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Deleting and Inserting Fixed Point Manifolds under the Weak Gap Condition

Dedicated to Professor Krzysztof Pawałowski on his 60th birthday

by

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Abstract

Let G be a finite group and X a compact smooth manifold. It is of interest which smooth manifolds can be the G-fixed point sets of smooth G-actions on X. The deleting-inserting theorem of this paper is related to this problem and has applications to one-fixed-point actions on spheres as well as to Smith equivalence.

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§1. Introduction

Let G be a finite group. In this paper, a *manifold* and a G-*manifold* mean a smooth manifold and a smooth G-manifold, respectively. Given a manifold X, it is a fundamental problem to study which manifolds and real vector bundles can be the G-fixed point sets and the normal bundles of G-fixed point sets, respectively, of smooth G-actions on X. This problem for the case where X is a disk was studied by B. Oliver [15], and for X a sphere in [11] under the gap condition. The Smith problem on tangential representations at fixed points on spheres is a part of the problem above and has been studied by various authors. It has been useful for the study of the problem to delete (or insert) manifolds from (or to) a given manifold X as G-fixed point sets. More precisely, for a given G-manifold Y having the diffeomorphism type of X and the G-fixed point set

$$Y^G = F_1 \amalg \cdots \amalg F_m$$

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and given integers $1 \le r_1 \le \cdots \le r_n \le m$, it is of interest whether there exists a G-manifold Z having the diffeomorphism type of X and the G-fixed point set

$$Z^G = F_{r_1} \amalg \cdots \amalg F_{r_n}.$$

A finite group G is called an *Oliver group* if there exists a smooth G-action on a disk without G-fixed points, or equivalently if there never exists a normal series $P \trianglelefteq H \trianglelefteq G$ such that P and G/H have prime power order and H/P is a cyclic group (cf. [16, 15, 6]). We studied such deleting-inserting methods for an Oliver group G invoking the gap condition for which the main requirement is

$$2 \dim Y^g < \dim Y$$

for all non-trivial elements g of G, i.e. $g \neq e$. In the current paper we give a deleting-inserting theorem (Theorem 5.1) for an Oliver group under the weak gap condition which allows the case that $2 \dim Y^g = \dim Y$ for $g \in G$. This theorem yields Theorems 1.3–1.10 below as applications.

Let $\mathcal{S}(G)$ denote the set of all subgroups of G, and $\mathcal{P}(G)$ the set of all prime-power-order subgroups of G, where by convention $\{e\} \in \mathcal{P}(G)$. For a prime p, let $G^{\{p\}}$ denote the smallest normal subgroup N of G such that $|G/N|$ is a power of p, possibly $|G/N| = 1$. Let $\mathcal{L}(G)$ denote the set of all subgroups H containing $G^{\{p\}}$ for some prime p. A (finite-dimensional) real G-module V is called *$\mathcal{L}$-free* if $V^L = 0$ for all $L \in \mathcal{L}(G)$. We define a G-submodule $V_{\mathcal{L}}$ of V by

$$V_{\mathcal{L}} = (V - V^G) - \bigoplus_{p \text{ prime}} (V^{G^{\{p\}}} - V^G).$$

Let $\mathbb{R}[G]$ denote the group ring of G with real coefficients having the canonical (left) G-action. Recall the following fact.

Lemma 1.1 ([6, Theorem 2.3]). *The real G-module $V = \mathbb{R}[G]_{\mathcal{L}}$ has the following properties:*

(1.1.1) $V^H = 0$ *if and only if* $H \in \mathcal{L}(G)$.

(1.1.2) $\dim V^H \ge |K : H| \dim V^K$ *for all* $H \le K \in \mathcal{S}(G)$.

(1.1.3) *The equality* $\dim V^H = 2 \dim V^K$ *holds, where* $H \le K \in \mathcal{S}(G)$*, if and only if* $|K : H| = 2$, $|KG^{\{2\}} : HG^{\{2\}}| = 2$, *and* $HG^{\{q\}} = G$ *for all odd primes* q.

By straightforward computation, we can show the next lemma.

Lemma 1.2 ([13, Proposition 1.9]). *If G is an Oliver group then* $\dim (\mathbb{R}[G]_{\mathcal{L}})^P \ge 2$ *for all* $P \in \mathcal{P}(G)$.

The following two theorems are an elaboration of [6, Theorem B]. In particular, for $m = 1$ they give smooth one-fixed-point actions on spheres.

Theorem 1.3. *Let G be an Oliver group and m a positive integer. Then for any integer $\ell \geq 3$ there exists a G-action on the standard sphere S of dimension*

$$d_\ell = \ell \cdot \Big\{(|G|-1) - \sum_{p||G|}(|G/G^{\{p\}}|-1)\Big\}$$

with exactly m G-fixed points $x_1, \ldots, x_m$ for which the tangential representations $T_{x_i}(S)$ are all isomorphic to the ℓ-fold direct sum $\mathbb{R}[G]_{\mathcal{L}}^{\oplus\ell}$ of $\mathbb{R}[G]_{\mathcal{L}}$.

Let $\mathcal{PH}(G)$ denote the set of all pairs (P, H) consisting of $P \in \mathcal{P}(G)$ and $H \in \mathcal{S}(G)$ with $P < H$. Let $\mathcal{PH}_2(G)$ denote the set of all pairs $(P, H) \in \mathcal{PH}(G)$ such that $|H : P| = 2$, $|HG^{\{2\}} : PG^{\{2\}}| = 2$, and $PG^{\{q\}} = G$ for all odd primes q. For a set $\mathcal{A}$ of pairs (H, K) with $H < K \in \mathcal{S}(G)$, we say that a real G-module V satisfies the *gap condition* (resp. the *weak gap condition*) on $\mathcal{A}$ if

$$\dim V^H > 2\dim V^K \quad (\text{resp. } \dim V^H \geq 2\dim V^K) \tag{1.1}$$

for any $(H, K) \in \mathcal{A}$. It should be remarked that if an $\mathcal{L}$-free real G-module V satisfies the weak gap condition on $\mathcal{PH}_2(G)$ then $V \oplus \mathbb{R}[G]_{\mathcal{L}}^{\oplus m}$ satisfies the weak gap condition on $\mathcal{PH}(G)$ for any $m \geq \dim V$.

Theorem 1.4. *Let G be an Oliver group, m a positive integer, and V an $\mathcal{L}$-free real G-module satisfying the weak gap condition on $\mathcal{PH}_2(G)$. Then there exists an integer N such that for every integer $\ell \geq N$ there exists a G-action on the standard sphere S with exactly m G-fixed points $x_1, \ldots, x_m$ for which the tangential representations $T_{x_i}(S)$ are all isomorphic to $V \oplus \mathbb{R}[G]_{\mathcal{L}}^{\oplus\ell}$.*

Let $\mathrm{RO}(G)$ denote the real representation ring. For a subset A of $\mathrm{RO}(G)$, $A_{\mathcal{P}}$ stands for the set

$$A \cap \bigcap_{P \in \mathcal{P}(G)} \mathrm{Ker}[\mathrm{res}^G_P : \mathrm{RO}(G) \to \mathrm{RO}(P)].$$

Real G-modules V and W are called *Smith equivalent* if there exists a homotopy sphere Σ with a G-action such that Σ^G consists of exactly two points a and b, and the tangential representations $T_a(\Sigma)$ and $T_b(\Sigma)$ are isomorphic to V and W, respectively. Let $\mathrm{Sm}(G)$ denote the *Smith set* of G, i.e.

$$\mathrm{Sm}(G) = \{[V] - [W] \in \mathrm{RO}(G) \mid V \text{ is Smith equivalent to } W\}.$$

The subset $\mathrm{Sm}(G)_{\mathcal{P}}$ is called the *primary Smith set* of G. For a subset A of $\mathrm{RO}(G)$, $A^{\mathcal{L}}$ stands for the set

$$\{[V]-[W]\in A \mid V^L=0 \text{ and } W^L=0 \text{ for all } L\in\mathcal{L}(G)\}.$$

We say that two real G-modules V and W are *$\mathcal{P}$-matched* if $\mathrm{res}^G_P V$ and $\mathrm{res}^G_P W$ are isomorphic for all $P\in\mathcal{P}(G)$.

Theorem 1.5. *Let G be an Oliver group. Let $V_1,\dots,V_m$ be $\mathcal{L}$-free real G-modules satisfying the weak gap condition on $\mathcal{PH}_2(G)$, of which arbitrary two are $\mathcal{P}$-matched. Then there exists an integer N such that for any integer $\ell\geq N$, there exists a smooth G-action on the standard sphere S with exactly m G-fixed points $x_1,\dots,x_m$ for which the tangential representation $T_{x_i}(S)$ is isomorphic to $V_i\oplus\mathbb{R}[G]^{\oplus\ell}_{\mathcal{L}}$, $1\leq i\leq m$.*

In the case $m=2$, we obtain the next theorem on Smith equivalence.

Theorem 1.6. *Let G be an Oliver group and let V and W be $\mathcal{P}$-matched and $\mathcal{L}$-free real G-modules both satisfying the weak gap condition on $\mathcal{PH}_2(G)$. Then there exists an integer N such that for any integer $\ell\geq N$ there exists a smooth G-action on the standard sphere S with exactly two G-fixed points x_1 and x_2 for which the tangential representations $T_{x_1}(S)$ and $T_{x_2}(S)$ are isomorphic to $V\oplus\mathbb{R}[G]^{\oplus\ell}_{\mathcal{L}}$ and $W\oplus\mathbb{R}[G]^{\oplus\ell}_{\mathcal{L}}$, respectively. In particular, V and W are stably Smith equivalent.*

Let X be a G-manifold and S a smooth G-action on the standard sphere with exactly one G-fixed point a and $T_a(S)\cong\mathbb{R}[G]^{\oplus\ell}_{\mathcal{L}}$. Then the cartesian product $Y=X\times S$ has the diagonal G-action and the G-fixed point set of Y is $X^G\times\{a\}$. For each $x\in X^G$, the tangential representation $T_{(x,a)}(Y)$ is isomorphic to $T_x(X)\oplus\mathbb{R}[G]^{\oplus\ell}_{\mathcal{L}}$. The next theorem follows from Theorems 1.3 and 1.6.

