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Biadjointness in Cyclotomic Khovanov–Lauda–Rouquier Algebras

by

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Abstract

We prove that a pair of functors E_i^Λ and F_i^Λ appearing in the categorification of irreducible highest weight modules of quantum groups via cyclotomic Khovanov–Lauda–Rouquier algebras is a biadjoint pair.

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§1. Introduction

Lascoux–Leclerc–Thibon ([13]) conjectured that the irreducible representations of Hecke algebras of type A are controlled by the upper global basis ([8, 9]) (or the dual canonical basis [16]) of the basic representation of the affine quantum group $U_q(A_\ell^{(1)})$. Then Ariki ([1]) proved this conjecture by generalizing it to cyclotomic affine Hecke algebras. The crucial ingredient there was the fact that the cyclotomic affine Hecke algebras categorify the irreducible highest weight representations of $U(A_\ell^{(1)})$. Because of the lack of grading on the cyclotomic affine Hecke algebras, these algebras do not categorify the representation of the quantum group.

Then Khovanov–Lauda and Rouquier introduced independently a new family of graded algebras, a generalization of affine Hecke algebras of type A, in order to categorify arbitrary quantum groups ([10, 11, 17]). These algebras are called *Khovanov–Lauda–Rouquier algebras* or *quiver Hecke algebras.*

Let $U_q(\mathfrak{g})$ be the quantum group associated with a symmetrizable Cartan datum and let $\{R(\beta)\}_{\beta\in Q^+}$ be the corresponding Khovanov–Lauda–Rouquier al-

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gebras. Then it was shown in [10, 11] that there exists an algebra isomorphism

$$U_{\mathbf{A}}^{-}(\mathfrak{g}) \simeq \bigoplus_{\beta\in Q^+} K(\mathrm{Proj}(R(\beta))),$$

where $U_{\mathbf{A}}^{-}(\mathfrak{g})$ is the integral form of the half $U_q^{-}(\mathfrak{g})$ of $U_q(\mathfrak{g})$ with $\mathbf{A} = \mathbb{Z}[q, q^{-1}]$, and $K(\mathrm{Proj}(R(\beta)))$ is the Grothendieck group of finitely generated projective graded $R(\beta)$-modules. Moreover, when the generalized Cartan matrix is a symmetric matrix, Varagnolo and Vasserot proved that the *lower global basis* introduced by the author or Lusztig's *canonical basis* corresponds to the isomorphism classes of indecomposable projective R-modules under this isomorphism ([18]).

For each dominant integral weight $\Lambda \in P^+$, the algebra $R(\beta)$ has a special quotient $R^\Lambda(\beta)$ which is called the *cyclotomic Khovanov–Lauda–Rouquier algebra*. In [10], Khovanov and Lauda conjectured that $\bigoplus_{\beta\in Q^+} K(\mathrm{Proj}(R^\Lambda(\beta)))$ has a $U_{\mathbf{A}}(\mathfrak{g})$-module structure and that there exists a $U_{\mathbf{A}}(\mathfrak{g})$-module isomorphism

$$V_{\mathbf{A}}(\Lambda) \simeq \bigoplus_{\beta\in Q^+} K(\mathrm{Proj}(R^\Lambda(\beta))),$$

where $V_{\mathbf{A}}(\Lambda)$ denotes the $U_{\mathbf{A}}(\mathfrak{g})$-module with highest weight Λ. After partial results of Brundan and Stroppel ([4]), Brundan and Kleshchev ([2, 3]) and Lauda and Vazirani ([15]), the conjecture was proved by Seok-Jin Kang and the author for *all* symmetrizable Kac–Moody algebras ([7]).

For each $i \in I$, let us consider the restriction functor and the induction functor:

$$\begin{aligned} E_i^\Lambda\colon\ &\mathrm{Mod}(R^\Lambda(\beta+\alpha_i)) \to \mathrm{Mod}(R^\Lambda(\beta)),\\ F_i^\Lambda\colon\ &\mathrm{Mod}(R^\Lambda(\beta)) \to \mathrm{Mod}(R^\Lambda(\beta+\alpha_i)) \end{aligned}$$

defined by

$$\begin{aligned} E_i^\Lambda(N) &= e(\beta,i)N = e(\beta,i)R^\Lambda(\beta+\alpha_i) \otimes_{R^\Lambda(\beta+\alpha_i)} N,\\ F_i^\Lambda(M) &= R^\Lambda(\beta+\alpha_i)e(\beta,i) \otimes_{R^\Lambda(\beta)} M, \end{aligned}$$

where $M \in \mathrm{Mod}(R^\Lambda(\beta))$, $N \in \mathrm{Mod}(R^\Lambda(\beta+\alpha_i))$. Then these functors categorify the root operators e_i and f_i in the quantum groups.

It is obvious that E_i^Λ is a right adjoint functor of F_i^Λ.

Khovanov–Lauda ([10, 11, 12, 14]) and Rouquier ([17]) conjectured that E_i^Λ and F_i^Λ are biadjoint to each other. Namely E_i^Λ is also a left adjoint of F_i^Λ. Furthermore they gave candidates for the unit and the counit of this adjunction explicitly from the first adjunction. In [12], Khovanov–Lauda proved it in the case of $\mathfrak{sl}_n$. Rouquier proved that the candidate for a counit (resp. unit) is the counit (resp. unit) of an adjunction in a more general framework ([17, Theorem 5.16]). In this paper we prove that these candidates are indeed the unit and the counit of an adjunction for an arbitrary cyclotomic Khovanov–Lauda–Rouquier algebra.

In order to prove this we use a method similar to that employed in [7]. Namely we use the module $e(\beta, i^2)R(\beta + 2\alpha_i)e(\beta + \alpha_i, i) \otimes_{R(\beta+\alpha_i)} R^\Lambda(\beta + \alpha_i)$ in order to study $e(\beta, i^2)R^\Lambda(\beta + 2\alpha_i)e(\beta + \alpha_i, i)$. We fully exploit the fact that this module is a free right module over the ring $\mathbf{k}[x_{n+2}]$ (Lemma 5.3).

Webster proved similar results in [19, Theorem 1.6] by a totally different method beyond the author's comprehension. We also mention that [5] is related to our results.

This paper is organized as follows. In Section 2, we recall the notions of Khovanov–Lauda–Rouquier algebras. In Section 3, we recall the definition of cyclotomic Khovanov–Lauda–Rouquier algebras and the results in [7], and then state our main result (Theorem 3.5). In Section 4, we interpret it in terms of algebras ((4.1), (4.2) and (4.3)), and we give the proof in Section 5.

§2. Khovanov–Lauda–Rouquier algebras

§2.1. Cartan data

Let I be a finite index set. An integral square matrix $A = (a_{ij})_{i,j\in I}$ is called a *symmetrizable generalized Cartan matrix* if (i) $a_{ii} = 2$ $(i \in I)$, (ii) $a_{ij} \le 0$ $(i \neq j)$, (iii) $a_{ij} = 0$ whenever $a_{ji} = 0$ $(i, j \in I)$, (iv) there is a diagonal matrix $D = \mathrm{diag}(d_i \in \mathbb{Z}_{>0} \mid i \in I)$ such that DA is symmetric.

A *Cartan datum* $(A, P, \Pi, P^\vee, \Pi^\vee)$ consists of

(1) a symmetrizable generalized Cartan matrix A,
(2) a free abelian group P of finite rank, called the *weight lattice*,
(3) $P^\vee := \mathrm{Hom}(P, \mathbb{Z})$, called the *co-weight lattice*,
(4) $\Pi = \{\alpha_i \mid i \in I\} \subset P$, called the set of *simple roots*,
(5) $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee$, called the set of *simple coroots*,

satisfying the condition $\langle h_i, \alpha_j\rangle = a_{ij}$ for all $i, j \in I$. We denote by

$$P^+ := \{\lambda \in P \mid \langle h_i, \lambda\rangle \in \mathbb{Z}_{\ge 0} \text{ for all } i \in I\}$$

the set of *dominant integral weights*. The free abelian group $Q := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ is called the *root lattice*. Set $Q^+ = \sum_{i\in I} \mathbb{Z}_{\ge 0}\alpha_i$. For $\alpha = \sum k_i\alpha_i \in Q^+$, we define the *height* $\mathrm{ht}(\alpha)$ of α to be $\mathrm{ht}(\alpha) = \sum k_i$. Let $\mathfrak{h} = \mathbb{Q} \otimes_{\mathbb{Z}} P^\vee$. Since A is symmetrizable, there is a symmetric bilinear form $(\ \mid\ )$ on $\mathfrak{h}^*$ satisfying

$$(\alpha_i|\alpha_j) = d_i a_{ij} \quad (i, j \in I) \quad \text{and} \quad \langle h_i, \lambda\rangle = \frac{2(\alpha_i|\lambda)}{(\alpha_i|\alpha_i)} \quad \text{for any } \lambda \in \mathfrak{h}^* \text{ and } i \in I.$$

§2.2. Definition of Khovanov–Lauda–Rouquier algebras

Let $(A, P, \Pi, P^\vee, \Pi^\vee)$ be a Cartan datum. In this section, we recall the construction of the Khovanov–Lauda–Rouquier algebra associated with $(A, P, \Pi, P^\vee, \Pi^\vee)$ and its properties. We take as a base ring a graded commutative ring $\mathbf{k} = \bigoplus_{n\in\mathbb{Z}} \mathbf{k}_n$ such that $\mathbf{k}_n = 0$ for any $n < 0$. Let us take a matrix $(Q_{ij})_{i,j\in I}$ in $\mathbf{k}[u, v]$ such that $Q_{ij}(u, v) = Q_{ji}(v, u)$ and $Q_{ij}(u, v)$ has the form

$$Q_{ij}(u,v) = \begin{cases} 0 & \text{if } i = j, \\ \sum\limits_{p,q\ge 0} t_{i,j;p,q} u^p v^q & \text{if } i \ne j, \end{cases} \tag{2.1}$$

where $t_{i,j;p,q} \in \mathbf{k}_{-2(\alpha_i|\alpha_j)-(\alpha_i|\alpha_i)p-(\alpha_j|\alpha_j)q}$ and $t_{i,j} := t_{i,j;-a_{ij},0} \in \mathbf{k}_0^\times$. In particular, we have $t_{i,j;p,q} = 0$ if $(\alpha_i|\alpha_i)p + (\alpha_j|\alpha_j)q > -2(\alpha_i|\alpha_j)$. Note that $t_{i,j;p,q} = t_{j,i;q,p}$.

We denote by $S_n = \langle s_1, \dots, s_{n-1}\rangle$ the symmetric group on n letters, where $s_i = (i, i + 1)$ is the transposition. Then S_n acts on I^n.

Definition 2.1 ([10, 17]). The *Khovanov–Lauda–Rouquier algebra* $R(n)$ of degree n associated with a Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$ and $(Q_{ij})_{i,j\in I}$ is the associative algebra over $\mathbf{k}$ generated by $e(\nu)$ $(\nu \in I^n)$, x_k $(1 \le k \le n)$, τ_l $(1 \le l \le n-1)$ satisfying the following defining relations:

$$e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \quad \sum_{\nu\in I^n} e(\nu) = 1,$$

$$x_k x_l = x_l x_k, \quad x_k e(\nu) = e(\nu)x_k,$$

$$\tau_l e(\nu) = e(s_l(\nu))\tau_l, \quad \tau_k\tau_l = \tau_l\tau_k \ \text{ if } |k-l| > 1,$$

$$\tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1})e(\nu),$$

$$(\tau_k x_l - x_{s_k(l)}\tau_k)e(\nu) = \begin{cases} -e(\nu) & \text{if } l = k,\ \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if } l = k+1,\ \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu) = \begin{cases} \dfrac{Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}) - Q_{\nu_k,\nu_{k+1}}(x_{k+2}, x_{k+1})}{x_k - x_{k+2}} e(\nu) & \text{if } \nu_k = \nu_{k+2}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $R(n)$ has an anti-involution ψ that fixes the generators x_k, τ_l and $e(\nu)$.

