u'}(\varphi_\lambda)$ *has a meromorphic continuation to* $\mathfrak{a}^*$*.*

(3) *If* $\operatorname{supp}\eta = \Pi$ *and* $i = r$ *then* $I_{f,u'}(\varphi_\lambda)$ *is holomorphic at any* $\lambda \in \mathfrak{a}^*$*.*

(4) *Let* $u' \in \mathrm{Wh}_{w_i^{-1}\eta}((\sigma \otimes e^{\lambda+\rho})')$*. If* $\langle \check{\alpha}, \lambda + \nu \rangle \notin \mathbb{Z}_{\le 0}$ *for all exponents* ν *of* σ *and* $\alpha \in \Sigma^+ \setminus w_i^{-1}(\Sigma^+ \cup \Sigma_\eta^-)$*, then* $I_{1,u'}(\varphi_\mu)$ *is holomorphic at* $\mu = \lambda$*.*

For a proof, we use the following notation. (It will also be used in Sections 7 and 8.)

Let $P_\eta \supset P_0$ be the parabolic subgroup corresponding to $\operatorname{supp}\eta \subset \Pi$ and $P_\eta = M_\eta A_\eta N_\eta$ its Langlands decomposition such that $A_\eta \subset A_0$. Denote the complexifications of the Lie algebras of P_η, M_η, A_η, N_η by $\mathfrak{p}_\eta$, $\mathfrak{m}_\eta$, $\mathfrak{a}_\eta$, $\mathfrak{n}_\eta$, respectively. Put $\mathfrak{l}_\eta = \mathfrak{m}_\eta \oplus \mathfrak{a}_\eta$, $\overline{N_\eta} = \theta(N_\eta)$ and $\overline{\mathfrak{n}_\eta} = \theta(\mathfrak{n}_\eta)$. Set $\Sigma_\eta^+ = \{\sum_{\alpha \in \operatorname{supp}\eta} n_\alpha \alpha \in \Sigma^+ \mid n_\alpha \in \mathbb{Z}_{\ge 0}\}$ and $\Sigma_\eta^- = -\Sigma_\eta^+$.

Proof. First we prove (1). If $f = 1$ then this is a well-known result. (See, for example, Knapp's book [Kna01, Theorem 7.22].) For a general f, extend f to a function on $w_i\overline{N}P/P$ by $f(nn'w_i) = f(nw_i)$ for $n \in w_i\overline{N}w_i^{-1} \cap N_0$ and $n' \in w_i\overline{N}w_i^{-1} \cap \overline{N_0}$. Then by Lemma 4.3 there exists a positive integer C such that $\overline{n} \mapsto e^{-C\rho(H(\overline{n}))}f(w_i\overline{n})$ extends to a C^∞-function h on G/P. Define $\kappa\colon G \to K$ by $g \in \kappa(g)A_0N_0$. Since

$$I_{f,u'}(\varphi_\lambda) = \int_{w_i\overline{N}w_i^{-1} \cap N_0} u'(\varphi(\kappa(nw)))e^{-(\lambda+\rho)(H(nw_r))}f(nw_r)\eta(n)^{-1}\,dn,$$

we have $I_{f,u'}(\varphi_\lambda) = I_{1,u'}((\varphi h)_{\lambda - C\rho})$.

We prove (3). By dualizing Casselman's subrepresentation theorem, there exist an irreducible representation σ_0 of M_0 and $\lambda_0 \in \mathfrak{a}_0^*$ such that σ is a quotient of $\mathrm{Ind}_{M\cap P_0}^M(\sigma_0 \otimes e^{\lambda_0})$. We may regard $u' \in J'_{w_r^{-1}\eta}(\mathrm{Ind}_{M\cap P_0}^M(\sigma_0 \otimes e^{\lambda_0}))$. By the proof of Lemma 3.11, there exist a polynomial f_0 on $(M \cap N_0)w_{M,0}(M \cap P_0)/(M \cap P_0)$ and $u_0' \in (\sigma_0 \otimes e^{\lambda_0})'$ such that u' is given by

$$\varphi_0 \mapsto \int_{M \cap N_0} u_0'(\varphi_0(n_0 w_{M,0}))f_0(n_0 w_{M,0})\eta(n_0)^{-1}\,dn_0.$$

Let $\pi\colon \mathrm{Ind}_{P_0}^G(\sigma_0 \otimes e^{\lambda+\lambda_0+\rho}) \to I(\sigma, \lambda)$ be the map induced from the quotient map $\mathrm{Ind}_{M\cap P_0}^M(\sigma_0 \otimes e^{\lambda_0}) \to \sigma$. Take $\widetilde{\varphi}\colon K \to \sigma_0$ with $\widetilde{\varphi}(km) = \sigma_0^{-1}(m)\widetilde{\varphi}(k)$ ($k \in K$, $m \in M_0$) and $\pi(\widetilde{\varphi}_{\lambda+\lambda_0}) = \varphi_\lambda$. Define a polynomial $\widetilde{f} \in \mathcal{P}(w_r w_{M,0}\overline{N_0}P_0/P_0)$ by

$$\widetilde{f}(w_r w_{M,0} n n_0 P_0/P_0) = f(w_r nP/P)f_0(w_{M,0}n_0(M \cap P_0)/(M \cap P_0))$$

for $n \in \overline{N}$ and $n_0 \in M \cap \overline{N_0}$. (Notice that $w_{M,0}(M \cap \overline{N_0}) = (M \cap N_0)w_{M,0}$, so f

is a polynomial on $w_{M,0}(M\cap\overline{N_0})(M\cap P_0)/(M\cap P_0)$.) We have

$$I_{f,u'}(\varphi_\lambda)=\int_{w_rw_{M,0}\overline{N_0}(w_rw_{M,0})^{-1}\cap N_0} u_0'(\widetilde{\varphi}_{\lambda+\lambda_0}(nw_rw_{M,0}))\\ \times\widetilde{f}(w_rw_{M,0}nP_0/P_0)\eta(n)^{-1}\,dn.$$

Hence, we may assume that P is minimal. By the same argument as in the proof of (1), we may assume $f=1$. If $f=1$ then this integral is known as a *Jacquet integral* and its analytic continuation is known [Jac67].

We prove (2) and (4). By the same argument in the proof of (1), we may assume that $f=1$. Using Casselman's subrepresentation theorem, there exist an irreducible representation σ_0 of M_0, $\nu\in\mathfrak{a}_0^*$ and a surjective homomorphism $\mathrm{Ind}_{P_0}^G(\sigma_0\otimes e^{\lambda+\nu+\rho_0})\to I(\sigma,\lambda)$. Moreover, ν is an exponent of σ. By the same argument as in the proof of (3), we may assume $P=P_0$. (Hence each exponent of σ is 0.)

Take $w'\in W_{M_\eta}$ and $w''\in W(M_\eta)^{-1}$ such that $w_i=w'w''$. Then we have $w_i\overline{N}w_i^{-1}\cap N_0=w_i\overline{N_0}w_i^{-1}\cap N_0=(w'\overline{N_0}(w')^{-1}\cap N_0)w'(w''\overline{N_0}(w'')^{-1}\cap N_0)(w')^{-1}$. The condition $w'\in W_{M_\eta}$ implies that $w'(\Sigma^+\setminus\Sigma_\eta^+)=\Sigma^+\setminus\Sigma_\eta^+$. Hence, $\operatorname{supp}\eta\cap w'\Sigma^+=\operatorname{supp}\eta\cap w'\Sigma_\eta^+$. This implies

$$\begin{aligned}\operatorname{supp}\eta\cap w'(w''\Sigma^-\cap\Sigma^+)&=\operatorname{supp}\eta\cap w_i\Sigma^-\cap w'\Sigma^+\\&=\operatorname{supp}\eta\cap w_i\Sigma^-\cap w_i(w'')^{-1}\Sigma_\eta^+\subset\operatorname{supp}\eta\cap w_i\Sigma^-\cap w_i\Sigma^+=\emptyset,\end{aligned}$$

i.e., η is trivial on $w'(w''\overline{N_0}(w'')^{-1}\cap N_0)(w')^{-1}$. Hence

$$I_{1,u'}(\varphi)=\int_{w'\overline{N_0}(w')^{-1}\cap N_0}\int_{w''\overline{N_0}(w'')^{-1}\cap N_0}u'(\varphi(n_1w'n_2w''))\eta(n_1)^{-1}\,dn_2\,dn_1.$$

Define a G-module homomorphism $A(\sigma,\lambda)\colon I(\sigma,\lambda)\to\mathrm{Ind}_{P_0}^G(w''(\sigma)\otimes e^{w''\lambda+\rho_0})$ by

$$(A(\sigma,\lambda)\psi)(x)=\int_{w''\overline{N_0}(w'')^{-1}\cap N_0}\psi(xnw'')\,dn.$$

By a result of Knapp and Stein [KS80], this homomorphism has a meromorphic continuation. We have

$$I_{1,u'}(\psi)=\int_{w'\overline{N_0}(w')^{-1}\cap N_0}u'((A(\sigma,\lambda)\psi)(nw'))\eta(n)^{-1}\,dn.$$

Notice that $w'\overline{N_0}(w')^{-1}\cap N_0\subset M_\eta$. Hence $I_{1,u'}$ is given by the composition

$$I(\sigma,\lambda)\xrightarrow{A(\sigma,\lambda)}\mathrm{Ind}_{P_0}^G(w''(\sigma)\otimes e^{w''\lambda+\rho_0})\\ \xrightarrow{\text{restriction to }M_\eta}\mathrm{Ind}_{M_\eta\cap P_0}^{M_\eta}(w''(\sigma)\otimes e^{w''\lambda+\rho_0})\to\mathbb{C}.$$

Here the last map is given by

$$\psi \mapsto \int_{w'\overline{N_0}w'^{-1}\cap N_0} u'(\psi(nw'))\eta(n)^{-1}\,dn.$$

By (3), this integral is holomorphic. Hence we get (2).

To prove (4), we calculate $(w'')^{-1}\Sigma^- \cap \Sigma^+$. Since $(w'')^{-1} \in W(M_\eta)$, we have $(w'')^{-1}\Sigma_\eta^- \subset \Sigma^-$. Hence $(w'')^{-1}\Sigma_\eta^- \cap \Sigma^+ = \emptyset$. Then

$$\begin{aligned}(w'')^{-1}\Sigma^- \cap \Sigma^+ &= (w'')^{-1}(\Sigma^- \setminus \Sigma_\eta^-) \cap \Sigma^+ = (w'')^{-1}(w')^{-1}(\Sigma^- \setminus \Sigma_\eta^-) \cap \Sigma^+ \\ &= w_i^{-1}(\Sigma^- \setminus \Sigma_\eta^-) \cap \Sigma^+ = \Sigma^+ \setminus w_i^{-1}(\Sigma^+ \cup \Sigma_\eta^-).\end{aligned}$$

Hence $\langle \check{\alpha}, \lambda\rangle \notin \mathbb{Z}_{\geq 0}$ for all $\alpha \in (w'')^{-1}\Sigma^- \cap \Sigma^+$. By an argument of Knapp and Stein [KS80], $A(\sigma, \mu)$ is holomorphic at $\mu = \lambda$ if λ satisfies the conditions of (4). Hence we get (4). □

In the rest of this section, we denote the Bruhat filtration $I_i \subset J'_\eta(I(\sigma,\lambda))$ by $I_i(\lambda)$. The following result is a corollary of Proposition 4.4.

Lemma 4.5. *Let $x \in I'_i$. Then there exists a distribution $x_t \in I_i(\lambda + t\rho)$ with a meromorphic parameter t such that $x_t|_{U_i}$ is a distribution with a holomorphic parameter t and $(x_t|_{U_i})|_{t=0} = x$. Moreover, for $E \in U(\mathfrak{g})$, $Ex = 0$ implies $Ex_t = 0$.*

Proof. By the definition of I'_i, we may assume $x = E((f\eta_i^{-1} \otimes u')\delta_i)$ for some $E \in U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0})$, $f \in \mathcal{P}(O_i)$ and $u' \in J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$. By (1) and (2) of the above proposition, $\varphi \mapsto I_{f,u'}(\varphi_{\lambda+t\rho})$ is a distribution with a meromorphic parameter t. Moreover, it does not have a pole near $t = 0$ by (4) of the proposition. Let x'_t be this distribution. Put $x_t = Ex'_t$. By construction, x_t is as desired. □

Let $C^\infty(K,\sigma)$ be the space of σ-valued C^∞-functions on K. For $X \in \mathfrak{g}$ and $\lambda \in \mathfrak{a}^*$, we define an operator $D(X,\lambda)$ on $C^\infty(K,\sigma)$ by

$$(D(X,\lambda)\varphi)(k) = \frac{d}{dt}(\sigma \otimes e^{\lambda+\rho})(\exp(-H(\exp(-tX)k)))\varphi(\kappa(\exp(-tX)k))\Big|_{t=0}$$

for $\varphi \in C^\infty(K,\sigma)$. If we regard $I(\sigma,\lambda)$ as a subspace of $C^\infty(K,\sigma)$, then $(X\varphi)(k) = (D(X,\lambda)\varphi)(k)$ for $\varphi \in I(\sigma,\lambda)$. It is easy to see that there exist differential operators D_1, D_2 on φ such that $D(X, \lambda + t\rho) = D_1 + tD_2$ for all $t \in \mathbb{C}$. (The operators D_1, D_2 may depend on X and λ, but do not depend on t.)

Lemma 4.6. *Assume that the following conditions hold.*

(1) *The character η is unitary.*

(2) *The character* η *is zero on* $\mathrm{Ad}(w_i)\mathfrak{n} \cap \mathfrak{n}_0$.

(3) *The module* $J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$ *is not zero.*

(*See Lemma* 3.6.) *For* $x \in I'_i$ *there exists a distribution* $x_t \in I_i(\lambda + t\rho)$ *with a holomorphic parameter* t *defined near* $t = 0$ *such that* $x_0 = x$ *on* U_i.

Proof. First we remark that $\eta = \eta'$ by (2).

We argue by induction on i. If $i = 1$, then $x \in I'_1$. Take a distribution $x_t \in I_1(\lambda + t\rho)$ as in Lemma 4.5. Then $x_t|_{U_1}$ is holomorphic with respect to t. Since $\operatorname{supp} x_t \subset O_1$, $x_t|_{(G/P)\setminus O_1}$ is holomorphic with respect to t. Hence x_t is holomorphic with respect to t on $U_1 \cup ((G/P) \setminus O_1) = G/P$ as desired.

Assume that $i > 1$. First we prove the following claim: for $y \in I_{i-1}(\lambda)$, there exists a distribution $y_t \in I_{i-1}(\lambda + t\rho)$ with a holomorphic parameter t defined near $t = 0$ such that $y_0 = y$. Applying the inductive hypothesis to $y|_{U_{i-1}}$, there exists a distribution $y_t^{(i-1)} \in I_{i-1}(\lambda + t\rho)$ with a holomorphic parameter t defined near $t = 0$ such that $y_0^{(i-1)} = y$ on U_{i-1}. Since the supports of both sides are contained in $\bigcup_{j\le i-1} N_0 w_j P/P$, we have $y_0^{(i-1)} = y$ on $\bigcup_{j\ge i-1} N_0 w_j P/P$. Applying the inductive hypothesis to $(y - y_0^{(i-1)})|_{U_{i-2}}$, there exists a distribution $y_t^{(i-2)} \in I_{i-2}(\lambda + t\rho)$ with a holomorphic parameter t defined near $t = 0$ such that $y_0^{(i-2)} = y - y_0^{(i-1)}$ on U_{i-2}. Since the supports of both sides are contained in $\bigcup_{j\le i-2} N_0 w_j P/P$, we have $y_0^{(i-1)} + y_0^{(i-2)} = y$ on $\bigcup_{j\ge i-2} N_0 w_j P/P$. Iterating this argument, for $j = 1, \ldots, i-1$ there exists a distribution $y_t^{(j)} \in I_j(\lambda + t\rho)$ with a holomorphic parameter t defined near $t = 0$ such that $y = y_0^{(1)} + \cdots + y_0^{(i-1)}$ on G/P. Hence we get the claim.

Now we prove the lemma. By Lemma 4.5, there exists a distribution $x'_t \in I_i(\lambda + t\rho)$ with a meromorphic parameter t such that $x'_t|_{U_i}$ is holomorphic and $(x'_t|_{U_i})|_{t=0} = x$. Let $x'_t = \sum_{s=-p}^{\infty} x^{(s)} t^s$ be the Laurent series of x'_t. Now we prove the following claim: if there exists a distribution $x'_t = \sum_{s=-p}^{\infty} x^{(s)} t^s \in I_i(\lambda + t\rho)$ with a meromorphic parameter t defined near $t = 0$ such that $x'_t|_{U_i}$ is holomorphic and $(x'_t|_{U_i})|_{t=0} = x$, then there exists $x_t \in I_i(\lambda + t\rho)$ with a holomorphic parameter t defined near $t = 0$ such that $x_0|_{U_i} = x$. We prove the claim by induction on p.

If $p = 0$, we have nothing to prove. Assume $p > 0$. Take $E \in \mathfrak{n}_0$ and define differential operators E_0 and E_1 by $D(E, \lambda + t\rho) = E_0 + tE_1$. There exists a positive integer k such that $(E_0 + tE_1 - \eta(E))^k x'_t = 0$. Hence $(E_0 - \eta(E))^k x^{(-p)} = 0$. Since $x_t|_{U_i}$ is holomorphic, $\operatorname{supp} x^{(-p)} \subset \bigcup_{j<i} N_0 w_j P/P$. Hence $x^{(-p)} \in I_{i-1}(\lambda)$. By the claim in the third paragraph of this proof, there exists $x''_t \in I_{i-1}(\lambda + t\rho)$ with a holomorphic parameter t defined near $t = 0$ such that $x''_0 = x^{(-p)}$. Using the inductive hypothesis for $x'_t - t^{-p} x''_t$, we get the claim and the assertion of the lemma. □

Theorem 4.7. (1) *The module I_i/I_{i-1} is non-zero if and only if the following conditions hold:*

(a) *The character η is unitary.*
(b) *The character η is zero on $\mathrm{Ad}(w_i)\mathfrak{n}\cap\mathfrak{n}_0$.*
(c) *The module $J'_{w_i^{-1}\eta}(\sigma\otimes e^{\lambda+\rho})$ is not zero.*

(2) *If $I_i/I_{i-1}\neq 0$ then $I_i/I_{i-1}\simeq I'_i$.*

Proof. Assume that conditions (a)–(c) hold. We prove that the homomorphism $\mathrm{Res}_i\colon I_i\to I'_i$ defined in Section 2 is surjective. Indeed, for $x\in I'_i$, take $x_t\in I_i(\lambda+t\rho)$ as in Lemma 4.6. Then $\mathrm{Res}_i(x_0)=(x_0)|_{U_i}=x$. □

§5. Twisting functors

Arkhipov defined the *twisting functor* for $\widetilde{w}\in\widetilde{W}$ [Ark04]. In this section, we define a modification of the twisting functor.

