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Asymptotic Behavior of the Transition Density of an Ergodic Linear Diffusion

by

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Abstract

Positive recurrent diffusions on the line are treated. We study the asymptotic behavior of the transition density in the long term. The problem is equivalent to the study of Krein's correspondence for bounded strings.

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§1. Introduction

This paper is a continuation of [6], where we studied Tauberian theorems for Krein's correspondence. We shall apply one of the results to linear diffusions.

Let $X = (X_t)_{t\geq 0}$ be a regular, conservative diffusion on an interval $I \subset \mathbb{R}$. We allow the case of generalized diffusions. It is well known as Feller's canonical representation that the local generator is of the form

$$\mathcal{L} = \frac{d}{dm(x)}\frac{d}{ds(x)}, \qquad x \in I, \tag{1.1}$$

where $s(x)$ is an increasing, continuous function and dm is a nonnegative Radon measure on I. $s(x)$ and $dm(x)$ are referred to as the *scale function* and the *speed measure*, respectively (see e.g., [3, Chap. 5]).

Let $p(t,x,y)$ be the transition density with respect to $dm(x)$. Then, for every $(x,y) \in I \times I$,

$$p(t,x,y) \to \frac{1}{\widehat{m}} \qquad (t \to \infty), \tag{1.2}$$

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where

$$\widehat{m} = \int_I dm(x) \qquad (\leq \infty) \tag{1.3}$$

(cf. [2, pp. 35–37]).

In the present article we shall evaluate the rate of convergence in (1.2) when $\widehat{m} < \infty$. Note that the condition $\widehat{m} < \infty$ is equivalent to the diffusion being *ergodic* (or *positive recurrent*) in the sense that the process has a finite expected first hitting time between every pair of points in the state space.

Our main result is as follows: Suppose that the diffusion is one-sided in the sense that $I = [0, \infty)$; then, for $0 < \beta < 1$,

$$p(t, x, y) - \frac{1}{\widehat{m}} \sim \text{const} \cdot t^{-\beta} \qquad (t \to \infty)$$

holds if and only if

$$\int_{s^{-1}(x)}^{\infty} dm(y) \sim \text{const} \cdot x^{-\beta/(\beta+1)} \qquad (x \to \infty)$$

(Theorem 3.1). Here, "$f(x) \sim g(x)$" means that $f(x)/g(x) \to 1$. A similar problem for the bilateral case (i.e., $I = (-\infty, \infty)$) will be treated in Section 5.

§2. Preliminaries

We start by explaining Krein's correspondence. By a (Krein) *string* we mean a function

$$m : (-\infty, \infty) \to \overline{\mathbb{R}}_+ := [0, \infty]$$

that is nondecreasing, right-continuous and vanishes on $(-\infty, 0)$ (i.e., $m(-0) = 0$)). We exclude the trivial case that $m(x) \equiv 0$. Throughout the paper $\mathcal{M}$ denotes the totality of strings. The Lebesgue–Stieltjes measure $dm(x)$ describes the mass distribution of the string and

$$\ell = \ell(m) := \sup\{x \mid m(x) < \infty\} \qquad (\in (0, \infty]) \tag{2.1}$$

is its "length".

With $m \in \mathcal{M}$ we associate the generalized Sturm–Liouville operator

$$\mathcal{L} = \frac{d}{dm(x)} \frac{d}{dx}, \qquad 0 \leq x < \ell. \tag{2.2}$$

Notice that we are considering only the special case $s(x) = x$ in (1.1) for the present. To treat the general case see Remark 3.7 in Section 3. The boundary condition for (2.2) is: 0 is regular and reflecting, while ℓ is absorbing if regular.

(Other boundary conditions at the right end can be reduced to this case using the notion of *inextensible measures* introduced in [8]. See Example 2.2.)

For each $\lambda \in \mathbb{C}$, we can define $\varphi_\lambda(x)$ and $\psi_\lambda(x)\,(-\infty < x < \ell)$ as the unique solutions of

$$-\mathcal{L}u = \lambda u$$

with the initial conditions $(u(0), u'(-0)) = (1, 0)$ and $(u(0), u'(-0)) = (0, 1)$, respectively; or, precisely, the solutions of the following integral equations:

$$\begin{cases} \varphi_\lambda(x) = 1 - \lambda \displaystyle\int_{-0}^{x} (x-y)\varphi_\lambda(y)\,dm(y) & (0 \le x < \ell), \\ \psi_\lambda(x) = x - \lambda \displaystyle\int_{-0}^{x} (x-y)\psi_\lambda(y)\,dm(y) & (0 \le x < \ell). \end{cases} \tag{2.3}$$