The $\mathbb{Z}$-grading on $R(n)$ is given by

$$\deg e(\nu) = 0, \quad \deg x_k e(\nu) = (\alpha_{\nu_k}|\alpha_{\nu_k}), \quad \deg \tau_l e(\nu) = -(\alpha_{\nu_l}|\alpha_{\nu_{l+1}}). \tag{2.2}$$

For $a, b, c \in \{1, \dots, n\}$, we define certain elements of $R(n)$ by

$$
\begin{aligned}
e_{a,b} &= \sum_{\nu \in I^n,\, \nu_a = \nu_b} e(\nu), \\
Q_{a,b} &= \sum_{\nu \in I^n} Q_{\nu_a,\nu_b}(x_a, x_b) e(\nu), \\
\overline{Q}_{a,b,c} &= \sum_{\nu \in I^n,\, \nu_a = \nu_c} \frac{Q_{\nu_a,\nu_b}(x_a, x_b) - Q_{\nu_a,\nu_b}(x_c, x_b)}{x_a - x_c} e(\nu) \quad \text{if } a \neq c.
\end{aligned}
\tag{2.3}
$$

Then we have

$$
\begin{aligned}
&Q_{a,b} = Q_{b,a}, \qquad \tau_a^2 = Q_{a,a+1}, \\
&\tau_{a+1}\tau_a\tau_{a+1} = \tau_a\tau_{a+1}\tau_a + \overline{Q}_{a,a+1,a+2}.
\end{aligned}
\tag{2.4}
$$

We define the operators $\partial_{a,b}$ on $\bigoplus_{\nu \in I^n} \mathbf{k}[x_1, \dots, x_n] e(\nu)$ by

$$
\partial_{a,b} f = \frac{s_{a,b} f - f}{x_a - x_b} e_{a,b}, \quad \partial_a = \partial_{a,a+1},
\tag{2.5}
$$

where $s_{a,b} = (a, b)$ is the transposition.

Thus we obtain

$$
\begin{aligned}
&\tau_a e_{b,c} = e_{s_a(b), s_a(c)} \tau_a, \\
&\tau_a f - (s_a f)\tau_a = f\tau_a - \tau_a(s_a f) = (\partial_a f) e_{a,a+1}.
\end{aligned}
\tag{2.6}
$$

For $n \in \mathbb{Z}_{\geq 0}$ and $\beta \in Q^+$ such that $\mathrm{ht}(\beta) = n$, we set

$$
I^\beta = \{\nu = (\nu_1, \dots, \nu_n) \in I^n \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta\}.
$$

We define

$$
\begin{aligned}
e(\beta) &= \sum_{\nu \in I^\beta} e(\nu), \\
R(\beta) &= R(n) e(\beta) = \bigoplus_{\nu \in I^\beta} R(n) e(\nu).
\end{aligned}
\tag{2.7}
$$

The algebra $R(\beta)$ is called the *Khovanov–Lauda–Rouquier algebra at* β.

For $\ell \geq 0$, we set

$$
e(\beta, i^\ell) = \textstyle\sum_\nu e(\nu) \in R(\beta + \ell\alpha_i)
\tag{2.8}
$$

where ν ranges over the set of $\nu \in I^{\beta+\ell\alpha_i}$ such that $\nu_k = i$ for $n+1 \leq k \leq n+\ell$. We sometimes regard $R(\beta)$ as a $\mathbf{k}$-subalgebra of the $\mathbf{k}$-algebra $e(\beta, i^\ell) R(\beta+\ell\alpha_i) e(\beta, i^\ell)$.

Theorem 2.2 ([7]). *Let $\beta \in Q^+$ with $\mathrm{ht}(\beta) = n$ and $i \in I$. Then there exists a natural isomorphism*

$$R(\beta)e(\beta-\alpha_i,i)\otimes_{R(\beta-\alpha_i)}\mathbf{k}\tau_n\otimes e(\beta-\alpha_i,i)R(\beta)\oplus\mathbf{k}[x_{n+1}]\otimes R(\beta) \xrightarrow{\sim} e(\beta,i)R(\beta+\alpha_i)e(\beta,i). \tag{2.9}$$

Here $R(\beta)e(\beta-\alpha_i,i)\otimes_{R(\beta-\alpha_i)}\mathbf{k}\tau_n\otimes e(\beta-\alpha_i,i)R(\beta)\to e(\beta,i)R(\beta+\alpha_i)e(\beta,i)$ is given by $a\otimes\tau_n\otimes b\mapsto a\tau_n b$.

Here, τ_n in $\mathbf{k}\tau_n$ is a symbolical basis of a free $\mathbf{k}$-module of rank one. We sometimes use such notation in order to make morphisms more explicit.

Note that if $\beta-\alpha_i\notin Q^+$ then $R(\beta)e(\beta-\alpha_i,i)\otimes_{R(\beta-\alpha_i)}\mathbf{k}\tau_n\otimes e(\beta-\alpha_i,i)R(\beta)$ should be understood to be zero.

§3. Cyclotomic Khovanov–Lauda–Rouquier algebras

§3.1. Definition of cyclotomic Khovanov–Lauda–Rouquier algebras

Let $\Lambda\in P^+$ be a dominant integral weight. For each $i\in I$, we choose a monic polynomial of degree $\langle h_i,\Lambda\rangle$,

$$a_i^\Lambda(u)=\sum_{k=0}^{\langle h_i,\Lambda\rangle}c_{i;k}u^{\langle h_i,\Lambda\rangle-k}, \tag{3.1}$$

with $c_{i;k}\in\mathbf{k}_{k(\alpha_i|\alpha_i)}$ and $c_{i;0}=1$.

For k $(1\le k\le n)$ and $\beta\in Q^+$ with $\mathrm{ht}(\beta)=n$, we set

$$a^\Lambda(x_k)=\sum_{\nu\in I^\beta}a^\Lambda_{\nu_k}(x_k)e(\nu)\in R(\beta). \tag{3.2}$$

Hence $a^\Lambda(x_k)e(\nu)$ is a homogeneous element of $R(\beta)$ with degree $2(\alpha_{\nu_k}|\Lambda)$.

Definition 3.1. For $\beta\in Q^+$ the *cyclotomic Khovanov–Lauda–Rouquier algebra* $R^\Lambda(\beta)$ *at* β is defined to be the quotient algebra

$$R^\Lambda(\beta)=\frac{R(\beta)}{R(\beta)a^\Lambda(x_1)R(\beta)}.$$

In this paper we *forget the grading*, and we denote by $\mathrm{Mod}(R^\Lambda(\beta))$ the abelian category of $R^\Lambda(\beta)$-modules.

For each $i\in I$, we define the functors

$$\mathsf{E}_i^\Lambda\colon \mathrm{Mod}(R^\Lambda(\beta+\alpha_i))\to\mathrm{Mod}(R^\Lambda(\beta)),$$
$$\mathsf{F}_i^\Lambda\colon \mathrm{Mod}(R^\Lambda(\beta))\to\mathrm{Mod}(R^\Lambda(\beta+\alpha_i))$$

by

$$\begin{aligned} \mathsf{E}_i^\Lambda(N) &= e(\beta,i)N \simeq e(\beta,i)R^\Lambda(\beta+\alpha_i) \otimes_{R^\Lambda(\beta+\alpha_i)} N \\ &\simeq \mathrm{Hom}_{R^\Lambda(\beta+\alpha_i)}(R^\Lambda(\beta+\alpha_i)e(\beta,i), N), \\ \mathsf{F}_i^\Lambda(M) &= R^\Lambda(\beta+\alpha_i)e(\beta,i) \otimes_{R^\Lambda(\beta)} M, \end{aligned} \tag{3.3}$$

where $M \in \mathrm{Mod}(R^\Lambda(\beta))$ and $N \in \mathrm{Mod}(R^\Lambda(\beta+\alpha_i))$.

The following result is proved in [7].

Theorem 3.2 ([7]). *The module $R^\Lambda(\beta+\alpha_i)e(\beta,i)$ is a projective right $R^\Lambda(\beta)$-module. Similarly, $e(\beta,i)R^\Lambda(\beta+\alpha_i)$ is a projective left $R^\Lambda(\beta)$-module.*

Corollary 3.3. (i) *The functor E_i^Λ sends finitely generated projective modules to finitely generated projective modules.*

(ii) *The functor F_i^Λ is exact.*

§3.2

The pair $(\mathsf{F}_i^\Lambda, \mathsf{E}_i^\Lambda)$ has a canonical adjunction: the unit $\eta\colon \mathrm{id} \to \mathsf{E}_i^\Lambda\mathsf{F}_i^\Lambda$ and the counit $\varepsilon\colon \mathsf{F}_i^\Lambda\mathsf{E}_i^\Lambda \to \mathrm{id}$.

For $\beta \in Q^+$ with $\mathrm{ht}(\beta) = n$, the functors

$$\mathrm{Mod}(R^\Lambda(\beta)) \underset{\mathsf{E}_i^\Lambda}{\overset{\mathsf{F}_i^\Lambda}{\rightleftarrows}} \mathrm{Mod}(R^\Lambda(\beta+\alpha_i))$$

are represented by the kernel bimodules $R^\Lambda(\beta+\alpha_i)e(\beta,i)$ and $e(\beta,i)R^\lambda(\beta+\alpha_i)$ as in (3.3). In what follows, we denote by $\mathbf{1}_\beta$ the identity functor of the category $\mathrm{Mod}(R^\Lambda(\beta))$, and we denote by $\mathbf{1}_\beta\mathsf{E}_i^\Lambda = \mathsf{E}_i^\Lambda\mathbf{1}_{\beta+\alpha_i}$ the restriction functor $\mathsf{E}_i^\Lambda\colon \mathrm{Mod}(R^\Lambda(\beta+\alpha_i)) \to \mathrm{Mod}(R^\Lambda(\beta))$. Similarly, $\mathsf{F}_i^\Lambda\mathbf{1}_\beta = \mathbf{1}_{\beta+\alpha_i}\mathsf{F}_i^\Lambda$ denotes the induction functor $\mathsf{F}_i^\Lambda\colon \mathrm{Mod}(R^\Lambda(\beta)) \to \mathrm{Mod}(R^\Lambda(\beta+\alpha_i))$.