Let $\mathfrak{g}^{\mathfrak{h}}_\alpha$ be the root space of $\alpha\in\Delta$. Set $\mathfrak{u}_0=\bigoplus_{\alpha\in\Delta^+}\mathfrak{g}^{\mathfrak{h}}_\alpha$, $\overline{\mathfrak{u}_0}=\bigoplus_{\alpha\in\Delta^+}\mathfrak{g}^{\mathfrak{h}}_{-\alpha}$ and $\mathfrak{u}_{0,\widetilde{w}}=\mathrm{Ad}(\widetilde{w})\overline{\mathfrak{u}_0}\cap\mathfrak{u}_0$. Let ψ be a character of $\mathfrak{u}_{0,\widetilde{w}}$. Let $\{e_1,\dots,e_l\}$ be a basis of $\mathfrak{u}_{0,\widetilde{w}}$ such that each e_i is a root vector and $\bigoplus_{s\le t-1}\mathbb{C}e_s$ is an ideal of $\bigoplus_{s\le t}\mathbb{C}e_s$ for each $t=1,\dots,l$. Notice that the multiplicative set $\{(e_k-\psi(e_k))^n\mid n\in\mathbb{Z}_{\ge 0}\}$ satisfies the Ore condition for $k=1,\dots,l$. Then we can consider the localization of $U(\mathfrak{g})$ with respect to $\{(e_k-\psi(e_k))^n\mid n\in\mathbb{Z}_{\ge 0}\}$. We denote the resulting algebra by $U(\mathfrak{g})_{e_k-\psi(e_k)}$. Put $S_{e_k-\psi(e_k)}=U(\mathfrak{g})_{e_k-\psi(e_k)}/U(\mathfrak{g})$. Then $S_{e_k-\psi(e_k)}$ is a $U(\mathfrak{g})$-bimodule.

Proposition 5.1. *The $U(\mathfrak{g})$-bimodule structure on*

$$S_{e_1-\psi(e_1)}\otimes_{U(\mathfrak{g})}\cdots\otimes_{U(\mathfrak{g})}S_{e_l-\psi(e_l)}$$

is independent of the choice of $e_1,\dots,e_l$.

We denote this module by $S_{\widetilde{w},\psi}$.

The proof of this proposition is similar to that of [Ark04, Theorem 2.1.6]. We omit it. Every element of $S_{\widetilde{w},\psi}$ can be written as a sum of elements of the form $(e_1-\psi(e_1))^{-(k_1+1)}\otimes\cdots\otimes(e_l-\psi(e_l))^{-(k_l+1)}E$ for $E\in U(\mathfrak{g})$. We denote this element by $(e_1-\psi(e_1))^{-(k_1+1)}\cdots(e_l-\psi(e_l))^{-(k_l+1)}E$ for short.

For any $U(\mathfrak{g})$-module V, we now define a $U(\mathfrak{g})$-module $T_{\widetilde{w},\psi}V$ by $T_{\widetilde{w},\psi}V=S_{\widetilde{w},\psi}\otimes_{U(\mathfrak{g})}(\widetilde{w}V)$. (Recall that $\widetilde{w}V$ is a $\mathfrak{g}$-module twisted by $\widetilde{w}$, see Notation.) This gives the twisting functor $T_{\widetilde{w},\psi}$. This is an endo-functor of the category of $\mathfrak{g}$-modules. If ψ is the trivial representation, $T_{\widetilde{w},\psi}$ is the twisting functor defined by Arkhipov. We put $T_{\widetilde{w}}=T_{\widetilde{w},0}$ where 0 is the trivial representation of $\mathfrak{u}_{0,\widetilde{w}}$.

Remark 5.2. Arkhipov [Ark04] denotes the twisting functor by Θ_w. We follow the notation of Andersen–Lauritzen [AL03].

We have a natural homomorphism $N_K(\mathfrak{h})/Z_K(\mathfrak{h}) \to N_K(\mathfrak{a}_0)/Z_K(\mathfrak{a}_0) = W$.

Lemma 5.3. *Let $w \in W$. Then there exists $\iota(w) \in N_K(\mathfrak{h})$ such that $\mathrm{Ad}(\iota(w))|_{\mathfrak{a}_0} = w$ and $\mathrm{Ad}(\iota(w))(\Delta^+_{M_0}) = \Delta^+_{M_0}$. If $\iota(w)$ and $\iota(w)'$ both satisfy these conditions, then $\iota(w) \in \iota(w)' Z_K(\mathfrak{h})$.*

Proof. Since $W = N_K(\mathfrak{a}_0)/Z_K(\mathfrak{a}_0)$, there is $k \in N_K(\mathfrak{a}_0)$ such that $\mathrm{Ad}(k)|_{\mathfrak{a}_0} = w$. Then k normalizes M_0. Hence there exists $m \in M_0$ such that km normalizes T_0. This implies $km \in N_K(A_0T_0)$. Take $w' \in N_{M_0}(\mathfrak{t}_0)$ such that $\mathrm{Ad}(kmw')(\Delta^+_{M_0}) = \Delta^+_{M_0}$ and put $\iota(w) = kmw'$. Then $\iota(w)$ satisfies the conditions of the lemma.

Assume that $\iota(w) \in N_K(\mathfrak{h})$ and $\iota(w)' \in N_K(\mathfrak{h})$ satisfy these conditions. Put $w_1 = \iota(w)^{-1}\iota(w)' \in N_K(\mathfrak{h})$. Then $\mathrm{Ad}(w_1)|_{\mathfrak{a}_0} = \mathrm{id}$, so $w_1 \in N_K(\mathfrak{a}_0) = M_0$. Hence w_1 gives an element of the Weyl group of M_0. Consequently, $\mathrm{Ad}(w_1)(\Delta^+_{M_0}) = \Delta^+_{M_0}$. Hence w_1 centralizes $\mathfrak{h}$. Therefore, $\iota(w) \in \iota(w)' Z_K(\mathfrak{h})$. □

The correspondence $w \mapsto \iota(w)$ gives a map $\iota \colon W \to N_K(\mathfrak{h})/Z_K(\mathfrak{h})$. By the characterization of $\iota(w)$, this map is injective. Since the group $N_K(\mathfrak{h})/Z_K(\mathfrak{h})$ is a subgroup of $\widetilde{W}$, we can regard W as a subgroup of $\widetilde{W}$. Hence we can define the twisting functor $T_{\iota(w),\psi}$ for $w \in W$ and the character ψ of $\mathrm{Ad}(w)\overline{\mathfrak{n}_0} \cap \mathfrak{n}_0$. For simplicity, we write w instead of $\iota(w)$. (We regard W as a subgroup of $\widetilde{W}$ via ι.)

Lemma 5.4. *Let e be a nilpotent element of $\mathfrak{g}$, $X \in \mathfrak{g}$ and $k \in \mathbb{Z}_{\geq 0}$. For $c \in \mathbb{C}$ we have the following equation in $U(\mathfrak{g})_{e-c}$:*

$$X(e-c)^{-(k+1)} = \sum_{n=0}^{\infty} \binom{n+k}{k} (e-c)^{-(n+k+1)} \operatorname{ad}(e)^n(X).$$

Proof. We prove the lemma by induction on k. If $k = 0$, the statement is well-known. Assume that $k > 0$. Then

$$\begin{aligned} X(e-c)^{-(k+1)} &= \sum_{k_0=0}^{\infty} (e-c)^{-(k_0+1)} \operatorname{ad}(e)^{k_0}(X)(e-c)^{-k} \\ &= \sum_{k_0=0}^{\infty} \sum_{k_1=0}^{\infty} \binom{k_1+k-1}{k-1} (e-c)^{-(k_0+k_1+k+1)} \operatorname{ad}(e)^{k_0+k_1}(X) \\ &= \sum_{n=0}^{\infty} \sum_{l'=0}^{n} \binom{l'+k-1}{k-1} (e-c)^{-(n+k+1)} \operatorname{ad}(e)^n(X) \\ &= \sum_{n=0}^{\infty} \binom{n+k}{k} (e-c)^{-(n+k+1)} \operatorname{ad}(e)^n(X). \end{aligned}$$

□

§6. The module I_i/I_{i-1}

Put $J_i = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$, where $\mathfrak{n}$ acts on $J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$ trivially. In this section, we prove the following theorem.

Theorem 6.1. *Assume that $I_i/I_{i-1} \neq 0$. Then $I_i/I_{i-1} \simeq T_{w_i,\eta} J_i$.*

Notice that $\mathfrak{u}_{0,w_i} = \mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0$ since $w_i(\Delta_M^+) \subset \Delta^+$. In this section we fix $i \in \{1, \dots, l\}$ and a basis $\{e_1, \dots, e_l\}$ of $\mathfrak{u}_{0,w_i}$ such that each vector e_s is a root vector and $\bigoplus_{s \le t-1} \mathbb{C}e_s$ is an ideal of $\bigoplus_{s \le t} \mathbb{C}e_s$. Let α_s be the restricted root corresponding to e_s. As in Section 3, for $\mathbf{k} = (k_1, \dots, k_l) \in \mathbb{Z}^l_{\ge 0}$ we denote $\mathrm{ad}(e_l)^{k_l} \cdots \mathrm{ad}(e_1)^{k_1}$ by $\mathrm{ad}(e)^{\mathbf{k}}$ and $((-x_1)^{k_1}/k_1!) \cdots ((-x_l)^{k_l}/k_l!)$ by $f_{\mathbf{k}}$.

Lemma 6.2. *We have*

$$I'_i = \left\{ \sum_{s=1}^{t} \delta_i(E_s, f_s \eta_i^{-1}, u'_s) \;\middle|\; \begin{array}{l} E_s \in U(\mathfrak{g}),\ f_s \in \mathcal{P}(O_i), \\ u'_s \in J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho}) \end{array} \right\}.$$

Proof. By Lemma 3.3,

$$E((f \otimes u')\delta_i) = \sum_{\mathbf{k} \in \mathbb{Z}^l_{\ge 0}} \delta_i(\mathrm{ad}(e)^{\mathbf{k}} E, f f_{\mathbf{k}}, u')$$

for $E \in U(\mathfrak{g})$, $f \in \mathcal{P}(O_i)\eta_i^{-1}$ and $u' \in \sigma'$. Hence, the left hand side in the statement is a subset of the right hand side. Define $f'_{\mathbf{k}} \in \mathcal{P}(O_i)$ by $f'_{\mathbf{k}} = (x_1^{k_1}/k_1!) \cdots (x_l^{k_l}/k_l!)$. By a similar calculation to the proof of Lemma 3.3, we have

$$\delta_i(E, f, u') = \sum_{\mathbf{k} \in \mathbb{Z}^l_{\ge 0}} (\mathrm{ad}(e)^{\mathbf{k}} E)(((f f'_{\mathbf{k}}) \otimes u')\delta_i).$$

This implies that the right hand side is contained in the left hand side. □

By the definition of the twisting functor and the Poincaré–Birkhoff–Witt theorem, we have the lemma below. For $\mathbf{k} = (k_1, \dots, k_l) \in \mathbb{Z}^l$ put $(e - \eta(e))^{\mathbf{k}} = (e_1 - \eta(e_1))^{k_1} \cdots (e_l - \eta(e_l))^{k_l} \in S_{w_i,\eta}$. Set $\mathbf{1} = (1, \dots, 1) \in \mathbb{Z}^l$. By multiplication from the right, the subspace $\sum_{\mathbf{k} \in \mathbb{Z}^l_{\ge 0}} \mathbb{C}(e - \eta(e))^{-(\mathbf{k}+\mathbf{1})} \subset S_{w_i,\eta}$ is a $U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0)$-submodule.

Lemma 6.3. *Let V be a $\mathfrak{p}$-module. Then*

$$\begin{aligned}
&\Bigl(\sum_{\mathbf{k} \in \mathbb{Z}^l_{\ge 0}} \mathbb{C}(e - \eta(e))^{-(\mathbf{k}+\mathbf{1})} \Bigr) \otimes U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0}) \otimes w_i V \\
&\quad \simeq \Bigl(\sum_{\mathbf{k} \in \mathbb{Z}^l_{\ge 0}} \mathbb{C}(e - \eta(e))^{-(\mathbf{k}+\mathbf{1})} \Bigr) \otimes_{U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0)} U(\mathfrak{g}) \otimes_{U(\mathrm{Ad}(w_i)\mathfrak{p})} w_i V \\
&\quad \simeq T_{w_i,\eta}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V).
\end{aligned}$$

The second isomorphism is given by $E \otimes F \otimes v \mapsto EF \otimes (1 \otimes v)$. (*Notice that* $EF \in S_{w_i,\eta}$.)

Proof of Theorem 6.1. By Lemmas 6.2 and 3.2, we have an isomorphism of vector spaces

$$I_i' \simeq \mathcal{P}(O_i) \otimes_{U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\mathfrak{n}_0)} U(\mathfrak{g}) \otimes_{U(\mathrm{Ad}(w_i)\mathfrak{p})} w_i J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$$

given by $\delta_i(E, f, u') \mapsto f \otimes E \otimes u'$.

Notice that $\mathfrak{u}_{0,w_i} = \mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0$ since $w_i \in W(M)$. By Lemma 6.3,

$$\begin{aligned} T_{w_i,\eta}(J_i) \simeq \Big(\sum_{\mathbf{k}\in\mathbb{Z}^l_{\geq 0}} \mathbb{C}(e-\eta(e))^{-(\mathbf{k}+\mathbf{1})} \Big) \otimes_{U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\mathfrak{n}_0)} U(\mathfrak{g}) \\ \otimes_{U(\mathrm{Ad}(w_i)\mathfrak{p})} w_i J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho}). \end{aligned}$$

Here $\sum_{\mathbf{k}\in\mathbb{Z}^l_{\geq 0}} \mathbb{C}(e-\eta(e))^{-(\mathbf{k}+\mathbf{1})}$ is an $\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\mathfrak{n}_0$-stable subspace of $S_{w_i,\eta}$. Hence, we can define a $\mathbb{C}$-vector space isomorphism $\Phi\colon T_{w_i,\eta}(J_i) \to I_i'$ by

$$\Phi((e-\eta(e))^{-(\mathbf{k}+\mathbf{1})} \otimes E \otimes u') = \delta_i(E, f_{\mathbf{k}}\eta_i^{-1}, u').$$

We now prove that Φ is a $\mathfrak{g}$-homomorphism. Fix $X \in \mathfrak{g}$. We will prove that

$$\Phi(X((e-\eta(e))^{-(\mathbf{k}+\mathbf{1})} \otimes E \otimes u')) = X\Phi((e-\eta(e))^{-(\mathbf{k}+\mathbf{1})} \otimes E \otimes u').$$

By Lemma 5.4,

$$\begin{aligned} &X((e-\eta(e))^{-(\mathbf{k}+\mathbf{1})} \otimes E \otimes u') \\ &= \sum_{p_s\geq 0} \binom{p_1+k_1}{k_1} \cdots \binom{p_l+k_l}{k_l} (e-\eta(e))^{-(\mathbf{k}+\mathbf{p}+\mathbf{1})} \otimes (\mathrm{ad}(e)^{\mathbf{p}}X)E \otimes u'. \end{aligned}$$

where $\mathbf{p} = (p_1, \dots, p_l)$. Hence,

$$\begin{aligned} &\Phi(X((e-\eta(e))^{-(\mathbf{k}+\mathbf{1})} \otimes E \otimes u')) \\ &\qquad = \sum_{p_s\geq 0} \delta_i\Big((\mathrm{ad}(e)^{\mathbf{p}}X)E, \Big(\frac{(-x_1)^{k_1+p_1}}{k_1!p_1!} \cdots \frac{(-x_l)^{k_l+p_l}}{k_l!p_l!} \Big) \eta_i^{-1}, u' \Big). \end{aligned}$$

By Lemma 3.3,

$$\begin{aligned} X\Phi((e-\eta(e))^{-(\mathbf{k}+1)} \otimes E \otimes u') &= X\delta_i(E, f_{\mathbf{k}}\eta_i^{-1}, u') \\ &= \sum_{\mathbf{p}\in\mathbb{Z}^l_{\geq 0}} \delta_i((\mathrm{ad}(e)^{\mathbf{p}}X)E, f_{\mathbf{k}}f_{\mathbf{p}}\eta_i^{-1}, u'). \end{aligned}$$

Hence the conclusion follows. □

§7. The module $J^*_\eta(I(\sigma,\lambda))$

Now we investigate the module $J^*_\eta(I(\sigma,\lambda))$. For a finite-length moderate growth Fréchet representation V of G, define a $\mathfrak{g}$-module $J(V)$ by

$$J(V) = (\varinjlim_{k\to\infty} (V_{K\text{-finite}}/\mathfrak{n}_0^k V_{K\text{-finite}}))_{\mathfrak{a}\text{-finite}}.$$

This is also called the *Jacquet module* of V [Cas80]. Define $\mathcal{O}'_{P_0}$ to be the full subcategory of finitely generated $\mathfrak{g}$-modules V satisfying the following conditions:

(1) The action of $\mathfrak{p}_0$ is locally finite. (In particular, the action of $\mathfrak{n}_0$ is locally nilpotent.)
(2) The module V is $Z(\mathfrak{g})$-finite.
(3) The group M_0 acts on V and its differential coincides with the action of $\mathfrak{m}_0 \subset \mathfrak{g}$.
(4) For $\nu \in \mathfrak{a}_0^*$ let V_ν be the generalized $\mathfrak{a}_0$-weight space with weight ν. Then $V = \bigoplus_{\nu\in\mathfrak{a}_0^*} V_\nu$ and $\dim V_\nu < \infty$.

We define $\mathcal{O}'_{\overline{P_0}}$ similarly. We write $\mathcal{O}'_{P_0,G}$ to emphasize the group G. Then for a finite-length Fréchet representation V of G we have $J(V) \in \mathcal{O}'_{\overline{P_0}}$ and $J^*(V) \in \mathcal{O}'_{P_0}$. For a $U(\mathfrak{g})$-module V, put $D'(V) = (V^*)_{\mathfrak{h}\text{-finite}}$ and $C(V) = (D'(V))^*$. The character $\eta\colon \mathfrak{n}_0 \to \mathbb{C}$ defines an algebra homomorphism $U(\mathfrak{n}_0) \to \mathbb{C}$ by the universality of the universal enveloping algebra. Let $\operatorname{Ker}\eta$ be the kernel of this algebra homomorphism and put $\Gamma_\eta(V) = \{v \in V \mid (\operatorname{Ker}\eta)^k v = 0 \text{ for some } k\}$. Then $J^*_\eta(V) = \Gamma_\eta((V_{K-\text{finite}})^*)$ by Remark 2.2. We will prove the following proposition.

Proposition 7.1. *Let V be a finite-length moderate growth Fréchet representation of G. Then $J^*_\eta(V) \simeq \Gamma_\eta(J(V)^*) \simeq \Gamma_\eta(C(J^*(V)))$.*

From this proposition, Theorem 6.1 and the automatic continuity theorem [Wal83, Theorem 4.8], we get the structure of $J^*_\eta(I(\sigma,\lambda))$.