Then

$$H(\lambda) := \lim_{x \uparrow \ell} \frac{\psi_\lambda(x)}{\varphi_\lambda(x)} \left(= \int_0^{\ell - 0} \frac{dx}{\varphi_\lambda(x)^2} \right) \qquad (\lambda \in \mathbb{C},\, \mathrm{Re}\,\lambda < 0)$$

is called the *characteristic function* of m and has the following representation:

$$H(\lambda) = a + \int_{[0,\infty)} \frac{\sigma(d\xi)}{\xi - \lambda}, \tag{2.4}$$

where $a = \inf\{x > 0 \mid m(x) > 0\}$ (≥ 0) and $\sigma(d\xi)$ is a Radon measure satisfying

$$\int_{[0,\infty)} \frac{\sigma(d\xi)}{\xi + 1} < \infty. \tag{2.5}$$

The function $\sigma(t) = \int_{[0,t]} \sigma(d\xi)$ $(t \ge 0)$ is referred to as the *spectral function*. For consistency of notation with other related papers such as [8] and [5], we put

$$h(s) = H(-s), \qquad s > 0,$$

throughout the paper. Conversely, to any function $H(\lambda)$ of the form (2.4) with $a \ge 0$ and $\sigma(d\xi)$ satisfying (2.5), there corresponds a unique $m \in \mathcal{M}$. This bijection $H(\lambda)$ (or $h(s)$) $\leftrightarrow m(x)$ is called *Krein's correspondence*. For further information on Krein's correspondence and its application to linear diffusions see Kotani–Watanabe [8]. See also Kotani [7] for an extension.

It is well known that the transition density $p(t, x, y)$ and the resolvent kernel $g_s(x, y)$ (with respect to $dm(x)$) of the diffusion corresponding to (2.2) can be expressed as follows (see e.g. [4]):

$$p(t, x, y) = \int_{-0}^{\infty} e^{-t\xi} \varphi_\xi(x)\varphi_\xi(y)\,d\sigma(\xi) \qquad (t > 0), \tag{2.6}$$

$$g_s(x, y) := \int_0^\infty e^{-st} p(t, x, y)\,dt = \int_{-0}^{\infty} \frac{\varphi_\xi(x)\varphi_\xi(y)}{s + \xi}\,d\sigma(\xi) \qquad (s > 0). \tag{2.7}$$

In particular, since $\varphi_\xi(0) = 1$, we have

$$p(t,0,0) = \int_{-0}^{\infty} e^{-t\xi}\, d\sigma(\xi) \qquad (t > 0), \tag{2.8}$$

$$g_s(0,0) = \int_{-0}^{\infty} \frac{d\sigma(\xi)}{s+\xi} \qquad (= h(s) - a) \qquad (s > 0). \tag{2.9}$$

Note that the asymptotic behavior of $p(t,x,y)$ is essentially the same as that of $p(t,0,0)$ in the following sense:

$$p(t,x,y) = p(t,0,0) + o(1/t) \qquad (t \to \infty). \tag{2.10}$$

Indeed,

$$\begin{aligned} t|p(t,x,y) - p(t,0,0)| &= t\left|\int_{+0}^{\infty} e^{-t\xi}(\varphi_\xi(x)\varphi_\xi(y) - 1)d\sigma(\xi)\right| \\ &\le \sup_{0<\xi\le 1} \frac{|\varphi_\xi(x)\varphi_\xi(y) - 1|}{\xi} \int_{+0}^{1} t\xi e^{-t\xi}\, d\sigma(\xi) + o(1), \end{aligned}$$

while

$$\int_{+0}^{1} t\xi e^{-t\xi}\, d\sigma(\xi) \le 2\int_{+0}^{1} \frac{d\sigma(\xi)}{1+t\xi} = o(1) \qquad (t \to \infty).$$

Since $p(t,0,0)$ and $h(s)$ are the Laplace transform and the Stieltjes transform of $\sigma(\xi)$, respectively, the study of the asymptotic behavior of $p(t,0,0)$ (or of $p(t,x,y)$) as $t \to \infty$ can be reduced to that of $\sigma(\xi)$ as $\xi \to +0$ or of $h(s)$ as $s \to +0$. The fundamental relationship is

$$\sigma(0) = \lim_{t\to\infty} p(t,0,0) = \lim_{s\to+0} sh(s) = \frac{1}{\widehat{m}} \tag{2.11}$$

with the convention that $1/\infty = 0$, where

$$\widehat{m} := \lim_{x\to\infty} m(x) = \int_{[0,\infty)} dm(x) \qquad (\le \infty)$$

is the total mass of the string. The first two equalities of (2.11) are elementary, while the last is well-known.

Remark 2.1. The diffusion is ergodic (positive recurrent) if and only if $\widehat{m} < \infty$. Note that, when $\widehat{m} < \infty$, we have $\ell = \infty$ because $\widehat{m} < \infty$ implies $m(x) < \infty$ (for all $x > 0$) so that $\ell = \infty$ (see (2.1)). On the other hand when $\widehat{m} = \infty$, the process is null-recurrent if $\ell = \infty$ and is transient or non-conservative if $\ell < \infty$.