Theorem 1.7. *Let G be an Oliver group and (V_i,W_i) a pair of $\mathcal{L}$-free $\mathcal{P}$-matched real G-modules V_i and W_i for each $1\leq i\leq m$. Suppose all V_i and W_i, $1\leq i\leq m$, satisfy the weak gap condition on $\mathcal{PH}_2(G)$. Let X be a G-manifold with G-fixed point set*

$$X^G=\{x_1\}\amalg\cdots\amalg\{x_m\}\amalg F \qquad (\textit{disjoint union})$$

such that for each $1\leq i\leq m$, the tangential representation $T_{x_i}(X)$ is isomorphic to V_i, where F is a union of connected components of X^G. Then there exists an integer N such that for any integer $\ell\geq N$ there exists a G-manifold Y with G-fixed point set X^G for which the underlying space is diffeomorphic to $X\times S(\mathbb{R}\oplus\mathbb{R}[G]^{\oplus\ell}_{\mathcal{L}})$ and the tangential representation $T_{x_i}(Y)$ is isomorphic to $W_i\oplus\mathbb{R}[G]^{\oplus\ell}_{\mathcal{L}}$ for each $1\leq i\leq m$.

A finite group G is called a *gap group* if each element x of $\mathrm{RO}(G)^{\mathcal{L}}_{\mathcal{P}}$ can be written in the form $x = [V] - [W]$ with $\mathcal{L}$-free real G-modules V and W satisfying the gap condition on $\mathcal{PH}(G)$. We remark that G with $\mathcal{L}(G) \cap \mathcal{P}(G) = \emptyset$ is a gap group if and only if there exists an $\mathcal{L}$-free real G-module V satisfying the gap condition on $\mathcal{PH}_2(G)$. An Oliver group G is a gap group if G is nilpotent, or $G = G^{\{2\}}$, or $G \neq G^{\{p\}}$ for at least two odd primes p. In the case where G is a gap Oliver group, we could determine the geometrically defined set $\mathrm{Sm}(G)^{\mathcal{L}}_{\mathcal{P}}$ in algebraic terms: $\mathrm{Sm}(G)^{\mathcal{L}}_{\mathcal{P}}$ coincides with $\mathrm{RO}(G)^{\mathcal{L}}_{\mathcal{P}}$ (cf. [17, p. 850, Realization Theorem]). But it is difficult to determine $\mathrm{Sm}(G)$ or even $\mathrm{Sm}(G)^{\mathcal{L}}_{\mathcal{P}}$ when G is not a gap group. Let us call a finite group G a *weak gap group* if each element x of $\mathrm{RO}(G)^{\mathcal{L}}_{\mathcal{P}}$ can be written in the form $x = [V]-[W]$ with $\mathcal{L}$-free real G-modules V and W satisfying the weak gap condition on $\mathcal{PH}(G)$. For example, $G = S_5 \times C_2 \times \cdots \times C_2$ is not a gap group but a weak gap group (cf. [4]), where S_5 is the symmetric group on five letters and C_2 is a group of order 2. Since $\mathrm{Sm}(G)^{\mathcal{L}}_{\mathcal{P}} \subset \mathrm{RO}(G)^{\mathcal{L}}_{\mathcal{P}}$, we obtain the next result.

Theorem 1.8. *If G is a weak gap Oliver group then $\mathrm{Sm}(G)^{\mathcal{L}}_{\mathcal{P}}$ coincides with $\mathrm{RO}(G)^{\mathcal{L}}_{\mathcal{P}}$.*

Let H be a subgroup of G. For a real H-module V, we denote by $\mathrm{ind}_H^G V$ the real G-module $\mathbb{R}[G] \otimes_{\mathbb{R}[H]} V$. If V satisfies the weak gap condition on $\mathcal{PH}(H)$ then $\mathrm{ind}_H^G V$ satisfies the weak gap condition on $\mathcal{PH}(G)$; if V is $\mathcal{L}$-free then $\mathrm{ind}_H^G V$ is also $\mathcal{L}$-free; and if V and W are $\mathcal{P}$-matched real H-modules then $\mathrm{ind}_H^G V$ and $\mathrm{ind}_H^G W$ are $\mathcal{P}$-matched real G-modules. Let ind_H^G denote the induction homomorphism $\mathrm{RO}(H) \to \mathrm{RO}(G)$. Then the inclusion $\mathrm{ind}_H^G(\mathrm{RO}(H)^{\mathcal{L}}_{\mathcal{P}}) \subset \mathrm{RO}(G)^{\mathcal{L}}_{\mathcal{P}}$ holds. Thus we obtain the next result from Theorem 1.6.

Theorem 1.9. *Let H be a subgroup of an Oliver group G.*

(1.9.1) *If V and W are $\mathcal{L}$-free $\mathcal{P}$-matched real H-modules satisfying the weak gap condition on $\mathcal{PH}_2(H)$ then $[\mathrm{ind}_H^G V] - [\mathrm{ind}_H^G W]$ belongs to $\mathrm{Sm}(G)^{\mathcal{L}}_{\mathcal{P}}$.*

(1.9.2) *If H is a weak gap group then*

$$\mathrm{ind}_H^G(\mathrm{Sm}(H)^{\mathcal{L}}_{\mathcal{P}}) \subset \mathrm{ind}_H^G(\mathrm{RO}(H)^{\mathcal{L}}_{\mathcal{P}}) \subset \mathrm{Sm}(G)^{\mathcal{L}}_{\mathcal{P}}.$$

Let $\mathcal{H}(G)$ denote the set of all subgroups H of G for which there exists $P \in \mathcal{P}(G)$ such that $P \leq H$ and $|H : P| \leq 2$. For a subset $A \subset \mathrm{RO}(G)$, we define $A_{\mathcal{H}}$ to be the set of all elements $x \in A$ such that $\mathrm{res}_H^G x = 0$ for all $H \in \mathcal{H}(G)$. It is obvious that $A^{\mathcal{L}}_{\mathcal{H}} \subset \mathrm{RO}(G)^{\mathcal{L}}_{\mathcal{P}}$.

Theorem 1.10. *If H is a subgroup of an Oliver group G then*

$$\mathrm{ind}_H^G(\mathrm{RO}(H)^{\mathcal{L}}_{\mathcal{H}}) \subset \mathrm{Sm}(G)^{\mathcal{L}}_{\mathcal{H}} \subset \mathrm{Sm}(G)^{\mathcal{L}}_{\mathcal{P}}.$$

This paper is organized as follows. Section 2 is devoted to preparation of basic terms and notation concerning G-manifolds and G-framed maps. In Section 3, we discuss equivariant surgery to obtain homology equivalences on even-dimensional manifolds satisfying the weak gap condition. Theorem 3.5 describes a surgery obstruction to $\mathbb{Z}_{(p)}$-homology equivalence in algebraic terms. Section 4 is devoted to the induction theory of equivariant surgery obstruction groups. In Section 5 we prove Theorem 5.1 which provides a method of deleting or inserting fixed point manifolds. Theorems 1.3–1.5 and 1.10 are proved in Section 6.

§2. Preliminaries

For families $\mathcal{A}$, $\mathcal{B}$ of sets closed under intersection, and a map $f : \mathcal{A} \to \mathcal{B}$, we say that f *preserves intersection* or is *intersection preserving* if

$$f(A_1 \cap A_2) = f(A_1) \cap f(A_2) \quad \text{for all } A_1, A_2 \in \mathcal{A}.$$

Let Θ be a G-set, $\rho : \Theta \to \mathcal{S}(G)$ a G-map, where G acts on $\mathcal{S}(G)$ by conjugation, and S a conjugation invariant subset of G consisting of elements of order 2. The group G acts on S by conjugation. The set Θ is called (ρ, S)-*simple* if for each $t \in \Theta$, the set $\rho(t)$ contains at most one element in S.

Definition 2.1. For a (ρ, S)-simple G-set Θ, we define the S-*contraction* $(\Theta/S, \rho/S)$ of (Θ, ρ) as follows. Let $\sim_S$ denote the equivalence relation on Θ such that $t \sim_S t'$ if and only if $\rho(t) \cap S = \rho(t') \cap S$. Denote by Θ/S the set of equivalence classes with respect to $\sim_S$. The map $\rho/S : \Theta/S \to \mathcal{S}(G)$ is defined by

$$\rho/S([t]) = \{e\} \cup (\rho(t) \cap S)$$

for the $\sim_S$-equivalence class $[t]$ of $t \in \Theta$. Then Θ/S has a canonical G-action and $\rho/S : \Theta/S \to \mathcal{S}(G)$ is a G-map.

A G-map $\rho : \Theta \to \mathcal{S}(G)$ is called S-*injective* (resp. S-*bijective*) if for each $s \in S$, there exists at most one (resp. exactly one) element $t \in \Theta$ such that $\rho(t)$ contains s.