Proposition 7.1 was proved by Matumoto [Mat90, Theorem 4.9.2] when $\operatorname{supp}\eta = \Pi$. We deduce the general case from his theorem. To do this, we need some lemmas. We use the following well-known properties (see Wallach's book [Wal88]):

Proposition 7.2. *Let V be a finite-length moderate growth Fréchet representation of G.*

(1) *$D'(J^*(V)) \simeq J(V)$.*
(2) *$V/\mathfrak{n}_0^k V \simeq J(V)/\mathfrak{n}_0^k J(V)$.*
(3) *The functor $\Gamma_\eta \circ C$ from $\mathcal{O}'_{P_0}$ or $\mathcal{O}'_{\overline{P_0}}$ to the category of $\mathfrak{g}$-modules is exact.*
(4) *$D'(\mathcal{O}'_{P_0}) \subset \mathcal{O}'_{\overline{P_0}}$ and $D'(\mathcal{O}'_{\overline{P_0}}) \subset \mathcal{O}'_{P_0}$. If $V \in \mathcal{O}'_{P_0}$ or $V \in \mathcal{O}'_{\overline{P_0}}$, then $D'D'V \simeq V$.*

Lemma 7.3. *Let $\mathfrak{c}$ be a nilpotent Lie algebra and ψ its character. Denote the corresponding $\mathbb{C}$-algebra homomorphism $U(\mathfrak{c}) \to \mathbb{C}$ again by ψ, and its kernel by* $\operatorname{Ker}\psi$. *Let V be a $\mathfrak{c}$-module and $\mathfrak{c}_1, \mathfrak{c}_2$ subalgebras such that $\mathfrak{c} = \mathfrak{c}_1 \oplus \mathfrak{c}_2$ and $\mathfrak{c}_2$ is an ideal of $\mathfrak{c}$. Set $\psi_i = \psi|_{U(\mathfrak{c}_i)}$. Then*

$$\bigcup_k \{v \in V \mid (\operatorname{Ker}\psi)^k v = 0\} = \bigcup_{k,l} \{v \in V \mid (\operatorname{Ker}\psi_1)^k v = 0,\ (\operatorname{Ker}\psi_2)^l v = 0\}.$$

Proof. Replacing V with $V \otimes (-\psi)$, we may assume ψ is trivial. By the same proof as in Remark 2.2, if $\mathfrak{c}_1^k v_0 = 0$, $\mathfrak{c}_2^l v_0 = 0$, then there exists k' such that $\mathfrak{c}^{k'} v_0 = 0$. Apply this to $v_0 = 1 \in V_0 = U(\mathfrak{c})/(U(\mathfrak{c})\mathfrak{c}_1^k + U(\mathfrak{c})\mathfrak{c}_2^l)$. Then there exists k' such that $\mathfrak{c}^{k'} v_0 = 0$.

Take v such that $\mathfrak{c}_1^k v = 0$, $\mathfrak{c}_2^l v = 0$. Then there exists a homomorphism $V_0 \to V$ such that $v_0 \mapsto v$. Hence $\mathfrak{c}^{k'} v = 0$. Therefore,

$$\{v \in V \mid \mathfrak{c}_1^k v = 0,\ \mathfrak{c}_2^l v = 0\} \subset \{v \in V \mid \mathfrak{c}^{k'} v = 0\}.$$

On the other hand,

$$\{v \in V \mid \mathfrak{c}^k v = 0\} \subset \{v \in V \mid \mathfrak{c}_1^k v = 0,\ \mathfrak{c}_2^k v = 0\}.$$

This implies the lemma. □

From the above lemma, we get the lemma below. Recall that $\mathfrak{p}_\eta = \mathfrak{m}_\eta \oplus \mathfrak{a}_\eta \oplus \mathfrak{n}_\eta$ is the complexification of the Lie algebra of the parabolic subgroup corresponding to $\operatorname{supp}\eta$ (Section 4).

Lemma 7.4. *Denote the $\mathbb{C}$-algebra homomorphism $U(\mathfrak{n}_0) \to \mathbb{C}$ corresponding to η again by η. Put $\eta_0 = \eta|_{U(\mathfrak{m}_\eta \cap \mathfrak{n}_0)}$. Then for any $\mathfrak{g}$-module V, we have*

$$\Gamma_\eta(V) = \bigcup_{k,l} \{v \in V \mid \mathfrak{n}_\eta^l v = 0,\ (\operatorname{Ker}\eta_0)^k v = 0\}.$$

Proof of Proposition 7.1. The second isomorphism follows from the definition of C, D' and Proposition 7.2(1).

We will prove $J_\eta^*(V) \simeq \Gamma_\eta(J(V)^*)$. If $\operatorname{supp}\eta = \Pi$, this was proved by Matumoto [Mat90, Theorem 4.9.2].

Put $I = V_{K\text{-finite}}$. Then I is a Harish-Chandra module. For a $U(\mathfrak{g})$-module V_0, put $Q(V_0) = (\varprojlim_k V_0/\mathfrak{n}_0^k V_0)_{\mathfrak{a}_0\text{-finite}}$. For a $U(\mathfrak{m}_\eta \oplus \mathfrak{a}_\eta)$-module V_1, put $Q_{M_\eta}(V_1) = (\varprojlim_k V_1/(\mathfrak{m}_\eta \cap \mathfrak{n}_0)^k V_1)_{(\mathfrak{m} \cap \mathfrak{a}_0)\text{-finite}}$. Let $\eta_0 \colon U(\mathfrak{m}_\eta \cap \mathfrak{n}_0) \to \mathbb{C}$ be the restriction of η to $U(\mathfrak{m}_\eta \cap \mathfrak{n}_0)$. Since $I/\mathfrak{n}_\eta^l I$ is a Harish-Chandra module of $\mathfrak{m}_\eta \oplus \mathfrak{a}_\eta$, by the result of Matumoto we have $\Gamma_{\eta_0}((I/\mathfrak{n}_\eta^l I)^*) = \Gamma_{\eta_0}(Q_{M_\eta}(I/\mathfrak{n}_\eta^l I)^*)$. Therefore,

$$\{v \in (I/\mathfrak{n}_\eta^l I)^* \mid (\operatorname{Ker}\eta_0)^k v = 0\} = \{v \in Q_{M_\eta}(I/\mathfrak{n}_\eta^l I)^* \mid (\operatorname{Ker}\eta_0)^k v = 0\}$$

for all $k \in \mathbb{Z}_{\geq 0}$.

We will prove $Q_{M_\eta}(I/\mathfrak{n}_\eta^l I) \simeq Q(I)/\mathfrak{n}_\eta^l Q(I)$. It is sufficient to show that $D'(Q_{M_\eta}(I/\mathfrak{n}_\eta^l I)) \simeq D'(Q(I)/\mathfrak{n}_\eta^l Q(I))$. By Proposition 7.2(1),

$$\begin{aligned} D'(Q_{M_\eta}(I/\mathfrak{n}_\eta^l I)) &\simeq \{v \in (I/\mathfrak{n}_\eta^l I)^* \mid (\mathfrak{m}_\eta \cap \mathfrak{n}_0)^k v = 0 \text{ for some } k\} \\ &\simeq \{v \in I^* \mid \mathfrak{n}_\eta^l v = 0,\ (\mathfrak{m}_\eta \cap \mathfrak{n}_0)^k v = 0 \text{ for some } k\} \\ &= \{v \in I^* \mid \mathfrak{n}_\eta^l v = 0,\ \mathfrak{n}_0^k v = 0 \text{ for some } k\}. \end{aligned}$$

Using Proposition 7.2(1) again, we obtain

$$\{v \in I^* \mid \mathfrak{n}_0^k v = 0 \text{ for some } k\} \simeq D'(Q(I)).$$

Hence

$$D'(Q_{M_\eta}(I/\mathfrak{n}_\eta^l I)) \simeq \{v \in D'(Q(I)) \mid \mathfrak{n}_\eta^l v = 0\}.$$

By its definition, D' is left exact. Hence we have an exact sequence

$$0 \to D'(Q(I)/\mathfrak{n}_\eta^l Q(I)) \to D'(Q(I)) \to D'(\mathfrak{n}_\eta^l Q(I)).$$

Therefore, $\{v \in D'(Q(I)) \mid \mathfrak{n}_\eta^l v = 0\} \simeq D'(Q(I)/\mathfrak{n}_\eta^l Q(I))$. Hence $Q_{M_\eta}(I/\mathfrak{n}_\eta^l I) \simeq Q(I)/\mathfrak{n}_\eta^l Q(I)$. This implies

$$\{v \in (I/\mathfrak{n}_\eta^l I)^* \mid (\operatorname{Ker}\eta_0)^k v = 0\} \simeq \{v \in (Q(I)/\mathfrak{n}_\eta^l Q(I))^* \mid (\operatorname{Ker}\eta_0)^k v = 0\}.$$

Hence

$$\{v \in I^* \mid \mathfrak{n}_\eta^l v = 0,\ (\operatorname{Ker}\eta_0)^k v = 0\} \simeq \{v \in Q(I)^* \mid \mathfrak{n}_\eta^l v = 0,\ (\operatorname{Ker}\eta_0)^k v = 0\}.$$

Therefore, by the previous lemma, we have

$$\Gamma_\eta(I^*) \simeq \Gamma_\eta(Q(I)^*).$$

By the definition and Remark 2.2, $Q(I) = J(V)$ and $\Gamma_\eta(I^*) = J_\eta^*(I)$. □

Combining Theorem 6.1, Proposition 7.1 and the automatic continuity theorem [Wal83, Theorem 4.8], we have the following theorem. Let I_i be the Bruhat filtration of $J'(I(\sigma,\lambda)) \simeq J^*(I(\sigma,\lambda))$. Put $\widetilde{I_i} = \Gamma_\eta(C(I_i)) \subset \Gamma_\eta(C(J^*(I(\sigma,\lambda)))) \simeq J_\eta^*(I(\sigma,\lambda))$.

Theorem 7.5. *The filtration* $0 = \widetilde{I_1} \subset \cdots \subset \widetilde{I_r} = J_\eta^*(I(\sigma,\lambda))$ *satisfies* $\widetilde{I_i}/\widetilde{I_{i-1}} \simeq \Gamma_\eta(C(T_{w_i}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} J^*(\sigma \otimes e^{\lambda+\rho}))))$.

Proof. This follows from Theorem 6.1 and Propositions 7.2 and 7.1. □

§8. Whittaker vectors

We now study the space of Whittaker vectors of $I(\sigma,\lambda)'$ and $(I(\sigma,\lambda)_{K\text{-finite}})^*$ (Definition 3.9) using the Bruhat filtration.

First, we consider $\mathrm{Wh}_\eta(I(\sigma,\lambda)')$. To calculate its dimension, we calculate $\dim \mathrm{Wh}_\eta(I_i/I_{i-1})$. The idea is to use the Harish-Chandra isomorphism. To explain the idea, recall a proof of the following fact: the Verma module has a unique *highest weight* if its infinitesimal character is generic. (Here, for a $\mathfrak{g}$-module V, we call $\widetilde{\lambda}\in\mathfrak{h}^*$ a highest weight of V if $\widetilde{\lambda}$ is the weight of a vector in V killed by the nilpotent radical of the Borel subalgebra.) The proof is the following. Let $\widetilde{\lambda}$ be the infinitesimal character of the Verma module and assume that the set of weights of the Verma module is $\widetilde{\lambda}+\mathbb{Z}_{\leq 0}\Delta$. Then by the Harish-Chandra isomorphism, each highest weight of the Verma module has the form $\widetilde{w}(\widetilde{\lambda}+\widetilde{\rho})-\widetilde{\rho}$. Therefore, $\widetilde{w}\widetilde{\lambda}-\widetilde{\lambda}\in\mathbb{Z}\Delta$. Since $\widetilde{\lambda}$ is generic, $\widetilde{w}=1$. We use an analogous proof. To do it, we decompose the Harish-Chandra homomorphism, using the following lemma.

Lemma 8.1. *If $I_i/I_{i-1}\neq 0$, then $\mathfrak{l}_\eta\cap\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\subset\mathfrak{n}_0$.*

Proof. By Lemma 3.6, the restriction of η to $\mathrm{Ad}(w_i)\mathfrak{n}\cap\mathfrak{n}_0$ is trivial. This is equivalent to $\operatorname{supp}\eta\cap w_i(\Sigma^+\setminus\Sigma_M^+)\cap\Sigma^+=\emptyset$. Thus, $(-\operatorname{supp}\eta\cap\Sigma^-)\cap w_i(\Sigma^-\setminus\Sigma_M^-)=\emptyset$, so $(\mathfrak{l}_\eta\cap\overline{\mathfrak{n}_0})\cap\mathrm{Ad}(w_i)\overline{\mathfrak{n}}=0$. ∎

For i such that $I_i/I_{i-1}\neq 0$, we define γ_1 to γ_4 to be the first projections with respect to the corresponding decompositions below:

$$\gamma_1\colon U(\mathfrak{g})=U(\mathfrak{l}_\eta)\oplus(\overline{\mathfrak{n}_\eta}U(\mathfrak{g})+U(\mathfrak{g})\mathfrak{n}_\eta)\to U(\mathfrak{l}_\eta),$$
$$\gamma_2\colon U(\mathfrak{l}_\eta)=U(\mathfrak{l}_\eta\cap\mathrm{Ad}(w_i)\mathfrak{p})\oplus U(\mathfrak{l}_\eta)\operatorname{Ker}\eta|_{\mathfrak{l}_\eta\cap\mathrm{Ad}(w_i)\overline{\mathfrak{n}}}\to U(\mathfrak{l}_\eta\cap\mathrm{Ad}(w_i)\mathfrak{p}),$$
$$\gamma_3\colon U(\mathfrak{l}_\eta\cap\mathrm{Ad}(w_i)\mathfrak{p})=U(\mathfrak{l}_\eta\cap\mathrm{Ad}(w_i)\mathfrak{l})\oplus(\mathfrak{l}_\eta\cap\mathrm{Ad}(w_i)\mathfrak{n})U(\mathfrak{l}_\eta\cap\mathrm{Ad}(w_i)\mathfrak{p})\to U(\mathfrak{l}_\eta\cap\mathrm{Ad}(w_i)\mathfrak{l}),$$
$$\gamma_4\colon U(\mathfrak{l}_\eta\cap\mathrm{Ad}(w_i)\mathfrak{l})=U(\mathfrak{h})\oplus((\overline{\mathfrak{u}_0}\cap\mathfrak{l}_\eta\cap\mathrm{Ad}(w_i)\mathfrak{l})U(\mathfrak{l}_\eta\cap\mathrm{Ad}(w_i)\mathfrak{l})\to U(\mathfrak{l}_\eta\cap\mathrm{Ad}(w_i)\mathfrak{l})(\mathfrak{l}_\eta\cap\mathrm{Ad}(w_i)\mathfrak{l}\cap\mathfrak{u}_0))\to U(\mathfrak{h}).$$

To define γ_2, we must check $\mathfrak{l}_\eta\cap\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\subset\mathfrak{n}_0$. This follows from $I_i/I_{i-1}\neq 0$ and the previous lemma. Then the restriction of $\gamma_4\circ\gamma_3\circ\gamma_2\circ\gamma_1$ to $Z(\mathfrak{g})$ is the (non-shifted) Harish-Chandra homomorphism. If $x\in\mathrm{Wh}_\eta(I_i/I_{i-1})$ then $Ex=\gamma_2\gamma_1(E)x$ for $E\in Z(\mathfrak{g})$.

Lemma 8.2. *Let V be a $U(\mathfrak{g})$-module with infinitesimal character $\widetilde{\lambda}$, and χ a character of $Z(\mathfrak{g})$ such that $z\in Z(\mathfrak{g})$ acts by $\chi(z)$ on V. Let $v\in V\setminus\{0\}$ and $\mu\in\mathfrak{a}^*$ be such that $(\gamma_3\gamma_2\gamma_1(z)-\chi(z))v=0$ and $Hv=(w_i\mu+\rho_0)(H)v$ for all $z\in Z(\mathfrak{g})$ and $H\in\mathrm{Ad}(w_i)\mathfrak{a}$. Then there exists $\widetilde{w}\in\widetilde{W}$ such that $\widetilde{w}\widetilde{\lambda}|_{\mathfrak{a}}=\mu$.*

Proof. Put $Z = \gamma_3\gamma_2\gamma_1(Z(\mathfrak{g}))U(\mathrm{Ad}(w_i)\mathfrak{a})$. By assumption, there exists a character χ_0 of Z such that $zv = \chi_0(z)v$ for all $z \in Z$. By a theorem of Harish-Chandra, $\gamma_4|_Z$ is injective and finite. Hence there exists $\widetilde{\lambda_1} \in \mathfrak{h}^*$ such that $\widetilde{\lambda_1} \circ \gamma_4 = \chi_0$ where we denote the algebra homomorphism $U(\mathfrak{h}) \to \mathbb{C}$ corresponding to $\widetilde{\lambda_1}$ again by $\widetilde{\lambda_1}$. Since V has infinitesimal character $\widetilde{\lambda}$, we have $\widetilde{\lambda_1} \in \widetilde{W}\widetilde{\lambda} + \widetilde{\rho}$. Since γ_4 is trivial on $U(\mathrm{Ad}(w_i)\mathfrak{a})$, $\widetilde{\lambda_1}|_{\mathrm{Ad}(w_i)\mathfrak{a}} = (w_i\mu + \rho_0)|_{\mathrm{Ad}(w_i)\mathfrak{a}}$. The restriction of $\widetilde{\rho}$ to $\mathfrak{a}_0$ is ρ_0. Hence $\widetilde{\rho}|_{\mathrm{Ad}(w_i)\mathfrak{a}} = \rho_0|_{\mathrm{Ad}(w_i)\mathfrak{a}}$. Then for some $\widetilde{w} \in \widetilde{W}$ we have $w_i\mu|_{\mathrm{Ad}(w_i)\mathfrak{a}} = \widetilde{w}\widetilde{\lambda}|_{\mathrm{Ad}(w_i)\mathfrak{a}}$, proving the lemma. □

Lemma 8.3. *Let $X_1, \dots, X_n \in \mathfrak{g}$, $f_1 \in C^\infty(O_i)$, $f_2 \in C^\infty(U_i)$, $u' \in (\sigma \otimes e^{\lambda+\rho})'$. Assume that $R(\mathrm{Ad}(w_i)^{-1}X_s)(f_2) = 0$ for all $s = 1, \dots, n$. Then*

$$\delta_i(X_1 \cdots X_n, f_1 f_2, u') = \delta_i(X_1 \cdots X_n, f_1, u')f_2.$$

Proof. Put $E = X_1 \cdots X_n$. By assumption and Leibniz's rule, we have

$$f_2(nw_i)(R(\mathrm{Ad}(w_i)^{-1}E)\varphi)(nw_i) = (R(\mathrm{Ad}(w_i)^{-1}E)(\varphi f_2))(nw_i).$$

Hence, by definition, for $\varphi \in C_c^\infty(U_i, \mathcal{L})$, we have

$$\begin{aligned}\langle \delta_i(E, f_1 f_2, u'), \varphi\rangle &= \int_{w_i\overline{N}w_i^{-1}\cap N_0} f_1(nw_i)f_2(nw_i)(u'(R(\mathrm{Ad}(w_i)^{-1}E)\varphi)(nw_i))\,dn \\ &= \int_{w_i\overline{N}w_i^{-1}\cap N_0} f_1(nw_i)(u'(R(\mathrm{Ad}(w_i)^{-1}E)(\varphi f_2))(nw_i))\,dn \\ &= \langle \delta_i(E, f_1, u'), f_2\varphi\rangle = \langle \delta_i(E, f_1, u')f_2, \varphi\rangle,\end{aligned}$$

and the lemma follows. □

Recall that the C^∞-function η_i on O_i is defined by $\eta_i(nw_iP/P) = \eta(n)$ for $n \in w_i\overline{N}w_i^{-1} \cap N_0$. For $\nu \in \mathfrak{a}^*$ put

$$V(\nu) = \left\{ \sum_s \delta_i(F_s, h_s, v'_s) \;\middle|\; \begin{array}{l} F_s \in U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0}),\ h_s \in \mathcal{P}(O_i), \\ v'_s \in J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho}), \\ (w_i^{-1}(\mathrm{wt}\, h_s + \mathrm{wt}\, F_s))|_{\mathfrak{a}} = \nu \end{array} \right\}.$$

Here, $\mathrm{wt}\, h_s$ is the $\mathfrak{a}_0$-weight of h_s with respect to D_i (see page 430) and $\mathrm{wt}\, F_s$ is the $\mathfrak{a}_0$-weight of F_s with respect to the adjoint action. We have no weight in I_i/I_{i-1}. The spaces $V(\nu)$ play the role of weight spaces.