For the convenience of the readers who are not familiar with the notation used here we give an example:

Example 2.2. (1) (absorbing boundary) Let $m(x) = x1_{[0,1)}(x) + \infty \cdot 1_{[1,\infty)}(x)$ so that $dm(x) = 1_{[0,1)}(x)dx + \infty \cdot \delta_1(dx)$. Then $\ell = 1$ and $\widehat{m} = \infty$. This string m corresponds to $\mathcal{L} = d^2/dx^2$ on $(0, 1)$ with the boundary condition $u'(0) = u(1) = 0$. The process is reflected at $x = 0$ and absorbed at $x = 1$. Since

$$\varphi_{-s}(x) = \cosh(\sqrt{s}\,x), \qquad \psi_{-s}(x) = \frac{1}{\sqrt{s}} \sinh(\sqrt{s}\,x), \qquad x \in [0, 1], \tag{2.12}$$

we have

$$h(s) = \lim_{x \uparrow 1} \frac{\psi_{-s}(x)}{\varphi_{-s}(x)} = \frac{1}{\sqrt{s}} \frac{\sinh \sqrt{s}}{\cosh \sqrt{s}}, \qquad s > 0.$$

(2) (reflecting boundary) Let $m(x) = x1_{[0,1)}(x) + 1_{[1,\infty)}(x)$. Then $\ell = \infty$ (not $\ell = 1$!) and $\widehat{m} = 1$. The string m corresponds to $\mathcal{L} = d^2/dx^2$ on $(0, 1)$ with the boundary condition $u'(0) = u'(1) = 0$. Both boundaries are reflecting and the process is ergodic. In this case $\varphi_{-s}(x)$ and $\psi_{-s}(x)$ are the same as (2.12) for $x \in [0, 1]$ and are linear on $[1, \infty)$:

$$\begin{cases} \varphi_{-s}(x) = \varphi'_{-s}(1)(x-1) + \varphi_{-s}(1), \\ \psi_{-s}(x) = \psi'_{-s}(1)(x-1) + \psi_{-s}(1), \end{cases} \qquad x \in [1, \infty).$$

Therefore,

$$h(s) = \lim_{x \uparrow \infty} \frac{\psi_{-s}(x)}{\varphi_{-s}(x)} = \frac{\psi'_{-s}(1)}{\varphi'_{-s}(1)} = \frac{\cosh \sqrt{s}}{\sqrt{s}\,\sinh \sqrt{s}}, \qquad s > 0.$$

(3) (elastic boundary) Let $c > 0$ and $m(x) = x1_{[0,1)}(x) + 1_{[1,1+c)}(x) + \infty \cdot 1_{[1+c,\infty)}(x)$ so that $dm(x) = 1_{[0,1)}(x)dx + \infty \cdot \delta_{1+c}(dx)$. Then $\ell = 1 + c$ and $\widehat{m} = \infty$. The string m corresponds to $\mathcal{L} = d^2/dx^2$ on $(0, 1)$ with the boundary condition $u'(0) = u(1 + c) = 0$ (i.e., $u'(0) = u'(1)c + u(1) = 0$). The process dies at $x = 1$ with a suitable probability. As in case (2), we have

$$h(s) = \lim_{x \uparrow 1+c} \frac{\psi_{-s}(x)}{\varphi_{-s}(x)} = \frac{\psi'_{-s}(1)c + \psi_{-s}(1)}{\varphi'_{-s}(1)c + \varphi_{-s}(1)} = \frac{c \cosh \sqrt{s} + (1/\sqrt{s}) \sinh \sqrt{s}}{c\sqrt{s}\,\sinh \sqrt{s} + \cosh \sqrt{s}}, \qquad s > 0.$$

§3. Main results

Let $m \in \mathcal{M}$, and let $h(s), \sigma(\xi), p(t, x, y), \widehat{m}$ be as in Section 2. Our main result is the following:

Theorem 3.1. *Assume that* $\widehat{m} < \infty$ *(and hence* $\ell = \infty$*). Let* $1 < \alpha < 2$ *and let* $\gamma = 1 - 1/\alpha$ *(so that* $0 < \gamma < 1/2$*). Then, for* $K > 0$*, the following conditions are* *equivalent:*

$$\widehat{m} - m(x) \sim Kx^{-\gamma} \qquad (x \to \infty); \tag{3.1}$$

$$h(s) - \frac{1}{\widehat{m}s} \sim \frac{K^\alpha}{\widehat{m}^2} C_\alpha s^{\alpha-2} \qquad (s \to +0); \tag{3.2}$$

$$\sigma(\xi) - \frac{1}{\widehat{m}} \sim \frac{K^\alpha}{\widehat{m}^2} \frac{C_\alpha}{\Gamma(2-\alpha)\,\Gamma(\alpha)} \xi^{\alpha-1} \qquad (\xi \to +0); \tag{3.3}$$

$$p(t,0,0) - \frac{1}{\widehat{m}} \sim \frac{K^\alpha}{\widehat{m}^2} \frac{C_\alpha}{\Gamma(2-\alpha)} t^{1-\alpha} \qquad (t \to \infty), \tag{3.4}$$

where

$$C_\alpha = \frac{\Gamma(2-\alpha)}{\Gamma(\alpha)} \{\alpha(\alpha-1)\}^{\alpha-1}. \tag{3.5}$$

Note that, thanks to (2.10), $p(t,0,0)$ in (3.4) may be replaced by $p(t,x,y)$ for any fixed (x,y).