Let us denote by x the endomorphism of $\mathbf{1}_\beta\mathsf{E}_i^\Lambda$ represented by left multiplication with x_{n+1} on $e(\beta,i)R^\Lambda(\beta+\alpha_i)$, and by τ the endomorphism of $\mathbf{1}_\beta\mathsf{E}_i^\Lambda\mathsf{E}_i^\Lambda\colon \mathrm{Mod}(R^\Lambda(\beta+2\alpha_i)) \to \mathrm{Mod}(R^\Lambda(\beta))$ represented by left multiplication with τ_{n+1} on $e(\beta,i)R^\Lambda(\beta+\alpha_i) \otimes_{R^\Lambda(\beta+\alpha_i)} e(\beta+\alpha_i,i)R^\Lambda(\beta+2\alpha_i) \simeq e(\beta,i^2)R^\Lambda(\beta+2\alpha_i)$. Similarly the endomorphism x of $\mathsf{F}_i^\Lambda\mathbf{1}_\beta$ is represented by right multiplication with x_{n+1} on $R^\Lambda(\beta+\alpha_i)e(\beta,i)$, and the endomorphism τ of $\mathsf{F}_i^\Lambda\mathsf{F}_i^\Lambda\mathbf{1}_\beta\colon \mathrm{Mod}(R^\Lambda(\beta)) \to \mathrm{Mod}(R^\Lambda(\beta+2\alpha_i))$ is represented by right multiplication with τ_{n+1} on $R^\Lambda(\beta+2\alpha_i)e(\beta+\alpha_i,i) \otimes_{R^\Lambda(\beta+\alpha_i)} R^\Lambda(\beta+\alpha_i)e(\beta,i) \simeq R^\Lambda(\beta+2\alpha_i)e(\beta,i^2)$. Then $x \in \mathrm{End}(\mathsf{F}_i^\Lambda\mathbf{1}_\beta)$ and $x \in \mathrm{End}(\mathbf{1}_\beta\mathsf{E}_i^\Lambda)$ are dual to each other, as also are $\tau \in \mathrm{End}(\mathsf{F}_i^\Lambda\mathsf{F}_i^\Lambda\mathbf{1}_\beta)$ and $\tau \in \mathrm{End}(\mathbf{1}_\beta\mathsf{E}_i^\Lambda\mathsf{E}_i^\Lambda)$.

By adjunction, $\tau \in \mathrm{End}(\mathsf{E}_i^\Lambda\mathsf{E}_i^\Lambda)$ induces a morphism

$$\sigma\colon \mathsf{F}_i^\Lambda\mathsf{E}_i^\Lambda\mathbf{1}_\beta \to \mathsf{E}_i^\Lambda\mathsf{F}_i^\Lambda\mathbf{1}_\beta. \tag{3.4}$$

It is represented by the morphism

$$R^{\Lambda}(\beta)e(\beta-\alpha_i,i)\otimes_{R^{\Lambda}(\beta-\alpha_i)} e(\beta-\alpha_i,i)R^{\Lambda}(\beta)\to e(\beta,i)R^{\Lambda}(\beta+\alpha_i)e(\beta,i)$$

given by $x\otimes y\mapsto x\tau_n y$.

The following theorem was formulated as one of the axioms for the categorification of representations of quantum groups ([6, 12, 14, 17]), and proved in [7] for an arbitrary Khovanov–Lauda–Rouquier algebra.

Theorem 3.4 ([7]). *Set $\lambda:=\Lambda-\beta$ and $\lambda_i:=\langle h_i,\lambda\rangle$.*

(a) *Assume $\lambda_i:=\langle h_i,\lambda\rangle\ge 0$. The the morphism of endofunctors on* $\mathrm{Mod}(R^{\Lambda}(\beta))$

$$\rho:\mathsf{F}_i^{\Lambda}\mathsf{E}_i^{\Lambda}\mathbf{1}_\beta\oplus\bigoplus_{k=0}^{\lambda_i-1}\mathbf{k}x^k\otimes\mathbf{1}_\beta\to\mathsf{E}_i^{\Lambda}\mathsf{F}_i^{\Lambda}\mathbf{1}_\beta$$

is an isomorphism. Here $\mathsf{F}_i^{\Lambda}\mathsf{E}_i^{\Lambda}\mathbf{1}_\beta\to\mathsf{E}_i^{\Lambda}\mathsf{F}_i^{\Lambda}\mathbf{1}_\beta$ is given by σ, and $\mathbf{k}x^k\otimes\mathbf{1}_\beta\to\mathsf{F}_i^{\Lambda}\mathsf{E}_i^{\Lambda}\mathbf{1}_\beta$ is given by $(x^k\mathsf{F}_i^{\Lambda})\circ\eta=(\mathsf{E}_i^{\Lambda}x^k)\circ\eta\colon\mathbf{1}_\beta\to\mathsf{E}_i^{\Lambda}\mathsf{F}_i^{\Lambda}\mathbf{1}_\beta$.

(b) *Assume that $\lambda_i\le 0$. Then the morphism*

$$\rho:\mathsf{F}_i^{\Lambda}\mathsf{E}_i^{\Lambda}\mathbf{1}_\beta\to\mathsf{E}_i^{\Lambda}\mathsf{F}_i^{\Lambda}\mathbf{1}_\beta\oplus\bigoplus_{k=0}^{-\lambda_i-1}\mathbf{k}(x^{-1})^k\otimes\mathbf{1}_\beta$$

is an isomorphism. Here $\mathsf{F}_i^{\Lambda}\mathsf{E}_i^{\Lambda}\mathbf{1}_\beta\to\mathsf{E}_i^{\Lambda}\mathsf{F}_i^{\Lambda}\mathbf{1}_\beta$ is given by σ, and $\mathsf{F}_i^{\Lambda}\mathsf{E}_i^{\Lambda}\mathbf{1}_\beta\to\mathbf{k}(x^{-1})^k\otimes\mathbf{1}_\beta$ is given by $\varepsilon\circ(x^k\mathsf{E}_i^{\Lambda})=\varepsilon\circ(\mathsf{F}_i^{\Lambda}x^k)\colon\mathsf{F}_i^{\Lambda}\mathsf{E}_i^{\Lambda}\mathbf{1}_\beta\to\mathbf{1}_\beta$.

In the theorem, x^k in $\mathbf{k}x^k$ and $(x^{-1})^k$ in $\mathbf{k}(x^{-1})^k$ are a symbolical basis of a free $\mathbf{k}$-module.

Now let us define a morphism $\widehat{\eta}\colon\mathbf{1}_\beta\to\mathsf{F}_i^{\Lambda}\mathsf{E}_i^{\Lambda}\mathbf{1}_\beta$ as follows.

(i) If $\lambda_i:=\langle h_i,\lambda\rangle\ge 0$, then $\widehat{\eta}$ is given by the commutativity of

$$\begin{array}{ccc}
\mathsf{F}_i^{\Lambda}\mathsf{E}_i^{\Lambda}\mathbf{1}_\beta & \xleftarrow{\text{projection}} & \mathsf{F}_i^{\Lambda}\mathsf{E}_i^{\Lambda}\mathbf{1}_\beta\oplus\bigoplus_{k=0}^{\lambda_i-1}\mathbf{k}x^k\otimes\mathbf{1}_\beta \\
\uparrow{\scriptstyle -\widehat{\eta}} & & \wr\downarrow{\scriptstyle\rho} \\
\mathbf{1}_\beta & \xrightarrow{x^{\lambda_i}F\circ\eta} & \mathsf{E}_i^{\Lambda}\mathsf{F}_i^{\Lambda}
\end{array}$$

Here the top horizontal arrow is the projection. Note the minus sign in front of $\widehat{\eta}$.

(ii) If $\lambda_i<0$, then $\widehat{\eta}$ is defined as the composition

$$\begin{array}{ccc}
\mathbf{1}_\beta & \xrightarrow{\widehat{\eta}} & \mathsf{F}_i^{\Lambda}\mathsf{E}_i^{\Lambda}\mathbf{1}_\beta \\
\wr\downarrow & & \wr\downarrow{\scriptstyle\rho} \\
\mathbf{k}(x^{-1})^{-\lambda_i-1}\otimes\mathbf{1}_\beta & \hookrightarrow & \mathsf{E}_i^{\Lambda}\mathsf{F}_i^{\Lambda}\mathbf{1}_\beta\oplus\bigoplus_{k=0}^{-\lambda_i-1}\mathbf{k}(x^{-1})^k\otimes\mathbf{1}_\beta
\end{array}$$

Here the bottom horizontal arrow is the canonical inclusion and the left vertical arrow is derived from $\mathbf{k} \xrightarrow{\sim} \mathbf{k}(x^{-1})^{-\lambda_i-1}$ $(1 \mapsto (x^{-1})^{-\lambda_i-1})$.

The morphism $\widehat{\varepsilon}\colon \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathbf{1}_\beta \to \mathbf{1}_\beta$ is defined as follows.

(i) If $\lambda_i > 0$, then $\widehat{\varepsilon}$ is defined as the composition

$$\begin{array}{ccc} \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathbf{1}_\beta \oplus \bigoplus_{k=0}^{\lambda_i-1} \mathbf{k}x^k \otimes \mathbf{1}_\beta & \xrightarrow{\text{projection}} & \mathbf{k}x^{\lambda_i-1} \otimes \mathbf{1}_\beta \\ \wr\downarrow\rho & & \wr\downarrow \\ \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathbf{1}_\beta & \xrightarrow{\widehat{\varepsilon}} & \mathbf{1}_\beta \end{array}$$

Here the top horizontal arrow is the canonical projection and the right vertical arrow is induced by $x^{\lambda_i-1} \mapsto 1$.

(ii) If $\lambda_i \leq 0$, then $\widehat{\varepsilon}$ is defined as the composition

$$\begin{array}{ccc} \mathbf{1}_\beta & \xleftarrow{\varepsilon\circ(x^{-\lambda_i}\mathsf{E}_i^\Lambda)} & \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathbf{1}_\beta \\ \uparrow\widehat{\varepsilon} & & \wr\downarrow\rho \\ \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda & \hookrightarrow & \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathbf{1}_\beta \oplus \bigoplus_{k=0}^{-\lambda_i-1} \mathbf{k}(x^{-1})^k \otimes \mathbf{1}_\beta \end{array}$$

Here the bottom horizontal arrow is the canonical inclusion.

Now our main result can be stated as follows.

Theorem 3.5. *The pair* $(\mathsf{E}_i^\Lambda, \mathsf{F}_i^\Lambda)$ *is an adjoint pair with* $(\widehat{\eta}, \widehat{\varepsilon})$ *as adjunction. Namely the compositions*

$$\mathsf{E}_i^\Lambda \xrightarrow{\mathsf{E}_i^\Lambda\,\widehat{\eta}} \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \xrightarrow{\widehat{\varepsilon}\,\mathsf{E}_i^\Lambda} \mathsf{E}_i^\Lambda \quad \textit{and} \quad \mathsf{F}_i^\Lambda \xrightarrow{\widehat{\eta}\,\mathsf{F}_i^\Lambda} \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \xrightarrow{\mathsf{F}_i^\Lambda\,\widehat{\varepsilon}} \mathsf{F}_i^\Lambda$$

are equal to the identities.

The rest of the paper is devoted to the proof of this theorem.

As mentioned in the Introduction, Rouquier ([17]) proved that there exists a morphism $\varepsilon'\colon \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \to \mathbf{1}_\beta$ such that

$$\mathsf{E}_i^\Lambda \xrightarrow{\mathsf{E}_i^\Lambda\,\widehat{\eta}} \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \xrightarrow{\varepsilon'\,\mathsf{E}_i^\Lambda} \mathsf{E}_i^\Lambda \quad \text{and} \quad \mathsf{F}_i^\Lambda \xrightarrow{\widehat{\eta}\,\mathsf{F}_i^\Lambda} \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \xrightarrow{\mathsf{F}_i^\Lambda\,\varepsilon'} \mathsf{F}_i^\Lambda$$

are the identities. Of course, such an ε' is uniquely determined. However, the identity $\varepsilon' = \widehat{\varepsilon}$ is non-trivial.