Remark 8.4. By Lemma 3.2(1), we have

$$V(\nu) = \left\{ \sum_s \delta_i(F_s, h_s, v'_s) \;\middle|\; \begin{array}{l} F_s \in U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}}),\ h_s \in \mathcal{P}(O_i), \\ v'_s \in J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho}), \\ (w_i^{-1}(\mathrm{wt}\, h_s + \mathrm{wt}\, F_s))|_{\mathfrak{a}} = \nu \end{array} \right\}.$$

Lemma 8.5. *Let $X \in U(\mathfrak{g})$ be an $\mathfrak{a}_0$-weight vector. Then*

$$XV(\nu) \subset V(\nu + w_i^{-1}\operatorname{wt}(X)|_{\mathfrak{a}}).$$

Proof. We may assume $X \in \mathfrak{g}$. Let $\delta_i(E, f, u') \in V(\nu)$. By Lemma 3.3, we have

$$X\delta_i(E, f, u') = \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^l} \delta_i((\operatorname{ad}(e)^{\mathbf{k}} X)E, ff_{\mathbf{k}}, u').$$

Assume $\operatorname{ad}(e)^{\mathbf{k}} X \in \operatorname{Ad}(w_i)\mathfrak{p}$. Then

$$\delta_i((\operatorname{ad}(e)^{\mathbf{k}} X)E, ff_{\mathbf{k}}, u') = \delta_i(E, ff_{\mathbf{k}}, \operatorname{Ad}(w_i)^{-1}((\operatorname{ad}(e)^{\mathbf{k}} X))u').$$

If $\operatorname{ad}(e)^{\mathbf{k}}(X) \in \operatorname{Ad}(w_i)\mathfrak{n}$, then this is 0. If $\operatorname{ad}(e)^{\mathbf{k}}(X) \in \operatorname{Ad}(w_i)\mathfrak{l}$, then we have $w_i^{-1}\operatorname{wt}(\operatorname{ad}(e)^{\mathbf{k}} X)|_{\mathfrak{a}} = 0$. Hence $w_i^{-1}\operatorname{wt}(X)|_{\mathfrak{a}} = w_i^{-1}\operatorname{wt}(f_{\mathbf{k}})|_{\mathfrak{a}}$. Therefore, $w_i^{-1}(\operatorname{wt}(E) + \operatorname{wt}(ff_{\mathbf{k}}))|_{\mathfrak{a}} = \nu + w_i^{-1}\operatorname{wt}(X)|_{\mathfrak{a}}$.

If $\operatorname{ad}(e)^{\mathbf{k}}(X) \in \operatorname{Ad}(w_i)\overline{\mathfrak{n}}$, then $(\operatorname{ad}(e)^{\mathbf{k}} X)E \in \operatorname{Ad}(w_i)\overline{\mathfrak{n}}$. We have

$$w_i^{-1}(\operatorname{wt}(\operatorname{ad}(e)^{\mathbf{k}} X)E + \operatorname{wt} ff_{\mathbf{k}}) = w_i^{-1}(\operatorname{wt} E + \operatorname{wt} f + \operatorname{wt} X).$$

This implies the lemma. □

Lemma 8.6. *Define $\widetilde{\eta}_i \in C^\infty(U_i)$ by $\widetilde{\eta}_i(nn_0w_iP/P) = \eta_i(n)$ for $n \in w_i\overline{N}w_i^{-1} \cap N_0$ and $n_0 \in w_i\overline{N}w_i^{-1} \cap \overline{N_0}$. Let $X \in U(\mathfrak{g})$. Assume that X is an $\mathfrak{a}_0$-weight vector. For $\delta_i(E, f, u') \in V(\nu)$, we have*

$$X\delta_i(E, f\eta_i^{-1}, u') - (X\delta_i(E, f, u'))\widetilde{\eta}_i^{-1} \in \sum_{\nu' > \nu} V(\nu' + w_i^{-1}\operatorname{wt} X|_{\mathfrak{a}})\widetilde{\eta}_i^{-1}.$$

Here, $\operatorname{wt} X$ *is the $\mathfrak{a}_0$-weight of X with respect to the adjoint action.*

Proof. Fix a basis $\{e_1, \ldots, e_l\}$ of $\mathfrak{u}_{0,w_i}$ such that each e_s is a root vector and $\bigoplus_{s \leq t-1} \mathbb{C}e_s$ is an ideal of $\bigoplus_{s \leq t} \mathbb{C}e_s$. Let α_s be the restricted root of e_s. As in Section 3, for $\mathbf{k} = (k_1, \ldots, k_l) \in \mathbb{Z}_{\geq 0}^l$ we denote $\operatorname{ad}(e_l)^{k_l} \cdots \operatorname{ad}(e_1)^{k_1}$ by $\operatorname{ad}(e)^{\mathbf{k}}$ and $((-x_1)^{k_1}/k_1!) \cdots ((-x_l)^{k_l}/k_l!)$ by $f_{\mathbf{k}}$. By Lemma 3.3,

$$X\delta_i(E, f\eta_i^{-1}, u') = \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^l} \delta_i((\operatorname{ad}(e)^{\mathbf{k}} X)E, ff_{\mathbf{k}}\eta_i^{-1}, u').$$

Take $a_{\mathbf{k}}^{(p)} \in U(\operatorname{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0)$, $b_{\mathbf{k}}^{(p)} \in U(\operatorname{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0})$ and $c_{\mathbf{k}}^{(p)} \in U(\operatorname{Ad}(w_i)\mathfrak{p})$ such that $(\operatorname{ad}(e)^{\mathbf{k}} X)E = \sum_p a_{\mathbf{k}}^{(p)} b_{\mathbf{k}}^{(p)} c_{\mathbf{k}}^{(p)}$ and $\operatorname{wt}((\operatorname{ad}(e)^{\mathbf{k}} X)E) = \operatorname{wt} a_{\mathbf{k}}^{(p)} + \operatorname{wt} b_{\mathbf{k}}^{(p)} + \operatorname{wt} c_{\mathbf{k}}^{(p)}$. Then

$$\begin{aligned}\delta_i((\operatorname{ad}(e)^{\mathbf{k}} X)E, ff_{\mathbf{k}}\eta_i^{-1}, u') &= \sum_p \delta_i(a_{\mathbf{k}}^{(p)} b_{\mathbf{k}}^{(p)} c_{\mathbf{k}}^{(p)}, ff_{\mathbf{k}}\eta_i^{-1}, u') \\ &= \sum_p \delta_i(b_{\mathbf{k}}^{(p)}, R_i'((a_{\mathbf{k}}^{(p)})^\vee)(ff_{\mathbf{k}}\eta_i^{-1}), \operatorname{Ad}(w_i)^{-1}(c_{\mathbf{k}}^{(p)})u').\end{aligned}$$

By the Leibniz rule, there is a finite subset $\mathcal{A}_{\mathbf{k}}^{(p)} \subset \{(a', a'') \in U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0)^2 \mid \mathrm{wt}\, a' + \mathrm{wt}\, a'' = \mathrm{wt}\, a_{\mathbf{k}}^{(p)},\ a'' \notin \mathbb{C}\}$ such that

$$\begin{aligned}
&\delta_i(b_{\mathbf{k}}^{(p)}, R_i'((a_{\mathbf{k}}^{(p)})^\vee)(f f_{\mathbf{k}} \eta_i^{-1}) - R_i'((a_{\mathbf{k}}^{(p)})^\vee)(f f_{\mathbf{k}})\eta_i^{-1}, \mathrm{Ad}(w_i)^{-1} c_{\mathbf{k}}^{(p)} u') \\
&\qquad = \sum_{(a',a'') \in \mathcal{A}_{\mathbf{k}}^{(p)}} \delta_i(b_{\mathbf{k}}^{(p)}, R_i'(a')(f f_{\mathbf{k}}) R_i'(a'')(\eta_i^{-1}), \mathrm{Ad}(w_i)^{-1} c_{\mathbf{k}}^{(p)} u') \\
&\qquad = \sum_{(a',a'') \in \mathcal{A}_{\mathbf{k}}^{(p)}} -\eta(a'')\delta_i(b_{\mathbf{k}}^{(p)}, R_i'(a')(f f_{\mathbf{k}})\eta_i^{-1}, \mathrm{Ad}(w_i)^{-1} c_{\mathbf{k}}^{(p)} u').
\end{aligned}$$

By the definition of $\widetilde{\eta}_i$, we have $R(\mathrm{Ad}(w_i)^{-1}X')\widetilde{\eta}_i = 0$ for $X' \in \mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0}$. Hence by Lemma 8.3,

$$\delta_i(b_{\mathbf{k}}^{(p)}, f'\eta_i^{-1}, \mathrm{Ad}(w_i)^{-1}(c_{\mathbf{k}}^{(p)})u') = \delta_i(b_{\mathbf{k}}^{(p)}, f', \mathrm{Ad}(w_i)^{-1}(c_{\mathbf{k}}^{(p)})u')\widetilde{\eta}_i^{-1}$$

for all $f' \in \mathcal{P}(O_i)$. Thus

$$\begin{aligned}
&\delta_i(b_{\mathbf{k}}^{(p)}, R_i'((a_{\mathbf{k}}^{(p)})^\vee)(f f_{\mathbf{k}} \eta_i^{-1}), \mathrm{Ad}(w_i)^{-1} c_{\mathbf{k}}^{(p)} u') \\
&\qquad - \delta_i(b_{\mathbf{k}}^{(p)}, R_i'((a_{\mathbf{k}}^{(p)})^\vee)(f f_{\mathbf{k}}), \mathrm{Ad}(w_i)^{-1} c_{\mathbf{k}}^{(p)} u')\widetilde{\eta}_i^{-1} \\
&\qquad = \sum_{(a',a'') \in \mathcal{A}_{\mathbf{k}}^{(p)}} -\eta(a'')\delta_i(b_{\mathbf{k}}^{(p)}, R_i'(a')(f f_{\mathbf{k}}), \mathrm{Ad}(w_i)^{-1} c_{\mathbf{k}}^{(p)} u')\widetilde{\eta}_i^{-1}.
\end{aligned}$$

By the Poincaré–Birkhoff–Witt theorem, we have a decomposition $U(\mathrm{Ad}(w_i)\mathfrak{p}) = U(\mathrm{Ad}(w_i)\mathfrak{p})(\mathrm{Ad}(w_i)\mathfrak{n}) \oplus U(\mathrm{Ad}(w_i)\mathfrak{l})$. Hence we may assume that $c_{\mathbf{k}}^{(p)} \in U(\mathrm{Ad}(w_i)\mathfrak{p})(\mathrm{Ad}(w_i)\mathfrak{n})$ or $c_{\mathbf{k}}^{(p)} \in U(\mathrm{Ad}(w_i)\mathfrak{l})$. If $c_{\mathbf{k}}^{(p)} \in U(\mathrm{Ad}(w_i)\mathfrak{p})(\mathrm{Ad}(w_i)\mathfrak{n})$ then $\mathrm{Ad}(w_i)^{-1} c_{\mathbf{k}}^{(p)} u' = 0$ since $\mathfrak{n}$ acts on $J'_{w_i^{-1}}(\sigma \otimes e^{\lambda+\rho})$ trivially. If $c_{\mathbf{k}}^{(p)} \in U(\mathrm{Ad}(w_i)\mathfrak{l})$ then $w_i^{-1}\, \mathrm{wt}\, c_{\mathbf{k}}^{(p)}|_{\mathfrak{a}} = 0$. Hence

$$\begin{aligned}
&w_i^{-1}(\mathrm{wt}\, b_{\mathbf{k}}^{(p)} + \mathrm{wt}(R_i'(a')(f f_{\mathbf{k}})))|_{\mathfrak{a}} \\
&\qquad = w_i^{-1}(\mathrm{wt}\, c_{\mathbf{k}}^{(p)} + \mathrm{wt}\, b_{\mathbf{k}}^{(p)} + \mathrm{wt}\, a' + \mathrm{wt}\, f + \mathrm{wt}\, f_{\mathbf{k}})|_{\mathfrak{a}} \\
&\qquad = w_i^{-1}(\mathrm{wt}\, a_{\mathbf{k}}^{(p)} + \mathrm{wt}\, b_{\mathbf{k}}^{(p)} + \mathrm{wt}\, c_{\mathbf{k}}^{(p)} + \mathrm{wt}\, f + \mathrm{wt}\, f_{\mathbf{k}} - \mathrm{wt}\, a'')|_{\mathfrak{a}} \\
&\qquad = w_i^{-1}(\mathrm{wt}((\mathrm{ad}(e)^{\mathbf{k}}X)E) + \mathrm{wt}\, f + \mathrm{wt}\, f_{\mathbf{k}} - \mathrm{wt}\, a'')|_{\mathfrak{a}} \\
&\qquad = w_i^{-1}(\mathrm{wt}\, X + \mathrm{wt}\, E + \mathrm{wt}\, f - \mathrm{wt}\, a'')|_{\mathfrak{a}} \\
&\qquad = \nu + w_i^{-1}(\mathrm{wt}\, X - \mathrm{wt}\, a'')|_{\mathfrak{a}} > \nu + w_i^{-1}\, \mathrm{wt}\, X|_{\mathfrak{a}}.
\end{aligned}$$

So we have

$$\begin{aligned}
&\delta_i(b_{\mathbf{k}}^{(p)}, R_i'((a_{\mathbf{k}}^{(p)})^\vee)(f f_{\mathbf{k}} \eta_i^{-1}), \mathrm{Ad}(w_i)^{-1} c_{\mathbf{k}}^{(p)} u') \\
&\quad - \delta_i(b_{\mathbf{k}}^{(p)}, R_i'((a_{\mathbf{k}}^{(p)})^\vee)(f f_{\mathbf{k}}), \mathrm{Ad}(w_i)^{-1} c_{\mathbf{k}}^{(p)} u')\widetilde{\eta}_i^{-1} \in \sum_{\nu' > \nu} V(\nu' + w_i^{-1}\, \mathrm{wt}\, X|_{\mathfrak{a}})\widetilde{\eta}_i^{-1}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&X\delta_i(E, f\eta_i^{-1}, u') + \sum_{\nu'>\nu} V(\nu' + w_i^{-1}\operatorname{wt} X|_{\mathfrak{a}})\widetilde{\eta_i}^{-1} \\
&\in \sum_{\mathbf{k},p} \delta_i(b_{\mathbf{k}}^{(p)}, R_i'((a_{\mathbf{k}}^{(p)})^{\vee})(ff_{\mathbf{k}}), \operatorname{Ad}(w_i)^{-1}(c_{\mathbf{k}}^{(p)})u')\widetilde{\eta_i}^{-1} \\
&= \sum_{\mathbf{k},p} \delta_i(a_{\mathbf{k}}^{(p)} b_{\mathbf{k}}^{(p)} c_{\mathbf{k}}^{(p)}, ff_{\mathbf{k}}, u')\widetilde{\eta_i}^{-1} + \sum_{\nu'>\nu} V(\nu' + w_i^{-1}\operatorname{wt} X|_{\mathfrak{a}})\widetilde{\eta_i}^{-1} \\
&= \sum_{\mathbf{k}} \delta_i(\operatorname{ad}(e)^{\mathbf{k}}(X)E, ff_{\mathbf{k}}, u')\widetilde{\eta_i}^{-1} + \sum_{\nu'>\nu} V(\nu' + w_i^{-1}\operatorname{wt} X|_{\mathfrak{a}})\widetilde{\eta_i}^{-1} \\
&= (X\delta_i(E, f, u'))\widetilde{\eta_i}^{-1} + \sum_{\nu'>\nu} V(\nu' + w_i^{-1}\operatorname{wt} X|_{\mathfrak{a}})\widetilde{\eta_i}^{-1}. \qquad \square
\end{aligned}
$$

Proposition 8.7. *Let $\widetilde{\mu} \in (\mathfrak{h} \cap \mathfrak{m})^*$ be the infinitesimal character of σ. Assume that $I_i/I_{i-1} \neq 0$ and for all $\widetilde{w} \in \widetilde{W}$,*

$$\lambda - \widetilde{w}(\lambda + \widetilde{\mu})|_{\mathfrak{a}} \notin \mathbb{Z}_{\leq 0}((\Sigma^+ \setminus \Sigma_M^+) \cap w_i^{-1}\Sigma^+)|_{\mathfrak{a}} \setminus \{0\}.$$

Then

$$\operatorname{Wh}_\eta(I_i') = \{(\eta_i^{-1} \otimes u')\delta_i \mid u' \in \operatorname{Wh}_{w_i^{-1}\eta}((\sigma \otimes e^{\lambda+\rho})')\}.$$

Proof. Let $x = \sum_s \delta_i(E_s, f_s\eta_i^{-1}, u_s') \in \operatorname{Wh}_\eta(I_i')$ where $E_s \in U(\operatorname{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0})$, $f_s \in \mathcal{P}(O_i)$ and $u_s' \in J'_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$. By Lemma 3.5, we have $(X - \eta(X))x = \sum_s \delta_i(E_s, L(X)(f_s)\eta_i^{-1}, u_s')$ for $X \in \operatorname{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0$. Hence, we may assume $f_s = 1$.

Let $z \in Z(\mathfrak{g})$. Since $J_\eta'(I(\sigma, \lambda))$ has infinitesimal character $-(\lambda+\widetilde{\mu})$, I_i' has the same character. Let $\chi(z)$ be a complex number such that z acts by $\chi(z)$ on I_i'. Take E_s and u_s' such that E_s are $\mathfrak{a}_0$-weight vectors and $\{E_s\}$ is linearly independent. Let $\nu = \min\{w_i^{-1}\operatorname{wt} E_s|_{\mathfrak{a}}\}_s$.

Since $\gamma_2\gamma_1(z) - \gamma_3\gamma_2\gamma_1(z) \in (\operatorname{Ad}(w_i)\mathfrak{n})U(\operatorname{Ad}(w_i)\mathfrak{p})$, we have

$$\gamma_2\gamma_1(z)x - \gamma_3\gamma_2\gamma_1(z)x \in \sum_{\nu'>\nu} V(\nu')\widetilde{\eta_i}^{-1}$$

by Lemmas 8.5 and 8.6. By Lemma 8.6,

$$\gamma_3\gamma_2\gamma_1(z)x \in \Big(\gamma_3\gamma_2\gamma_1(z) \sum_{w_i^{-1}\operatorname{wt} E_s|_{\mathfrak{a}}=\nu} \delta_i(E_s, 1, u_s')\Big)\widetilde{\eta_i}^{-1} + \sum_{\nu'>\nu} V(\nu')\widetilde{\eta_i}^{-1}.$$

Therefore,

$$
\begin{aligned}
\chi(z)x &= zx = \gamma_2\gamma_1(z)x \\
&\in \Big(\gamma_3\gamma_2\gamma_1(z) \sum_{w_i^{-1}\operatorname{wt} E_s|_{\mathfrak{a}}=\nu} \delta_i(E_s, 1, u_s')\Big)\widetilde{\eta_i}^{-1} + \sum_{\nu'>\nu} V(\nu')\widetilde{\eta_i}^{-1}.
\end{aligned}
$$

By Lemma 8.6 ($X=1$), we have

$$x \in \sum_{w_i^{-1}\,\mathrm{wt}\,E_s|_{\mathfrak{a}}=\nu} \delta_i(E_s,1,u'_s)\widetilde{\eta}_i^{-1} + \sum_{\nu'>\nu} V(\nu')\widetilde{\eta}_i^{-1}.$$

Hence

$$\Bigl((\chi(z)-\gamma_3\gamma_2\gamma_1(z))\Bigl(\sum_{w_i^{-1}\,\mathrm{wt}\,E_s|_{\mathfrak{a}}=\nu} \delta_i(E_s,1,u'_s)\Bigr)\Bigr)\widetilde{\eta}_i^{-1} \in \sum_{\nu'>\nu} V(\nu')\widetilde{\eta}_i^{-1}.$$

By Lemma 8.5, the left hand side is in $V(\nu)\widetilde{\eta}_i^{-1}$. Hence

$$(\chi(z)-\gamma_3\gamma_2\gamma_1(z))\delta_i(E_s,1,u'_s)=0$$

for all s such that $w_i^{-1}\,\mathrm{wt}\,E_s|_{\mathfrak{a}}=\nu$. By the same calculation as in the proof of Lemma 2.8, $H\delta_i(E_s,1,u'_s)=(-w_i\lambda+\mathrm{wt}\,E_s+\rho_0)(H)\delta_i(E_s,1,u'_s)$ for $H\in\mathrm{Ad}(w_i)\mathfrak{a}$. By Lemma 8.2, there exists $\widetilde{w}\in\widetilde{W}$ such that $-\widetilde{w}(\lambda+\widetilde{\mu})|_{\mathrm{Ad}(w_i)\mathfrak{a}}=-w_i\lambda+\mathrm{wt}\,E_s$. Then $\lambda-w_i^{-1}\widetilde{w}(\lambda+\widetilde{\mu})|_{\mathfrak{a}}=w_i^{-1}\,\mathrm{wt}\,E_s|_{\mathfrak{a}}\in\mathbb{Z}_{\le 0}((\Sigma^+\setminus\Sigma_M^+)\cap w_i^{-1}\Sigma^+)|_{\mathfrak{a}}$. By assumption, $w_i^{-1}\,\mathrm{wt}\,E_s|_{\mathfrak{a}}=0$, i.e., $E_s\in\mathbb{C}$. Hence, we may assume that x has the form $x=\delta_i(1,\eta_i^{-1},u')+\sum_{s\ge 2}\delta_i(E_s,\eta_i^{-1},u'_s)$ where $E_s\notin\mathbb{C}$ for all $s\ge 2$.