In fact we can generalize Theorem 3.1 as follows: A measurable function $L : (c,\infty) \to (0,\infty)$ (for some $c > 0$) is said to be *slowly varying* at infinity [or at 0] if

$$\lim_{\lambda\to\infty[+0]} \frac{L(\lambda x)}{L(\lambda)} = 1 \qquad (\forall x > 0).$$

Typical examples are $L(x) = \text{const}, \log(x+1), \exp\sqrt{x}$. Also a measurable function of the form $\psi(x) = x^\rho L(x)$ with a suitable slowly varying function L and a real ρ is said to be *regularly varying* (at infinity [or at 0]) with index ρ. In other words $\psi(x) > 0$ is regularly varying with index ρ if and only if

$$\lim_{\lambda\to\infty[+0]} \frac{\psi(\lambda x)}{\psi(\lambda)} = x^\rho \qquad (\forall x > 0).$$

If $\psi(x)$ is regular varying with index $\rho > 0$, then so is its asymptotic inverse $\psi^{-1}(x)$ with index $1/\rho$. We refer to [1] for properties of regularly varying functions.

Theorem 3.2. *Assume that $\widehat{m} < \infty$. Let α, γ and C_α be as in Theorem 3.1, and let $\psi(x) = x^{1/\alpha}L(x)$ be a regularly varying function at ∞ with index $1/\alpha$. Then the following conditions are equivalent:*

$$\widehat{m} - m(x) \sim \frac{\psi(x)}{x} \; (= x^{-\gamma}L(x)) \qquad (x \to \infty); \tag{3.6}$$

$$h(s) - \frac{1}{\widehat{m}s} \sim \frac{1}{\widehat{m}^2} \frac{C_\alpha}{s^2\psi^{-1}(1/s)} \qquad (s \to +0); \tag{3.7}$$

$$\sigma(\xi) - \frac{1}{\widehat{m}} \sim \frac{1}{\widehat{m}^2} \frac{C_\alpha}{\Gamma(\alpha)\Gamma(2-\alpha)} \frac{1}{\xi\psi^{-1}(1/\xi)} \qquad (\xi \to +0); \tag{3.8}$$

$$p(t,0,0) - \frac{1}{\widehat{m}} \sim \frac{1}{\widehat{m}^2} \frac{C_\alpha}{\Gamma(2-\alpha)} \frac{t}{\psi^{-1}(t)} \qquad (t \to \infty). \tag{3.9}$$

Since Theorem 3.1 is a special case of Theorem 3.2, we shall prove the latter only. The proof is based on the following result of [6]. We stress that Theorem 3.3 itself is not for ergodic diffusions but for transient processes (see Remark 2.1).

The reader should recall that $\ell = \lim_{s\to+0} h(s)$ in general.

Theorem 3.3 ([6, Thm. 2.2]). *Assume that $\ell<\infty$ and let $1<\alpha<2$. If $\varphi_0(x)>0$ is a function varying regularly at $+0$ with index $\alpha-1$, then the following two conditions are equivalent:*

$$m(\ell-\epsilon) \sim \left(\frac{\alpha}{\alpha-1}\right)^{\alpha/(\alpha-1)} \frac{1}{\epsilon\varphi_0^{-1}(\epsilon)} \qquad (\epsilon\downarrow 0); \tag{3.10}$$

$$\ell - h(s) \sim \frac{\Gamma(2-\alpha)}{(\alpha-1)\Gamma(1+\alpha)}\alpha^{2\alpha}\varphi_0(s) \qquad (s\downarrow 0). \tag{3.11}$$

Let us rewrite Theorem 3.3 as follows for our present use.

Lemma 3.4. *Assume that $\ell<\infty$ and let $1<\alpha<2$. If $\varphi(x)>0$ is a function varying regularly at $+0$ with index $\alpha-1$, then the following two conditions are equivalent:*

$$m(\ell-\varphi(s)) \sim \frac{1}{\varphi(s)s} \qquad (s\downarrow 0); \tag{3.12}$$

$$\ell - h(s) \sim C_\alpha \varphi(s) \qquad (s\downarrow 0), \tag{3.13}$$

where $C_\alpha = \frac{\Gamma(2-\alpha)}{\Gamma(\alpha)}\{\alpha(\alpha-1)\}^{\alpha-1}$ *as before* (see (3.5)).