Let $\mathfrak{P}(\Theta)$ denote the set of all subsets of Θ. Clearly $\mathfrak{P}(\Theta)$ has the induced G-action. A G-map $f : \mathcal{S}(G) \to \mathfrak{P}(\Theta)$ is called ρ-*compatible* if $\rho(f(H)) \subset \mathcal{S}(H)$ for all $H \in \mathcal{S}(G)$. A G-map $f : \mathcal{S}(G) \to \mathfrak{P}(\Theta)$ is called (ρ, S)-*saturated* if

$$f(H) \supset \{t \in \Theta \mid \rho(t) \cap S \cap H \neq \emptyset\} \quad \text{for all } H \in \mathcal{S}(G). \tag{2.1}$$

It is straightforward to verify the next lemma.

Lemma 2.2. *Let $f : \mathcal{S}(G) \to \mathfrak{P}(\Theta)$ be an intersection preserving ρ-compatible G-map and set $\Theta_H = f(H)$ and $\rho_H = \rho|_{\Theta_H} : \Theta_H \to \mathcal{S}(H)$.*

(2.2.1) *If Θ is (ρ, S)-simple, then Θ_H is $(\rho_H, S\cap H)$-simple for $H \in \mathcal{S}(G)$ and the associated map $\rho/S : \Theta/S \to \mathcal{S}(G)$ is S-injective.*

(2.2.2) *If $\rho : \Theta \to \mathcal{S}(G)$ is S-injective then $\rho_H : \Theta_H \to \mathcal{S}(H)$ is $(S \cap H)$-injective for $H \in \mathcal{S}(G)$.*

(2.2.3) *If $\rho : \Theta \to \mathcal{S}(G)$ is S-bijective and $f : \mathcal{S}(G) \to \mathfrak{P}(\Theta)$ is (ρ, S)-saturated then $\rho_H : \Theta_H \to \mathcal{S}(H)$ is $(S \cap H)$-bijective for $H \in \mathcal{S}(G)$.*

Let X be a compact, connected G-manifold, possibly with boundary ∂X. The *singular set* X_{sing} of X is defined by

$$X_{\mathrm{sing}} = \bigcup_{g\in G\smallsetminus\{e\}} X^g.$$

We say that X satisfies the *weak gap condition* if

$$\dim X_{\mathrm{sing}} \le \frac{1}{2}\dim X. \tag{2.2}$$

In the case where X has even dimension $2k$ and satisfies the weak gap condition, we say that X satisfies the k-*tame condition* if

$$\begin{gathered}\dim X^K \le k-2\\ \text{whenever } H < K \in \mathcal{S}(G),\ \dim X^H = k, \text{ and } H = \bigcap_{x\in X^H} G_x,\end{gathered} \tag{2.3}$$

where G_x stands for the isotropy subgroup of G at the point x. Let $G(2)$ denote the set of all elements of G of order 2. In the case where X has even dimension $2k$ and satisfies the weak gap condition, we say that X satisfies the $G(2)$-*condition* if

$$|H| = 2 \quad \text{whenever} \quad H \in \mathcal{S}(G) \text{ and } 2\dim X^H = \dim X. \tag{2.4}$$

For a subgroup H and an integer $\ell \ge 0$, let $\pi_0(X^H, \ell)$ denote the set of all connected components of dimension ℓ of X^H. For $\alpha \in \pi_0(X^H, \ell)$, we denote by X_α or X^H_α the underlying space of α. Each $\alpha \in \pi_0(X^H, \ell)$ determines the group

$$\rho_X(\alpha) = \bigcap_{x\in X_\alpha} G_x.$$

Definition 2.3. Let X be a compact, connected G-manifold, possibly with boundary, satisfying the weak gap condition. Then we set

$$\begin{aligned}
S(X) &= \{g \in G \mid 2\dim X^g = \dim X\},\\
Q(X) &= \{g \in G \mid \dim X^g = [(\dim X - 1)/2]\},\\
\Sigma(X) &= \{\alpha \mid H \in \mathcal{S}(G),\ \alpha \in \pi_0(X^H, \dim X/2), \text{ and } \rho_X(\alpha) = H\},
\end{aligned}$$

where for a real number x, $[x]$ denotes the greatest integer not exceeding x. The $(\dim X/2)$-*dimensional singular structure* $\mathfrak{S}(X)$ associated with X is defined to be the set of all X_s, $s \in \Sigma(X)$. For each $s \in \Sigma(X)$, the manifold X_s has the unique orientation class t_s in $H_k(X_s, \partial X_s; \mathbb{Z}_2)$. The G-set $\Theta^{(2)}(X)$ is defined to be the set of all t_s, where s runs over $\Sigma(X)$. The correspondence $s \mapsto t_s$ gives a bijection $\Sigma(X) \to \Theta^{(2)}(X)$. The map $\rho_X^{(2)} : \Theta^{(2)}(X) \to \mathcal{S}(G)$ is defined by $\rho_X^{(2)}(t_s) = \rho_X(s)$ for $s \in \Sigma(X)$.

The proof of the next lemma is straightforward.

Lemma 2.4. *Let X be a G-manifold as in Definition 2.3. Suppose that X has even dimension $n = 2k$ and satisfies the $G(2)$-condition. Then the following hold:*

(2.4.1) $\Theta^{(2)}(X)$ *is* $(\rho_X^{(2)}, S(X))$*-simple.*

(2.4.2) $\rho_X^{(2)}/S(X) : \Theta^{(2)}(X)/S(X) \to \mathcal{S}(G)$ *is* $S(X)$*-bijective.*

(2.4.3) *For* $H \in \mathcal{S}(G)$, $S(\mathrm{res}_H^G X)$ *coincides with* $S(X) \cap H$. *Thus the map* $H \mapsto S(\mathrm{res}_H^G X)$ *is intersection preserving.*

(2.4.4) *For* $H \in \mathcal{S}(G)$, $\Theta^{(2)}(\mathrm{res}_H^G X)$ *coincides with* $\{t \in \Theta^{(2)}(X) \mid \rho_X^{(2)}(t) \subset H\}$. *Hence the map* $f : \mathcal{S}(G) \to \mathfrak{P}(\Theta^{(2)}(X))$; $H \mapsto \Theta^{(2)}(\mathrm{res}_H^G X)$, *is intersection preserving,* $\rho_X^{(2)}$*-compatible, and* $(\rho_X^{(2)}, S(X))$*-saturated, and furthermore* $f(G) = \Theta^{(2)}(X)$.

(2.4.5) *The canonical map* $\gamma : \Theta^{(2)}(X) \to \Theta^{(2)}(X)/S(X)$ *is a* G*-map, the diagram*

$$\begin{array}{ccc} \Theta^{(2)}(X) & \xrightarrow{\rho_X^{(2)}} & \mathcal{S}(G) \\ {\scriptstyle\gamma}\downarrow & \nearrow{\scriptstyle \rho_X^{(2)}/S(X)} & \\ \Theta^{(2)}(X)/S(X) & & \end{array}$$

commutes, and

$$\gamma(\Theta^{(2)}(X)) = \Theta^{(2)}(X)/S(X).$$

Let X be a compact, connected, oriented G-manifold of dimension $n \geq 5$, possibly with boundary ∂X. Let R be a commutative ring with 1 and with trivial anti-involution $\bar{\cdot}$. The group ring $R[G]$ has the anti-involution $\bar{\cdot}$ derived from the orientation homomorphism $w_X : G \to \{\pm 1\}$ of X, i.e.

$$\Big(\sum_{g \in G} r_g g\Big)^{-} = \sum_{g \in G} r_g w_X(g) g^{-1},$$

where $r_g \in R$. Let $\widetilde{X}$ denote the universal covering space of X. Let $\widetilde{G}$ denote the fundamental group $\pi_1(EG \times_G X)$, where EG is a contractible G-CW complex with a free G-action. We have the exact sequence

$$1 \to \pi_1(X) \to \widetilde{G} \to G \to 1.$$

If X^G is nonempty then this sequence splits, i.e. $\widetilde{G} = \pi_1(X) \rtimes G$.

Let Y be a compact, connected, oriented G-manifold of dimension n, possibly with boundary ∂Y. Let $\boldsymbol{f} = (f, b)$ be a G-framed map, where $f : (X, \partial X) \to (Y, \partial Y)$ is a G-map such that $f : X \to Y$ is 1-connected, and $b : T(X) \oplus f^*\eta \to f^*\xi$ is a real G-vector bundle isomorphism for real G-vector bundles η and ξ over Y such that $\eta \supset \varepsilon_Y(\mathbb{R}^n)$ (cf. [2, Lemma 6.1]). Then $\boldsymbol{f}$ is covered by the induced $\widetilde{G}$-framed map $\widetilde{\boldsymbol{f}} = (\widetilde{f}, \widetilde{b})$ consisting of a $\widetilde{\varphi}$-map $\widetilde{f} : (\widetilde{X}, \partial\widetilde{X}) \to (\widetilde{Y}, \partial\widetilde{Y})$ and a real $\widetilde{G}$-vector bundle isomorphism $\widetilde{b} : T(\widetilde{X}) \oplus \widetilde{f}^*\widetilde{\eta} \to \widetilde{f}^*\widetilde{\xi}$, where $\widetilde{Y}$ is the universal covering space of Y, $\widetilde{\varphi}$ is the canonical homomorphism $\widetilde{G} = \pi_1(EG \times_G X) \to \pi_1(EG \times_G Y) = \widehat{G}$, and $\widetilde{\eta}$ and $\widetilde{\xi}$ are the real $\widehat{G}$-vector bundles over $\widetilde{Y}$ induced from η and ξ, respectively:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\ {\scriptstyle \pi_{\widetilde{X},X}} \downarrow & & \downarrow {\scriptstyle \pi_{\widetilde{Y},Y}} \\ X & \xrightarrow{f} & Y \end{array}$$

We note that the map $\widetilde{f} : (\widetilde{X}, \partial\widetilde{X}) \to (\widetilde{Y}, \partial\widetilde{Y})$ is not necessarily of degree one.