Take $X\in\mathfrak{n}_0\cap\mathrm{Ad}(w_i)\mathfrak{m}$. Then by Lemmas 3.5 and 8.6,

$$0=(X-\eta(X))x\in\delta_i(1,1,(\mathrm{Ad}(w_i)^{-1}X-\eta(X))u')\widetilde{\eta}_i^{-1}+\sum_{\nu'>0}V(\nu')\widetilde{\eta}_i^{-1}.$$

Therefore, $\delta_i(1,1,(\mathrm{Ad}(w_i)^{-1}X-\eta(X))u')=0$. Hence $u'\in\mathrm{Wh}_{w_i^{-1}\eta}((\sigma\otimes e^{\lambda+\rho})')$. This implies that $x-\delta_i(1,\eta_i^{-1},u')\in\mathrm{Wh}_\eta(I'_i)$. If $x-\delta_i(1,\eta_i^{-1},u')\neq 0$, then $\min\{w_i^{-1}\,\mathrm{wt}\,E_s|_{\mathfrak{a}}\}_{s\ge 2}=0$ by the above argument. This is a contradiction. □

Theorem 8.8. *Assume that for all $w\in W(M)$ with $w(\Sigma^+\setminus\Sigma_M^+)\cap\operatorname{supp}\eta=\emptyset$, the following two conditions hold:*

(a) *$\langle\check{\alpha},\lambda+\nu\rangle\notin\mathbb{Z}_{\le 0}$ for each exponent ν of σ and $\alpha\in\Sigma^+\setminus w^{-1}(\Sigma^+\cup\Sigma_\eta^-)$.*

(b) *$\lambda-\widetilde{w}(\lambda+\widetilde{\mu})|_{\mathfrak{a}}\notin\mathbb{Z}_{\le 0}((\Sigma^+\setminus\Sigma_M^+)\cap w^{-1}\Sigma^+)|_{\mathfrak{a}}\setminus\{0\}$ for all $\widetilde{w}\in\widetilde{W}$, where $\widetilde{\mu}$ is the infinitesimal character of σ.*

Moreover, assume that η is unitary. Then

$$\dim\mathrm{Wh}_\eta(I(\sigma,\lambda)')=\sum_{w\in W(M),\,w(\Sigma^+\setminus\Sigma_M^+)\cap\operatorname{supp}\eta=\emptyset}\dim\mathrm{Wh}_{w^{-1}\eta}((\sigma\otimes e^{\lambda+\rho})').$$

Remark 8.9. We have $w(\Sigma^+\setminus\Sigma_M^+)\cap\operatorname{supp}\eta=\emptyset$ if and only if η is trivial on $wNw^{-1}\cap N_0$. About this condition, see Theorem 4.7.

Proof of Theorem 8.8. By the exact sequence $0 \to I_{i-1} \to I_i \to I_i/I_{i-1} \to 0$, we have $0 \to \mathrm{Wh}_\eta(I_{i-1}) \to \mathrm{Wh}_\eta(I_i) \to \mathrm{Wh}_\eta(I_i/I_{i-1})$. By Proposition 8.7, it is sufficient to prove that the last map $\mathrm{Wh}_\eta(I_i) \to \mathrm{Wh}_\eta(I_i/I_{i-1})$ is surjective.

Take $x \in \mathrm{Wh}_\eta(I_i') \simeq \mathrm{Wh}_\eta(I_i/I_{i-1})$. Then x is $(\eta_i^{-1} \otimes u')\delta_i$ for some $u' \in \mathrm{Wh}_{w_i^{-1}\eta}(\sigma \otimes e^{\lambda+\rho})$ by Proposition 8.7. By Lemma 4.5, there exists a distribution $x_t \in I_i(\lambda + t\rho)$ with a meromorphic parameter t such that $x_t|_{U_i}$ is holomorphic and $(x_t|_{U_i})|_{t=0} = x$. Moreover, $(X - \eta(X))x_t = 0$ for $X \in \mathfrak{n}_0$. By Proposition 4.4 and (a), the distribution x_t is holomorphic at $t = 0$. (See the proof of Lemma 4.5.) Hence $x_0|_{U_i} = x$, so $\mathrm{Wh}_\eta(I_i) \to \mathrm{Wh}_\eta(I_i/I_{i-1})$ is surjective. □

Next we consider the module $\mathrm{Wh}_\eta((I(\sigma,\lambda)_{K\text{-finite}})^*)$.

Lemma 8.10. *Let V be an object of the category $\mathcal{O}'_{P_0}$. Then $C(H^0(\mathfrak{n}_\eta, V)) = H^0(\mathfrak{n}_\eta, C(V))$ where $H^0(\mathfrak{n}_\eta, V) = \{v \in V \mid \mathfrak{n}_\eta v = 0\}$ is the 0-th $\mathfrak{n}_\eta$-cohomology.*

Proof. This follows from (we use Proposition 7.2(4))

$$\begin{aligned} H^0(\mathfrak{n}_\eta, C(V)) &= H^0(\mathfrak{n}_\eta, D'(V)^*) = (D'(V)/\mathfrak{n}_\eta D'(V))^* \\ &= CD'(D'(V)/\mathfrak{n}_\eta D'(V)) = C(H^0(\mathfrak{n}_\eta, D'(V)^*)_{\mathfrak{h}\text{-finite}}) \\ &= C(H^0(\mathfrak{n}_\eta, D'D'(V))) = C(H^0(\mathfrak{n}_\eta, V)). \end{aligned}$$ □

By Proposition 7.1, we have

$$\mathrm{Wh}_\eta((I(\sigma,\lambda)_{K\text{-finite}})^*) = \mathrm{Wh}_\eta(C(J^*(I(\sigma,\lambda)))).$$

By the above lemma,

$$\begin{aligned} \mathrm{Wh}_\eta(C(J^*(I(\sigma,\lambda)))) &= \mathrm{Wh}_{\eta|_{\mathfrak{l}_\eta\cap\mathfrak{n}_0}}(H^0(\mathfrak{n}_\eta, C(J^*(I(\sigma,\lambda))))) \\ &= \mathrm{Wh}_{\eta|_{\mathfrak{l}_\eta\cap\mathfrak{n}_0}}(C(H^0(\mathfrak{n}_\eta, J^*(I(\sigma,\lambda))))). \end{aligned}$$

Since $\eta|_{\mathfrak{l}_\eta\cap\mathfrak{n}_0}$ is nondegenerate, a theorem of Lynch [Lyn79] shows that the dimension of the above space is determined by the character of $H^0(\mathfrak{n}_\eta, J^*(I(\sigma,\lambda)))$. To calculate $H^0(\mathfrak{n}_\eta, J^*(I(\sigma,\lambda)))$, we use the following lemma.

Lemma 8.11. *Let $e_1, \ldots, e_l$ be a basis of $\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0$ such that each e_s is a root vector and $\bigoplus_{s\le t-1} \mathbb{C}e_s$ is an ideal of $\bigoplus_{s\le t} \mathbb{C}e_s$. In $S_{w_i,0}$, where 0 is the trivial representation of $\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0$, we have the following formulas:*

(1) *For all $t = 1, \ldots, l$,*

$$\begin{aligned} &e_t(e_1^{-1} \cdots e_{t-1}^{-1} e_t^{-(k_t+1)} e_{t+1}^{-(k_{t+1}+1)} \cdots e_l^{-(k_l+1)}) \\ &\qquad = e_1^{-1} \cdots e_{t-1}^{-1} e_t^{-k_t} e_{t+1}^{-(k_{t+1}+1)} \cdots e_l^{-(k_l+1)}. \end{aligned}$$

(2) *Fix* $t \in \{1, \dots, l\}$ *such that* $e_t \in \mathfrak{n}_\eta$. *Assume that* $k_s = 0$ *for all* $s < t$ *such that* $e_s \in \mathfrak{n}_\eta$. *Then*

$$e_t(e_1^{-(k_1+1)} \cdots e_l^{-(k_l+1)}) = e_1^{-(k_1+1)} \dots e_{t-1}^{-(k_{t-1}+1)} e_t^{-k_t} e_{t+1}^{-(k_t+1)} \dots e_l^{-(k_l+1)}.$$

(3) $X(e_1^{-1} \cdots e_l^{-1}) = (e_1^{-1} \cdots e_l^{-1})X$ *for* $X \in \mathrm{Ad}(w_i)\mathfrak{m} \cap \mathfrak{n}_0$.

Proof. Let α_s be the restricted root corresponding to e_s.

(1) It is sufficient to prove the equality $e_t(e_1^{-1} \cdots e_{t-1}^{-1}) = (e_1^{-1} \cdots e_{t-1}^{-1})e_t$ in $S_{e_1} \otimes_{U(\mathfrak{g})} \cdots \otimes_{U(\mathfrak{g})} S_{e_{t-1}}$. Since $\bigoplus_{s=1}^{t-1} \mathbb{C}e_s$ is an ideal of $\bigoplus_{s=1}^{t} \mathbb{C}e_s$, we have

$$e_t(e_1^{-1} \cdots e_{t-1}^{-1}) - (e_1^{-1} \cdots e_{t-1}^{-1})e_t \in \bigoplus_{p_s \geq 0} \mathbb{C}e_1^{-(p_1+1)} \cdots e_{t-1}^{-(p_{t-1}+1)}.$$

The $\mathfrak{a}_0$-weight of the left hand side is $-\alpha_1 - \cdots - \alpha_{t-1} + \alpha_t$. However, the set of $\mathfrak{a}_0$-weights of the right hand side is $\{-(p_1+1)\alpha_1 - \cdots - (p_{t-1}+1)\alpha_{t-1} \mid p_s \in \mathbb{Z}_{\geq 0}\}$. Hence each $\mathfrak{a}_0$-weight appearing in the right hand side is less than that of the left hand side. This implies $e_t(e_1^{-1} \dots e_{t-1}^{-1}) - (e_1^{-1} \dots e_{t-1}^{-1})e_t = 0$.

(2) We will prove $e_t(e_1^{-(k_1+1)} \cdots e_{t-1}^{-(k_{t-1}+1)}) = (e_1^{-(k_1+1)} \dots e_{t-1}^{-(k_{t-1}+1)})e_t$ in $S_{e_1} \otimes_{U(\mathfrak{g})} \cdots \otimes_{U(\mathfrak{g})} S_{e_{t-1}}$. As in the proof of (1), we have

$$e_t(e_1^{-(k_1+1)} \cdots e_{t-1}^{-(k_{t-1}+1)}) - (e_1^{-(k_1+1)} \cdots e_{t-1}^{-(k_{t-1}+1)})e_t \in \bigoplus_{p_s \geq 0} \mathbb{C}e_1^{-(p_1+1)} \cdots e_{t-1}^{-(p_{t-1}+1)}.$$

The $\mathfrak{a}_\eta$-weight of the left hand side is $\sum_{e_s \in \mathfrak{n}_\eta,\, s<t} -\alpha_s + \alpha_t$. However, the set of $\mathfrak{a}_\eta$-weights of the right hand side is $\{\sum_{e_s \in \mathfrak{n}_\eta,\ s<t} -(p_s+1)\alpha_s \mid p_s \in \mathbb{Z}_{\geq 0}\}$. Hence each $\mathfrak{a}_\eta$-weight appearing in the right hand side is less than that of the left hand side. This implies the assertion.

(3) We may assume X is a restricted root vector. Let α be the restricted root corresponding to X. Since X normalizes $\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0$, we have

$$X(e_1^{-1} \cdots e_l^{-1}) - (e_1^{-1} \cdots e_l^{-1})X \in \bigoplus_{p_s \geq 0} \mathbb{C}e_1^{-(p_1+1)} \cdots e_l^{-(p_l+1)}.$$

Then $X(e_1^{-1} \cdots e_l^{-1}) - (e_1^{-1} \cdots e_l^{-1})X$ has the $\mathfrak{a}_0$-weight $-(\alpha_1 + \cdots + \alpha_s) + \alpha$. However, $e_1^{-(p_1+1)} \cdots e_l^{-(p_l+1)}$ has the $\mathfrak{a}_0$-weight $-((p_1+1)\alpha_1 + \cdots + (p_l+1)\alpha_l)$. If $-((p_1+1)\alpha_1 + \cdots + (p_l+1)\alpha_l) = -(\alpha_1 + \cdots + \alpha_s) + \alpha$, then $((p_1+1)\alpha_1 + \cdots + (p_l+1)\alpha_l)|_{\mathrm{Ad}(w_i)\mathfrak{a}} = (\alpha_1 + \cdots + \alpha_l)|_{\mathrm{Ad}(w_i)\mathfrak{a}}$. Hence $p_1 = \cdots = p_l = 0$. Therefore, $\alpha = 0$, a contradiction. Hence $X(e_1^{-1} \cdots e_l^{-1}) - (e_1^{-1} \cdots e_l^{-1})X = 0$. □

Lemma 8.12. *Let* $e_1, \dots, e_l$ *be a basis of* $\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \mathfrak{n}_0$ *such that each* e_s *is a root vector and* $\bigoplus_{s \leq t-1} \mathbb{C}e_s$ *is an ideal of* $\bigoplus_{s \leq t} \mathbb{C}e_s$. *Let* V *be a* $U(\mathfrak{m} \oplus \mathfrak{a})$*-representation. Regard* V *as a* $\mathfrak{p}$*-representation by* $\mathfrak{n}V = 0$. *By Lemma 6.3,*

$T_{w_i}(U(\mathfrak{g})\otimes_{U(\mathfrak{p})}V) \simeq (\bigoplus_{k_s\geq 0}\mathbb{C}e_1^{-(k_1+1)}\cdots e_l^{-(k_l+1)})\otimes U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\overline{\mathfrak{n}_0})\otimes w_iV$. *Then* $\{v\in e_1^{-1}\cdots e_l^{-1}\otimes 1\otimes w_iV \mid \mathfrak{n}_\eta v=0\} = e_1^{-1}\cdots e_l^{-1}\otimes 1\otimes H^0(\mathrm{Ad}(w_i)\mathfrak{m}\cap\mathfrak{n}_\eta, w_iV)$.

Proof. Take $v = e_1^{-1}\cdots e_l^{-1}\otimes 1\otimes v_0 \in H^0(\mathfrak{n}_\eta, T_{w_i}(U(\mathfrak{g})\otimes_{U(\mathfrak{p})}V))$. Then for $X\in \mathrm{Ad}(w_i)\mathfrak{m}\cap\mathfrak{n}_\eta$ we have $X(e_1^{-1}\cdots e_l^{-1}\otimes 1\otimes v_0) = 0$. By Lemma 8.11, we have $e_1^{-1}\cdots e_l^{-1}\otimes 1\otimes Xv_0 = 0$. Hence $Xv_0 = 0$. □

By the definition of the Harish-Chandra homomorphism, we get the following.

Lemma 8.13. *Let* $\mathfrak{q}$ *be a parabolic subalgebra of* $\mathfrak{g}$ *containing* $\mathfrak{h}\oplus\mathfrak{u}_0$. *Take the Levi decomposition* $\mathfrak{l}_\mathfrak{q}\oplus\mathfrak{u}_\mathfrak{q}$ *of* $\mathfrak{q}$ *such that* $\mathfrak{h}\subset\mathfrak{l}_\mathfrak{q}$. *Let* $\widetilde{W_{\mathfrak{l}_\mathfrak{q}}}\subset\widetilde{W}$ *be the Weyl group of* $\mathfrak{l}_\mathfrak{q}$, *and* V *a* $\mathfrak{g}$*-module with infinitesimal character* $\widetilde{\mu}$. *Put* $V' = H^0(\mathfrak{u}_\mathfrak{q}, V)$ *and* $\widetilde{\rho_{\mathfrak{u}_\mathfrak{q}}}(H) = (1/2)\,\mathrm{Tr}\,\mathrm{ad}(H)|_{\mathfrak{u}_\mathfrak{q}}$ *for* $H\in\mathfrak{h}$. *Then* V' *is* $\mathfrak{l}_\mathfrak{q}$*-stable and* $V' = \bigoplus_{\widetilde{w}\in\widetilde{W_{\mathfrak{l}_\mathfrak{q}}}\backslash\widetilde{W}}(V')_{[\widetilde{w}\widetilde{\mu}-\widetilde{\rho_{\mathfrak{u}_\mathfrak{q}}}]}$ *where* $(V')_{[\widetilde{w}\widetilde{\mu}-\widetilde{\rho_{\mathfrak{u}_\mathfrak{q}}}]}$ *is the maximal* $\mathfrak{l}_\mathfrak{q}$*-submodule whose infinitesimal character is* $\widetilde{w}\widetilde{\mu}-\widetilde{\rho_{\mathfrak{u}_\mathfrak{q}}}$. *In particular, for every* $\mathfrak{l}_\mathfrak{q}$*-submodule* V'' *of* V', *all highest weights of* V'/V'' *belong to* $\{\widetilde{w}\widetilde{\mu}-\widetilde{\rho}\mid\widetilde{w}\in\widetilde{W}\}$.

The following lemma is well-known.

Lemma 8.14. *Let* $V\in\mathcal{O}'_{P_0}$. *Assume that* V *has infinitesimal character* $\widetilde{\lambda}\in\mathfrak{h}^*$. *Then all* $\mathfrak{h}$*-weights appearing in* V *belong to* $\{\widetilde{w}\widetilde{\lambda}-\widetilde{\rho}-\alpha\mid\widetilde{w}\in\widetilde{W},\ \alpha\in\mathbb{Z}_{\geq 0}\Delta^+\}$.

Take a filtration $\widetilde{I}_i\subset J^*_\eta(I(\sigma,\lambda))$ as in Theorem 7.5. Now we determine the dimension of the space of Whittaker vectors of $\widetilde{I}_i/\widetilde{I_{i-1}}$ under some conditions.