Proof. Let $\varphi_0(x) = \varphi(Kx)$ with

$$K = \left(\frac{\alpha-1}{\alpha}\right)^{\alpha/(\alpha-1)}.$$

Then we can rewrite (3.12) as

$$m(\ell-\epsilon) \sim \frac{1}{\epsilon\varphi^{-1}(\epsilon)} = \frac{1}{K\epsilon\varphi_0^{-1}(\epsilon)} = \left(\frac{\alpha}{\alpha-1}\right)^{\alpha/(\alpha-1)} \frac{1}{\epsilon\varphi_0^{-1}(\epsilon)} \qquad (\epsilon\downarrow 0).$$

Therefore, applying Theorem 3.3 we see that (3.12) is equivalent to

$$\begin{aligned} \ell - h(s) &\sim \frac{\Gamma(2-\alpha)}{(\alpha-1)\Gamma(1+\alpha)}\alpha^{2\alpha}\varphi_0(s) = \frac{\Gamma(2-\alpha)}{(\alpha-1)\Gamma(1+\alpha)}\alpha^{2\alpha}\varphi(Ks) \\ &\sim \frac{\Gamma(2-\alpha)}{(\alpha-1)\Gamma(1+\alpha)}\alpha^{2\alpha}K^{\alpha-1}\varphi(s) = C_\alpha\varphi(s). \end{aligned}$$

□

We next rewrite Lemma 3.4 in terms of the *dual* string: The dual string $m^* \in \mathcal{M}$ of $m \in \mathcal{M}$ is defined by

$$m^*(x) = \begin{cases} m^{-1}(x) = \inf\{u > 0 \mid m(u) > x\} & (x \geq 0), \\ 0 & (x < 0). \end{cases} \tag{3.14}$$

Note that $\ell^* = \ell(m^*)$ $(:= \sup\{x \mid m^*(x) < \infty\}) = \widehat{m}$ (draw the picture). That is, probabilistically, the diffusion X is ergodic if and only if the 'dual' process X^* corresponding to m^* is transient. Thus Lemma 3.4 is applicable to X^*. This is the basic idea of the present article.

The fundamental relationship between $h(s)$ and $h^*(s)$ (the characteristic function of m^*) is the following (see e.g. [8]):

$$h^*(s) = \frac{1}{sh(s)}.$$

Lemma 3.5. *If $\widehat{m}$ $(= \ell^*) < \infty$, then*

$$h(s) - \frac{1}{\widehat{m}s} \sim \frac{1}{\widehat{m}^2 s}\{\ell^* - h^*(s)\} \qquad (s \to +0). \tag{3.15}$$

Proof. Consider the ratio of both sides:

$$\left(h(s) - \frac{1}{\widehat{m}s}\right)\Big/\left(\frac{1}{s}\{\ell^* - h^*(s)\}\right) = \left(h(s) - \frac{1}{\widehat{m}s}\right)\Big/\left(\frac{1}{s}\left\{\widehat{m} - \frac{1}{sh(s)}\right\}\right) = \frac{sh(s)}{\widehat{m}},$$

and recall that $sh(s) \to 1/\widehat{m}$ (see (2.11)). □

Lemma 3.6. *Let $g(x) > 0\,(x > 0)$ be a function varying regularly at infinity with index $\rho > 0$ so that $g(x) \to \infty\,(x \to \infty)$. Then*

$$\widehat{m} - m(x) \sim \frac{1}{g(x)} \qquad (x \to \infty) \tag{3.16}$$

if and only if

$$m^*\left(\widehat{m} - \frac{1}{g(x)}\right) \sim x \qquad (x \to \infty). \tag{3.17}$$

Proof. Let $f(x) = 1/(\widehat{m} - m(x))$. Then, by the definition of m^*, we have $f^{-1}(x) = m^*(\widehat{m} - 1/x)$. Therefore, (3.16) and (3.17) can be rewritten as $f(x) \sim g(x)$ $(x \to \infty)$ and $f^{-1}(g(x)) \sim x\,(x \to \infty)$, respectively. Now we easily see the equivalence of these two under the assumption that $g(x)$ varies regularly (cf. [1, p. 28]). □

We are now ready to prove Theorem 3.2:

Proof of Theorem 3.2. Since $\psi(x)$ varies regularly at infinity with index $1/\alpha$ (>0), so does $\psi^{-1}(x)$ with index α. That is, there exists a slowly varying function $L^*(x)$ such that

$$\psi^{-1}(x) = x^{\alpha} L^*(x) \tag{3.18}$$

(see [1, p. 28]). By Lemma 3.6, (3.6) is equivalent to

$$m^*(\widehat{m} - \psi(x)/x) \sim x \qquad (x \to \infty), \tag{3.19}$$

because $g(x) := x/\psi(x) = x^{1-1/\alpha}/L(x)$ varies regularly at infinity with index $1 - 1/\alpha > 0$. Changing the variable $x = \psi^{-1}(1/s)$ (i.e., $s = 1/\psi(x)$), (3.19) may be rewritten as

$$m^*\left(\widehat{m} - \frac{1}{s\psi^{-1}(1/s)}\right) \sim \psi^{-1}(1/s) = \frac{s\psi^{-1}(1/s)}{s} \qquad (s \downarrow 0),$$

or, equivalently,

$$m^*(\ell^* - \varphi(s)) \sim \frac{1}{\varphi(s)s} \qquad (s \downarrow 0), \tag{3.20}$$