§4. Proof of Theorem 3.5

§4.1

We shall first prove that the composition

$$\mathbf{1}_\beta \mathsf{E}_i^\Lambda \xrightarrow{\mathsf{E}_i^\Lambda\,\widehat{\eta}} \mathbf{1}_\beta \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \xrightarrow{\widehat{\varepsilon}\,\mathsf{E}_i^\Lambda} \mathbf{1}_\beta \mathsf{E}_i^\Lambda$$

is the identity. Here $\beta \in Q^+$ with $\mathrm{ht}(\beta) = n$ and we set $\lambda := \Lambda - \beta$ and $\lambda_i := \langle h_i, \lambda\rangle$.

4.1.1. Case $\lambda_i \geq 2$. We shall first assume that $\lambda_i \geq 2$. Then the above composition can be described by kernel bimodules as follows. The morphism $\mathbf{1}_\beta \mathsf{E}_i^\Lambda \xrightarrow{\mathsf{E}_i^\Lambda\,\widehat{\eta}} \mathbf{1}_\beta \mathsf{E}_i^\Lambda(\mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathbf{1}_{\beta+\alpha_i})$ is given by the $(R^\Lambda(\beta), R^\Lambda(\beta+\alpha_i))$-bilinear homomorphism

$$\begin{array}{c}
e(\beta,i)R^\Lambda(\beta+\alpha_i)\\
\Big\downarrow{\scriptstyle -x_{n+2}^{\lambda_i-2}}\\
e(\beta,i^2)R^\Lambda(\beta+2\alpha_i)e(\beta+\alpha_i,i)\\
\Big\uparrow{\scriptstyle \rho\ \wr}\\
e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i)\otimes_{R^\Lambda(\beta)}\mathbf{k}\tau_{n+1}\otimes e(\beta,i)R^\Lambda(\beta+\alpha_i)\\
\qquad\oplus\bigoplus_{k=0}^{k=\lambda_i-3}\mathbf{k}x_{n+2}^k\otimes e(\beta,i)R^\Lambda(\beta+\alpha_i)\\
\Big\downarrow{\scriptstyle \text{projection}}\\
e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i)\otimes_{R^\Lambda(\beta)}\mathbf{k}\tau_{n+1}\otimes e(\beta,i)R^\Lambda(\beta+\alpha_i)
\end{array}$$

The morphism $(\mathbf{1}_\beta \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda)\mathsf{E}_i^\Lambda \xrightarrow{\widehat{\varepsilon}\,\mathsf{E}_i^\Lambda} \mathbf{1}_\beta \mathsf{E}_i^\Lambda$ is given by the $(R^\Lambda(\beta), R^\Lambda(\beta+\alpha_i))$-bilinear homomorphism

$$\begin{array}{c}
e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i)\otimes_{R^\Lambda(\beta)}\mathbf{k}\tau_{n+1}\otimes e(\beta,i)R^\Lambda(\beta+\alpha_i)\\
\Big\uparrow{\scriptstyle \rho\ \wr}\\
\big(R^\Lambda(\beta)e(\beta-\alpha_i,i)\otimes_{R^\Lambda(\beta-\alpha_i)}\mathbf{k}\tau_n\otimes e(\beta-\alpha_i,i)R^\Lambda(\beta)\\
\qquad\oplus\bigoplus_{k=0}^{\lambda_i-1}\mathbf{k}x_{n+1}^k\otimes R^\Lambda(\beta)\big)\otimes_{R^\Lambda(\beta)}\mathbf{k}\tau_{n+1}\otimes e(\beta,i)R^\Lambda(\beta+\alpha_i)\\
\Big\downarrow{\scriptstyle \text{projection}}\\
\mathbf{k}x_{n+1}^{\lambda_i-1}\otimes\mathbf{k}\tau_{n+1}\otimes e(\beta,i)R^\Lambda(\beta+\alpha_i)\\
\Big\downarrow{\scriptstyle \wr}\\
e(\beta,i)R^\Lambda(\beta+\alpha_i)
\end{array}$$

Hence in order to see that the composition is the identity, it is enough to show that

$$(4.1)\qquad x_{n+2}^{\lambda_i-2}e(\beta,i^2)+x_{n+1}^{\lambda_i-1}\tau_{n+1}e(\beta,i^2)\\ \in R^\Lambda(\beta)\tau_n\tau_{n+1}e(\beta-\alpha_i,i^3)R^\Lambda(\beta+\alpha_i)\\ +\sum_{k=0}^{\lambda_i-2}x_{n+1}^k\tau_{n+1}e(\beta,i^2)R^\Lambda(\beta+\alpha_i)+\sum_{k=0}^{\lambda_i-3}x_{n+2}^k e(\beta,i^2)R^\Lambda(\beta+\alpha_i)$$

as an element of $e(\beta,i^2)R^\Lambda(\beta+2\alpha_i)e(\beta+\alpha_i,i)$.

This inclusion is proved in §5.

4.1.2. Now consider the case $\lambda_i=1$. The morphism $\mathbf{1}_\beta\mathsf{E}_i^\Lambda \xrightarrow{\mathsf{E}_i^\Lambda\,\widehat{\eta}} \mathbf{1}_\beta\mathsf{E}_i^\Lambda(\mathsf{F}_i^\Lambda\mathsf{E}_i^\Lambda\mathbf{1}_{\beta+\alpha_i})$ is given by

$$\begin{array}{c} e(\beta,i)R^\Lambda(\beta+\alpha_i)\\ \big\downarrow{\scriptstyle \text{inclusion}}\\ e(\beta,i^2)R^\Lambda(\beta+2\alpha_i)e(\beta+\alpha_i,i)\oplus e(\beta,i)R^\Lambda(\beta+\alpha_i)\\ \wr\big\uparrow{\scriptstyle \rho=\Sigma\oplus\widetilde{E}}\\ e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i)\otimes_{R^\Lambda(\beta)}e(\beta,i)R^\Lambda(\beta+\alpha_i)\ni u \end{array}$$

(See below for Σ and $\widetilde{E}$.)

The morphism $(\mathbf{1}_\beta\mathsf{E}_i^\Lambda\mathsf{F}_i^\Lambda)\mathsf{E}_i^\Lambda \xrightarrow{\widehat{\varepsilon}\,\mathsf{E}_i^\Lambda} \mathbf{1}_\beta\mathsf{E}_i^\Lambda$ is given by

$$\begin{array}{c} e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i)\otimes_{R^\Lambda(\beta)}e(\beta,i)R^\Lambda(\beta+\alpha_i)\ni u\\ \wr\big\uparrow{\scriptstyle \rho}\\ \big(R^\Lambda(\beta)e(\beta-\alpha_i,i)\otimes_{R^\Lambda(\beta-\alpha_i)}e(\beta-\alpha_i,i)R^\Lambda(\beta)\oplus R^\Lambda(\beta)\big)\\ \otimes_{R^\Lambda(\beta)}e(\beta,i)R^\Lambda(\beta+\alpha_i)\\ \big\downarrow{\scriptstyle \text{projection}}\\ e(\beta,i)R^\Lambda(\beta+\alpha_i) \end{array}$$

Hence to see that the composition is the identity, it is enough to show the following:

$$(4.2)\qquad \begin{cases} \text{There exists } u\in e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i)\otimes_{R^\Lambda(\beta)}e(\beta,i)R^\Lambda(\beta+\alpha_i) \text{ such that}\\ \text{(a)}\ \ \Sigma(u)=0,\\ \text{(b)}\ \ \widetilde{E}(u)=e(\beta,i),\\ \text{(c)}\ \ u-e(\beta,i)\otimes e(\beta,i)\in\big(R^\Lambda(\beta)e(\beta-\alpha_i,i^2)\tau_n e(\beta-\alpha_i,i^2)R^\Lambda(\beta)\big)\\ \qquad\qquad\otimes_{R^\Lambda(\beta)}e(\beta,i)R^\Lambda(\beta+\alpha_i). \end{cases}$$

Here

$$\Sigma\colon e(\beta,i)R^{\Lambda}(\beta+\alpha_i)e(\beta,i)\otimes_{R^{\Lambda}(\beta)} e(\beta,i)R^{\Lambda}(\beta+\alpha_i) \\ \to e(\beta,i^2)R^{\Lambda}(\beta+2\alpha_i)e(\beta+\alpha_i,i)$$

is given by $\Sigma(a\otimes b)=a\tau_{n+1}b$, and

$$\widetilde{E}\colon e(\beta,i)R^{\Lambda}(\beta+\alpha_i)e(\beta,i)\otimes_{R^{\Lambda}(\beta)} e(\beta,i)R^{\Lambda}(\beta+\alpha_i)\to e(\beta,i)R^{\Lambda}(\beta+\alpha_i)$$

is given by $\widetilde{E}(a\otimes b)=ab$.

The proof of (4.2) will be given in §5.

4.1.3. Now we assume that $\lambda_i\le 0$. Then the composition under study can be described by the kernel bimodules as follows.

The morphism $\mathbf{1}_\beta\mathsf{E}_i^{\Lambda}\xrightarrow{\mathsf{E}_i^{\Lambda}\,\widehat{\eta}}\mathbf{1}_\beta\mathsf{E}_i^{\Lambda}(\mathsf{F}_i^{\Lambda}\mathsf{E}_i^{\Lambda}\mathbf{1}_{\beta+\alpha_i})$ is given by

$$\begin{array}{c} e(\beta,i)R^{\Lambda}(\beta+\alpha_i)\\ \big\downarrow\wr\\ \mathbf{k}(x_{n+1}^{-1})^{1-\lambda_i}\otimes e(\beta,i)R^{\Lambda}(\beta+\alpha_i)\\ \big\downarrow\text{inclusion}\\ e(\beta,i^2)R^{\Lambda}(\beta+2\alpha_i)e(\beta+\alpha_i,i)\oplus\bigoplus_{k=0}^{1-\lambda_i}\mathbf{k}(x_{n+1}^{-1})^{k}\otimes e(\beta,i)R^{\Lambda}(\beta+\alpha_i)\\ {\scriptstyle\rho=f\oplus\bigoplus_k H'_k}\big\uparrow\wr\\ e(\beta,i)R^{\Lambda}(\beta+\alpha_i)e(\beta,i)\otimes_{R^{\Lambda}(\beta)} e(\beta,i)R^{\Lambda}(\beta+\alpha_i) \end{array}$$

The morphism $(\mathbf{1}_\beta\mathsf{E}_i^{\Lambda}\mathsf{F}_i^{\Lambda})\mathsf{E}_i^{\Lambda}\xrightarrow{\widehat{\varepsilon}\,\mathsf{E}_i^{\Lambda}}\mathbf{1}_\beta\mathsf{E}_i^{\Lambda}$ is given by

$$\begin{array}{c} e(\beta,i)R^{\Lambda}(\beta+\alpha_i)e(\beta,i)\otimes_{R^{\Lambda}(\beta)} e(\beta,i)R^{\Lambda}(\beta+\alpha_i)\\ \big\downarrow\text{inclusion}\\ e(\beta,i)R^{\Lambda}(\beta+\alpha_i)e(\beta,i)\otimes_{R^{\Lambda}(\beta)} e(\beta,i)R^{\Lambda}(\beta+\alpha_i)\\ \oplus\bigoplus_{k=0}^{-\lambda_i-1}\mathbf{k}(x_n^{-1})^{k}\otimes e(\beta,i)R^{\Lambda}(\beta+\alpha_i)\\ {\scriptstyle\rho=g\oplus\bigoplus_k T_k}\big\uparrow\wr\\ R^{\Lambda}(\beta)e(\beta-\alpha_i,i)\otimes_{R^{\Lambda}(\beta-\alpha_i)} e(\beta-\alpha_i,i^2)R^{\Lambda}(\beta+\alpha_i)\ni v\\ \big\downarrow T_{-\lambda_i}\\ e(\beta,i)R^{\Lambda}(\beta+\alpha_i) \end{array}$$