Lemma 8.15. *Let* $\widetilde{\mu}$ *be the infinitesimal character of* σ. *Assume that* $(\lambda+\widetilde{\mu}) - \widetilde{w}(\lambda+\widetilde{\mu})\notin\mathbb{Z}\Delta$ *for all* $\widetilde{w}\in\widetilde{W}\setminus\widetilde{W_M}$. *Then*

$$\dim\mathrm{Wh}_\eta(T_{w_i}(U(\mathfrak{g})\otimes_{U(\mathfrak{p})}J^*(\sigma\otimes e^{\lambda+\rho}))) = \dim\mathrm{Wh}_{w_i^{-1}\eta}((\sigma_{M\,\cap\,K\text{-finite}})^*).$$

Proof. Put $V = T_{w_i}(U(\mathfrak{g})\otimes_{U(\mathfrak{p})}J^*(\sigma\otimes e^{\lambda+\rho}))$. Let $e_1,\dots,e_l$ be a basis of $\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\mathfrak{n}_0$ such that $\bigoplus_{s\leq t-1}\mathbb{C}e_s$ is an ideal of $\bigoplus_{s\leq t}\mathbb{C}e_s$. Moreover, assume that each e_s is a root vector. For $\mathbf{k}=(k_1,\dots,k_l)\in\mathbb{Z}^l$, put $e^{\mathbf{k}} = e_1^{k_1}\cdots e_l^{k_l}$. Set $\mathbf{1} = (1,\dots,1)\in\mathbb{Z}^l$. Then

$$V = \bigoplus_{\mathbf{k}\in\mathbb{Z}^l_{\geq 0}}\mathbb{C}e^{-(\mathbf{k}+\mathbf{1})}\otimes U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\overline{\mathfrak{n}_0})\otimes w_iJ^*(\sigma\otimes e^{\lambda+\rho}).$$

Put

$$V' = \bigoplus_{\mathbf{k}\in\mathcal{A}}\mathbb{C}e^{-(\mathbf{k}+\mathbf{1})}\otimes U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\overline{\mathfrak{n}_0}\cap\mathfrak{m}_\eta)\otimes H^0(\mathfrak{m}\cap\mathfrak{n}_\eta, w_iJ^*(\sigma\otimes e^{\lambda+\rho}))$$

where $\mathcal{A} = \{(k_1,\dots,k_l)\in\mathbb{Z}^l_{\geq 0}\mid \text{if } e_s\in\mathfrak{n}_\eta \text{ then } k_i = 0\}$. It is easy to see that V' is $\mathfrak{m}_\eta\oplus\mathfrak{a}_\eta$-stable. By Lemma 8.11, $V'\subset H^0(\mathfrak{n}_\eta, V)$. We first prove that $V' = H^0(\mathfrak{n}_\eta, V)$.

It is sufficient to prove that there exists no highest weight vector in $H^0(\mathfrak{n}_\eta, V)/V'$. Let $v \in H^0(\mathfrak{n}_\eta, V)$ be such that $(\mathfrak{m}_\eta \cap \mathfrak{u})v \in V'$.

First, we prove that $v \in e^{-\mathbf{1}} \otimes U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0}) \otimes w_i J^*(\sigma \otimes e^{\lambda+\rho}) + V'$. Take $y_{\mathbf{k}} \in U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0}) \otimes w_i J^*(\sigma \otimes e^{\lambda+\rho})$ such that $v = \sum_{\mathbf{k}} e^{-(\mathbf{k}+\mathbf{1})} \otimes y_{\mathbf{k}}$. We prove that if $k_t \neq 0$ and $e_t \in \mathfrak{n}_\eta$ then $y_{\mathbf{k}} = 0$ by induction on t where $\mathbf{k} = (k_1, \dots, k_l)$. Put $\mathbf{1}_t = (\delta_{st})_{1 \le s \le l} \in \mathbb{Z}^l$ (δ_{st} is Kronecker's delta). By inductive hypothesis, for $s < t$ such that $e_s \in \mathfrak{n}_\eta$, if $y_{\mathbf{k}} \neq 0$ then $k_s = 0$. By Lemma 8.11(2), we have $e_t v = \sum_{\mathbf{k} \in \mathbb{Z}^l_{\ge 0}} e^{-(\mathbf{k}+\mathbf{1})+\mathbf{1}_t} \otimes y_{\mathbf{k}}$. Since $v \in H^0(\mathfrak{n}_\eta, V)$, we have $e_t v = 0$. Hence if $e^{-(\mathbf{k}+\mathbf{1})+\mathbf{1}_t} \neq 0$ then $y_{\mathbf{k}} = 0$. Since $e^{-(\mathbf{k}+\mathbf{1})+\mathbf{1}_t} = 0$ is equivalent to $k_t = 0$, $k_t \neq 0$ implies $y_{\mathbf{k}} = 0$.

We now prove that if $k_t \neq 0$ then $e^{-(\mathbf{k}+\mathbf{1})} \otimes y_{\mathbf{k}} \in V'$ by induction on t. If $e_t \in \mathfrak{n}_\eta$ then this claim is already proved. We may assume that $e_t \in \mathfrak{m}_\eta$. Hence $e_t V' \subset V'$. By inductive hypothesis, if $k_s \neq 0$ for some $s < t$ then $e^{-(\mathbf{k}+\mathbf{1})} \otimes y_{\mathbf{k}} \in V'$. Then $e_t v \in \sum_{\mathbf{k} \in \mathbb{Z}^l_{\ge 0}} e^{-(\mathbf{k}+\mathbf{1})+\mathbf{1}_t} \otimes y_{\mathbf{k}} + V'$ by Lemma 8.11(1). Since $e_t v \in V'$, we have $\sum_{\mathbf{k} \in \mathbb{Z}^l_{\ge 0}} e^{-(\mathbf{k}+\mathbf{1})+\mathbf{1}_t} \otimes y_{\mathbf{k}} \in V'$. By the definition of V', if $e^{-(\mathbf{k}+\mathbf{1})+\mathbf{1}_t} \neq 0$ then $e^{-(\mathbf{k}+\mathbf{1})} \otimes y_{\mathbf{k}} \in V'$. Notice that $e^{-(\mathbf{k}+\mathbf{1})+\mathbf{1}_t} \neq 0$ if and only if $k_t \neq 0$. Hence we get the claim.

We now prove $v \in V'$. We may assume that v is a weight vector with respect to $\mathfrak{h}$. We can take $\widetilde{w} \in \widetilde{W}$ such that $-\widetilde{w}(\lambda + \widetilde{\mu}) - \widetilde{\rho}$ is the $\mathfrak{h}$-weight of v by Lemma 8.13. Put $\widetilde{\rho_M} = \sum_{\alpha \in \Delta_M^+} (1/2)\alpha$. Since $J^*(\sigma \otimes e^{\lambda+\rho})$ has infinitesimal character $-(\widetilde{\mu} + \lambda + \rho)$, all $\mathfrak{h}$-weights appearing in $J^*(\sigma \otimes e^{\lambda+\rho})$ are contained in $\{-\widetilde{w}(\widetilde{\mu} + \lambda + \rho) - \widetilde{\rho_M} + \alpha \mid \widetilde{w} \in \widetilde{W_M},\ \alpha \in \mathbb{Z}\Delta_M\}$ by Lemma 8.14. Since $\rho \in \mathfrak{a}^*$, we have $\widetilde{w}\rho = \rho$ for $\widetilde{w} \in \widetilde{W_M}$. Hence $-\widetilde{w}\rho - \widetilde{\rho_M} = -\rho - \widetilde{\rho_M} = -\widetilde{\rho}$. Notice that $w_i\widetilde{\rho} - \widetilde{\rho} \in \mathbb{Z}\Delta$. Therefore all $\mathfrak{h}$-weights appearing in V belong to

$$\begin{aligned} &- w_i\widetilde{W_M}(\widetilde{\mu} + \lambda) - w_i\widetilde{\rho} + w_i\mathbb{Z}\Delta_M + \mathbb{Z}_{\ge 0}(w_i\Delta^- \cap \Delta^-) - \mathbb{Z}_{\ge 1}(w_i\Delta^- \cap \Delta^+) \\ &\qquad \subset -w_i\widetilde{W_M}(\widetilde{\mu} + \lambda) - \widetilde{\rho} + \mathbb{Z}\Delta \end{aligned}$$

by Lemma 6.3. This implies that for some $\widetilde{w'} \in \widetilde{W_M}$, we have $\widetilde{w}(\widetilde{\mu}+\lambda) - w_i\widetilde{w'}(\lambda+\widetilde{\mu}) \in \mathbb{Z}\Delta$. By assumption we have $\widetilde{w} \in w_i\widetilde{W_M}$. This implies $(\mathrm{wt}\, v)(\mathrm{Ad}(w_i)H) = -(\lambda(H) + w_i^{-1}\widetilde{\rho}(H))$ for all $H \in \mathfrak{a}$ where $\mathrm{wt}\, v$ is the $\mathfrak{h}$-weight of v.

Take $E_p \in U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}} \cap \overline{\mathfrak{n}_0})$ and $x_p \in w_i J^*(\sigma \otimes e^{\lambda+\rho})$ such that $v \in \sum_p e^{-\mathbf{1}} \otimes E_p \otimes x_p + V'$. We may assume that E_p and x_p are $\mathfrak{h}$-weight vectors. We denote their $\mathfrak{h}$-weights by $\mathrm{wt}\, E_p$ and $\mathrm{wt}\, x_p$. Fix $H \in \mathfrak{a}$. Then $\alpha(H) = 0$ for all $\alpha \in \Delta_M$. Since $\mathrm{wt}\, x_p \in -w_i(\widetilde{W_M}(\widetilde{\mu}+\lambda+\rho) - \widetilde{\rho_M} + \mathbb{Z}\Delta_M)$, $(\mathrm{wt}\, x_p)(\mathrm{Ad}(w_i)H) = -(\lambda+\rho)(H)$. Hence

$$\begin{aligned} (\mathrm{wt}\, v)(\mathrm{Ad}(w_i)H) &= (\mathrm{wt}(e^{-\mathbf{1}}) + \mathrm{wt}(E_p) + \mathrm{wt}(x_p))(\mathrm{Ad}(w_i)(H)) \\ &= (\mathrm{wt}(e^{-\mathbf{1}})(\mathrm{Ad}(w_i)H) + (\mathrm{wt}\, E_p)(\mathrm{Ad}(w_i)H) - (\lambda + \rho)(H). \end{aligned}$$

We calculate $\mathrm{wt}(e^{-\mathbf{1}})(\mathrm{Ad}(w_i)H)$. By definition,

$$\mathrm{wt}(e^{-\mathbf{1}})(\mathrm{Ad}(w_i)H) = -\operatorname{Tr}\operatorname{ad}(\mathrm{Ad}(w_i)H)|_{\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\mathfrak{n}_0}.$$

Since $\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\mathfrak{n}_0 = \mathrm{Ad}(w_i)\overline{\mathfrak{n}_0}\cap\mathfrak{n}_0$, we have

$$\begin{aligned}\operatorname{Tr}\operatorname{ad}(\mathrm{Ad}(w_i)H)|_{\mathrm{Ad}(w_i)\overline{\mathfrak{n}}\cap\mathfrak{n}_0} &= \operatorname{Tr}\operatorname{ad}(\mathrm{Ad}(w_i)H)|_{\mathrm{Ad}(w_i)\overline{\mathfrak{n}_0}\cap\mathfrak{n}_0}\\ &= \operatorname{Tr}\operatorname{ad}(H)|_{\mathrm{Ad}(w_i)^{-1}\mathfrak{n}_0\cap\overline{\mathfrak{n}_0}} = (-\widetilde{\rho}+w_i^{-1}\widetilde{\rho})(H).\end{aligned}$$

Since $H\in\mathfrak{a}$, $\widetilde{\rho}(H)=\rho(H)$. Hence

$$(\mathrm{wt}\, v)(\mathrm{Ad}(w_i)H) = (\mathrm{wt}\, E_p)(\mathrm{Ad}(w_i)H) - (\lambda + w_i^{-1}\widetilde{\rho})(H).$$

We have already proved that $(\mathrm{wt}\, v)(\mathrm{Ad}(w_i)H) = -(\lambda+w_i^{-1}\widetilde{\rho})(H)$. Therefore we get $(\mathrm{wt}\, E_p)(\mathrm{Ad}(w_i)H) = 0$ for all $H\in\mathfrak{a}$. Since $E_p\in U(\mathrm{Ad}(w_i)\overline{\mathfrak{n}})$, this implies $E_p\in\mathbb{C}$, i.e., there exist $v'\in e^{-\mathbf{1}}\otimes 1\otimes w_iJ^*(\sigma\otimes e^{\lambda+\rho})$ and $v''\in V'$ such that $v=v'+v''$. Therefore $\mathfrak{n}_\eta(v')=\mathfrak{n}_\eta(v-v'')=0$. Hence $v'\in V'$ by Lemma 8.12. Therefore $H^0(\mathfrak{n}_\eta, V)=V'$.

We now prove the lemma. For an $\mathfrak{m}_0\oplus\mathfrak{a}_0$-module τ and a subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ containing $\mathfrak{m}_0\oplus\mathfrak{a}_0$, put $M_{\mathfrak{c}}(\tau)=U(\mathfrak{c})\otimes_{U(\mathfrak{c}\cap\overline{\mathfrak{p}_0})}(\tau\otimes\rho')$ where $\mathfrak{c}\cap\overline{\mathfrak{n}_0}$ acts on $\tau\otimes\rho'$ trivially and $\rho'(H)=(1/2)(\mathrm{Tr}(\mathrm{ad}(H)|_{\mathfrak{c}\cap\overline{\mathfrak{n}_0}}))$ for $H\in\mathfrak{a}_0$.

We give some notation and facts about $\mathcal{O}'_{P_0}$. All facts are well-known. For $\widetilde{\lambda}\in\mathfrak{h}^*$ such that $\widetilde{\lambda}|_{\mathfrak{m}_0\cap\mathfrak{h}}$ is a regular dominant integral, let $\sigma_{M_0A_0,\widetilde{\lambda}}$ be the finite-dimensional representation of M_0A_0 with infinitesimal character $\widetilde{\lambda}$. Let L' be a Levi sugbgoup of a parabolic subgroup such that $M_0A_0\subset L'$. Let $\mathrm{ch}\, V_0$ be the character of $V_0\in\mathcal{O}'_{P_0\cap L',L'}$ and $K_0(\mathcal{O}'_{P_0\cap L',L'})$ the Grothendieck group of $\mathcal{O}'_{P_0\cap L',L'}$. Then we can define $\mathrm{ch}\, V_0$ for $V_0\in K_0(\mathcal{O}'_{P_0\cap L',L'})$ (namely, ch is additive) and $\mathrm{ch}\, V_0 = \mathrm{ch}\, V_1$ if and only if $V_0=V_1$ for $V_0,V_1\in K_0(\mathcal{O}'_{P_0\cap L',L'})$. A basis of $K_0(\mathcal{O}'_{\overline{P_0}\cap L',L'})$ is given by $\{M_{\mathfrak{l}}(\sigma_{M_0A_0,\widetilde{\lambda}})\}$. Let P'' be a parabolic subgroup of L' containing $P_0\cap L'$, L'' its Levi subgroup and $\mathfrak{n}''$ the nilpotent radical of the Lie algebra of P''. Then for $V_0\in\mathcal{O}'_{P_0\cap L',L'}$, we have $H^0(\mathfrak{n}'',V_0)\in\mathcal{O}'_{P_0\cap L'',L''}$.

By Remark 2.5, $\mathrm{Ad}(w_i)(\mathfrak{m}\cap\mathfrak{p}_0) = \mathrm{Ad}(w_i)\mathfrak{m}\cap\mathfrak{p}_0\subset\mathrm{Ad}(w_i)\mathfrak{m}\cap\mathfrak{p}_\eta$. Therefore, $\mathfrak{m}\cap\mathfrak{p}_0\subset\mathfrak{m}\cap\mathrm{Ad}(w_i)^{-1}\mathfrak{p}_\eta$. Hence $\mathfrak{m}\cap\mathrm{Ad}(w_i)^{-1}\mathfrak{p}_\eta$ is a parabolic subalgebra of $\mathfrak{m}$. Therefore, $H^0(\mathrm{Ad}(w_i)^{-1}\mathfrak{n}_\eta\cap\mathfrak{m}, J^*(\sigma\otimes e^{\lambda+\rho}))\in\mathcal{O}'_{P_0\cap M\cap w_i^{-1}M_\eta w_i, M\cap w_i^{-1}M_\eta w_i}$. Recall that we have a functor w_i. (It twists the action of $\mathfrak{g}$ by w_i.) Since $w_i(P_0\cap M\cap w_i^{-1}M_\eta w_i)w_i^{-1} = P_0\cap w_iMw_i^{-1}\cap M_\eta$ (Remark 2.5), we deduce that $w_i(\mathcal{O}'_{P_0\cap M\cap w_i^{-1}M_\eta w_i, M\cap w_i^{-1}M_\eta w_i}) = \mathcal{O}'_{P_0\cap w_iMw_i^{-1}\cap M_\eta, w_iMw_i^{-1}\cap M_\eta}$. (This follows from the definition.) Therefore,

$$H^0(\mathfrak{n}_\eta\cap\mathrm{Ad}(w_i)\mathfrak{m}, w_iJ^*(\sigma\otimes e^{\lambda+\rho}))\in\mathcal{O}'_{P_0\cap w_iMw_i^{-1}\cap M_\eta, w_iMw_i^{-1}\cap M_\eta}.$$

Hence we can take $c_{\widetilde{\lambda}}$ such that

$$\operatorname{ch} D'H^0(\mathfrak{n}_\eta \cap \operatorname{Ad}(w_i)\mathfrak{m}, w_i J^*(\sigma \otimes e^{\lambda+\rho})) = \sum_{\widetilde{\lambda}} c_{\widetilde{\lambda}} \operatorname{ch} M_{(\mathfrak{m}_\eta \cap \operatorname{Ad}(w_i)\mathfrak{m})+\mathfrak{a}_0}(\sigma_{M_0 A_0, \widetilde{\lambda}}).$$

Then it is straightforward to prove $\operatorname{ch} D'V' = \sum_{\widetilde{\lambda}} c_{\widetilde{\lambda}} \operatorname{ch} M_{\mathfrak{m}_\eta \oplus \mathfrak{a}_\eta}(\sigma_{M_0 A_0, \widetilde{\lambda}})$. By a result of Lynch [Lyn79], the functor $X \mapsto \operatorname{Wh}_{\eta|_{\mathfrak{m}_\eta \cap \mathfrak{n}_0}}(X^*)$ from the category $\mathcal{O}'_{\overline{P_0} \cap M_\eta, M_\eta}$ to the category of vector spaces is exact. Therefore,

$$\dim \operatorname{Wh}_{\eta|_{\mathfrak{m}_\eta \cap \mathfrak{n}_0}}(C(V')) = \sum_{\widetilde{\lambda}} c_{\widetilde{\lambda}} \dim \operatorname{Wh}_{\eta|_{\mathfrak{m}_\eta \cap \mathfrak{n}_0}}(M_{\mathfrak{m}_\eta \oplus \mathfrak{a}_\eta}(\sigma_{M_0 A_0, \widetilde{\lambda}})^*).$$