where $\varphi(s) = 1/\{s\psi^{-1}(1/s)\}$ $(= s^{\alpha-1}/L^*(1/s))$. (Recall that $\widehat{m} = \ell^*$.) Thus, (3.6) is equivalent to (3.20). Since $\varphi(s)$ varies regularly at 0 with index $\alpha - 1$, Lemma 3.4 is applicable to $\{m^*, h^*\}$ in place of $\{m, h\}$ and hence (3.20) is equivalent to

$$\ell^* - h^*(s) \sim C_\alpha\, \varphi(s) = \frac{C_\alpha}{s\psi^{-1}(1/s)} \qquad (s \downarrow 0), \tag{3.21}$$

which is, by Lemma 3.5, equivalent to

$$h(s) - \frac{1}{s\widehat{m}} \left(\sim \frac{1}{\widehat{m}^2 s}(\ell^* - h^*(s))\right) \sim \frac{1}{\widehat{m}^2}\, \frac{C_\alpha}{s^2\psi^{-1}(1/s)} \qquad (s \downarrow 0).$$

Thus we have proved the equivalence of (3.6) and (3.7).

We next show that (3.7) and (3.8) are equivalent. Recall that $\sigma(0) = 1/\widehat{m}$ (see (2.11)). Then, by the representation (2.4), we have

$$h(s) - \frac{1}{s\widehat{m}} = \left(a + \int_{-0}^{\infty} \frac{d\sigma(\xi)}{s+\xi}\right) - \int_{\{0\}} \frac{d\sigma(\xi)}{s+\xi} = a + \int_{+0}^{\infty} \frac{d\sigma(\xi)}{s+\xi} \tag{3.22}$$

and hence (3.7) may be written as

$$\int_{+0}^{\infty} \frac{d\sigma(\xi)}{s+\xi} \sim \frac{C_\alpha}{\widehat{m}^2}\, \frac{1}{s^2\psi^{-1}(1/s)} \qquad (s \to +0).$$

Here, notice that the right-hand side varies regularly at $+0$ with index $\alpha - 2$. Indeed, by (3.18),

$$\frac{1}{s^2\psi^{-1}(1/s)} = s^{\alpha-2}/L^*\left(\frac{1}{s}\right)$$

Therefore, the equivalence of (3.7) and (3.8) follows from Karamata's Tauberian theorem for Stieltjes transform; that is, if $L_0(x)\,(>0)$ varies slowly at 0 and if $0<\beta<1$, then

$$\int_{+0}^{\infty}\frac{d\sigma(\xi)}{s+\xi}\sim s^{\beta-1}L_0(s)\qquad(s\to+0)$$

if and only if

$$\sigma(\xi)-\sigma(0)\sim\frac{1}{\Gamma(1+\beta)\Gamma(1-\beta)}\,\xi^{\beta}L_0(\xi)\qquad(\xi\to+0)$$

(see, e.g., [6, Appendix]).

Similarly, recall the formula (2.8). Then we have

$$p(t,0,0)-\frac{1}{\widehat{m}}=\int_{-0}^{\infty}e^{-t\xi}\,d\sigma(\xi)-\sigma(0)=\int_{+0}^{\infty}e^{-t\xi}\,d\sigma(\xi)\qquad(t>0). \tag{3.23}$$

Thus, applying the Tauberian theorem for the Laplace transform (e.g., [1, p. 38]) we deduce the equivalence of (3.8) and (3.9). □

Remark 3.7. So far we have considered only the case where $s(x)=x$ in (1.1). In order to apply the results of Theorems 3.1 and 3.2 to the general case, we only need to change the scale: Consider $\widetilde{m}(x):=m(s^{-1}(x))$ instead of $m(x)$ itself. Indeed, if $(X_t)_{t\geq0}$ is a diffusion corresponding to $\frac{d}{dm(x)}\frac{d}{ds(x)}$, then the scaled process $(s(X_t))_{t\geq0}$ corresponds to $\frac{d}{d\widetilde{m}(x)}\frac{d}{dx}$.

Example 3.8. The diffusion on $I=[0,\infty)$ with the local generator

$$\mathcal{L}=\frac{1}{2}\left(\frac{d^2}{dx^2}+\frac{\rho-1}{x}\frac{d}{dx}\right),\qquad x>0, \tag{3.24}$$

is called the *ρ-dimensional Bessel diffusion*. If $\rho>0$, it is easy to see that

$$p(t,x,y)\sim\text{const}\cdot t^{-\rho/2}\qquad(t\to\infty)$$

because the explicit formula for $p(t,x,y)$ is known (see e.g. [2, p. 75]). Now let us consider the case where $-2<\rho<0$. Since $x=0$ becomes an exit-not-entrance boundary in this case, we need to exclude a suitable neighborhood of 0. So let us consider $I=[1,\infty)$ with the reflecting boundary condition at $x=1$. Then Feller's canonical representation of $\mathcal{L}$ is, for example,

$$\mathcal{L}=\frac{d}{dm(x)}\frac{d}{ds(x)},\qquad x\geq1,$$

with

$$m(x)=\frac{2}{(2-\rho)\rho}(x^{\rho}-1),\qquad s(x)=x^{2-\rho}-1\qquad(x\geq1).$$

