Publ. RIMS Kyoto Univ. **48** (2012), 615–622
DOI 10.2977/PRIMS/83

Commutator Length of Leaf Preserving Diffeomorphisms

by

Kazuhiko Fukui

Abstract

We consider the group of leaf preserving C^∞-diffeomorphisms for a C^∞-foliation on a manifold which are isotopic to the identity through leaf preserving C^∞-diffeomorphisms with compact support. We show that for a one-dimensional C^∞-foliation $\mathcal{F}$ on the torus, this group is uniformly perfect if and only if $\mathcal{F}$ has no compact leaves. Moreover we consider the group of leaf preserving C^∞-diffeomorphisms for the product foliation on $S^1 \times S^n$ which are isotopic to the identity through leaf preserving C^∞-diffeomorphisms. Here the product foliation has leaves of the form $\{\mathrm{pt}\} \times S^n$. We show that this group is uniformly perfect for $n \geq 2$.

2010 Mathematics Subject Classification: Primary 58D05; Secondary 57R30, 57R50.
Keywords: commutator length, leaf preserving diffeomorphism, uniformly perfect.

§1. Introduction and statement of results

Let M be a connected C^∞-manifold without boundary and let $D_c^\infty(M)$ denote the group of all C^∞-diffeomorphisms of M which are isotopic to the identity through C^∞-diffeomorphisms with compact support. It is well known by the results of M. Herman [8] and W. Thurston [13] that $D_c^\infty(M)$ is perfect, that is, coincides with its commutator subgroup. There are many analogous results on the group of diffeomorphisms preserving a geometric structure of M (for examples, J. Mather [9], A. Banyaga [5], K. Abe–K. Fukui [1]–[3], T. Rybicki [11], T. Tsuboi [14] etc.). Let $\mathcal{F}$ be a C^∞-foliation on M. A diffeomorphism $f : M \to M$ is said to be *leaf preserving* if f maps each leaf of $\mathcal{F}$ to itself. We denote by $D_{L,c}^\infty(M, \mathcal{F})$ the group of leaf preserving C^∞-diffeomorphisms of $(M, \mathcal{F})$ which are isotopic to the identity through leaf preserving C^∞-diffeomorphisms with compact support. By T. Tsuboi [14] and T. Rybicki [11], it is known that $D_{L,c}^\infty(M, \mathcal{F})$ is perfect.

Communicated by by T. Ohtsuki. Received May 13, 2011. Revised September 15, 2011.

K. Fukui: Department of Mathematics, Kyoto Sangyo University, Kyoto 603-8555, Japan;
e-mail: fukui@cc.kyoto-su.ac.jp

© 2012 Research Institute for Mathematical Sciences, Kyoto University. All rights reserved.

Given a group G, each element g of the commutator subgroup $[G, G]$ of G can be written as a product $g = \prod_{i=1}^{k}[a_i, b_i]$ $(a_i, b_i \in G)$. The smallest integer k for which such an expression exists is called the *commutator length* of g and is denoted by $\ell(g)$. We call $\ell : [G, G] \to \mathbb{N}$ the *commutator length function* of G. A group is said to be *uniformly perfect* if it is perfect and its commutator length function of G is bounded.

In this paper we consider the commutator length of leaf preserving diffeomorphisms for foliations. First we consider the group of leaf preserving diffeomorphisms for one-dimensional foliations and give a sufficient condition for the group to be non-uniformly perfect (Theorem 2.1). As an application, we have the following main result.

Theorem A (Theorem 3.1). *Let $\mathcal{F}$ be a one-dimensional transversely orientable C^∞-foliation on the torus T^2.*

(1) *If $\mathcal{F}$ has some compact leaves, then the commutator length function of $D_L^\infty(T^2, \mathcal{F})$ is unbounded, hence $D_L^\infty(T^2, \mathcal{F})$ is not a uniformly perfect group.*
(2) *If every leaf of $\mathcal{F}$ is dense, then $D_L^\infty(T^2, \mathcal{F})$ is a uniformly perfect group. Indeed, any $f \in D_L^\infty(T^2, \mathcal{F})$ can be represented by a product of at most six commutators of elements in $D_L^\infty(T^2, \mathcal{F})$.*

Secondly we consider the group of leaf preserving diffeomorphisms for the product foliation $\mathcal{F}$ on $S^1 \times S^n$ with leaves of the form $\{\mathrm{pt}\} \times S^n$. Then we have the following.