Hence to see that the composition is the identity, it is enough to show

$$(4.3)\quad \begin{cases} \text{There exists } v \in R^\Lambda(\beta)e(\beta-\alpha_i)\otimes_{R^\Lambda(\beta-\alpha_i)} e(\beta,i)R^\Lambda(\beta+\alpha_i) \text{ such that} \\ \text{(a)}\ \ T_k(v)=0 \text{ for } 0\le k\le -\lambda_i-1, \\ \text{(b)}\ \ T_{-\lambda_i}(v)=e(\beta,i), \\ \text{(c)}\ \ G(v)=0, \\ \text{(d)}\ \ H_k(v)=0 \text{ for } 0\le k\le -\lambda_i, \\ \text{(e)}\ \ H_{1-\lambda_i}(v)=e(\beta,i). \end{cases}$$

Here the homomorphism

$$f\colon e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i)\otimes_{R^\Lambda(\beta)}e(\beta,i)R^\Lambda(\beta+\alpha_i)\to e(\beta,i^2)R^\Lambda(\beta+2\alpha_i)e(\beta+\alpha_i,i)$$

is given by $f(a\otimes b)=a\tau_{n+1}b$;

$$\begin{aligned} H'_k\colon e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i)\otimes_{R^\Lambda(\beta)} e(\beta,i)R^\Lambda(\beta+\alpha_i) \\ \to \mathbf{k}(x_{n+1}^{-1})^k\otimes e(\beta,i)R^\Lambda(\beta+\alpha_i)\simeq e(\beta,i)R^\Lambda(\beta+\alpha_i) \end{aligned}$$

is given by $H'_k(a\otimes b)=ax_{n+1}^k b$;

$$\begin{aligned} g\colon R^\Lambda(\beta)e(\beta-\alpha_i)\otimes_{R^\Lambda(\beta-\alpha_i)} e(\beta,i)R^\Lambda(\beta+\alpha_i) \\ \to e(\beta,i)R^\Lambda(\beta+\alpha_i)e(\beta,i)\otimes_{R^\Lambda(\beta)} e(\beta,i)R^\Lambda(\beta+\alpha_i) \end{aligned}$$

is given by $g(a\otimes b)=a\tau_n\otimes b$;

$$T_k\colon R^\Lambda(\beta)e(\beta-\alpha_i)\otimes_{R^\Lambda(\beta-\alpha_i)} e(\beta,i)R^\Lambda(\beta+\alpha_i)\to e(\beta,i)R^\Lambda(\beta+\alpha_i)$$

is given by $T_k(a\otimes b)=ax_n^k b$;

$$\begin{aligned} G=f\circ g\colon R^\Lambda(\beta)e(\beta-\alpha_i)\otimes_{R^\Lambda(\beta-\alpha_i)} e(\beta,i)R^\Lambda(\beta+\alpha_i) \\ \to e(\beta,i^2)R^\Lambda(\beta+2\alpha_i)e(\beta+\alpha_i,i) \end{aligned}$$

is given by $G(a\otimes b)=a\tau_n\tau_{n+1}b$; and

$$H_k=H'_k\circ g\colon R^\Lambda(\beta)e(\beta-\alpha_i)\otimes_{R^\Lambda(\beta-\alpha_i)} e(\beta,i)R^\Lambda(\beta+\alpha_i)\to e(\beta,i)R^\Lambda(\beta+\alpha_i)$$

is given by $H_k(a\otimes b)=a\tau_n x_{n+1}^k b$.

The statement (4.3) is proved in §5.

§4.2

Let us show that the composition $\mathsf{F}_i^\Lambda \xrightarrow{\widehat{\eta}\,\mathsf{F}_i^\Lambda} \mathsf{F}_i^\Lambda\mathsf{E}_i^\Lambda\mathsf{F}_i^\Lambda \xrightarrow{\mathsf{F}_i^\Lambda\,\widehat{\varepsilon}} \mathsf{F}_i^\Lambda$ is equal to the identity by reducing it to the corresponding statement for $\mathsf{E}_i^\Lambda \xrightarrow{\mathsf{E}_i^\Lambda\,\widehat{\eta}} \mathsf{E}_i^\Lambda\mathsf{F}_i^\Lambda\mathsf{E}_i^\Lambda \xrightarrow{\widehat{\varepsilon}\,\mathsf{E}_i^\Lambda} \mathsf{E}_i^\Lambda$.

Let us recall that ψ is the anti-involution of $R^\Lambda(\beta)$ sending the generators $e(\nu)$, x_k, τ_k to themselves. For an $R^\Lambda(\beta)$-module M, we denote by M^ψ the $R^\Lambda(\beta)^{\mathrm{opp}}$-module induced by ψ from M, where $R^\Lambda(\beta)^{\mathrm{opp}}$ is the opposite ring of $R^\Lambda(\beta)$. We define the bifunctor

$$\Psi_\beta\colon\ \mathrm{Mod}(R^\Lambda(\beta)) \times \mathrm{Mod}(R^\Lambda(\beta)) \to\ \mathrm{Mod}(\mathbf{k})$$

by

$$\Psi_\beta(M,N) := M^\psi \otimes_{R^\Lambda(\beta)} N.$$

We have an isomorphism

$$\Psi_\beta(M,N) \simeq \Psi_\beta(N,M) \qquad \text{functorial in } M,N \in \mathrm{Mod}(R^\Lambda(\beta)).$$

For two $\mathbf{k}$-linear categories $\mathscr{C}$ and $\mathscr{C}'$, let us denote by $\mathrm{Fct}_{\mathbf{k}}(\mathscr{C},\mathscr{C}')$ the category of $\mathbf{k}$-linear functors from $\mathscr{C}$ to $\mathscr{C}'$. Then Ψ_β induces a functor

$$\mathsf{H}_\beta\colon\ \mathrm{Mod}(R^\Lambda(\beta)) \to\ \mathrm{Fct}_{\mathbf{k}}\big(\mathrm{Mod}(R^\Lambda(\beta)), \mathrm{Mod}(\mathbf{k})\big)$$

by assigning to $M \in \mathrm{Mod}(R^\Lambda(\beta))$ the functor $N \mapsto \Psi_\beta(M,N)$. The following lemma similar to the Yoneda lemma is easily proved, and its proof is omitted.

Lemma 4.1. *The functor H_β is fully faithful.*

For $\beta,\beta' \in Q^+$ and a pair of $\mathbf{k}$-linear functors $F\colon\ \mathrm{Mod}(R^\Lambda(\beta)) \to \mathrm{Mod}(R^\Lambda(\beta'))$ and $G\colon\ \mathrm{Mod}(R^\Lambda(\beta')) \to\ \mathrm{Mod}(R^\Lambda(\beta))$, we say that F and G are *Ψ-adjoint*, or G is a *Ψ-adjoint* of F, if there exists an isomorphism

$$\Psi_{\beta'}(F(M),N) \simeq \Psi_\beta(M,G(N))$$

functorial in $M \in \mathrm{Mod}(R^\Lambda(\beta))$ and $N \in \mathrm{Mod}(R^\Lambda(\beta'))$. For a given F, a Ψ-adjoint of F is unique up to a unique isomorphism if it exists. We shall denote by $F^\vee$ the Ψ-adjoint of F (if it exists).

If $\mathrm{Mod}(R^\Lambda(\beta)) \xrightarrow{F} \mathrm{Mod}(R^\Lambda(\beta')) \xrightarrow{F'} \mathrm{Mod}(R^\Lambda(\beta''))$ are functors which admit Ψ-adjoints, then $F^\vee \circ F'^\vee$ is a Ψ-adjoint of $F' \circ F$.

Now let $F_k\colon\ \mathrm{Mod}(R^\Lambda(\beta)) \to\ \mathrm{Mod}(R^\Lambda(\beta'))$ $(k=1,2)$ be two functors. Then Lemma 4.1 implies

$$\mathrm{Hom}(F_1,F_2) \simeq \mathrm{Hom}(F_1^\vee, F_2^\vee).$$

For $f \in \mathrm{Hom}(F_1,F_2)$, the corresponding morphism in $\mathrm{Hom}(F_1^\vee,F_2^\vee)$ is called the Ψ-adjoint of f, denoted by $f^\vee$, and we have a commutative diagram

$$\begin{array}{ccc}
\Psi_{\beta'}(F_1(M),N) & \overset{\sim}{\text{———}} & \Psi_\beta(M,F_1^\vee(N)) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^\vee} \\
\Psi_{\beta'}(F_2(M),N) & \overset{\sim}{\text{———}} & \Psi_\beta(M,F_2^\vee(N))
\end{array}$$

Then $(f\circ g)^\vee = f^\vee \circ g^\vee$ for $F_1 \xrightarrow{g} F_2 \xrightarrow{f} F_3$.

The following lemma is elementary and its proof is omitted.

Lemma 4.2. (i) *Let K be an $(R^\Lambda(\beta'), R^\Lambda(\beta))$-bimodule and let the functor $F\colon \mathrm{Mod}(R^\Lambda(\beta)) \to \mathrm{Mod}(R^\Lambda(\beta'))$ be given by $K \otimes_{R^\Lambda(\beta)} \bullet$. Then F admits a Ψ-adjoint.*

(ii) *Conversely if a $\mathbf{k}$-linear functor $F\colon \mathrm{Mod}(R^\Lambda(\beta)) \to \mathrm{Mod}(R^\Lambda(\beta'))$ admits a Ψ-adjoint, then F is isomorphic to $F(R^\Lambda(\beta)) \otimes_{R^\Lambda(\beta)} \bullet$, and $F^\vee(R^\Lambda(\beta')) \simeq F(R^\Lambda(\beta))^\psi$ as $(R^\Lambda(\beta), R^\Lambda(\beta'))$-bimodules.*

We can easily see that E_i^Λ and F_i^Λ are Ψ-adjoint. Moreover, $x \in \mathrm{End}(\mathsf{E}_i^\Lambda)$ and $x \in \mathrm{End}(\mathsf{F}_i^\Lambda)$ as well as $\tau \in \mathrm{End}(\mathsf{E}_i^\Lambda \circ \mathsf{E}_i^\Lambda)$ and $\tau \in \mathrm{End}(\mathsf{F}_i^\Lambda \circ \mathsf{F}_i^\Lambda)$ are Ψ-adjoint. We can also see that $\eta \in \mathrm{Hom}(\mathbf{1}_\beta, \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathbf{1}_\beta)$ is a Ψ-adjoint of itself. Similarly $\varepsilon \in \mathrm{Hom}(\mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathbf{1}_\beta, \mathbf{1}_\beta)$ and $\sigma \in \mathrm{Hom}(\mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda, \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda)$ are Ψ-adjoints of themselves. Note that $\mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda$ and $\mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda$ are Ψ-adjoints of themselves. Hence $\widehat{\eta}$ and $\widehat{\varepsilon}$ are also Ψ-adjoints of themselves.