Lynch also proved $\dim \operatorname{Wh}_{\eta|_{\mathfrak{m}_\eta \cap \mathfrak{n}_0}}(M_{\mathfrak{m}_\eta}(\sigma_{M_0 A_0, \widetilde{\lambda}})^*) = \dim \sigma_{M_0 A_0, \widetilde{\lambda}}$. Therefore, by Lemma 8.10 and $V' = H^0(\mathfrak{n}_\eta, V)$,

$$\begin{aligned} \dim \operatorname{Wh}_\eta(\widetilde{I_i}/\widetilde{I_{i-1}}) &= \dim \operatorname{Wh}_\eta(C(V)) = \dim \operatorname{Wh}_{\eta|_{\mathfrak{m}_\eta \cap \mathfrak{n}_0}}(H^0(\mathfrak{n}_\eta, C(V))) \\ &= \dim \operatorname{Wh}_{\eta|_{\mathfrak{m}_\eta \cap \mathfrak{n}_0}}(C(V')) = \sum_{\widetilde{\lambda}} c_{\widetilde{\lambda}} \dim \sigma_{M_0 A_0, \widetilde{\lambda}}. \end{aligned}$$

By the same argument,

$$\begin{aligned} \sum_{\widetilde{\lambda}} c_{\widetilde{\lambda}} \dim \sigma_{M_0 A_0, \widetilde{\lambda}} &= \sum_{\widetilde{\lambda}} c_{\widetilde{\lambda}} \dim \operatorname{Wh}_{\eta|_{\mathfrak{m}_\eta \cap \operatorname{Ad}(w_i)\mathfrak{m} \cap \mathfrak{n}_0}}(M_{(\mathfrak{m}_\eta \cap \operatorname{Ad}(w_i)\mathfrak{m})+\mathfrak{a}_0}(\sigma_{M_0 A_0, \widetilde{\lambda}})^*) \\ &= \dim \operatorname{Wh}_{\eta|_{\mathfrak{m}_\eta \cap \operatorname{Ad}(w_i)\mathfrak{m} \cap \mathfrak{n}_0}}(CH^0(\mathfrak{n}_\eta \cap \operatorname{Ad}(w_i)\mathfrak{m}, w_i J^*(\sigma \otimes e^{\lambda+\rho}))) \\ &= \dim \operatorname{Wh}_{\eta|_{\operatorname{Ad}(w_i)\mathfrak{m} \cap \mathfrak{n}_0}}(C(w_i J^*(\sigma \otimes e^{\lambda+\rho}))) \\ &= \dim \operatorname{Wh}_{w_i^{-1}\eta}(C(J^*(\sigma \otimes e^{\lambda+\rho}))) \\ &= \dim \operatorname{Wh}_{w_i^{-1}\eta}((\sigma_{M \cap K\text{-finite}})^*). \end{aligned}$$

This implies the conclusion. ∎

Theorem 8.16. *Let $\widetilde{\mu}$ be an infinitesimal character of σ. Assume that $(\lambda + \widetilde{\mu}) - \widetilde{w}(\lambda + \widetilde{\mu}) \notin \mathbb{Z}\Delta$ for all $\widetilde{w} \in \widetilde{W} \setminus \widetilde{W_M}$. Then*

$$\dim \operatorname{Wh}_\eta((I(\sigma, \lambda)_{K\text{-finite}})^*) = \sum_{w \in W(M)} \dim \operatorname{Wh}_{w^{-1}\eta}((\sigma_{M \cap K\text{-finite}})^*).$$

Proof. Let I_i be the Bruhat filtration of $J'(I(\sigma, \lambda)) = J^*(I(\sigma, \lambda))$. Since all $\mathfrak{h}$-weights appearing in $I_i/I_{i-1} \simeq T_{w_i}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} J^*(\sigma \otimes e^{\lambda+\rho}))$ belong to $\{-w_i \widetilde{w}(\lambda + \widetilde{\mu}) - \widetilde{\rho} + \alpha \mid \widetilde{w} \in \widetilde{W_M},\ \alpha \in \Delta\}$, we have

$$\operatorname{wt}(I_i/I_{i-1}) \cap (\operatorname{wt}(I_j/I_{j-1}) + \mathbb{Z}\Delta) = \emptyset$$

if $i \neq j$, where $\operatorname{wt}(I_i/I_{i-1})$ is the set of $\mathfrak{h}$-weights in I_i/I_{i-1}. Therefore, the exact sequence $0 \to I_{i-1} \to I_i \to I_i/I_{i-1} \to 0$ splits by the block decomposition of $\mathcal{O}'_{P_0}$. Hence $J^*_\eta(I(\sigma,\lambda)) = \bigoplus_i \Gamma_\eta(C(T_{w_i}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} J^*(\sigma\otimes e^{\lambda+\rho}))))$. Therefore, the conclusion follows from Lemma 8.15. □

Finally we study the case where σ is finite-dimensional. Then $\mathfrak{m}\cap\mathfrak{n}_0$ acts on σ as nilpotent operators. Therefore, $\mathrm{Wh}_{w_i^{-1}\eta}(\sigma^*) \neq 0$ if and only if $w_i^{-1}\eta = 0$ on $\mathfrak{m}\cap\mathfrak{n}_0$.

Definition 8.17. Let $\Theta, \Theta_1, \Theta_2$ be subsets of Π.

(1) Put $W(\Theta) = \{w \in W \mid w(\Theta) \subset \Sigma^+\}$ and $\Sigma_\Theta = \mathbb{Z}\Theta \cap \Sigma$.

(2) Put $W(\Theta_1,\Theta_2) = \{w \in W(\Theta_1) \cap W(\Theta_2)^{-1} \mid w(\Sigma_{\Theta_1}) \cap \Sigma_{\Theta_2} = \emptyset\}$.

(3) Let W_Θ be the Weyl group of Σ_Θ.

Lemma 8.18. *Let Θ be the subset of Π corresponding to P.*

(1) $\#W(\operatorname{supp}\eta,\Theta) = \#\{w \in W(M) \mid w(\Sigma^+) \cap \Sigma^+_\eta = \emptyset\}$.

(2) $\#W(\operatorname{supp}\eta,\Theta) \times \#W_{\operatorname{supp}\eta} = \#\{w \in W(M) \mid \operatorname{supp}\eta \cap w(\Sigma^+_M) = \emptyset\}$.

Proof. (1) Put $\mathcal{W} = \{w \in W(M) \mid w(\Sigma^+) \cap \Sigma^+_\eta = \emptyset\}$. Let $w_{\eta,0}$ be the longest Weyl element of W_{M_η}. We will prove that the map $\mathcal{W} \to W(\operatorname{supp}\eta,\Theta)$ defined by $w \mapsto (w_{\eta,0}w)^{-1}$ is well-defined and bijective.

To prove that the map is well-defined, let $w \in \mathcal{W}$. The equality $w(\Sigma^+)\cap\Sigma^+_\eta = \emptyset$ implies $(w_{\eta,0}w)^{-1}(\Sigma^+_\eta) = w^{-1}(-\Sigma^+_\eta) \subset \Sigma^+$. Hence $(w_{\eta,0}w)^{-1} \in W(\operatorname{supp}\eta)$. Moreover, $w(\Sigma^+_M) \subset \Sigma^+$ and $w(\Sigma^+)\cap\Sigma^+_\eta = \emptyset$ imply that $w(\Sigma^+_M) \subset \Sigma^+ \cap (\Sigma\setminus\Sigma^+_\eta) = \Sigma^+\setminus\Sigma^+_\eta$. Hence $(w_{\eta,0}w)(\Sigma^+_M) \subset \Sigma^+\setminus\Sigma^+_\eta \subset \Sigma^+$. We have $(w_{\eta,0}w)^{-1} \in W(\Theta)^{-1}$. Finally $w(\Sigma^+_M) \subset \Sigma^+\setminus\Sigma^+_\eta$ implies $w(\Sigma_M) = w(\Sigma^+_M)\cup(-w(\Sigma^+_M)) \subset \Sigma\setminus\Sigma_\eta$. Hence $(w_{\eta,0}w)^{-1}\Sigma_\eta \cap \Sigma_M = w^{-1}\Sigma_\eta\cap\Sigma_M = \emptyset$.

Assume that $(w_{\eta,0}w)^{-1} \in W(\operatorname{supp}\eta,\Theta)$. From $(w_{\eta,0}w)^{-1}(\Sigma^+_\eta) \subset \Sigma^+$, we have $w^{-1}(\Sigma^-_\eta) \subset \Sigma^+$. Hence $\Sigma^+_\eta = -\Sigma^-_\eta \subset -w(\Sigma^+) = w(\Sigma^-)$. Thus $w(\Sigma^+)\cap\Sigma^+_\eta = \emptyset$. Since $(w_{\eta,0}w)^{-1}\Sigma_\eta\cap\Sigma_M = \emptyset$ we have $w(\Sigma_M)\cap\Sigma_\eta = \emptyset$. As $(w_{\eta,0}w)(\Sigma^+_M) \subset \Sigma^+$ and $w(\Sigma^+)\cap\Sigma^+_\eta = \emptyset$, it follows that $w(\Sigma^+_M) \subset w_{\eta,0}^{-1}(\Sigma^+)\cap(\Sigma\setminus\Sigma^-_\eta) = ((\Sigma^+\setminus\Sigma^+_\eta)\cup\Sigma^-_\eta)\cap(\Sigma\setminus\Sigma^-_\eta) = (\Sigma^+\setminus\Sigma^+_\eta)$. Consequently, $w \in W(M)$.

(2) Put $\mathcal{W} = \{w \in W(M) \mid \operatorname{supp}\eta \cap w(\Sigma^+_M) = \emptyset\}$. Define a map $W(\operatorname{supp}\eta,\Theta)\times W_{\operatorname{supp}\eta} \to \mathcal{W}$ by $(w_1,w_2) \mapsto w_2w_1^{-1}$. This map is injective since $W(\operatorname{supp}\eta,\Theta) \subset W(\operatorname{supp}\eta)$. We prove that it is well-defined and surjective. Since $w_1 \in W(\operatorname{supp}\eta,\Theta) \subset W(M)^{-1}$, $w_1^{-1}(\Sigma^+_M) = w_1^{-1}(\Sigma^+_M)\cap\Sigma^+$. As $w_1(\Sigma_\eta)\cap\Sigma_M = \emptyset$, we have $w_1^{-1}(\Sigma^+_M)\cap\Sigma^+ \subset (\Sigma\setminus\Sigma_\eta)\cap\Sigma^+ = \Sigma^+\setminus\Sigma^+_\eta$. Therefore $w_2w_1^{-1}(\Sigma^+_M) \subset \Sigma^+\setminus\Sigma^+_\eta$, so the map is well-defined. Next let $w \in \mathcal{W}$. Let $w_1 \in W(\operatorname{supp}\eta)^{-1}$ and $w_2 \in W_{\operatorname{supp}\eta}$ be such that $w = w_2w_1^{-1}$. Then $w_1^{-1}(\Sigma^+_M) = w_2^{-1}w(\Sigma^+_M) \subset w_2^{-1}(\Sigma^+\setminus\Sigma^+_\eta) = \Sigma^+\setminus\Sigma^+_\eta \subset \Sigma^+$. Hence $w_1 \in W(M)^{-1}$. Moreover, $w_1^{-1}(\Sigma^+_M) \subset \Sigma^+\setminus\Sigma^+_\eta$ implies $w_1^{-1}(\Sigma_M) \subset \Sigma\setminus\Sigma_\eta$. Hence $\Sigma_\eta\cap w_1^{-1}(\Sigma_M) = \emptyset$. Therefore, $w_1\Sigma_\eta\cap\Sigma_M = \emptyset$. This implies $w_1 \in W(\operatorname{supp}\eta,\Theta)$. □

Lemma 8.19. *Assume that σ is irreducible and finite-dimensional. Let $\widetilde{\mu}$ be the highest weight of σ and V the irreducible finite-dimensional representation of M_0A_0 with highest weight $\lambda+\widetilde{\mu}$. Then $\mathrm{Wh}_0(\sigma^*)\simeq V^*$ as M_0A_0-modules. In particular,* $\dim \mathrm{Wh}_0(\sigma')=\dim V$.

Proof. Let $\widetilde{w}_{M,0}$ be the longest element of $\widetilde{W_M}$. Then both sides have highest weight $-\widetilde{w}_{M,0}(\widetilde{\mu}+\lambda)$ and the spaces of highest weight vectors are 1-dimensional. □

As a corollary to Theorems 8.8 and 8.16, we have the following theorem announced by T. Oshima. Define $\widetilde{\rho_M}\in\mathfrak{h}^*$ by $\widetilde{\rho_M}=(1/2)\sum_{\alpha\in\Delta_M^+}\alpha$.

Theorem 8.20. *Assume that σ is the irreducible finite-dimensional representation of M with highest weight $\widetilde{\nu}$. Let $\dim_{M_0}(\lambda+\widetilde{\nu})$ be the dimension of the finite-dimensional irreducible representation of M_0A_0 with highest weight $\lambda+\widetilde{\nu}$.*

(1) *Assume that for all $w\in W$ such that $w(\Sigma^+\setminus\Sigma_M^+)\cap\operatorname{supp}\eta=\emptyset$ the following two conditions hold:*

  (a) $\langle\check{\alpha},\lambda+w_0\widetilde{\nu}\rangle\notin\mathbb{Z}_{\leq 0}$ *for all* $\alpha\in\Sigma^+\setminus w^{-1}(\Sigma_M^+\cup\Sigma_\eta^+)$.

  (b) $\lambda-\widetilde{w}(\lambda+\widetilde{\nu}+\widetilde{\rho_M})|_{\mathfrak{a}}\notin\mathbb{Z}_{\leq 0}((\Sigma^+\setminus\Sigma_M^+)\cap w^{-1}\Sigma^+)|_{\mathfrak{a}}\setminus\{0\}$ *for all* $\widetilde{w}\in\widetilde{W}$.

  Then
$$\dim \mathrm{Wh}_\eta(I(\sigma,\lambda)')=\#W(\operatorname{supp}\eta,\Theta)\times(\dim_{M_0}(\lambda+\widetilde{\nu})).$$

(2) *Assume that $(\lambda+\widetilde{\nu})-\widetilde{w}(\lambda+\widetilde{\nu})\notin\Delta$ for all $\widetilde{w}\in\widetilde{W}\setminus\widetilde{W_M}$. Then*
$$\dim \mathrm{Wh}_\eta((I(\sigma,\lambda)_{K\text{-finite}})^*)=\#W(\operatorname{supp}\eta,\Theta)\times\#W_{\operatorname{supp}\eta}\times\dim_{M_0}(\lambda+\widetilde{\nu}).$$

Proof. Recall that $\mathrm{Wh}_{w^{-1}\eta}(\sigma^*)\neq 0$ if and only if $w^{-1}\eta=0$ on $\mathfrak{m}\cap\mathfrak{n}_0$. This is equivalent to $\operatorname{supp}\eta\cap w(\Sigma_M^+)=\emptyset$.

(1) By Theorem 8.8, we have
$$\mathrm{Wh}_\eta(I(\sigma,\lambda)')=\sum_{w\in W(M),\,w(\Sigma^+\setminus\Sigma_M^+)\cap\operatorname{supp}\eta=\emptyset}\dim \mathrm{Wh}_{w^{-1}\eta}((\sigma\otimes e^{\lambda+\rho})').$$

Since σ is finite-dimensional, $(\sigma\otimes e^{\lambda+\rho})'=(\sigma\otimes e^{\lambda+\rho})^*$. Then by the above remark, $\mathrm{Wh}_{w^{-1}\eta}((\sigma\otimes e^{\lambda+\rho})^*)\neq 0$ if and only if $\operatorname{supp}\eta\cap w_i(\Sigma_M^+)=\emptyset$. Moreover, if $\operatorname{supp}\eta\cap w_i(\Sigma_M^+)=\emptyset$, then $\dim \mathrm{Wh}_\eta((\sigma\otimes e^{\lambda+\rho})^*)=\dim \mathrm{Wh}_0((\sigma\otimes e^{\lambda+\rho})^*)=\dim_{M_0}(\lambda+\widetilde{\nu})$ by Lemma 8.19. Hence we get

$$\begin{aligned}&\dim \mathrm{Wh}_\eta(I(\sigma,\lambda)')\times\dim_{M_0}(\lambda+\widetilde{\nu})\\&\quad=\#\{w\in W(M)\mid w(\Sigma\setminus\Sigma_M^+)\cap\operatorname{supp}\eta=\emptyset,\ w(\Sigma_M^+)\cap\operatorname{supp}\eta=\emptyset\}\\&\quad=\#\{w\in W(M)\mid w(\Sigma^+)\cap\operatorname{supp}\eta=\emptyset\}\times\dim_{M_0}(\lambda+\widetilde{\nu}).\end{aligned}$$

By the definition of Σ_η^+, we have $w(\Sigma^+)\cap\operatorname{supp}\eta=\emptyset$ if and only if $w(\Sigma^+)\cap\Sigma_\eta^+=\emptyset$. Hence we get (1) by Lemma 8.18(1).

(2) By the above argument, we have

$$\dim \mathrm{Wh}_\eta(I(\sigma,\lambda)') = \#\{w\in W(M) \mid w(\Sigma_M^+)\cap\operatorname{supp}\eta=\emptyset\}\times \dim_{M_0}(\lambda+\widetilde{\nu}).$$

Hence we get (2) by Lemma 8.18(2). □

Appendix A. C^∞-functions with values in Fréchet spaces

Appendix A.1. $\mathcal{L}$-distributions and tempered $\mathcal{L}$-distributions

Let M be a C^∞-manifold, V a Fréchet space and $\mathcal{L}$ a vector bundle on M with fibers V. We define the sheaf of $\mathcal{L}$-distributions as follows.

First we assume that $\mathcal{L}$ is trivial on M. Then the definition of $\mathcal{L}$-distributions is found in Kolk–Varadarajan [KV96]. (It is the continuous dual space of the space of C^∞-functions $G\to V$ with compact support.) It is easy to see that the spaces of $\mathcal{L}$-distributions form a sheaf on M.

In general, let $M=\bigcup_{\lambda\in\Lambda}U_\lambda$ be an open covering of M such that the vector bundle $\mathcal{L}$ is trivial on each U_λ. For an arbitrary open subset U of M, put

$$\mathcal{D}'(U,\mathcal{L})=\Big\{(x_\lambda)\in\prod_{\lambda\in\Lambda}\mathcal{D}'(U\cap U_\lambda,\mathcal{L})\ \Big|\ x_\lambda=x_{\lambda'} \text{ on } U_\lambda\cap U_{\lambda'}\Big\}.$$

This is independent of the choice of an open covering $\{U_\lambda\}$ and defines the sheaf of *$\mathcal{L}$-distributions* on M.

Let X be a compact C^∞-manifold such that M is an open dense submanifold of X. Assume that there exists a vector bundle on X whose restriction to M is $\mathcal{L}$. (We denote this vector bundle again by $\mathcal{L}$.) In this case, we define a subspace $\mathcal{T}(M,\mathcal{L})$ of $\mathcal{D}'(M,\mathcal{L})$ by

$$\mathcal{T}(M,\mathcal{L})=\{x\in\mathcal{D}'(M,\mathcal{L})\mid x=z|_M \text{ for some } z\in\mathcal{D}'(X,\mathcal{L})\}.$$

An element of $\mathcal{T}(M,\mathcal{L})$ is called a *tempered $\mathcal{L}$-distribution* (cf. [Sch66]).

Remark A.1. The author does not know whether this space depends on the choice of X or not. Hence, in this paper, we specify X when we use the notion of a tempered $\mathcal{L}$-distribution. For example, in the main part of this paper, we consider the space of tempered distributions on U_i (Section 2). In this case, we take G/P as X.