Theorem B (Theorem 4.1). *$D_L^\infty(S^1 \times S^n, \mathcal{F})$ is a uniformly perfect group for $n \geq 2$. Indeed, any $f \in D_L^\infty(S^1 \times S^n, \mathcal{F})$ can be represented by a product of at most eight commutators of elements in $D_L^\infty(S^1 \times S^n, \mathcal{F})$.*

§2. The commutator length of leaf preserving diffeomorphisms for a one-dimensional foliation with a compact leaf

Let $\mathcal{F}$ be a one-dimensional foliation on a manifold M with compact leaf L_0 $(= S^1)$. We consider the commutator length function of $D_{L,c}^\infty(M, \mathcal{F})$ and show that it is unbounded if the compact leaf L_0 has infinite holonomy.

The strategy to show that the commutator length function of a group G is unbounded is to construct non-trivial quasimorphisms. Here a *quasimorphism* on G is a function $\psi : G \to \mathbb{R}$ having a constant $D_\psi > 0$, called the defect of ψ, such that

$$|\psi(ab) - \psi(a) - \psi(b)| < D_\psi$$

for $a, b \in G$. Then we can see that if one can construct a non-trivial quasimorphism, then the commutator length function is unbounded (cf. J.-M. Gambaudo–É. Ghys [7]). We have the following.

Theorem 2.1. *Let $\mathcal{F}$ be a one-dimensional C^∞-foliation on a manifold M. If there exists a compact leaf L_0 with infinite holonomy, then the commutator length function of $D^\infty_{L,c}(M,\mathcal{F})$ is unbounded, hence the group is not uniformly perfect.*

Proof. We have only to construct a non-trivial quasimorphism of $G = D^\infty_{L,c}(M,\mathcal{F})$ to $\mathbb{R}$. Let $L_0 = S^1$ $(= \mathbb{R}/\mathbb{Z})$ and $\pi : \mathbb{R} \to S^1$ be the covering projection. Take $0 = \pi(0) \in S^1$. For any $f \in G$, take an isotopy $\{f_t\}_{0\le t\le 1}$ from $f_0 = \mathrm{id}$ to $f_1 = f$. Then $f_t(0)$ $(0 \le t \le 1)$ is a path on S^1 from 0 to $f(0)$. Let $\tilde{f}_t(0)$ $(0 \le t \le 1)$ be the lift of the path $f_t(0)$ $(0 \le t \le 1)$ to $\mathbb{R}$ satisfying $\tilde{f}_0(0) = 0$. Then the integer part of the value $\tilde{f}_1(0)$ represents the winding number of the path $f_t(0)$ $(0 \le t \le 1)$ around $L_0 = S^1$. Since L_0 has infinite holonomy, the value $\tilde{f}_1(0)$ does not depend on the choice of isotopies of f. Thus we can define a map $\psi : G \to \mathbb{R}$ by $\psi(f) = \tilde{f}_1(0)$ for any $f \in G$.