Therefore,

$$\widetilde{m}(x) := m(s^{-1}(x)) = \frac{2}{(2-\rho)|\rho|}\{1-(x+1)^{\rho/(2-\rho)}\} \qquad (x \geq 0).$$

Hence,

$$\widehat{m} = \frac{2}{(2-\rho)|\rho|} \quad \text{and} \quad \widehat{m} - \widetilde{m}(x) \sim \frac{2}{(2-\rho)|\rho|}\, x^{\rho/(2-\rho)} \qquad (x \to \infty).$$

Thus, we can apply Theorem 3.1 with $\alpha = (2-\rho)/2$, $\gamma = -\rho/(2-\rho)$ to obtain

$$p(t,x,y) - \frac{1}{\widehat{m}} \sim \text{const} \cdot t^{-|\rho|/2} \qquad (t \to \infty).$$

This argument can be generalized as follows: If

$$\mathcal{L} = \frac{1}{2}\left(\frac{d^2}{dx^2} + b(x)\,\frac{d}{dx}\right), \qquad x > 0, \tag{3.25}$$

for a bounded, measurable function $b(x)$ such that $b(x) \sim (\rho - 1)/x\,(x \to \infty)$ for some $\rho \in (-2, 0)$, then there exists a slowly varying function $L(t)$ such that

$$p(t,x,y) - \frac{1}{\widehat{m}} \sim \text{const} \cdot t^{-|\rho|/2} L(t) \qquad (t \to \infty).$$

Note that $L(t)$ cannot be replaced by a constant in general.

§4. The case of oscillations

Throughout $f(x) \asymp g(x)$ means that

$$0 < \liminf f(x)/g(x) \leq \limsup f(x)/g(x) < \infty.$$

Theorem 4.1. *Assume that $\widehat{m} < \infty$. Let $1 < \alpha < 2$ and let $\gamma = 1 - 1/\alpha$ (so that $0 < \gamma < 1/2$). Then the following conditions are equivalent:*

$$\widehat{m} - m(x) \asymp x^{-\gamma} \qquad (x \to \infty); \tag{4.1}$$

$$h(s) - \frac{1}{\widehat{m}s} \asymp s^{\alpha-2} \qquad (s \to +0); \tag{4.2}$$

$$\sigma(\xi) - \frac{1}{\widehat{m}} \asymp \xi^{\alpha-1} \qquad (\xi \to +0); \tag{4.3}$$

$$p(t,0,0) - \frac{1}{\widehat{m}} \asymp t^{1-\alpha} \qquad (t \to \infty). \tag{4.4}$$

More generally, we have

Theorem 4.2. *Assume that* $\widehat{m} < \infty$. *Let* α *and* γ *be as in Theorem 4.1 and let* $\psi(x) = x^{1/\alpha}L(x)$ *vary regularly at* $+\infty$ *with index* α. *Then the following conditions are equivalent:*

$$\widehat{m} - m(x) \asymp \frac{\psi(x)}{x} = x^{-\gamma}L(x) \qquad (x \to \infty); \tag{4.5}$$

$$h(s) - \frac{1}{\widehat{m}s} \asymp \frac{1}{s^2\psi^{-1}(1/s)} \qquad (s \to 0); \tag{4.6}$$

$$\sigma(\xi) - \frac{1}{\widehat{m}} \asymp \frac{1}{\xi\psi^{-1}(1/\xi)} \qquad (\xi \to +0); \tag{4.7}$$

$$p(t, 0, 0) - \frac{1}{\widehat{m}} \asymp \frac{t}{\psi^{-1}(t)} \qquad (t \to \infty). \tag{4.8}$$

The proof can be carried out completely in parallel to the previous section. The only difference is that we use [6, Theorem 3.2] instead of [6, Theorem 2.2].

§5. Bilateral case

We next consider the case where $X = (X_t)_{t\geq 0}$ is a (generalized) diffusion on the whole line $(-\infty, \infty)$ with the local generator

$$\mathcal{L} = \frac{d}{dm(x)}\frac{d}{dx}, \qquad -\infty < x < \infty,$$

where $dm(x)$ is a finite Borel measure on $\mathbb{R}$ so that the diffusion is ergodic. We assume that $0 \in \operatorname{Supp} dm$ for simplicity.