For a subset $M_0\subset M$, put $\mathcal{D}'_{M_0}(U,\mathcal{L})=\{x\in\mathcal{D}'(U,\mathcal{L})\mid \operatorname{supp}x\subset M_0\}$ and $\mathcal{T}_{M_0}(M,\mathcal{L})=\{x\in\mathcal{T}(M,\mathcal{L})\mid \operatorname{supp}x\subset M_0\}$. Assume that M_0 is a closed submanifold of M. Then dualizing the restriction map $C_c^\infty(M,\mathcal{L}) \to C_c^\infty(M_0,\mathcal{L})$, we have an injective map $\mathcal{D}'(M_0,\mathcal{L}) \to \mathcal{D}'_{M_0}(M,\mathcal{L})$. Via this map, we regard $\mathcal{D}'(M_0,\mathcal{L})$ as a subspace of $\mathcal{D}'_{M_0}(M,\mathcal{L})$.

Appendix A.2. $\mathcal{L}$-distributions with support in a subspace

Let M be the Euclidean space $\mathbb{R}^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n\}$ and M_0 the subspace $\mathbb{R}^{n-m}$ of M defined by $x_1 = \cdots = x_m = 0$. Assume that there exists a compact C^∞-manifold X which satisfies the condition of the previous section. Let $E_1,\dots,E_m$ be vector fields on M such that:

(1) $(E_i\varphi)|_{M_0} = (\frac{\partial}{\partial x_i}\varphi)|_{M_0}$ for all $\varphi \in C^\infty(M)$.
(2) The space $\sum_{i=1}^m \mathbb{C}E_i$ is a Lie algebra.

Set $D_i = \partial/\partial x_i$. Condition (1) implies that $D_iT = E_iT$ for all $T \in \mathcal{D}'(M_0,\mathcal{L})$. We define $U_n(E_1,\dots,E_m) = \sum_{k_1+\cdots+k_m\le n} \mathbb{C}E_1^{k_1}\cdots E_m^{k_m}$ and $U(E_1,\dots,E_m) = \sum_n U_n(E_1,\dots,E_m)$. Then the algebra $U(E_1,\dots,E_m)$ is isomorphic to the universal enveloping algebra of $\sum_{i=1}^m \mathbb{C}E_i$. For $\alpha = (\alpha_1,\dots,\alpha_m)$, put $E^\alpha = E_1^{\alpha_1}\cdots E_m^{\alpha_m}$ where $E_i^0 = 1$.

Lemma A.2. *Let $E'_1,\dots,E'_m$ be vector fields on M which satisfy the same conditions as $E_1,\dots,E_m$. Then*

$$E^\alpha T \in (E')^\alpha T + U_{|\alpha|-1}(E'_1,\dots,E'_m)\mathcal{D}'(M_0,\mathcal{L})$$

for $T \in \mathcal{D}'(M_0,\mathcal{L})$ and $\alpha \in \mathbb{Z}_{\ge 0}^m$.

Proof. First we remark that if the order of a differential operator P is at most k, then $P(\mathcal{D}'(M_0,\mathcal{L})) \subset U_k(D_1,\dots,D_m)\mathcal{D}'(M_0,\mathcal{L})$. Take $P \in U_{k-1}(E_1,\dots,E_m)$. Then

$$\begin{aligned} E_iPT = PE_iT + [E_i,P]T = PD_iT + [E_i,P]T &= D_iPT + [E_i - D_i,P]T \\ &\in D_iPT + U_{k-1}(D_1,\dots,D_m)\mathcal{D}'(M_0,\mathcal{L}) \end{aligned}$$

since the order of $[E_i - D_i, P]$ is less than or equal to $k-1$. Hence, using induction on $|\alpha|$, we have $E^\alpha T \in D^\alpha T + U_{|\alpha|-1}(D_1,\dots,D_m)\mathcal{D}'(M_0,\mathcal{L})$.

Hence $U_k(E_1,\dots,E_m)\mathcal{D}'(M_0,\mathcal{L}) \subset U_k(D_1,\dots,D_m)\mathcal{D}'(M_0,\mathcal{L})$. Therefore,

$$E^\alpha T + U_{|\alpha|-1}(E_1,\dots,E_m)\mathcal{D}'(M_0,\mathcal{L}) \subset D^\alpha T + U_{|\alpha|-1}(D_1,\dots,D_m)\mathcal{D}'(M_0,\mathcal{L}).$$

By the same argument,

$$E^\alpha T + U_{|\alpha|-1}(E_1,\dots,E_m)\mathcal{D}'(M_0,\mathcal{L}) \supset D^\alpha T + U_{|\alpha|-1}(D_1,\dots,D_m)\mathcal{D}'(M_0,\mathcal{L}).$$

Hence

$$E^\alpha T + U_{|\alpha|-1}(E_1,\dots,E_m)\mathcal{D}'(M_0,\mathcal{L}) = D^\alpha T + U_{|\alpha|-1}(D_1,\dots,D_m)\mathcal{D}'(M_0,\mathcal{L}).$$

The same formulas hold for $E_1',\dots,E_m'$. Consequently,

$$\begin{aligned} E^\alpha T \in D^\alpha T &+ U_{|\alpha|-1}(D_1,\dots,D_m)\mathcal{D}'(M_0,\mathcal{L}) \\ &= (E')^\alpha T + U_{|\alpha|-1}(D_1,\dots,D_m)\mathcal{D}'(M_0,\mathcal{L}) \\ &= (E')^\alpha T + U_{|\alpha|-1}(E_1',\dots,E_m')\mathcal{D}'(M_0,\mathcal{L}). \qquad \square \end{aligned}$$

Proposition A.3. (1) *The map* $\Phi\colon U(E_1,\dots,E_m)\otimes\mathcal{D}'(M_0,\mathcal{L}) \to \mathcal{D}'_{M_0}(M,\mathcal{L})$ *defined by* $P\otimes T\mapsto PT$ *is injective.*

(2) $\mathcal{T}_{M_0}(M,\mathcal{L})\subset \operatorname{Im}\Phi$. *Hence we have an injective homomorphism*

$$\mathcal{T}_{M_0}(M,\mathcal{L}) \hookrightarrow U(E_1,\dots,E_m)\mathcal{D}'(M_0,\mathcal{L}) \simeq U(E_1,\dots,E_m)\otimes\mathcal{D}'(M_0,\mathcal{L}).$$

Proof. (1) Let $\sum_{\alpha\in\mathbb{Z}^m_{\geq 0}} E^\alpha\otimes T_\alpha$ be an element of $U(E_1,\dots,E_m)\otimes\mathcal{D}'(M_0,\mathcal{L})$. Set $T=\sum_{\alpha\in\mathbb{Z}^m_{\geq 0}} E^\alpha T_\alpha$ and assume that $T=0$. Put $k=\max\{|\alpha| \mid T_\alpha\neq 0\}$. We will prove that $k=-\infty$. Assume that $k\geq 0$. By Lemma A.2, if $|\alpha|=k$ then $E^\alpha T_\alpha\in D^\alpha T_\alpha + U_{k-1}(D_1,\dots,D_m)\mathcal{D}'(M_0,\mathcal{L})$. There exist T'_α ($|\alpha|<k$) such that $\sum_{\alpha\in\mathbb{Z}^m_{\geq 0}} E^\alpha T_\alpha = \sum_{\alpha\in\mathbb{Z}^m_{\geq 0},\ |\alpha|<k} D^\alpha T'_\alpha + \sum_{\alpha\in\mathbb{Z}^m_{\geq 0},\ |\alpha|=k} D^\alpha T_\alpha$. Fix $\beta\in\mathbb{Z}^m_{\geq 0}$ such that $|\beta|=k$ and $f\in C_c^\infty(M_0,\mathcal{L})$. Define a function $\varphi_{\beta,f}$ on M by

$$\varphi_{\beta,f}(x_1,\dots,x_n) = x_1^{\beta_1}\cdots x_m^{\beta_m} f(0,\dots,0,x_{m+1},\dots,x_n).$$

Then $0=\langle T,\varphi\rangle = \beta_1!\cdots\beta_m!\langle T_\beta, f\rangle$. Since f is arbitrary, we have $T_\beta=0$ for all β such that $|\beta|=k$. This is a contradiction.

(2) For a differential operator P, let $\operatorname{ord}(P)$ be its order. Let $S\in\mathcal{T}_{M_0}(M,\mathcal{L})$. By [KV96, (2.8)], for any $p\in M_0$ there exist an open subset $U_p\ni p$ and $T_{\alpha,p}\in\mathcal{D}'(U_p\cap M_0,\mathcal{L})$ such that $S|_{U_p}=\sum_{\alpha\in\mathbb{Z}^m_{\geq 0}} E^\alpha T_{\alpha,p}$ (finite sum). Let $\widetilde{S}\in\mathcal{D}'(M,\mathcal{L})$ be such that $\widetilde{S}|_M = S$. Since the support of $\widetilde{S}$ is compact, there exists $r\in\mathbb{Z}_{\geq 0}$ such that if $\varphi\in C_c^\infty(X,\mathcal{L})$ satisfies $P\varphi|_{\operatorname{supp}\widetilde{S}}=0$ for each differential operator P with $\operatorname{ord}(P)\leq r$, then $\langle\widetilde{S},\varphi\rangle=0$. (When $\mathcal{L}=\mathbb{C}$, this is [Sch66, Ch. 3, §7, Th. XXVIII]. The same proof applies.) Then S has the same property. Fix $p\in M_0$. Set $k=\max\{|\alpha| \mid T_{\alpha,p}\neq 0\}$. Assume that $k>r$. Then for $\beta\in\mathbb{Z}^m_{\geq 0}$ such that $|\beta|=k$, $P\varphi_{\beta,f}|_{M_0}=0$ for each differential operator P with $\operatorname{ord}(P)\leq r$. However, by the proof of (1), we have $\langle S,\varphi_{\beta,f}\rangle\neq 0$ for some f. This is a contradiction. Hence $k\leq r$ for each $p\in M_0$. By the proof of (1), $T_{\alpha,p}=T_{\alpha,p'}$ on $U_p\cap U_{p'}$. Hence $\{T_{\alpha,p}\}_p$ defines a distribution T_α on M_0 and $S=\Phi(\sum_{|\alpha|\leq r} E^\alpha\otimes T_\alpha)$. $\square$

Appendix A.3. Distributions on a nilpotent Lie group

Let N be a connected, simply connected nilpotent Lie group. Put $\mathfrak{n} = \mathrm{Lie}(N)_{\mathbb{C}}$. Then the exponential map $\exp\colon \mathrm{Lie}(N) \to N$ is a diffeomorphism. It induces the structure of a vector space on N. Let $\mathcal{P}(N)$ be the ring of polynomials with respect to this vector space structure (cf. Corwin and Greenleaf [CG90, §1.2]).

Let $\mathcal{L}$ be a vector bundle on N whose fiber is V. Since N is simply connected, $\mathcal{L}$ is trivial, i.e., $\mathcal{L} = N \times V$. Fix a Haar measure dn on N. Let V' be the continuous dual space of V. For $F \in C^\infty(N, V')$, we define a distribution $F\delta$ by $\langle F\delta, \varphi\rangle = \int_N F(n)(\varphi(n))\, dn$ where $\varphi \in C_c^\infty(N, \mathcal{L})$. Thus we can regard $C^\infty(N, V')$ as a subspace of $\mathcal{D}'(N, \mathcal{L})$. Let $\mathcal{P}_k(N)$ be the space of polynomials of degree less than or equal to k. Then $\mathcal{P}(N) = \sum_k \mathcal{P}_k(N)$.

Let η be a character of N and denote its differential $\mathfrak{n} \to \mathbb{C}$ again by η. Then η can be extended to a $\mathbb{C}$-algebra homomorphism $U(\mathfrak{n}) \to \mathbb{C}$ where $U(\mathfrak{n})$ is the universal enveloping algebra of $\mathfrak{n}$. We denote this $\mathbb{C}$-algebra homomorphism again by η. Let $\operatorname{Ker}\eta$ be its kernel. For $X \in \mathrm{Lie}(N)$ and a C^∞-function ψ on N, put $(X\psi)(n) = \frac{d}{dt}\psi(\exp(-tX)n)|_{t=0}$.

The algebraic tensor product $C_c^\infty(N) \otimes V$ is canonically identified with a linear subspace of $C_c^\infty(N, \mathcal{L})$ via $\varphi \otimes v \mapsto (x \mapsto \varphi(x)v)$. This subspace is dense [KV96, (2.1)].

Proposition A.4. *For all $k \in \mathbb{Z}_{>0}$, there exists a positive integer l such that if $T \in \mathcal{D}'(N, \mathcal{L})$ satisfies $(\operatorname{Ker}\eta)^k T = 0$ then $T \in (\mathcal{P}_l(N)\eta^{-1} \otimes V')\delta$. Conversely, for all $l \in \mathbb{Z}_{>0}$ there exists $k > 0$ such that $(\operatorname{Ker}\eta)^k(\mathcal{P}_l(N)\eta^{-1} \otimes V')\delta = 0$.*

As a corollary, we get the following.

Corollary A.5. *Let $T \in \mathcal{D}'(N, \mathcal{L})$. Assume that there exists a positive integer k such that $(\operatorname{Ker}\eta)^k T \in (\mathcal{P}(N)\eta^{-1} \otimes V')\delta$. Then $T \in (\mathcal{P}(N)\eta^{-1} \otimes V')\delta$.*

Proof. By the second part of Proposition A.4, there exists $k' > 0$ such that $(\operatorname{Ker}\eta)^{k'} T = 0$. Hence $T \in (\mathcal{P}(N)\eta^{-1} \otimes V')\delta$ by the first part. □

Proof of Proposition A.4. For $T \in \mathcal{D}'(N, \mathcal{L})$, it is easy to see that $\mathfrak{n}^k(T\eta) = 0$ if and only if $(\operatorname{Ker}\eta)^k T = 0$. Therefore, we may assume that η is trivial.

First assume that $V = \mathbb{C}$. We argue by induction on $\dim N$. Take an element Z of the center of $\mathrm{Lie}(N)$ and a subspace $\mathfrak{n}_{0,\mathbb{R}}$ such that $\mathrm{Lie}(N) = \mathbb{R}Z \oplus \mathfrak{n}_{0,\mathbb{R}}$. Put $\mathfrak{n}_0 = \mathfrak{n}_{0,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathfrak{n}' = \mathfrak{n}/\mathbb{C}Z$ and $N' = N/\exp(\mathbb{R}Z)$. Then the projection $\mathfrak{n} \to \mathfrak{n}'$ gives an isomorphism $\Phi\colon \mathfrak{n}_0 \to \mathfrak{n}'$ of vector spaces. Set $\Psi = \Phi^{-1}$. We have an isomorphism $\tau\colon \mathbb{R} \times N' \simeq \mathbb{R} \times \mathrm{Lie}(N') \simeq \mathbb{R} \times \mathfrak{n}_{0,\mathbb{R}} \simeq \mathbb{R}Z \oplus \mathfrak{n}_{0,\mathbb{R}} = \mathrm{Lie}(N) \simeq N$. An element of $\mathfrak{n}$ gives a vector field on N. We consider the corresponding vector field on $\mathbb{R} \times N'$. Define a differential operator D_0 on $\mathbb{R} \times N'$ by $(D_0 f)(z, n') = (\partial f/\partial z)(z, n')$.

The action of Z is given by $-D_0$. Let D'_Y be the differential operator on N' given by $Y \in \mathfrak{n}'$. For $Y_0, Y \in \mathfrak{n}_{0,\mathbb{R}}$ and $z, t \in \mathbb{R}$, by the Campbell–Hausdorff formula, there exists a polynomial $P_t(Y_0, Y')$ on $\mathbb{R} \times (\mathfrak{n}_{0,\mathbb{R}})^2$ such that

$$\exp(-tY_0)\exp(zZ + Y) = \exp((z + P_t(Y_0, Y))Z + \Psi(Y'(\Phi(-tY_0), \Phi(Y))),$$

where $Y'\colon \operatorname{Lie}(N') \times \operatorname{Lie}(N') \to \operatorname{Lie}(N')$ is given by

$$\exp(Y_0)\exp(Y) = \exp(Y'(Y_0, Y)).$$

Hence the action of Y_0 is given by $P(Y_0, n')D_0 + D'_{\Phi(Y_0)}$ for a polynomial P.

Now we prove the first part of the proposition when $V = \mathbb{C}$. Since $(-D_0)^l T = 0$ for some l, we have $T(z, n') = \sum_{p=0}^{l} z^p T_p(n')$ for some distributions T_p on N'. By inductive hypothesis and Remark 2.2, it is sufficient to prove that for all $Y \in \mathfrak{n}'$ there exists a positive integer k' such that $Y^{k'} T_p$ is a polynomial. (See also the proof of Corollary A.5.) We prove this by induction on p. Set $Y_0 = \Psi(Y)$. Since the action of Y_0 is given by $P(Y_0, n')D_0 + D'_Y$, we have

$$Y_0^k(z^s T_s) \in \sum_{s_0 < s} z^{s_0} C^\infty(N') + z^s (D'_Y)^k (T_s) \qquad (s \le p),$$
$$Y_0^k(z^s T_s) \in \sum_{s_0 < s} z^{s_0} \mathcal{P}(N') + z^s (D'_Y)^k (T_s) \qquad (s > p),$$

since T_s is a polynomial if $s > p$ (inductive hypothesis). Take k such that $Y_0^k T = 0$. Then

$$0 = Y_0^k T \in \sum_{s<p} z^s C^\infty(N') + \sum_{s=0}^{l} z^s \mathcal{P}(N') + z^p (D'_Y)^k (T_s).$$

Therefore, $(D'_Y)^k(T_s) \in \mathcal{P}(N')$. The second part follows from [Goo76, 2.3, Corollary 2].

Fix a basis $\{e_1, \ldots, e_n\}$ of $\operatorname{Lie}(N)$. The map $\mathbb{R}^n \to N$ defined by $(x_1, \ldots, x_n) \mapsto \exp(x_1 e_1 + \cdots + x_n e_n)$ is an isomorphism. Using this map, we introduce a coordinate $(x_1, \ldots, x_n)$ of N.

Fix $v \in V$ and consider an ordinary distribution $T_v\colon \varphi \mapsto \langle T, \varphi \otimes v\rangle$ for $\varphi \in C_c^\infty(N)$. If $\mathfrak{n}^k T = 0$, then $\mathfrak{n}^k T_v = 0$. Hence for some l, we have $T_v = \sum_{\alpha_1+\cdots+\alpha_n \le l} (x_1^{\alpha_1} \cdots x_n^{\alpha_n} \otimes c_{v,\alpha_1,\ldots,\alpha_n})\delta$, where $c_{v,\alpha_1,\ldots,\alpha_n} \in \mathbb{C}$. The map $v \mapsto c_{v,\alpha_1,\ldots,\alpha_n}$ is continuous linear. Hence it defines an element of V'; denote it by $v'_{\alpha_1,\ldots,\alpha_n}$. Then for $\varphi \in C_c^\infty(N)$ and $v \in V$ we have $\langle T, \varphi \otimes v\rangle = \langle (\sum_{\alpha_1+\cdots+\alpha_n \le l} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \otimes v'_{\alpha_1,\ldots,\alpha_n})\delta, \varphi \otimes v\rangle$. Since $C_c^\infty(N) \otimes V$ is dense in $C_c^\infty(N, \mathcal{L})$, we have $T = (\sum_{\alpha_1+\cdots+\alpha_n \le l} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \otimes v'_{\alpha_1,\ldots,\alpha_n})\delta$.

We now prove the second part of the proposition. For $X \in \mathfrak{n}$, $f \in \mathcal{P}_l(N)$ and $v' \in V'$, we have $X((f \otimes v')\delta) = ((Xf) \otimes v')\delta$. Hence we may assume that $V = \mathbb{C}$. □

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