Now we prove that $\psi : G \to \mathbb{R}$ is a non-trivial quasimorphism. It is easy to see that ψ is non-trivial. To show that ψ is a quasimorphism, note that for any $g, h \in G$ and their isotopies g_t, h_t, the path $(g \cdot h)_t(0) = (g_t(0)) \cdot (h_t(0))$ is homotopic to the composition of the paths $(h_t(0))$ and $(g_t(0)) \cdot (h_1(0))$ $(0 \le t \le 1)$ fixing the starting point 0 and the end point $g(0) \cdot h(0)$. For any number a, let $\tilde{g}_t(0)$ $(0 \le t \le 1)$ and $\tilde{g}_t(a)$ $(0 \le t \le 1)$ be the lifts of the paths $g_t(0)$ $(0 \le t \le 1)$ and $g_t(\pi(a))$ $(0 \le t \le 1)$ to $\mathbb{R}$ satisfying $\tilde{g}_0(0) = 0$ and $\tilde{g}_0(a) = a$ respectively. Then we have $a - 1 < \tilde{g}_t(a) - \tilde{g}_t(0) < a + 1$ for any t $(0 \le t \le 1)$ (see also [4] for the proof). Indeed, assume that $\tilde{g}_{t_1}(a) - \tilde{g}_{t_1}(0) \ge a + 1$ for some t_1 $(0 < t_1 \le 1)$. Since $\tilde{g}_0(a) - \tilde{g}_0(0) = 0$, there is t_0 $(0 < t_0 < t_1)$ satisfying $\tilde{g}_{t_0}(a) - \tilde{g}_{t_0}(0) \in \mathbb{Z}$. Thus we have $\pi(\tilde{g}_{t_0}(a) - \tilde{g}_{t_0}(0)) = 0$, hence $g_{t_0}(\pi(a)) = g_{t_0}(0)$. Since g_{t_0} is a homeomorphism, $\pi(a) = 0$, hence $\tilde{g}_t(a) - \tilde{g}_t(0) = a$ for any t. This is a contradiction. When $a - 1 < \tilde{g}_t(a) - \tilde{g}_t(0)$, we argue similarly.

Putting $a = \tilde{h}_1(0)$, we have

$$\tilde{h}_1(0) - 1 < \tilde{g}_1(\tilde{h}_1(0)) - \tilde{g}_1(0) < \tilde{h}_1(0) + 1.$$

Since $\psi(g \cdot h) = (\widetilde{g \cdot h})_1(0) = \tilde{g}_1(\tilde{h}_1(0))$, we have

$$|\psi(g \cdot h) - \psi(g) - \psi(h)| < 1.$$

This completes the proof.

Let $\mathcal{F}$ be the product foliation of $\mathbb{R}^n \times S^1$ with leaves of the form $\{\mathrm{pt}\} \times S^1$. We consider the commutator length function of $D^\infty_{L,c}(\mathbb{R}^n \times S^1, \mathcal{F})$. By the same argument as in the proof of Theorem 2.1 we have:

Corollary 2.2. *The commutator length function of $D^\infty_{L,c}(\mathbb{R}^n \times S^1, \mathcal{F})$ is unbounded, hence the group is not uniformly perfect.*

Proof. Take a compact leaf L_0. As in the proof of Theorem 2.1, we can construct a non-trivial quasimorphism $\psi_{L_0} : D^\infty_{L,c}(\mathbb{R}^n \times S^1, \mathcal{F}) \to \mathbb{R}$ because any element of $D^\infty_{L,c}(\mathbb{R}^n \times S^1, \mathcal{F})$ is compactly supported. The rest of the proof is the same as that of Theorem 2.1.

For the product foliation $\mathcal{F}$ on the torus T^2 we have the following.

Theorem 2.3. *Let $\mathcal{F}$ be the product foliation on the torus T^2. Then the commutator length function of $D^\infty_L(T^2, \mathcal{F})$ is unbounded, hence the group is not uniformly perfect.*

Proof. Take two compact leaves L_1 and L_2 of $\mathcal{F}$. As in the proof of Theorem 2.1, for any $f \in G$ and its isotopy $\{f_t\}_{0 \le t \le 1}$ from $f_0 = \mathrm{id}$ to $f_1 = f$, we can lift the path $f_t(0)$ $(0 \le t \le 1)$ on S^1 $(= L_1 = L_2)$ to the path $\tilde{f}_t(0)$ $(0 \le t \le 1)$ on $\mathbb{R}$ satisfying $\tilde{f}_0(0) = 0$. Thus we can construct functions $\psi_{L_0}, \psi_{L_1} : D^\infty_L(T^2, \mathcal{F}) \to \mathbb{R}$ simultaneously. The functions ψ_{L_1} and ψ_{L_2} depend on the choice of isotopies of an element of $D^\infty_L(T^2, \mathcal{F})$ but the difference of ψ_{L_1} and ψ_{L_2} does not depend on the choice of the isotopies. Thus we define a new function $\varphi : D^\infty_L(T^2, \mathcal{F}) \to \mathbb{R}$ by

$$\varphi(a) = \psi_{L_1}(a) - \psi_{L_2}(a)$$

for any $a \in D^\infty_L(T^2, \mathcal{F})$. Then we can easily see in the same way as in the proof of Theorem 2.1 that φ is a non-trivial quasimorphism with defect $D_\varphi = 2$. This completes the proof.