Therefore $\mathsf{F}_i^\Lambda \xrightarrow{\widehat{\eta}\,\mathsf{F}_i^\Lambda} \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \xrightarrow{\mathsf{F}_i^\Lambda\,\widehat{\varepsilon}} \mathsf{F}_i^\Lambda$ is a Ψ-adjoint of $\mathsf{E}_i^\Lambda \xrightarrow{\mathsf{E}_i^\Lambda\,\widehat{\eta}} \mathsf{E}_i^\Lambda \mathsf{F}_i^\Lambda \mathsf{E}_i^\Lambda \xrightarrow{\widehat{\varepsilon}\,\mathsf{E}_i^\Lambda} \mathsf{E}_i^\Lambda$. Hence if the latter composition is the identity, so is the former.

Thus we have reduced Theorem 3.5 to the three statements (4.1), (4.2) and (4.3), which will be proved in the next section.

§5. Proof of the three statements

§5.1. Intertwiner

Let us define $\varphi_a \in R(n)$ as follows:

$$\begin{aligned}\varphi_a e(\nu) &= (x_a\tau_a - \tau_a x_a)e(\nu) = (\tau_a x_{a+1} - x_{a+1}\tau_a)e(\nu)\\ &= ((x_a - x_{a+1})\tau_a + 1)e(\nu) = (\tau_a(x_{a+1} - x_a) - 1)e(\nu)\end{aligned}$$

if $\nu_a = \nu_{a+1}$ and $\varphi_a e(\nu) = \tau_a e(\nu)$ if $\nu_a \neq \nu_{a+1}$. It is called the *intertwiner*.

The following lemma is well-known (for example, it easily follows from the polynomial representation of Khovanov–Lauda–Rouquier algebras [10, Proposition 2.3], [17, Proposition 3.12]).

Lemma 5.1. (i) *For $1 \le a \le n$, we have*

$$x_{s_a(b)}\varphi_a = \varphi_a x_b \qquad (1 \le b \le n+1).$$

(ii) $\varphi_a^2 = Q_{a,a+1} + e_{a,a+1}$.

(iii) *$\{\varphi_k\}_{1 \le k < n}$ satisfies the braid relation.*

(iv) *For $w \in S_n$ and $1 \le k < n$, if $w(k+1) = w(k)+1$, then $\varphi_w \tau_k = \tau_{w(k)}\varphi_w$.*

(v) *In particular*

$$\tau_a\varphi_{a+1}\varphi_a = \varphi_{a+1}\varphi_a\tau_{a+1}, \qquad \tau_{a+1}\varphi_a\varphi_{a+1} = \varphi_a\varphi_{a+1}\tau_a,$$
$$\tau_k\varphi_a\cdots\varphi_{n-1} = \varphi_a\cdots\varphi_{n-1}\tau_{k-1} \qquad \text{for } a < k \le n-1.$$

§5.2

Let us take $\beta \in Q^+$ with $\mathrm{ht}(\beta) = n$ and $i \in I$. Let p be the number of times that α_i appears in β. The following lemma is proved by repeated use of Theorem 2.2.

Lemma 5.2. *We have*

$$\begin{aligned} &e(\beta,i^2)R(\beta+2\alpha_i)e(\beta+\alpha_i,i)\otimes_{R(\beta+\alpha_i)} R^\Lambda(\beta+\alpha_i)\\ &\quad\simeq R(\beta)e(\beta-\alpha_i,i)\otimes \mathbf{k}\tau_n\tau_{n+1}\otimes_{R(\beta-\alpha_i)} e(\beta-\alpha_i,i)R^\Lambda(\beta+\alpha_i)\\ &\qquad\oplus \tau_{n+1}\mathbf{k}[x_{n+2}]\otimes e(\beta,i)R^\Lambda(\beta+\alpha_i)\oplus \mathbf{k}[x_{n+2}]\otimes e(\beta,i)R^\Lambda(\beta+\alpha_i). \end{aligned}$$

Proof. We have

$$\begin{aligned} &e(\beta,i^2)R(\beta+2\alpha_i)e(\beta+\alpha_i,i)\otimes_{R(\beta+\alpha_i)} R^\Lambda(\beta+\alpha_i)\\ &\quad= e(\beta,i^2)\big(R(\beta+\alpha_i)e(\beta,i)\tau_{n+1}\otimes_{R(\beta)} e(\beta,i)R(\beta+\alpha_i)\oplus \mathbf{k}[x_{n+2}]\otimes_{\mathbf{k}} R(\beta+\alpha_i)\big)\\ &\qquad\otimes_{R(\beta+\alpha_i)} R^\Lambda(\beta+\alpha_i)\\ &\quad= e(\beta,i^2)\big(R(\beta)e(\beta-\alpha_i,i)\tau_n\otimes_{R(\beta-\alpha_i)} e(\beta-\alpha_i,i)R(\beta)\oplus \mathbf{k}[x_{n+1}]\otimes R(\beta)\big)\tau_{n+1}\\ &\qquad\otimes_{R(\beta)} e(\beta,i)R^\Lambda(\beta+\alpha_i)\oplus \mathbf{k}[x_{n+2}]\otimes_{\mathbf{k}} R^\Lambda(\beta+\alpha_i)\\ &\quad= e(\beta,i^2)R(\beta)e(\beta-\alpha_i,i)\tau_n\tau_{n+1}\otimes_{R(\beta-\alpha_i)} e(\beta-\alpha_i,i^2)R^\Lambda(\beta+\alpha_i)\\ &\qquad\oplus \mathbf{k}[x_{n+1}]\tau_{n+1}\otimes e(\beta,i^2)R^\Lambda(\beta+\alpha_i)\oplus \mathbf{k}[x_{n+2}]\otimes_{\mathbf{k}} e(\beta,i^2)R^\Lambda(\beta+\alpha_i). \end{aligned}$$

Then the lemma follows from $\mathbf{k}[x_{n+1}]\tau_{n+1}\oplus\mathbf{k}[x_{n+2}] = \tau_{n+1}\mathbf{k}[x_{n+2}]\oplus\mathbf{k}[x_{n+2}]$. □

We set

$$\begin{aligned} K &:= e(\beta,i^2)R(\beta+2\alpha_i)e(\beta+\alpha_i,i)\otimes_{R(\beta+\alpha_i)} R^\Lambda(\beta+\alpha_i)\\ &\simeq \frac{e(\beta,i^2)R(\beta+2\alpha_i)e(\beta+\alpha_i,i)}{e(\beta,i^2)R(\beta+2\alpha_i)a^\Lambda(x_1)R(\beta+\alpha_i)e(\beta+\alpha_i,i)}. \end{aligned}$$

Then K is an $(e(\beta,i^2)R(\beta+2\alpha_i)e(\beta,i^2), R^\Lambda(\beta+\alpha_i)\otimes\mathbf{k}[x_{n+2}])$-bimodule.

The preceding lemma says

$$\begin{aligned} K &= R(\beta)\tau_n\tau_{n+1}e(\beta-\alpha_i,i^3)R^\Lambda(\beta+\alpha_i)+\tau_{n+1}\mathbf{k}[x_{n+2}]e(\beta,i^2)R^\Lambda(\beta+\alpha_i)\\ &\quad+\mathbf{k}[x_{n+2}]e(\beta,i^2)R^\Lambda(\beta+\alpha_i). \end{aligned}$$

We define the filtration $\{\Gamma_k\}_{k\in\mathbb{Z}}$ of K by

$$\Gamma_k = \begin{cases} 0 & \text{if } k < -1, \\ R(\beta)\tau_n\tau_{n+1}e(\beta-\alpha_i, i^3)R^\Lambda(\beta+\alpha_i) + e(\beta, i^2)\tau_{n+1}R^\Lambda(\beta+\alpha_i) & \text{if } k = -1, \\ \Gamma_{k-1} + e(\beta, i^2)x_{n+2}^k R^\Lambda(\beta+\alpha_i) + e(\beta, i^2)\tau_{n+1}x_{n+2}^{k+1}R^\Lambda(\beta+\alpha_i) & \text{if } k \geq 0. \end{cases}$$

Note that $\Gamma_k = \Gamma_{k-1} + e(\beta, i^2)x_{n+2}^k R^\Lambda(\beta+\alpha_i) + e(\beta, i^2)x_{n+1}^{k+1}\tau_{n+1}R^\Lambda(\beta+\alpha_i)$ for $k \geq 0$.

Recall that $\mathrm{Gr}_k^\Gamma K := \Gamma_k/\Gamma_{k-1}$. Then we have the following lemma that will be used frequently.

Lemma 5.3. (i) *The Γ_k's are $(R(\beta), R^\Lambda(\beta+\alpha_i))$-bimodules.*

(ii) $\Gamma_k x_{n+2} \subset \Gamma_{k+1}$ *for any k.*

(iii) *Right multiplication by x_{n+2} induces an isomorphism $\mathrm{Gr}_k^\Gamma K \xrightarrow{\sim} \mathrm{Gr}_{k+1}^\Gamma K$ for any $k \geq 0$.*

(iv) $\mathrm{Ker}(x_{n+2}\colon \Gamma_{-1} \to \mathrm{Gr}_0^\Gamma K) = R(\beta)\tau_n\tau_{n+1}e(\beta-\alpha_i, i^3)R^\Lambda(\beta+\alpha_i)$.

Proof. (i) is obvious. (ii) follows from

$$\tau_n\tau_{n+1}x_{n+2} = \tau_n(x_{n+1}\tau_{n+1}+1) = (x_n\tau_n+1)\tau_{n+1}+\tau_n. \tag{5.1}$$

(iii) follows from Lemma 5.2.

Let us prove (iv). Define $S := R(\beta)\tau_n\tau_{n+1}e(\beta-\alpha_i, i^3)R^\Lambda(\beta+\alpha_i)$. Then $Sx_{n+2} \subset \Gamma_{-1} + e(\beta, i^2)R^\Lambda(\beta+\alpha_i)$ by (5.1). The homomorphism $\Gamma_{-1}/S \to (\mathrm{Gr}_0^\Gamma K)/(e(\beta, i^2)R^\Lambda(\beta+\alpha_i))$ is an isomorphism since it is isomorphic to $\mathbf{k}\tau_{n+1} \otimes e(\beta, i)R^\Lambda(\beta+\alpha_i) \xrightarrow[x_{n+2}]{\sim} \mathbf{k}\tau_{n+1}x_{n+2} \otimes e(\beta, i)R^\Lambda(\beta+\alpha_i)$. □

As a corollary, we obtain the following

Lemma 5.4. *Let $m \in \mathbb{Z}$ and let $f(x_{n+2}) \in R^\Lambda(\beta+\alpha_i) \otimes \mathbf{k}[x_{n+2}]$ be a monic polynomial of degree $r \geq 0$ in x_{n+2} and $u \in K$. Assume that $uf(x_{n+2}) \in \Gamma_m$. Then we have:*

(i) *If $m \geq r-1$, then $u \in \Gamma_{m-r}$.*

(ii) $ux_{n+2}^k \in \Gamma_{\max(-1, m-r+k)}$ *for any $k \geq 0$.*

(iii) $uf(x_{n+2}) \equiv ux_{n+2}^r \mod \Gamma_{\max(-1, m-1)}$.

(iv) *If $m < r-1$, then $u \in R(\beta)\tau_n\tau_{n+1}e(\beta-\alpha_i, i^3)R^\Lambda(\beta+\alpha_i)$.*

Proof. (i) It is enough to show that if $u \in \Gamma_k$ and $k > m-r$, then $u \in \Gamma_{k-1}$. For such a u we have $uf(x_{n+2}) \in \Gamma_m \subset \Gamma_{k+r-1}$, and the injectivity of $\mathrm{Gr}_k^\Gamma K \xrightarrow{f(x_{n+2})=x_{n+2}^r} \mathrm{Gr}_{r+k}^\Gamma K$ implies $u \in \Gamma_{k-1}$.