§3. *G*-surgery maps on even-dimensional manifolds

Let X be a compact, connected, oriented G-manifold of even dimension $n = 2k \geq 6$, possibly with boundary ∂X. Throughout this section, we assume that X satisfies the weak gap condition and the k-tame condition. Let R be a commutative ring with 1 and with trivial anti-involution $\bar{\cdot}$. We set $\lambda = (-1)^k$, $S = S(X)$ and $Q = Q(X)$; further define

$$(Q)_R = R[Q] + \{x - \lambda\overline{x} \mid x \in R[G]\}, \qquad (S)_R = R[S] + \{x + \lambda\overline{x} \mid x \in R[G]\}.$$

Then

$$\boldsymbol{A}_X = (R[G], (\bar{\cdot}, \lambda), (S)_R, G, R[S], (Q)_R + R[S])$$

is a double parameter algebra in the sense of [2, Definition 2.5].

Let $\mathfrak{S} = \{X_s \mid s \in \Sigma\}$ be a set of compact connected k-dimensional neat submanifolds of X, where Σ is a G-set, such that $gX_s = X_{gs}$ for all $g \in G$ and $s \in \Sigma$. Set

$$X_{\mathfrak{S}} = \bigcup_{s \in \Sigma} X_s.$$

In this paper, we assume that $\mathfrak{S}$ satisfies the *k-tame condition*, i.e.

$$\text{(3.1)} \qquad X_s \cap X_t \text{ is a neat submanifold of } X_s \text{ of dimension } \leq k-2$$

for all $s, t \in \Sigma$, $s \neq t$. If $\mathfrak{S} \supset \mathfrak{S}(X)$ then we call $\mathfrak{S}$ a *k-singular structure* of X. The index set Σ decomposes into the disjoint union of Σ_+ and Σ_- consisting of all elements $s \in \Sigma$ such that X_s is orientable and non-orientable, respectively. Let $\Theta^{(0)}(\mathfrak{S})$ denote the set of all generators of $H_k(X_s, \partial X_s; \mathbb{Z})$, where s runs over Σ_+, and let $\Theta^{(2)}(\mathfrak{S})$ denote the set of all generators of $H_k(X_s, \partial X_s; \mathbb{Z}_2)$, where s runs over Σ. The sets $\Theta^{(0)}(\mathfrak{S})$ and $\Theta^{(2)}(\mathfrak{S})$ have canonical actions of $G \times \{\pm 1\}$ and G, respectively. In addition, there is a canonical map $p_{\mathfrak{S}} : \Theta^{(0)}(\mathfrak{S}) \to \Theta^{(2)}(\mathfrak{S})$; for a generator x of $H_k(X_s, X_s; \mathbb{Z})$, $p_{\mathfrak{S}}(x)$ is the generator of $H_k(X_s, X_s; \mathbb{Z}_2)$. We have a natural one-to-one correspondence from Σ to $\Theta^{(2)}(\mathfrak{S})$. Thus we often identify $\Theta^{(2)}(\mathfrak{S})$ with Σ as G-sets. On the other hand, we may not have a $(G \times \{\pm 1\})$-bijection from $\Theta^{(0)}(\mathfrak{S})$ to $\Sigma_+ \times \{\pm 1\}$, although there is a non-equivariant bijection between these sets. Let $\rho_{\mathfrak{S}}$ denote the map $\Theta^{(2)}(\mathfrak{S}) = \Sigma \to \mathcal{S}(G)$ defined by

$$\rho_{\mathfrak{S}}(s) = \bigcap_{x \in X_s} G_x \qquad (s \in \Sigma).$$

Let $\boldsymbol{\Theta}(\mathfrak{S})$ denote the datum

$$(\Theta^{(0)}(\mathfrak{S}), \Theta^{(2)}(\mathfrak{S}), p_{\mathfrak{S}}, \rho_{\mathfrak{S}}).$$

Set

$$\begin{aligned} \widetilde{Q} &= Q_{\widetilde{X}} \ (= \{g \in \widetilde{G}(2) \mid \dim \widetilde{X}^g = k-1\}), \\ \widetilde{S} &= S_{\widetilde{X}} \ (= \{g \in \widetilde{G}(2) \mid \dim \widetilde{X}^g = k\}), \\ (\widetilde{Q})_R = R[\widetilde{Q}] + \{x - \lambda\overline{x} \mid x \in R[\widetilde{G}]\}, &\qquad (\widetilde{S})_R = R[\widetilde{S}] + \{x + \lambda\overline{x} \mid x \in R[\widetilde{G}]\}. \end{aligned}$$

Then

$$\widetilde{\boldsymbol{A}} = \boldsymbol{A}_{\widetilde{X}} = (R[\widetilde{G}], (\bar{\cdot}, \lambda), (\widetilde{S})_R, \widetilde{G}, R[\widetilde{S}], (\widetilde{Q})_R + R[\widetilde{S}])$$

is a double parameter algebra.

Let $\mathfrak{S} = \{X_s \mid s \in \Sigma\}$ be a k-singular structure of X as above. Consider the set

$$\widetilde{\mathfrak{S}} = \{\widetilde{X}_t \mid t \in \widetilde{\Sigma}\}$$

of all connected components $\widetilde{X}_t$ of $\pi^{-1}_{\widetilde{X},X}(X_s)$, $s \in \Sigma$, where $\pi_{\widetilde{X},X}$ is the canonical projection $\widetilde{X} \to X$. Here we have canonical surjections $\widetilde{\mathfrak{S}} \to \mathfrak{S}$ and $\widetilde{\Sigma} \to \Sigma$. We call $\widetilde{\mathfrak{S}}$ the *k-singular structure of* $\widetilde{X}$ *induced from* $\mathfrak{S}$. Note that $\widetilde{X}$ and $\widetilde{X}_t$ are possibly non-compact. The index set $\widetilde{\Sigma}$ decomposes into the disjoint union of $\widetilde{\Sigma}_+$ and $\widetilde{\Sigma}_-$ consisting of all elements $t \in \widetilde{\Sigma}$ such that $\widetilde{X}_t$ is orientable and non-orientable, respectively. Let $\Theta^{(0)}(\widetilde{\mathfrak{S}})$ denote the set of all generators of

$H_k^{\text{loc.fin.}}(\widetilde{X}_t, \partial\widetilde{X}_t; \mathbb{Z})$, where t runs over $\widetilde{\Sigma}_+$, and let $\Theta^{(2)}(\widetilde{\mathfrak{S}})$ denote the set of all generators of $H_k^{\text{loc.fin.}}(\widetilde{X}_t, \partial\widetilde{X}_t; \mathbb{Z}_2)$, where t runs over $\widetilde{\Sigma}$. The sets $\Theta^{(0)}(\widetilde{\mathfrak{S}})$ and $\Theta^{(2)}(\widetilde{\mathfrak{S}})$ have canonical actions of $\widetilde{G} \times \{\pm 1\}$ and $\widetilde{G}$, respectively. In addition, we have the canonical map $p_{\widetilde{\mathfrak{S}}} : \Theta^{(0)}(\widetilde{\mathfrak{S}}) \to \Theta^{(2)}(\widetilde{\mathfrak{S}})$. Define the map

$$\rho_{\widetilde{\mathfrak{S}}} : \Theta^{(2)}(\widetilde{\mathfrak{S}}) = \widetilde{\mathfrak{S}} = \widetilde{\Sigma} \to \mathcal{S}(\widetilde{G}) \quad \text{by} \quad \rho_{\widetilde{\mathfrak{S}}}(t) = \bigcap_{x \in \widetilde{X}_t} \widetilde{G}_x.$$

Let $\boldsymbol{\Theta}(\widetilde{\mathfrak{S}})$ denote the datum

$$(\Theta^{(0)}(\widetilde{\mathfrak{S}}), \Theta^{(2)}(\widetilde{\mathfrak{S}}), p_{\widetilde{\mathfrak{S}}}, \rho_{\widetilde{\mathfrak{S}}}).$$

The next lemma is well-known.

Lemma 3.1. *Let $\boldsymbol{f} = (f, b)$ be a G-framed map and $\mathfrak{S}$ a k-singular structure of X as above. Suppose the map $f : (X, \partial X) \to (Y, \partial Y)$ has degree one. Then $\boldsymbol{f}$ can be converted to a G-framed map $\boldsymbol{f}' = (f', b')$, where $f' : (X', \partial X') \to (Y, \partial Y)$ and $b' : T(X') \oplus f'^*\eta \to f'^*\xi$, such that $f' : X' \to Y$ is k-connected, by a G-surgery on X relative to $X_{\text{sing}} \cup X_{\mathfrak{S}} \cup \partial X$.*

First, note that the degree of the resulting map $f' : (X', \partial X') \to (Y, \partial Y)$ above is 1. Second, note that if $f : X \to Y$ is k-connected then the mapping cylinder $M_{\widetilde{f}}$ of $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is the universal covering space of the mapping cylinder M_f of $f : X \to Y$, the group $\pi_{k+1}(\widetilde{f})$ can be identified with $\pi_{k+1}(f)$, and the canonical homomorphism $\pi_{k+1}(\widetilde{f}) \to K_k(\widetilde{f}; \mathbb{Z})$ is an isomorphism, where $\pi_{k+1}(\widetilde{f}) = \pi_{k+1}(M_{\widetilde{f}}, \widetilde{X})$ and

$$K_k(\widetilde{f}; \mathbb{Z}) = \operatorname{Ker}[\widetilde{f}_* : H_k(\widetilde{X}; \mathbb{Z}) \to H_k(\widetilde{Y}; \mathbb{Z})].$$

Now let R be $\mathbb{Z}$ or $\mathbb{Z}_{(p)}$ for a prime p. We denote by $\mathcal{P}_p(G)$ the set of all subgroups of G with p-power order. Thus we have

$$\mathcal{P}(G) = \bigcup_{p \text{ prime}} \mathcal{P}_p(G).$$

Let $\boldsymbol{f} = (f, b)$, $f : (X, \partial X) \to (Y, \partial Y)$ be a G-framed map and $\mathfrak{S}$ a k-singular structure of X as above. Then let $\widetilde{\boldsymbol{f}} = (\widetilde{f}, \widetilde{b})$, $\widetilde{f} : (\widetilde{X}, \partial\widetilde{X}) \to (\widetilde{Y}, \partial\widetilde{Y})$, denote the $\widetilde{G}$-framed map induced from $\boldsymbol{f}$, where $\widetilde{X}$ and $\widetilde{Y}$ are the universal covering spaces of X and Y, respectively. Let $\widetilde{\mathfrak{S}}$ denote the induced k-singular structure of $\widetilde{X}$.