§3. The commutator length of leaf preserving diffeomorphisms for foliations on the torus

Let X be a non-singular C^∞-vector field on the torus T^2 and $\mathcal{F}_X$ be the foliation on T^2 constructed from X. Then we have the following.

Theorem 3.1. (1) *If X has some periodic orbits, then the commutator length function of $D^\infty_L(T^2, \mathcal{F}_X)$ is unbounded, hence $D^\infty_L(T^2, \mathcal{F}_X)$ is not a uniformly perfect group.*

(2) *If every orbit of X is ergodic, then $D^\infty_L(T^2, \mathcal{F}_X)$ is a uniformly perfect group. Indeed, any $f \in D^\infty_L(T^2, \mathcal{F}_X)$ can be represented by a product of at most six commutators of elements in $D^\infty_L(T^2, \mathcal{F}_X)$.*

Proof. C. L. Siegel [12] generalized the theorem of A. Denjoy [6] by proving that exactly one of the following happens:

(i) X has some periodic orbits.

(ii) Every orbit of X is ergodic.

(1) If X has a periodic orbit, then $\mathcal{F}_X$ has a compact leaf. If every leaf of $\mathcal{F}_X$ is compact, then $\mathcal{F}_X$ is a bundle foliation with leaf homeomorphic to S^1, and the result follows from Theorem 2.3. If $\mathcal{F}_X$ has some compact leaves but not all leaves are compact, then $\mathcal{F}_X$ has a compact leaf with infinite holonomy, and the result follows from Theorem 2.1.

(2) If every orbit of X is ergodic, then all leaves in $\mathcal{F}_X$ are dense. Fix $p \in T^2$ and $f \in D^\infty_L(T^2, \mathcal{F}_X)$. We need the following lemma and proposition.

Lemma 3.2. *There exist a small neighborhood V of p, a neighborhood U of p ($V \subset U$), and $f_1, f_2 \in D^\infty_L(T^2, \mathcal{F}_X)$ such that*

- *U and V are diffeomorphic to an open disk,*
- *$f = f_2 \circ f_1$,*
- *$f_1 = \mathrm{id}$ on V, and*
- *$\mathrm{supp}(f_2) \subset U$.*

Proof. When $f(p) = p$, take small neighborhoods U, V ($V \subset U$) of p which are diffeomorphic to an open disk. We can deform f in U to $f_1 \in D^\infty_L(T^2, \mathcal{F}_X)$ satisfying $f_1 = \mathrm{id}$ on V. Then $f_2 = f \circ f_1^{-1}$ satisfies the desired conditions.

Now consider the case when $f(p) \neq p$. Let ℓ be the (shortest) part of a leaf of $\mathcal{F}_X$ joining p and $f(p)$. Then we can take a small neighborhood V of p and a thin neighborhood U ($V \subset U$) of ℓ such that U and V are diffeomorphic to an open disk. By the diffeotopy extension theorem, there exists $h \in D^\infty_L(T^2, \mathcal{F}_X)$ satisfying $h = f$ on V and $\mathrm{supp}(h) \subset U$. Put $f_1 = h^{-1} \circ f$ and $f_2 = h$. Then U, V and f_1, f_2 satisfy the desired conditions. This completes the proof.

Let N be a manifold and $\mathcal{F}$ be the product foliation on the product manifold $N \times \mathbb{R}^m$ with leaves of the form $\{\mathrm{pt}\} \times \mathbb{R}^m$. Then we have the following.