(ii) We have $ux_{n+2}^k f(x_{n+2}) \in \Gamma_{m+k} \subset \Gamma_{r+\max(-1,m-r+k)}$. Hence (i) implies that $ux_{n+2}^k \in \Gamma_{\max(-1,m-r+k)}$.

(iii) follows from (ii).

(iv) By (ii), $u, ux_{n+2} \in \Gamma_{-1}$. Then the assertion follows from Lemma 5.3(iv). □

Our goal in this subsection is to prove Proposition 5.7 below, and the following lemma is the starting point.

Lemma 5.5. *For $\nu \in I^\beta$ we have, as an element of K,*

$$\tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\varphi_1\cdots\varphi_{n+1}e(\nu,i^2)\prod_{a\le n,\,\nu_a=i}(x_a-x_{n+2}) \equiv -\tau_{n+1}a^\Lambda(x_{n+2})\prod_{\nu_a\neq i}Q_{i,\nu_a}(x_{n+2},x_a)e(\nu,i,i) \bmod \Gamma_{-1}.$$

Proof. We have $\tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\varphi_1\cdots\varphi_{n+1} = \tau_{n+1}\cdots\tau_1\varphi_1\cdots\varphi_{n+1}a^\Lambda(x_{n+2})$. We shall show, for $a \le n$,

$$\tag{5.2} \begin{aligned}\tau_{n+1}\cdots\tau_a\varphi_a\cdots\varphi_{n+1}a^\Lambda(x_{n+2})e(\nu,i^2)&\prod_{a\le k\le n,\,\nu_k=i}(x_k-x_{n+2}) \cdot\prod_{k<a,\,\nu_k\neq i}Q_{i,\nu_a}(x_{n+2},x_k)\\ \equiv \tau_{n+1}\cdots\tau_{a+1}\varphi_{a+1}\cdots\varphi_{n+1}a^\Lambda(x_{n+2})e(\nu,i^2)&\cdot\prod_{a+1\le k\le n,\,\nu_k=i}(x_k-x_{n+2})\prod_{k<a+1,\,\nu_k\neq i}Q_{i,\nu_a}(x_{n+2},x_k).\end{aligned}$$

If $\nu_a \neq i$, this is obvious. Assume that $\nu_a = i$. Then

$$\begin{aligned}\tau_{n+1}\cdots\tau_a\varphi_a\cdots\varphi_{n+1}&a^\Lambda(x_{n+2})e(\nu,i^2)(x_a-x_{n+2})\\ &=\tau_{n+1}\cdots\tau_a(x_{a+1}-x_a)\varphi_a\cdots\varphi_{n+1}a^\Lambda(x_{n+2})e(\nu,i^2)\\ &=\tau_{n+1}\cdots\tau_{a+1}(\varphi_a+1)\varphi_a\varphi_{a+1}\cdots\varphi_{n+1}a^\Lambda(x_{n+2})e(\nu,i^2)\\ &=\tau_{n+1}\cdots\tau_{a+1}(\varphi_a+1)\varphi_{a+1}\cdots\varphi_{n+1}a^\Lambda(x_{n+2})e(\nu,i^2)\\ &=\tau_{n+1}\cdots\tau_{a+1}\varphi_a\varphi_{a+1}\cdots\varphi_{n+1}a^\Lambda(x_{n+2})e(\nu,i^2)\\ &\quad+\tau_{n+1}\cdots\tau_{a+1}\varphi_{a+1}\cdots\varphi_{n+1}a^\Lambda(x_{n+2})e(\nu,i^2).\end{aligned}$$

We shall show that for any $f(x_{n+2})$ and $g = g(x_1,\dots,x_n)$,

$$\tag{5.3} \tau_{n+1}\cdots\tau_{a+1}\varphi_a\varphi_{a+1}\cdots\varphi_{n+1}e(\nu,i^2)f(x_{n+2})g \in \Gamma_{-1}.$$

Indeed,

$$\begin{aligned}\tau_{n+1}\cdots\tau_{a+1}\varphi_a\varphi_{a+1}\cdots\varphi_{n+1}e(\nu,i^2)f(x_{n+2})\\ &= \tau_{n+1}\cdots\tau_{a+1}f(x_a)\varphi_a\varphi_{a+1}\cdots\varphi_{n+1}e(\nu,i^2)\\ &= f(x_a)\tau_{n+1}\cdots\tau_{a+1}\varphi_a\varphi_{a+1}\cdots\varphi_{n+1}e(\nu,i^2)\\ &= f(x_a)\varphi_a\varphi_{a+1}\cdots\varphi_{n+1}\tau_n\cdots\tau_a e(\nu,i^2).\end{aligned}$$

We have

$$\begin{aligned}\varphi_n\varphi_{n+1} &= \varphi_n(x_{n+1}\tau_{n+1}-\tau_{n+1}x_{n+1})\\ &= x_n(x_n\tau_n-\tau_n x_n)\tau_{n+1}-(x_n\tau_n-\tau_n x_n)\tau_{n+1}x_{n+1}\\ &= x_n^2\tau_n\tau_{n+1}-x_n\tau_n\tau_{n+1}x_n-x_n\tau_n\tau_{n+1}x_{n+1}+\tau_n\tau_{n+1}x_n x_{n+1},\end{aligned}$$

and this belongs to Γ_{-1}. Hence we obtain (5.3). Then the repeated use of (5.2) implies that

$$\begin{aligned}\tau_{n+1}\cdots\tau_1\varphi_1\cdots\varphi_{n+1}a^\Lambda(x_{n+2})e(\nu,i^2)\prod_{k\le n,\,\nu_k=i}(x_k-x_{n+2})\\ &\equiv \tau_{n+1}\varphi_{n+1}a^\Lambda(x_{n+2})e(\nu,i^2)\prod_{\nu_k\neq i}Q_{i,\nu_a}(x_{n+2},x_k).\end{aligned}$$

Finally $\tau_{n+1}\varphi_{n+1}e(\nu,i^2)=\tau_{n+1}(\tau_{n+1}(x_{a+1}-x_a)-1)e(\nu,i^2)=-\tau_{n+1}e(\nu,i^2)$. □

Lemma 5.6. *The following equality holds as an equality in K:*

$$\begin{aligned}\tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\varphi_1\cdots\varphi_{n+1}e(\nu,i^2)\\ &= \tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\tau_1\cdots\tau_{n+1}e(\nu,i^2)\prod_{k=n+1\ or\ \nu_k=i}(x_{n+2}-x_a).\end{aligned}$$

Proof. It is enough to show that

$$\begin{aligned}\tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\tau_1\cdots\tau_{a-1}\varphi_a\cdots\varphi_{n+1}e(\nu,i^2)\\ = \tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\tau_1\cdots\tau_a\varphi_{a+1}\cdots\varphi_{n+1}e(\nu,i^2)(x_{n+2}-x_k)^{\delta(a=n+1\,\mathrm{or}\,\nu_a=i)}.\end{aligned}\tag{5.4}$$

If $\nu_a\neq i$ this is trivial. If $\nu_a=i$ or $a=n+1$ then

$$\begin{aligned}\varphi_a\cdots\varphi_{n+1}e(\nu,i^2) &= (\tau_a(x_{a+1}-x_a)-1)\varphi_{a+1}\cdots\varphi_{n+1}e(\nu,i^2)\\ &= \tau_a\varphi_{a+1}\cdots\varphi_{n+1}(x_{n+2}-x_a)e(\nu,i^2)-\varphi_{a+1}\cdots\varphi_{n+1}e(\nu,i^2).\end{aligned}$$

Since

$$\begin{aligned}\tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\tau_1\cdots\tau_{a-1}\varphi_{a+1}\cdots\varphi_{n+1}e(\nu,i^2)\\ &= \tau_{n+1}\cdots\tau_1\varphi_{a+1}\cdots\varphi_{n+1}a^\Lambda(x_1)\tau_1\cdots\tau_{a-1}e(\nu,i^2)\end{aligned}$$

vanishes as an element of K for $a\le n+1$, we obtain (5.4). □

Thus we have

$$(-1)^p\tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\tau_1\cdots\tau_{n+1}e(\nu,i^2)\prod_{a=n+1\text{ or }\nu_a=i}(x_{n+2}-x_a)^2$$
$$\equiv -\tau_{n+1}a^\Lambda(x_{n+2})\prod_{\nu_a\neq i}Q_{i,\nu_a}(x_{n+2},x_a)e(\nu,i^2)(x_{n+2}-x_{n+1}) \bmod \Gamma_{-1}.$$

Since $\tau_{n+1}(x_{n+2}-x_{n+1})\in\Gamma_0$, Lemma 5.4 implies

$$\tau_{n+1}a^\Lambda(x_{n+2})\prod_{\nu_a\neq i}Q_{i,\nu_a}(x_{n+2},x_a)e(\nu,i^2)(x_{n+2}-x_{n+1})$$
$$\equiv \tau_{n+1}x_{n+2}^{\langle h_i,\Lambda-\beta\rangle+2p+1}e(\nu,i^2)\prod_{\nu_a\neq i}t_{i\nu_a} \bmod \Gamma_{\langle h_i,\Lambda-\beta\rangle+2p-1}.$$

In particular

$$\tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\tau_1\cdots\tau_{n+1}e(\nu,i^2)\prod_{a=n+1\text{ or }\nu_a=i}(x_{n+2}-x_a)^2\in\Gamma_{\langle h_i,\Lambda-\beta\rangle+2p}.$$

Hence this element is equivalent to $\tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\tau_1\cdots\tau_{n+1}e(\nu,i,i)x_{n+2}^{2p+2}$ modulo $\Gamma_{\langle h_i,\Lambda-\beta\rangle+2p-1}$.

Thus we obtain the following proposition.

Proposition 5.7. *For $\beta\in Q^+$, let p be the number of times that α_i appears in β, and set $\lambda:=\Lambda-\beta$, $\lambda_i:=\langle h_i,\lambda\rangle$. Then there exists $c\in\mathbf{k}_0^{*\times}$ such that*

$$\tau_{n+1}x_{n+2}^{\lambda_i+2p+1}e(\beta,i^2)\equiv c\tau_{n+1}\cdots\tau_1 a^\Lambda(x_1)\tau_1\cdots\tau_{n+1}e(\beta,i^2)x_{n+2}^{2p+2} \bmod \Gamma_{\lambda_i+2p-1}.$$

Note that $\lambda_i+2p\geq 0$.

§5.3

Let us define two homomorphisms

$$P\colon R(\beta)e(\beta-\alpha_i,i)\otimes_{R(\beta-\alpha_i)}e(\beta-\alpha_i,i^2)R^\Lambda(\beta+\alpha_i)\to K,$$
$$E\colon R(\beta)e(\beta-\alpha_i,i)\otimes_{R(\beta-\alpha_i)}e(\beta-\alpha_i,i^2)R^\Lambda(\beta+\alpha_i)\to e(\beta,i)R^\Lambda(\beta+\alpha_i)$$

by $P(a\otimes b)=a\tau_n\tau_{n+1}\otimes b$ and $E(a\otimes b)=ab$. Then P is injective and Lemma 5.4 implies

$$\operatorname{Im}(P)=\operatorname{Ker}(x_{n+2}\colon\Gamma_{-1}\to\operatorname{Gr}_0^\Gamma K). \tag{5.5}$$

We can see that $R(\beta)e(\beta-\alpha_i,i)\otimes_{R(\beta-\alpha_i)}e(\beta-\alpha_i,i^2)R^\Lambda(\beta+\alpha_i)$ has a structure of $(R(\beta)\otimes \mathbf{k}\langle x_n,x_{n+1},\tau_n\rangle, \mathbf{k}[x_n]\otimes R^\Lambda(\beta+\alpha_i))$-bimodule by

$$\begin{aligned}
&(a\otimes b)(x_n\otimes 1)=ax_n\otimes b,\\
&(1\otimes \tau_n)(a\otimes b)=a\otimes \tau_n b,\\
&(1\otimes x_k)(a\otimes b)=a\otimes x_k b \quad \text{for } k=n,n+1.
\end{aligned}$$

Here $\mathbf{k}\langle x_n,x_{n+1},\tau_n\rangle$ is the $\mathbf{k}$-subalgebra of $e(\beta-\alpha_i,i^2)R^\Lambda(\beta+\alpha_i)e(\beta-\alpha_i,i^2)$ generated by x_n,x_{n+1},τ_n, and it is isomorphic to the nil affine Hecke algebra $R(2\alpha_i)$.