Definition 3.2. Let $\boldsymbol{f}$ be the G-framed map above. We define the $R[\widetilde{G}]$-module $M(\widetilde{f}; R)$ by

$$M(\widetilde{f}; R) = \pi_{k+1}(\widetilde{f}) \otimes R.$$

We call $\boldsymbol{f}$ a (G, R)-*surgery map* if the following conditions are fulfilled:

(3.2.1) $f : X \to Y$ is of degree one.

(3.2.2) $f : X \to Y$ is 1-connected.

(3.2.3) $f_* : H_j(X; R) \to H_j(Y; R)$, $j < k$, are all isomorphisms, and $f_* : H_k(X; R) \to H_k(Y; R)$ is surjective.

(3.2.4) $\partial f_* : H_j(\partial X; R) \to H_j(\partial Y; R)$, $j \leq n - 1$, are all isomorphisms.

(3.2.5) $f : X \to Y$ is k-connected, or the canonical map $M(\widetilde{f}; R) \otimes_{R[\widetilde{G}]} R[G] \to K_k(f; R)$ is an isomorphism, where

$$K_k(f; R) = \mathrm{Ker}[f_* : H_k(X; R) \to H_k(Y; R)].$$

(3.2.6) In the case $R = \mathbb{Z}$, $f^P : X^P \to Y^P$ are $\mathbb{Z}_q$-homology equivalences for all subgroups $P \in \mathcal{P}(G)$ with $P \neq \{e\}$, and primes q dividing $|P|$. In the case $R = \mathbb{Z}_{(p)}$, $f^P : X^P \to Y^P$ are $\mathbb{Z}_p$-homology equivalences for all $P \in \mathcal{P}_p(G)$ with $P \neq \{e\}$.

(3.2.7) $\chi(X^g) = \chi(Y^g)$ for all $g \in G$, $g \neq e$.

We have the Poincaré pairing

$$H_k^{\text{loc.fin.}}(\widetilde{X}, \partial\widetilde{X}; \mathbb{Z}) \times H_k(\widetilde{X}; \mathbb{Z}) \to \mathbb{Z}.$$

Passing along the canonical homomorphisms

$$\pi_{k+1}(\widetilde{f}) \to H_{k+1}(M_{\widetilde{f}}, \widetilde{X}; \mathbb{Z}) \to K_k(\widetilde{f}; \mathbb{Z}) \subset H_k(\widetilde{X}; \mathbb{Z}) \to H_k^{\text{loc.fin.}}(\widetilde{X}, \partial\widetilde{X}; \mathbb{Z})$$

we obtain the intersection form $\widetilde{B}_0 : M(\widetilde{f}; R) \times M(\widetilde{f}; R) \to R$, and hence the $\widetilde{G}$-equivariant intersection form

$$\widetilde{B} : M(\widetilde{f}; R) \times M(\widetilde{f}; R) \to R[\widetilde{G}];\ \widetilde{B}(x, y) = \sum_{g \in \widetilde{G}} \widetilde{B}_0(x, g^{-1}y)g.$$

Let $x \in \pi_{k+1}(\widetilde{f})$. Then x is represented by a commutative diagram

$$\begin{array}{ccc} S^k & \xrightarrow{\alpha} & \widetilde{X} \\ \downarrow & & \downarrow{\scriptstyle \widetilde{f}} \\ D^{k+1} & \longrightarrow & \widetilde{Y} \end{array}$$

By virtue of this diagram and the bundle isomorphism b, the induced bundle $\alpha^*T(\widetilde{X})$ is stably trivial. Thus x is represented by an immersion $\alpha : S^k \to \widetilde{X}$ with trivial normal bundle. Let g be an element in $\widetilde{G}$ of order 2 satisfying $\dim \widetilde{X}^g \leq k-2$. Then the regular homotopy classes of immersions $S^k \to \widetilde{X}$ correspond in a one-to-one way to the regular homotopy classes of immersions $S^k \to \widetilde{X} \smallsetminus \widetilde{X}^g$. Hence

Theorem 5.2 of [18] provides the $\langle g\rangle$-equivariant self-intersection form

$$\widetilde{q}_{\langle g\rangle} : \pi_{k+1}(\widetilde{f}) \to \mathbb{Z}[\langle g\rangle]/\{a - \lambda\overline{a} \mid a \in \mathbb{Z}[\langle g\rangle]\}.$$

Assembling the data of the $\widetilde{G}$-equivariant intersection form $\widetilde{B}$ and the $\langle g\rangle$-equivariant self-intersection forms $\widetilde{q}_{\langle g\rangle}$ (cf. [2, Definition 4.11]), we obtain the $\widetilde{G}$-equivariant self-intersection form

$$\widetilde{q} : M(\widetilde{f}; R) \to R[\widetilde{G}]/((Q_{\widetilde{X}})_R + R[S_{\widetilde{X}}]) \qquad \text{(cf. [2, p. 567, } \ell. \text{ 3]).}$$

For a generator $\alpha \in H_k^{\text{loc.fin.}}(\widetilde{X}_t, \partial\widetilde{X}_t; \mathbb{Z})$, where $t \in \widetilde{\Sigma}_+$, we have the element

$$j_*\alpha \in H_k^{\text{loc.fin.}}(\widetilde{X}, \partial\widetilde{X}; \mathbb{Z}),$$

where $j_* : H_k^{\text{loc.fin.}}(\widetilde{X}_t, \partial\widetilde{X}_t; \mathbb{Z}) \to H_k^{\text{loc.fin.}}(\widetilde{X}, \partial\widetilde{X}; \mathbb{Z})$ is the canonical homomorphism. Via the intersection paring (or the Poincaré pairing up to sign)

$$H_k^{\text{loc.fin.}}(\widetilde{X}, \partial\widetilde{X}; \mathbb{Z}) \times H_k(\widetilde{X}; \mathbb{Z}) \to \mathbb{Z}$$

and the canonical map $M(\widetilde{f}; \mathbb{Z}) \to H_k(\widetilde{X}; \mathbb{Z})$, $j_*\alpha$ determines an element

$$\widetilde{\theta}^{(0)}(\alpha) \in M(\widetilde{f}; \mathbb{Z})^{\#}, \quad \text{where} \quad M(\widetilde{f}; \mathbb{Z})^{\#} = \mathrm{Hom}_{\mathbb{Z}[\widetilde{G}]}(M(\widetilde{f}; \mathbb{Z}), \mathbb{Z}[\widetilde{G}]).$$

Thus we obtain the $(\widetilde{G} \times \{\pm 1\})$-map

$$\widetilde{\theta}^{(0)} : \Theta^{(0)}(\widetilde{\mathfrak{S}}) \to M(\widetilde{f}; R)^{\#}, \quad \text{where} \quad M(\widetilde{f}; R)^{\#} = \mathrm{Hom}_{R[\widetilde{G}]}(M(\widetilde{f}; R), R[\widetilde{G}]).$$

Similarly we obtain the $\widetilde{G}$-map

$$\widetilde{\theta}^{(2)} : \Theta^{(2)}(\widetilde{\mathfrak{S}}) \to M(\widetilde{f}; R/2R)^{\#},$$

where

$$M(\widetilde{f}; R/2R)^{\#} = \mathrm{Hom}_{R/2R[\widetilde{G}]}(M(\widetilde{f}; R/2R), R/2R[\widetilde{G}]).$$

Putting all this together, we obtain the surgery module

$$\boldsymbol{M}_{\widetilde{\boldsymbol{f}}, \widetilde{\mathfrak{S}}} = (M(\widetilde{f}; R), \widetilde{B}, \widetilde{q}, \widetilde{\theta}^{(0)}, \widetilde{\theta}^{(2)}).$$

By the hypothesis $M(\widetilde{f}; R) \otimes_{R[\widetilde{G}]} R[G] = K_k(f; R)$, we obtain the commutative diagram

$$\begin{array}{ccccc}
\Theta^{(0)}(\widetilde{\mathfrak{S}}) & \longrightarrow & H_k^{\text{loc.fin.}}(\widetilde{X}, \partial\widetilde{X}; R) & \longrightarrow & M(\widetilde{f}; R)^{\#} \\
\downarrow & & \downarrow & & \downarrow \\
\Theta^{(2)}(\widetilde{\mathfrak{S}}) & \longrightarrow & H_k^{\text{loc.fin.}}(\widetilde{X}, \partial\widetilde{X}; R/2R) & \longrightarrow & M(\widetilde{f}; R/2R)^{\#} \\
\downarrow & & & & \downarrow \\
\Theta^{(2)}(\mathfrak{S}) & \longrightarrow & K_k(f, \partial f; R/2R) & \longrightarrow & K(f; R/2R)^{\#}
\end{array}$$