It is well known that the transition density can be computed as follows (see [4]). Consider two strings $m_+(x) = \int_{[0,x\vee 0]} dm(y)$ and $m_-(x) = \int_{[-(x\vee 0),0)} dm(y)$, and let $h_+(s), h_-(s)$ correspond to m_+, m_-, respectively. Now define φ_λ and ψ_λ as in (2.3) with the convention that $\int_{-0}^{x} = -\int_{x}^{-0}$ if $x < 0$. For every $s > 0$, we put

$$u_1(s; x) = \varphi_{-s}(x) - \frac{1}{h_+(s)}\psi_{-s}(x), \qquad u_2(s; x) = \varphi_{-s}(x) + \frac{1}{h_-(s)}\psi_{-s}(x).$$

These two are nonnegative solutions of

$$-\mathcal{L}u(x) = su(x) \quad (-\infty < x < \infty), \qquad u(0) = 1,$$

such that u_1 is nonincreasing and u_2 nondecreasing. Their Wronskian is

$$W[u_1(s;\cdot), u_2(s;\cdot)] \; (:= u_1u_2' - u_1'u_2) = \frac{1}{h_-(s)} + \frac{1}{h_+(s)}. \tag{5.1}$$

So the Green function is given by

$$G_s(x,y)=\begin{cases} h(s)u_2(s;x)u_1(s;y) & (x\le y),\\ h(s)u_1(s;x)u_2(s;y) & (x>y),\end{cases}$$

where $h(s)=1/W[u_1(s;\cdot),u_2(s;\cdot)]$; that is,

$$\frac{1}{h(s)}=\frac{1}{h_+(s)}+\frac{1}{h_-(s)}. \tag{5.2}$$

Therefore, the transition density $p(t,x,y)$ (with respect to $dm(x)$) can be specified by the following relationship:

$$\int_0^\infty e^{-st}p(t,x,y)\,dt=G_s(x,y)\qquad (s>0)\qquad dm(x)dm(y)\text{-a.e.}$$

In particular,

$$\int_0^\infty e^{-st}p(t,0,0)\,dt=G_s(0,0)=h(s)\qquad (s>0). \tag{5.3}$$

Theorem 5.1. *Let* $1<\alpha<2$ *and let* $\gamma=1-1/\alpha$ *(so that* $0<\gamma<1/2$*). Assume that* $\widehat{m}_\pm:=m_\pm(\infty)<\infty$ *and put* $\widehat{m}:=\widehat{m}_++\widehat{m}_-$ $(=\int_{\mathbb{R}}dm(x))$*. Then*

$$\begin{cases}\widehat{m}_+-m_+(x)\sim K_+x^{-\gamma}\\ \widehat{m}_--m_-(x)\sim K_-x^{-\gamma}\end{cases}\qquad (x\to\infty) \tag{5.4}$$

for $K_+,K_->0$ *implies that*

$$h(s)-\frac{1}{\widehat{m}s}\sim\frac{1}{\widehat{m}^2}(K_+^\alpha+K_-^\alpha)C_\alpha s^{\alpha-2}\qquad (s\to+0), \tag{5.5}$$

$$p(t,0,0)-\frac{1}{\widehat{m}}\sim\frac{1}{\widehat{m}^2}(K_+^\alpha+K_-^\alpha)\frac{C_\alpha}{\Gamma(2-\alpha)}t^{\alpha-1}\qquad (t\to\infty), \tag{5.6}$$

where C_α *is as in* (3.5).

Proof. By Theorem 3.1, the condition (5.4) implies

$$h_\pm(s)-\frac{1}{\widehat{m}_\pm s}\sim\frac{K_\pm^\alpha}{\widehat{m}_\pm^2}C_\alpha s^{\alpha-2}\qquad (s\to+0),$$

which is also equivalent to

$$\widehat{m}_\pm-\frac{1}{sh_\pm(s)}\sim K_\pm^\alpha C_\alpha s^{\alpha-1}\qquad (s\to+0) \tag{5.7}$$

(see Lemma 3.5). Combining (5.7) with (5.2), we have

$$\widehat{m}-\frac{1}{sh(s)}\sim(K_+^\alpha+K_-^\alpha)C_\alpha s^{\alpha-1}\qquad (s\to+0). \tag{5.8}$$

Since

$$sh(s) - \frac{1}{\widehat{m}} = \left(\widehat{m} - \frac{1}{sh(s)}\right)\frac{sh(s)}{\widehat{m}} \sim \left(\widehat{m} - \frac{1}{sh(s)}\right)\frac{1}{\widehat{m}^2},$$

(5.8) implies

$$sh(s) - \frac{1}{\widehat{m}} \sim \frac{1}{\widehat{m}^2}(K_+^\alpha + K_-^\alpha)\, C_\alpha s^{\alpha-1} \qquad (s \to +0).$$

Thus we have (5.5). The equivalence of (5.5) and (5.6) can be shown as in the proof of Theorem 3.1: $h(s) - 1/(\widehat{m}s)$ is the Laplace transform of the monotone function $p(t,0,0) - 1/\widehat{m}$ (see (5.3)). □

Remark 5.2. (i) Theorem 5.1 can be extended to the case where a slowly varying function is involved as in Theorem 3.2. Also the assumption that "$K_+, K_- > 0$" may be replaced by "$K_+, K_- \geq 0$" with a slight modification of the proof.

(ii) The converse of Theorem 5.1 fails in general unless we add some suitable balancing conditions for $h_+(s)$ and $h_-(s)$.

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