Lemma 5.8. *For any $z\in R(\beta)e(\beta-\alpha_i,i)\otimes_{R(\beta-\alpha_i)}e(\beta-\alpha_i,i^2)R^\Lambda(\beta+\alpha_i)$, we have*

$$P(z)x_{n+2}=P(z(x_n\otimes 1))+\tau_{n+1}E(z)+E((1\otimes\tau_n)z).$$

Proof. For $z=a\otimes b$, we have

$$\begin{aligned}
P(a\otimes b)x_{n+2}&=a\tau_n\tau_{n+1}x_{n+2}\otimes b=a\tau_n(x_{n+1}\tau_{n+1}+1)\otimes b\\
&=a(x_n\tau_n+1)\tau_{n+1}\otimes b+1\otimes a\tau_n b.
\end{aligned}$$

□

Corollary 5.9. *If $z\in R(\beta)e(\beta-\alpha_i,i)\otimes_{R(\beta-\alpha_i)}e(\beta-\alpha_i,i^2)R^\Lambda(\beta+\alpha_i)$ satisfies $P(z)x_{n+2}\in\Gamma_{-1}$, then $E((1\otimes\tau_n)z)=0$.*

Indeed, $P(z)x_{n+2}\equiv E((1\otimes\tau_n)z) \mod \Gamma_{-1}$.

Set $K^\Lambda=e(\beta,i^2)R^\Lambda(\beta^+2\alpha_i)e(\beta+\alpha_i,i)$. Hence we have

$$\begin{aligned}
K&\simeq\frac{e(\beta,i^2)R(\beta+2\alpha_i)e(\beta+\alpha_i,i)}{e(\beta,i^2)R(\beta^+2\alpha_i)a^\Lambda(x_1)R(\beta+\alpha_i)e(\beta+\alpha_i,i)},\\
K^\Lambda&\simeq\frac{e(\beta,i^2)R(\beta+2\alpha_i)e(\beta+\alpha_i,i)}{e(\beta,i^2)R(\beta^+2\alpha_i)a^\Lambda(x_1)R(\beta+2\alpha_i)e(\beta+\alpha_i,i)}.
\end{aligned}$$

Then there exists a surjective homomorphism $p\colon K\to K^\Lambda$. Note that

$$p(\tau_{n+1}\cdots\tau_1a^\Lambda(x_1)\tau_1\cdots\tau_{n+1}e(\beta,i^2))=0. \tag{5.6}$$

Let us denote by $\{\Gamma^\Lambda_k\}_{k\in\mathbb{Z}}$ the filtration of K^Λ induced by the filtration Γ of K.

§5.4. Proof of (4.1)

Assume that $\lambda_i\geq 2$. The statement (4.1) can be read as

$$x_{n+2}^{\lambda_i-2}e(\beta,i^2)+x_{n+1}^{\lambda_i-1}\tau_{n+1}e(\beta,i^2)\in\Gamma^\Lambda_{\lambda_i-3}\quad\text{as an element of } K^\Lambda.$$

By Proposition 5.7 and Lemma 5.3, we have

$$\tau_{n+1}x_{n+2}^{\lambda_i-1}e(\beta,i^2)\equiv c\tau_{n+1}\cdots\tau_1 a^{\Lambda}(x_1)\tau_1\cdots\tau_{n+1}e(\beta,i^2) \bmod \Gamma_{\lambda_i-3}$$

as an element of K. Then the desired result holds since

$$\tau_{n+1}x_{n+2}^{\lambda_i-1}e(\beta,i^2)\equiv (x_{n+1}^{\lambda_i-1}\tau_{n+1}+x_{n+2}^{\lambda_i-2})e(\beta,i^2) \bmod \Gamma_{\lambda_i-3}.$$

§5.5. Proof of (4.2)

Assume that $\lambda_i=1$. Set

$$w:=\tau_{n+1}e(\beta,i^2)-c\tau_{n+1}\cdots\tau_1 a^{\Lambda}(x_1)\tau_1\cdots\tau_{n+1}e(\beta,i^2)\in K.$$

Then Proposition 5.7 together with Lemma 5.4(iv) implies that $w, wx_{n+2}\in\Gamma_{-1}$ and $w\in\operatorname{Im}(P)$. Hence we can write $w=P(z)$ for some $z\in R(\beta)e(\beta-\alpha_i,i)\otimes_{R(\beta-\alpha_i)} e(\beta-\alpha_i,i^2)R^{\Lambda}(\beta+\alpha_i)$. Then Corollary 5.9 implies that

$$E(1\otimes\tau_n)z)=0.$$

Let us define the morphism

$$T\colon R(\beta)e(\beta-\alpha_i,i)\otimes_{R(\beta-\alpha_i)} e(\beta-\alpha_i,i^2)R^{\Lambda}(\beta+\alpha_i) \to e(\beta,i)R^{\Lambda}(\beta+\alpha_i)\otimes_{R^{\Lambda}(\beta)} e(\beta,i)R^{\Lambda}(\beta+\alpha_i)$$

by $T(a\otimes b)=(a\tau_n)\otimes b$. Then we have

$$\Sigma(T(z))=p(P(z)),\qquad \widetilde{E}(T(z))=E((1\otimes\tau_n)z)=0.$$

Let us show that $u:=e(\beta,i)\otimes e(\beta,i)-T(z)$ satisfies the condition (4.2).

(a) $\Sigma(T(z))=p(P(z))=\tau_{n+1}e(\beta,i^2)$ as an element of $R^{\Lambda}(\beta+2\alpha_i)$.
(b) $\widetilde{E}(u)=e(\beta,i)-E(T(z))=e(\beta,i)$.
(c) is obvious.

§5.6. Proof of (4.3)

Assume that $\lambda_i\le 0$. Note that $\ell:=-\lambda_i\le 2p$. Then Proposition 5.7 says that, by setting

$$w:=c\tau_{n+1}\cdots\tau_1 a^{\Lambda}(x_1)\tau_1\cdots\tau_{n+1}e(\beta,i^2),$$

the element $(wx_{n+2}^{2+\ell}-\tau_{n+1}x_{n+2}e(\beta,i^2))x_{n+2}^{-\ell+2p}$ of K belongs to $\Gamma_{-\ell+2p-1}$.

Hence we have

$$wx_{n+2}^{\ell+2}-\tau_{n+1}x_{n+2}e(\beta,i^2)\in\Gamma_{-1}.$$

Since $\tau_{n+1}x_{n+2}e(\beta,i^2)\in\Gamma_0$, we have $wx_{n+2}^{\ell+2}\in\Gamma_0$. Hence Lemma 5.4 implies that $wx_{n+2}^{k}\in\Gamma_{-1}$ for $0\le k\le\ell+1$. We set

$$wx_{n+2}^{k}=P(z_k)+\tau_{n+1}y_k\qquad\text{for } 0\le k\le\ell+1$$

with $z_k\in R(\beta)e(\beta-\alpha_i,i)\otimes_{R(\beta-\alpha_i)}e(\beta-\alpha_i,i^2)R^{\Lambda}(\beta+\alpha_i)$ and $y_k\in e(\beta,i)R^{\Lambda}(\beta+\alpha_i)$.

Then for $1 \le k \le \ell+2$ we have

$$\begin{aligned} wx_{n+2}^k &= (P(z_{k-1}) + \tau_{n+1}y_{k-1})x_{n+2} \\ &= P(z_{k-1}(x_n \otimes 1)) + \tau_{n+1}E(z_{k-1}) + E((1\otimes\tau_n)z_{k-1}) + \tau_{n+1}x_{n+2}y_{k-1}. \end{aligned}$$

Hence Lemma 5.2 implies, for $1 \le k \le \ell+1$,

$$z_k = z_{k-1}(x_n \otimes 1), \qquad y_k = E(z_{k-1}), \qquad E((1\otimes\tau_n)z_{k-1}) = 0, \qquad y_{k-1} = 0.$$

Since $wx_{n+2}^{\ell+2} \equiv \tau_{n+1}x_{n+2}e(\beta,i^2) \mod \Gamma_{-1}$, it follows that $y_{\ell+1} = e(\beta,i)$ and $E((1\otimes\tau_n)z_{\ell+1}) = 0$. Thus we obtain $z_k = z_0(x_n^k \otimes 1)$ for $0 \le k \le 1+\ell$, and

$$E(z_0(x_n^k \otimes 1)) = \begin{cases} 0 & \text{for } 0 \le k \le \ell-1, \\ e(\beta,i) & \text{for } k = \ell, \end{cases} \tag{5.7}$$

$$E((1\otimes\tau_n)z_0(x_n^k\otimes 1)) = 0 \quad 0 \le k \le \ell+1. \tag{5.8}$$

Let us denote by

$$\begin{aligned} q\colon R(\beta)e(\beta-\alpha_i,i) \otimes_{R(\beta-\alpha_i)} e(\beta-\alpha_i,i^2)R^\Lambda(\beta+\alpha_i) \\ \to R^\Lambda(\beta)e(\beta-\alpha_i,i) \otimes_{R^\Lambda(\beta-\alpha_i)} e(\beta-\alpha_i,i^2)R^\Lambda(\beta+\alpha_i) \end{aligned}$$

the canonical homomorphism, and set $v = q(z_0)$. Then (a) and (b) in (4.3) follow from $T_k(v) = E(z_0(x_n^k \otimes 1))$. The equality $G(v) = 0$ follows from $G(v) = p(P(z_0)) = p(w) = 0$.

Finally let us prove (d) and (e). We have

$$E((1\otimes x_n^k\tau_n)z_0) = E((1\otimes\tau_n)z_0(x_n^k\otimes 1)) = 0 \quad \text{for } 0 \le k \le \ell+1$$

by (5.8). On the other hand we have $H_k(v) = E((1\otimes\tau_n x_{n+1}^k)z_0)$. Since $\tau_n x_{n+1}^k = x_n^k\tau_n + \sum_{a+b=k-1} x_{n+1}^a x_n^b$, we obtain

$$\begin{aligned} H_k(v) &= E((1\otimes x_n^k\tau_n)z_0) + \sum_{a+b=k-1} x_{n+1}^a E((1\otimes x_n^b)z_0) \\ &= \sum_{a+b=k-1} x_{n+1}^a E(z_0(x_n^b \otimes 1)). \end{aligned}$$

Hence (5.7) implies that $H_k(v) = 0$ for $0 \le k \le \ell$ and $H_{\ell+1}(v) = e(\beta,i)$.

Thus the proof of (4.1)–(4.3) is complete.

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