Moreover we note

$$K_k(f,\partial f;R)=K_k(f;R),\qquad K_k(f,\partial f;R/2R)=K_k(f;R/2R),$$
$$K_k(f;R)^{\#}=K_k(f;R)\quad\text{and}\quad K_k(f;R/2R)^{\#}=K_k(f;R/2R).$$

Thus by the hypothesis $M(\widetilde{f};R)\otimes_{R[\widetilde{G}]}R[G]=K_k(f;R)$, we obtain the surgery module

$$\boldsymbol{M}_{\boldsymbol{f},\mathfrak{S}}=\boldsymbol{M}_{\widetilde{\boldsymbol{f}},\widetilde{\mathfrak{S}}}\otimes_{R[\widetilde{G}]}R[G]=(K_k(f;R),B_f,q_{\boldsymbol{f}},\theta^{(0)},\theta^{(2)}),$$

where

$$B_f:K_k(f;R)\times K_k(f;R)\to R[G]$$

is the G-equivariant intersection form,

$$q_{\boldsymbol{f}}:K_k(f;R)\to R[G]/((Q)_R+R[S])$$

is the G-equivariant self-intersection form, and

$$\theta^{(0)}:\Theta^{(0)}(\mathfrak{S})\to K_k(f;R)^{\#}=K_k(f;R),$$
$$\theta^{(2)}:\Theta^{(2)}(\mathfrak{S})\to K_k(f;R/2R)^{\#}=K_k(f;R/2R)$$

are positioning maps (cf. [2, §5, pp. 563–564]).

By similar arguments to [2, p. 575, ℓ. 24 – p. 578, ℓ. 2], we obtain the next lemma.

Lemma 3.3. *Let $\boldsymbol{f}$ be a (G,R)-surgery map and $\mathfrak{S}$ a k-singular structure as above. If there exists an $R[\widetilde{G}]$-submodule $\widetilde{L}$ of $M(\widetilde{f};R)$ satisfying the conditions below then $\boldsymbol{f}$ can be converted to a (G,R)-surgery map $\boldsymbol{f}'=(f',b')$, where $f':(X',\partial X')\to(Y,\partial Y)$ and $b':T(X')\oplus f'^*\eta\to f'^*\xi$, such that $f':X'\to Y$ is an R-homology equivalence via G-surgery on X relative to $X_{\text{sing}}\cup X_{\mathfrak{S}}\cup\partial X$.*

(3.3.1) $\widetilde{\theta}^{(0)}(\alpha)(\widetilde{L})=0$ *for all* $\alpha\in\Theta^{(0)}(\widetilde{\mathfrak{S}})$ *and* $\widetilde{\theta}^{(2)}(\beta)(\widetilde{L})=0$ *for all* $\beta\in\Theta^{(2)}(\widetilde{\mathfrak{S}})$.

(3.3.2) $\widetilde{B}(\widetilde{L},\widetilde{L})=0$.

(3.3.3) $\widetilde{q}(\widetilde{L})=0$.

(3.3.4) *The canonical image L in $K_k(f;R)$ of $\widetilde{L}$ is an $R[G]$-free direct summand of $K_k(f;R)$ of half the rank, i.e. $2\cdot\mathrm{rank}_R L=\mathrm{rank}_R K_k(f;R)$.*

Lemma 3.4. *Let p be a prime and $R=\mathbb{Z}_{(p)}$. Let $\boldsymbol{f}=(f,b)$ be a (G,R)-surgery map and $\mathfrak{S}$ a k-singular structure of X as above. Suppose the following.*

(3.4.1) $\pi_1(X)$ *is finite and* $|\pi_1(X)|$ *is prime to* p.

(3.4.2) *the canonical homomorphism* $\widetilde{G}\to G$ *has a splitting, i.e.* $\widetilde{G}=\pi_1(X)\rtimes G$.

(3.4.3) $f:X\to Y$ *is k-connected.*

(3.4.4) $\pi_{\widetilde{X},X}(\widetilde{X}_t)$ *are orientable for all* $t\in\widetilde{\Sigma}_+$.

If the module

$$\boldsymbol{M}_{\boldsymbol{f},\mathfrak{S}} = (K_k(f;R), B_f, q_{\boldsymbol{f}}, \theta^{(0)}, \theta^{(2)})$$

has an $R[G]$*-free Lagrangian* L*, then there exists a submodule* $\widetilde{L}$ *of* $M(f;R)$ *satisfying the conditions* (3.3.1)–(3.3.4).

Before proving this lemma, we give an important application of the two lemmas above. Let

$$W_n(\boldsymbol{A}_X, \boldsymbol{\Theta}(\mathfrak{S}))_{\text{free}}$$

denote the surgery obstruction group

$$W_n(R, G, Q_X, S_X, \boldsymbol{\Theta}(\mathfrak{S}))_{\text{free}}$$

defined in [2, p. 545, Definition 3.33]. In the case $R = \mathbb{Z}_{(p)}$, a (G, R)-surgery map $\boldsymbol{f}$ with k-singular structure $\mathfrak{S}$ determines the module $\boldsymbol{M}_{\boldsymbol{f},\mathfrak{S}}$ above, and further the element $\sigma(\boldsymbol{f}, \mathfrak{S})$ of $W_n(\boldsymbol{A}_X, \boldsymbol{\Theta}(\mathfrak{S}))_{\text{free}}$ as the equivalence class of $\boldsymbol{M}_{\boldsymbol{f},\mathfrak{S}}$. By Lemmas 3.3, 3.4 and [18, Lemma 5.5], we obtain the next theorem.

Theorem 3.5. *Let* $R = \mathbb{Z}_{(p)}$ *for a prime* p*,* $\boldsymbol{f}$ *a* (G, R)*-surgery map and* $\mathfrak{S}$ *a* k*-singular structure satisfying the conditions* (3.4.1)–(3.4.4)*. If* $\sigma(\boldsymbol{f}, \mathfrak{S}) = 0$ *in* $W_n(\boldsymbol{A}_X, \boldsymbol{\Theta}(\mathfrak{S}))_{\text{free}}$ *then* $\boldsymbol{f}$ *can be converted to* $\boldsymbol{f}' = (f', b')$ *such that* $f' : X' \to Y$ *is an* R*-homology equivalence via a* G*-surgery on* X *relative to* $X_{\text{sing}} \cup X_{\mathfrak{S}} \cup \partial X$*.*

Proof of Lemma 3.4. Let L be an $R[G]$-free Lagrangian of $\boldsymbol{M}_{\boldsymbol{f},\mathfrak{S}}$. Let $\{x_1, \dots, x_m\}$ be an $R[G]$-basis of L and $\{y_1, \dots, y_m\}$ be elements of $K_k(f;R)$ such that

$$B_f(x_i, y_j) = \delta_{ij}$$

for $1 \le i, j \le m$. Thus $\{x_1, \dots, x_m, y_1, \dots, y_m\}$ is an $R[G]$-basis of $K_k(f;R)$. Arbitrarily choose liftings $\widetilde{x}_1, \dots, \widetilde{x}_m, \widetilde{y}_1, \dots, \widetilde{y}_m \in M(\widetilde{f};R)$ of $x_1, \dots, x_m, y_1, \dots, y_m$, respectively. Define a map $\tau : K_k(f;R) \to M(\widetilde{f};R)$ by

$$\tau\Big(\sum_i (a_i x_i + b_i y_i)\Big) = \frac{1}{|\pi_1(X)|} \sum_i \sum_{h \in \pi_1(X)} (h a_i \widetilde{x}_i + h b_i \widetilde{y}_i).$$

This map is an $R[G]$-splitting of the canonical map $M(\widetilde{f};R) \to K_k(f;R)$. Clearly, $\pi_1(X)$ acts trivially on the image of τ. Set

$$\widetilde{L} = \tau(L).$$

That $\widetilde{B}(\widetilde{L}, \widetilde{L}) = 0$ and $\widetilde{q}(\widetilde{L}) = 0$ follows from Steps 1 and 2 in the proof of [11, Theorem 2.6].

Thus it suffices to show that $\widetilde{\theta}^{(0)}(\alpha)(\widetilde{L}) = 0$ for $\alpha \in \Theta^{(0)}(\widetilde{\mathfrak{S}})$, and $\widetilde{\theta}^{(2)}(\beta)(\widetilde{L}) = 0$ for $\beta \in \Theta^{(2)}(\widetilde{\mathfrak{S}})$. Let $\widetilde{\varepsilon} : R[\widetilde{G}] \to R$ and $\varepsilon : R[G] \to R$ be the homomorphisms of taking the coefficients of the identity elements of $\widetilde{G}$ and G, respectively. For $\alpha \in \Theta^{(0)}(\widetilde{\mathfrak{S}})$, let $[\alpha]$ denote the canonical image of α in $\Theta^{(0)}(\mathfrak{S})$ and let $\pi_1(X)_\alpha$ denote the isotropy subgroup of the $\pi_1(X)$-action on $\Theta^{(0)}(\widetilde{\mathfrak{S}})$ at the point α. Then the canonical map $M(\widetilde{f}; R) \to K_k(f; R)$ assigns $m\theta^{(0)}([\alpha])$ to $\widetilde{\theta}^{(0)}(\alpha)$ with $m = |\pi_1(X)_\alpha|$. Thus for $x \in L$, we get

$$\begin{aligned}\varepsilon(\theta^{(0)}([\alpha])(x)) &= \sum_{h\in\pi_1(X)} \widetilde{\varepsilon}\left(\frac{1}{m}\widetilde{\theta}^{(0)}(\alpha)(h^{-1}\tau(x))\right)\\ &= \sum_{h\in\pi_1(X)} \widetilde{\varepsilon}\left(\frac{1}{m}\widetilde{\theta}^{(0)}(\alpha)(\tau(x))\right)\\ &= |\pi_1(X) : \pi_1(X)_\alpha|\widetilde{\varepsilon}\big(\widetilde{\theta}^{(0)}(\alpha)(\tau(x))\big),\end{aligned}$$

and hence

$$\widetilde{\varepsilon}\big(\widetilde{\theta}^{(0)}(\alpha)(\tau(x))\big) = \frac{|\pi_1(X)_\alpha|}{|\pi_1(X)|}\varepsilon\big(\theta^{(0)}([\alpha])(x)\big) = 0.$$

Since

$$\widetilde{\theta}^{(0)}(\alpha)(\tau(x)) = \sum_{g\in\widetilde{G}} \widetilde{\varepsilon}\big(\widetilde{\theta}^{(0)}(\alpha)(\tau(g^{-1}x))\big)g,$$

the triviality $\theta^{(0)}([\alpha])(L) = 0$ implies $\widetilde{\theta}^{(0)}(\alpha)(\tau(x)) = 0$.