Proposition 3.3. *Any $f \in D^\infty_{L,c}(N \times \mathbb{R}^m, \mathcal{F})$ can be represented by a product of two commutators of elements in $D^\infty_{L,c}(N \times \mathbb{R}^m, \mathcal{F})$.*

Proof. We prove Proposition 3.3 in the parallel way to Tsuboi [15]. Take $f \in D^\infty_{L,c}(N \times \mathbb{R}^m, \mathcal{F})$. By the theorem of Tsuboi [14] and Rybicki [11], f can be represented as a product of commutators

$$f = \prod_{i=1}^{k} [a_i, b_i], \qquad \text{where } a_i, b_i \in D^\infty_{L,c}(N \times \mathbb{R}^m, \mathcal{F}).$$

Let U be a bounded open subset of $N \times \mathbb{R}^m$ containing the supports of a_i's and b_i's. Take $\phi \in D^\infty_{L,c}(N \times \mathbb{R}^m, \mathcal{F})$ such that $\{\phi^i(U)\}_{i=1}^k$ are disjoint. This is possible by sliding U along a direction of $\mathbb{R}^m$. We put $F = \prod_{j=1}^k \phi^j(\prod_{i=j}^k [a_i, b_i])\phi^{-j}$, which defines an element in $D^\infty_{L,c}(N \times \mathbb{R}^m, \mathcal{F})$. Then we have

$$\begin{aligned}\phi^{-1} \circ F \circ \phi \circ F^{-1} &= f \circ \Big(\prod_{j=1}^{k} \phi^j [a_j, b_j]^{-1} \phi^{-j}\Big) \\ &= f \circ \Big[\prod_{j=1}^{k} \phi^j b_j \phi^{-j}, \prod_{j=1}^{k} \phi^j a_j \phi^{-j}\Big].\end{aligned}$$

Thus

$$f = [\phi^{-1}, F] \circ \Big[\prod_{j=1}^{k} \phi^j a_j \phi^{-j}, \prod_{j=1}^{k} \phi^j b_j \phi^{-j}\Big],$$

so f is a product of two commutators of elements in $D^\infty_{L,c}(N \times \mathbb{R}^m, \mathcal{F})$. This completes the proof.

Proof of Theorem 3.1(2) continued. The map f_2 in Lemma 3.2 satisfies $\mathrm{supp}(f_2) \subset U$ ($\cong \operatorname{int} D^2$). We may assume that $\mathcal{F}_X$ is a product foliation on U. From Proposition 3.3, f_2 can be represented by a product of at most two commutators of elements in $D^\infty_{L,c}(U, \mathcal{F}_X \mid_U)$.

Next we consider f_1. Choose $V' \subset V$ ($\subset U$) as open flow boxes with common transverse component and take A to be the union of a leaf of $\mathcal{F}_X|_{T^2 \setminus \overline{V'}}$ and two leaves of $\mathcal{F}_X|_V$, which is connected. This is possible because all leaves of $\mathcal{F}_X$ are dense. Take small neighborhoods P, Q ($P \subset Q$) of A such that P, Q are unions of connected parts of leaves of $\mathcal{F}_X$ and are diffeomorphic to an open disk. Then we may assume that $\mathcal{F}_X$ is a product foliation on Q. Thus by deforming f_1 in Q, we obtain $f_3 \in D^\infty_L(T^2, \mathcal{F}_X)$ with $f_3 = \mathrm{id}$ on P. Putting $f_4 = f_3^{-1} \circ f_1$, we have $f_1 = f_3 \circ f_4$. Since f_4 is supported in Q ($\cong \operatorname{int} D^2$) and $\mathcal{F}_X$ is a product foliation on Q, f_4 can be represented by a product of at most two commutators of elements in $D^\infty_{L,c}(Q, \mathcal{F}_X|_Q)$ by the above argument. Since $V \cup Q$ is homeomorphic to an open cylinder, a small open neighborhood W of the complement of $V \cup Q$ is also homeomorphic to $S^1 \times (-1, 1)$. Furthermore we may assume that $\mathcal{F}_X$ is a product foliation on W. From Proposition 3.3, $f_3 \in D^\infty_{L,c}(W, \mathcal{F}_X|_W)$ can be represented by a product of at most two commutators of elements in $D^\infty_{L,c}(W, \mathcal{F}_X|_W)$. Hence f can be represented by a product of at most six commutators of elements in $D^\infty_L(T^2, \mathcal{F}_X)$. This completes the proof.