We can similarly show that $\widetilde{\theta}^{(2)}(\beta)(\tau(x)) = 0$.

§4. The Mackey structure of surgery obstruction groups

In this section, let R be a principal ideal domain, hence necessarily a commutative ring, with 1 satisfying the square condition, i.e.

$$r \equiv r^2 \bmod 2R \quad \text{for each } r \in R. \tag{4.1}$$

Let Θ be a finite G-set, $\rho : \Theta \to \mathcal{S}(G)$ a G-map, and S a conjugation invariant subset of $G(2)$. The map $\mathcal{S}(G) \to \mathfrak{P}(S)$; $H \mapsto S_H = S \cap H$, preserves intersection. Let $\mathrm{SGW}_0(R, G, S, \Theta)$ denote the special Grothendieck–Witt group defined in [10, p. 2358].

Lemma 4.1 ([10, Proposition 5.4]). *If ρ is S-injective then $\mathrm{SGW}_0(R, G, S, \Theta)$ is a commutative ring possibly without 1, and moreover the canonical map*

$$\mathrm{SGW}_0(\mathbb{Z}, G, S, \Theta) \to \mathrm{SGW}_0(R, G, S, \Theta)$$

of ring change is a ring homomorphism. If ρ is S-bijective then $\mathrm{SGW}_0(R, G, S, \Theta)$ possesses the unit 1.

Let

$$f : \mathcal{S}(G) \to \mathfrak{P}(\Theta);\ H \mapsto \Theta_H,$$

be an intersection preserving ρ-compatible G-map and let $w : G \to \{\pm 1\}$ be a homomorphism. We denote by w_H the restriction $w|_H : H \to \{\pm 1\}$.

Definition 4.2 (cf. [10, p. 2357]). For a Θ-positioning Hermitian form $\boldsymbol{M} = (M, B, \theta)$, where M is an R-free $R[G]$-module, $B : M \times M \to R$ is a G-invariant (or w-invariant) symmetric bilinear form, and $\theta : \Theta \to M$ is a G-map, and for $s \in S$, $x \in M$, we define the *trace* $\Delta_\theta(s) \in M$ of (θ, ρ) at s and the ∇-*invariant* $\nabla_{\boldsymbol{M}}(x)(s) \in R/2R$ of $\boldsymbol{M}$ at (x, s) by

$$\Delta_\theta(s) = \sum_{t \in \Theta} \{\theta(t) \mid \rho(t) \ni s\}, \qquad \nabla_{\boldsymbol{M}}(x)(s) = [B(\Delta_\theta(s) - x, sx)].$$

We remark that what we precisely need for the definition is $B : M \times M \to R/2R$ rather than $B : M \times M \to R$.

Lemma 4.3. *Let H and K be subgroups of G and let $\boldsymbol{\varphi} = (\varphi, \psi)$ be a pair consisting of a monomorphism $\varphi : H \to K$ which is a composition of inclusion and conjugation and the associated injective φ-map $\psi : \Theta_H \to \Theta_K$. Let $g_1, \dots, g_m \in K$ be a complete set of representatives of $K/\varphi(H)$. Further let $\boldsymbol{M} = (M, B, \alpha)$ be a positioning Hermitian module, where M is an R-free $R[H]$-module, $B : M \times M \to R$ is an H-invariant (or w_H-invariant) symmetric bilinear form, and $\alpha : \Theta_H \to M$ is an H-map. Then the ∇-invariant of the induced module $\boldsymbol{M}' = \boldsymbol{\varphi}_\# M$ satisfies*

$$\nabla_{\boldsymbol{M}'}(g_i \otimes_\varphi x)(s') = \begin{cases} \nabla_{\boldsymbol{M}}(x)(\varphi^{-1}(g_i^{-1} s' g_i)) & (g_i^{-1} s' g_i \in \varphi(H)), \\ 0 & (g_i^{-1} s' g_i \notin \varphi(H)), \end{cases}$$

for $x \in M$ and $s' \in S_K = S \cap K$.

Proof. By definition, $\boldsymbol{M}' = (M', B', \alpha')$ is given by $M' = R[K] \otimes_{R[H], \varphi} M$,

$$B'(g_j \otimes_\varphi x, g_k \otimes_\varphi y) = \delta_{jk} B(x, y), \quad \text{and}$$
$$\alpha'(t') = \sum_{(i,t)} \{g_i \otimes \alpha(t) \mid t \in \Theta_H,\ g_i \psi(t) = t'\},$$

where $x, y \in M$, $t' \in \Theta_K$. Let $s' \in S_K$. We have

$$\nabla_{\boldsymbol{M}'}(g_i \otimes_\varphi x)(s') = B'(\Delta_{\alpha'}(s') - g_i \otimes_\varphi x, s'(g_i \otimes_\varphi x)).$$

Moreover the following equalities hold:

$$\begin{aligned}
&B'(\Delta_{\alpha'}(s'), s'(g_i \otimes_\varphi x)) = B'(\Delta_{\alpha'}(s'), g_i \otimes_\varphi x)\\
&\quad = \sum_{t' \in \Theta_K} \{B'(\psi_\# \alpha(t'), g_i \otimes_\varphi x) \mid \rho_K(t') \ni s'\}\\
&\quad = \sum_{t' \in \Theta_K} \sum_{j,t} \{B'(g_j \otimes_\varphi \alpha(t), g_i \otimes_\varphi x) \mid t \in \Theta_H,\ g_j\psi(t) = t',\ g_j\varphi(\rho_H(t))g_j^{-1} \ni s'\}\\
&\quad = \sum_{t' \in \Theta_K} \sum_{t} \{B(\alpha(t), x) \mid t \in \Theta_H,\ g_i\psi(t) = t',\ g_i\varphi(\rho_H(t))g_i^{-1} \ni s'\}\\
&\quad = \sum_{t \in \Theta_H} \{B(\alpha(t), x) \mid \varphi(\rho_H(t)) \ni g_i^{-1}s'g_i\}\\
&\quad = \sum_{t \in \Theta_H} \{B(\alpha(t), x) \mid \rho_H(t) \ni \varphi^{-1}(g_i^{-1}s'g_i)\}.
\end{aligned}$$

On the other hand, we have

$$B'(g_i \otimes_\varphi x, s'(g_i \otimes_\varphi x)) = \begin{cases} B(x, \varphi^{-1}(g_i^{-1}s'g_i)x) & (g_i^{-1}s'g_i \in \varphi(H)),\\ 0 & (g_i^{-1}s'g_i \notin \varphi(H)). \end{cases}$$

Thus we obtain

$$\nabla_{\boldsymbol{M}'}(g_i \otimes_\varphi x)(s') = \begin{cases} \nabla_{\boldsymbol{M}}(x)(\varphi^{-1}(g_i^{-1}s'g_i)) & (g_i^{-1}s'g_i \in \varphi(H)),\\ 0 & (g_i^{-1}s'g_i \notin \varphi(H)). \end{cases}$$

Lemma 4.4. *If $f : \mathcal{S}(G) \to \mathfrak{P}(\Theta)$ is (ρ, S)-saturated then the correspondence*

$$H \mapsto \mathrm{SGW}_0(R, H, S_H, \Theta_H) \qquad (H \in \mathcal{S}(G))$$

affords a Mackey functor.

Proof. This follows from the proof of [10, Proposition 11.2] with a modification using Lemma 4.3.

Lemma 4.5 ([10, Theorem 11.3]). *If $\rho : \Theta \to \mathcal{S}(G)$ is S-bijective and f is (ρ, S)-saturated then the correspondence*

$$H \mapsto \mathrm{SGW}_0(R, H, S_H, \Theta_H) \qquad (H \in \mathcal{S}(G))$$

affords a Green functor. Moreover, the canonical homomorphisms

$$\mathrm{SGW}_0(\mathbb{Z}, H, S_H, \Theta_H) \to \mathrm{SGW}_0(R, H, S_H, \Theta_H)$$

of ring change afford a natural transformation of Green functors.