Remark 3.4. Any transversely orientable C^∞-foliation $\mathcal{F}$ on the torus T^2 comes from a foliation constructed from a non-singular C^∞-vector field. Thus Theorem A follows from Theorem 3.1. On the other hand, any transversely non-orientable C^∞-foliation $\mathcal{F}$ on T^2 has a compact leaf with infinite holonomy. Thus Theorem 2.1 implies that $D^\infty_L(T^2, \mathcal{F})$ is not uniformly perfect.

§4. The uniform perfectness of leaf preserving diffeomorphisms for the product foliation on $S^1 \times S^n$ $(n \geq 2)$

Let S^n be the unit n-sphere in $\mathbb{R}^{n+1}$ and let $\mathcal{F}$ be the product foliation on $S^1 \times S^n$ $(n \geq 1)$ with leaves of the form $\{\mathrm{pt}\} \times S^n$, and $D_L^\infty(S^1 \times S^n, \mathcal{F})$ be the group of leaf preserving C^∞-diffeomorphisms of $(S^1 \times S^n, \mathcal{F})$ which are isotopic to the identity through leaf preserving C^∞-diffeomorphisms.

In this section we show the uniform perfectness of $D_L^\infty(S^1 \times S^n, \mathcal{F})$ $(n \geq 2)$. Put $G = D_L^\infty(S^1 \times S^n, \mathcal{F})$. Fix a point $p \in S^n$. Let p^* be the antipodal point of p.

Theorem 4.1. *$D_L^\infty(S^1 \times S^n, \mathcal{F})$ is uniformly perfect for $n \geq 2$. Indeed, any $f \in D_L^\infty(S^1 \times S^n, \mathcal{F})$ can be represented by a product of at most eight commutators of elements in $D_L^\infty(S^1 \times S^n, \mathcal{F})$.*

Proof. Fix any $f \in G$. If $f(S^1 \times \{p\}) \cap (S^1 \times \{p^*\}) \neq \emptyset$, then by sliding f on a neighborhood of $S^1 \times \{p^*\}$, we can assume $f(S^1 \times \{p\}) \cap (S^1 \times \{p^*\}) = \emptyset$ because $n \geq 2$. That is, there are $f_1, f_2 \in G$ such that (1) $f = f_1 \circ f_2$ and (2) $f_i(S^1 \times \{p\}) \cap (S^1 \times \{p^*\}) = \emptyset$ $(i = 1, 2)$. Thus we may assume $f(S^1 \times \{p\}) \cap (S^1 \times \{p^*\}) = \emptyset$. Then for any $s \in S^1$, there exists a continuous family of shortest geodesics $\ell(s, p)$ joining (s, p) to $f(s, p) \in \{s\} \times S^n$. Put $L = \bigcup_{s \in S^1} \ell(s, p)$, which may be considered as a (singular) surface with $S^1 \times \{p\}$ and $f(S^1 \times \{p\})$ as boundary. Take an open neighborhood U of L such that $U \cap (S^1 \times \{p^*\}) = \emptyset$ and U is diffeomorphic to $S^1 \times \operatorname{int} D^n$. Let $U(s, p) = U \cap (\{s\} \times S^n)$. Take a sufficiently small disk $V(s, p)$ around (s, p) in $U(s, p)$ satisfying $W(f(s, p)) = f(V(s, p)) \subset U(s, p)$. Then we can take a constant disk $V = V(s, p)$ not depending on s because of the compactness of S^1. Then by using the diffeotopy extension theorem (see Theorem 2.3 of Milnor [10]), for each $s \in S^1$, there exists a leaf preserving C^∞-diffeomorphism $h : (S^1 \times S^n, \mathcal{F}) \to (S^1 \times S^n, \mathcal{F})$ such that for each s, $h(s, \cdot) : \{s\} \times S^n \to \{s\} \times S^n$ satisfies

- $h(s, x) = f(s, x)$ for $(s, x) \in V = V(s, p)$,
- $h(s, x) = (s, x)$ for $(s, x) \notin U(s, p)$.

Put $f = h \circ (h^{-1} \circ f) = h_1 \circ h_2$, where $h_1 = h$ and $h_2 = h^{-1} \circ f$. Then $\operatorname{supp}(h_1) \subset U \cong S^1 \times \operatorname{int} D^n$ and $\operatorname{supp}(h_2) \subset S^1 \times (S^n - \bar{V}) \cong S^1 \times \operatorname{int} D^n$. Thus we may consider that $h_1, h_2 \in D_{L,c}^\infty(S^1 \times \operatorname{int} D^n, \mathcal{F}_0)$, where $\mathcal{F}_0$ is the product foliation with leaves of the form $\{\mathrm{pt}\} \times \operatorname{int} D^n$. From Proposition 3.3, h_1, h_2 can each be represented by a product of at most two commutators of elements in $D_{L,c}^\infty(S^1 \times \operatorname{int} D^n, \mathcal{F}_0)$. Hence f can be represented by a product of at most eight commutators. This completes the proof.

Corollary 4.2. *$D_L^\infty(S^1 \times S^n, \mathcal{F})$ is uniformly perfect if and only if $n \neq 1$.*

Proof. This is an immediate consequence of Theorems 2.3 and 4.1.

Acknowledgments

The author expresses his gratitude to the referee for some comments.

This research was partially supported by Grant-in-Aid for Scientific Research (No. 20540098), Japan Society for the Promotion of Science.

References

[1] K. Abe and K. Fukui, On commutators of equivariant diffeomorphisms, Proc. Japan Acad. **54** (1978), 52–54. Zbl 0398.57032 MR 0474334

[2] ———, The first homology of the group equivariant diffeomorphisms and its applications, J. Topology **1** (2008), 461–476. Zbl 1145.58005 MR 2399140

[3] ———, Commutators of diffeomorphisms preserving a submanifold, J. Math. Soc. Japan **61** (2009), 427–436. Zbl 1169.57029 MR 2532896

[4] ———, Commutators of diffeomorphisms preserving a submanifold II, preprint.

[5] A. Banyaga, *The structure of classical diffeomorphism groups*, Math. Appl. 400, Kluwer, Dordrecht, 1997. Zbl 0874.58005 MR 1445290

[6] A. Denjoy, Sur les courbes définies par les équations différentielles à la surface du tore, J. Math. Pures Appl. **11** (1932), 333–375. Zbl 0006.30501

[7] J.-M. Gambaudo and É. Ghys, Commutators and diffeomorphisms of surfaces, Ergodic Theory Dynam. Systems **24** (1980), 1591–1617. Zbl 1088.37018 MR 2104597

[8] M. Herman, Simplicité du groupe des difféomorphismes de classe C^∞, isotopes à l'identité, du tore de dimension n, C. R. Acad. Sci. Paris **273** (1971), 232–234. Zbl 0217.49602 MR 0287585

[9] J. Mather, Commutators of diffeomorphisms, Comment. Math. Helv. **49** (1974), 512–518; II **50** (1975), 33–40; III **60** (1985), 122–124. Zbl 0289.57014(I) MR 0356129(I) Zbl 0299.58008(II) MR 0375382(II) Zbl 0575.58011(III) MR 0787665(III)

[10] J. Milnor, Lecture on differentiable structures, in *Collected papers of John Milnor, III, Differential topology*, Amer. Math. Soc., 2007, 177–190.

[11] T. Rybicki, The identity component of the leaf preserving diffeomorphism group is perfect, Monatsh. Math. **120** (1995), 289–305. Zbl 0847.57033 MR 1363142

[12] C. L. Siegel, Note on the differential equations on the torus, Ann. of Math. **46** (1945), 423–428. Zbl 0061.19510 MR 0013397

[13] W. Thurston, Foliations and groups of diffeomorphisms, Bull. Amer. Math. Soc. **80** (1974), 304–307. Zbl 0295.57014 MR 0339267

[14] T. Tsuboi, On the group of foliation preserving diffeomorphisms, in *Foliations 2005*, P. Walczak et al. (eds.), World Sci., Singapore, 2006, 411–430. Zbl 1222.57031 MR 2284795

[15] ———, On the uniform perfectness of diffeomorphism groups, in *Groups of diffeomorphisms*, Adv. Stud. Pure Math. 52, Math. Soc. Japan, 2008, 505–524. Zbl 1183.57024 MR 2509724