Let $w : G \to \{\pm 1\}$ be a homomorphism and let $\bar{\cdot}$ denote the anti-involution on $\mathbb{Z}[G]$ associated with w. Let $\lambda = (-1)^k$. Then $(\bar{\cdot}, \lambda)$ is an anti-structure of $\mathbb{Z}[G]$. Let Q be a conjugation invariant subset of $G(2)$. Suppose

$$S \subset G(2)^{\lambda} = \{g \in G(2) \mid g = \lambda \overline{g}\}, \qquad Q \subset G(2)^{-\lambda} = \{g \in G(2) \mid g = -\lambda \overline{g}\}.$$

Then we obtain the double parameter algebra

$$\boldsymbol{A} = (R[G], (\bar{\cdot}, \lambda), (S)_R, G, R[S], (Q)_R + R[S])$$

in the sense of [2, Definition 2.5]. Let $\Theta^{(0)}$ and $\Theta^{(2)}$ be a finite $(G \times \{\pm 1\})$-set and a finite G-set, respectively and let $p_{\Theta^{(0)}} : \Theta^{(0)} \to \Theta^{(2)}$ be a G-map. Throughout this paper we assume that $\{\pm 1\}$ acts freely on $\Theta^{(0)}$ and $p^{-1}_{\Theta^{(0)}}(p_{\Theta^{(0)}}(t))$ coincides with $\{t, -t\}$ for all $t \in \Theta^{(0)}$. Let $\rho_{\Theta^{(2)}} : \Theta^{(2)} \to \mathcal{S}(G)$ be a G-map and set $\boldsymbol{\Theta} = (\Theta^{(0)}, \Theta^{(2)}, p_{\Theta^{(0)}}, \rho_{\Theta^{(2)}})$. We use the notation

$$W_n(\boldsymbol{A}, \boldsymbol{\Theta})_{\text{free}} = W_n(R, G, Q, S, \boldsymbol{\Theta})_{\text{free}}, \qquad W_n(\boldsymbol{A}, \boldsymbol{\Theta})_{\text{proj}} = W_n(R, G, Q, S, \boldsymbol{\Theta})_{\text{proj}},$$

where $n = 2k$, defined in [2, Definition 3.33].

Let Θ be a finite G-set and $\rho : \Theta \to \mathcal{S}(G)$ a G-map. Let $\gamma : \Theta^{(2)} \to \Theta$ be a G-map such that the diagram

$$\begin{array}{ccc} \Theta^{(2)} & \xrightarrow{\rho^{(2)}} & \mathcal{S}(G) \\ {\scriptstyle\gamma}\downarrow & \nearrow{\scriptstyle\rho} & \\ \Theta & & \end{array} \tag{4.2}$$

commutes and

$$\gamma(\Theta^{(2)}) = \Theta. \tag{4.3}$$

Lemma 4.6. *If $\rho : \Theta \to \mathcal{S}(G)$ is S-bijective then $W_n(\boldsymbol{A}, \boldsymbol{\Theta})_{\text{free}}$ is a module over* $\mathrm{SGW}_0(R, G, S, \Theta)$.

Proof. Let $\boldsymbol{M}_1 = (M_1, B_1, \alpha_1)$ be a Θ-positioning, non-singular Hermitian $R[G]$-module with trivial ∇-invariant, where M is an R-free $R[G]$-module, $B_1 : M_1 \times M_1 \to R$ and $\alpha_1 : \Theta \to M$. Let $\boldsymbol{M}_2 = (M_2, B_2, q_2, \alpha^{(0)}, \alpha^{(2)})$ be an object in $\nabla\mathcal{Q}(\boldsymbol{A}, \boldsymbol{\Theta})$ defined in [2, p. 535] such that M_2 is a stably $R[G]$-free module, where $B_2 : M_2 \times M_2 \to R[G]$, $q_2 : M_2 \to R[G]/((Q)_R + R[S])$, $\alpha^{(0)} : \Theta^{(0)} \to M_2$, and $\alpha^{(2)} : \Theta^{(2)} \to M_2/2M_2$. Then we define

$$\boldsymbol{M} = \boldsymbol{M}_1 \cdot \boldsymbol{M}_2 = (M, B, q, \theta^{(0)}, \theta^{(2)}) \in \mathcal{Q}(\boldsymbol{A}, \boldsymbol{\Theta})$$

as follows. The triple (M, B, q) is described in [10, §9]. The map $\theta^{(0)} : \Theta^{(0)} \to M = M_1 \otimes_R M_2$ is given by

$$\theta^{(0)}(t) = \alpha_1(\gamma(p_{\Theta^{(0)}}(t))) \otimes_R \alpha^{(0)}(t) \quad \text{for } t \in \Theta^{(0)},$$

and the map $\theta^{(2)} : \Theta^{(2)} \to M/2M$ is given by

$$\theta^{(2)}(t) = \alpha_1(\gamma(t)) \otimes_R \alpha^{(2)}(t)$$

for $t \in \Theta^{(2)}$. It is easy to verify the ∇-triviality of $\boldsymbol{M}$, i.e. $\boldsymbol{M} \in \nabla\mathcal{Q}(\boldsymbol{A}, \boldsymbol{\Theta})$. The correspondence $(\boldsymbol{M}_1, \boldsymbol{M}_2) \mapsto \boldsymbol{M}$ affords the module structure

$$\mathrm{SGW}_0(R, G, S, \Theta) \times W_n(\boldsymbol{A}, \boldsymbol{\Theta})_{\mathrm{free}} \mapsto W_n(\boldsymbol{A}, \boldsymbol{\Theta})_{\mathrm{free}}.$$

In this section we set

$$Q_H = Q \cap H \quad \text{for } H \in \mathcal{S}(G).$$

Then the map $\mathcal{S}(G) \to \mathfrak{P}(Q)$; $H \mapsto Q_H$, preserves intersection.

We regard $\mathcal{S}(G)$ as a $(G \times \{\pm 1\})$-set, with the trivial $\{\pm 1\}$-action. Let $\boldsymbol{f_\Theta} = (f_{\Theta^{(0)}}, f_{\Theta^{(2)}})$ be a pair of an intersection preserving $(G \times \{\pm 1\})$-map $f_{\Theta^{(0)}} : \mathcal{S}(G) \to \mathfrak{P}(\Theta^{(0)})$; $H \mapsto \Theta_H^{(0)}$, and an intersection preserving $\rho_{\Theta^{(2)}}$-compatible G-map $f_{\Theta^{(2)}} : \mathcal{S}(G) \to \mathfrak{P}(\Theta^{(2)})$; $H \mapsto \Theta_H^{(2)}$, satisfying

$$p_{\Theta^{(0)}}(\Theta_H^{(0)}) \subset \Theta_H^{(2)}$$

for $H \in \mathcal{S}(G)$. Define $p_{\Theta_H^{(0)}} : \Theta_H^{(0)} \to \Theta_H^{(2)}$ as the restriction of $p_{\Theta^{(0)}}$, and $\rho_{\Theta_H^{(2)}} : \Theta_H^{(2)} \to \mathcal{S}(H)$ as the restriction of $\rho_{\Theta^{(2)}}$. Then we obtain the double parameter algebras

$$\boldsymbol{A}_H = (R[H], (\bar{\cdot}, \lambda), (S_H)_R, H, R[S_H], (Q_H)_R + R[S_H]),$$

where $S_H = S \cap H$, and the positioning data

$$\boldsymbol{\Theta}_H = (\Theta_H^{(0)}, \Theta_H^{(2)}, p_{\Theta_H^{(0)}}, \rho_{\Theta_H^{(2)}}), \quad \text{where } H \in \mathcal{S}(G).$$

Lemma 4.7. *If $f_{\Theta^{(2)}} : \mathcal{S}(G) \to \mathfrak{P}(\Theta^{(2)})$ is $(\rho_{\Theta^{(2)}}, S)$-saturated then the correspondences*

$$H \mapsto W_n(\boldsymbol{A}_H, \boldsymbol{\Theta}_H)_{\mathrm{proj}} \quad (H \in \mathcal{S}(G))$$

and

$$H \mapsto W_n(\boldsymbol{A}_H, \boldsymbol{\Theta}_H)_{\mathrm{free}} \quad (H \in \mathcal{S}(G))$$

afford Mackey functors, respectively.

Proof. Recalling [10, Proposition 10.3], we will prove the lemma by showing that

$$H \mapsto W_n(\boldsymbol{A}_H, \boldsymbol{\Theta}_H)_{\mathrm{proj}}, W_n(\boldsymbol{A}_H, \boldsymbol{\Theta}_H)_{\mathrm{free}} \quad (H \in \mathcal{S}(G))$$

are w-Mackey functors. Most of the proof is already given in the proof of Theorem 12.10 of [10]. It suffices to discuss the part concerning the $(H \times \{\pm 1\})$-sets $\Theta_H^{(0)}$, where $H \in \mathcal{S}(G)$.

Let H and K be subgroups of G. Given an injective homomorphism $\varphi : H \to K$, we have the canonical injective homomorphism $\varphi_\pm : H \times \{\pm 1\} \to K \times \{\pm 1\}$ defined by $\varphi_\pm(h, \epsilon) = (\varphi(h), \epsilon)$ for $h \in H$ and $\epsilon \in \{\pm 1\}$. The sets $\Theta_H^{(0)}$ and $\Theta_K^{(0)}$ are an $(H \times \{\pm 1\})$-set and a $(K \times \{\pm 1\})$-set, respectively, on which the group $\{\pm 1\}$ acts freely. Let $\psi : \Theta_H^{(0)} \to \Theta_K^{(0)}$ be a $\varphi_\pm$-map, i.e.

$$\psi((h, \epsilon)t) = \varphi_\pm(h, \epsilon)\psi(t) \ (= (\varphi(h), \epsilon)\psi(t))$$

for $h \in H$, $\epsilon \in \{\pm 1\}$, and $t \in \Theta_H^{(0)}$. Let $\boldsymbol{\varphi}$ denote the pair (φ, ψ).

An $R[K]$-module N is usually regarded as an $R[K \times \{\pm 1\}]$-module via $(k, \epsilon)x = \epsilon(kx)$ for $k \in K$, $\epsilon \in \{\pm 1\}$, and $x \in N$. For a pair $\boldsymbol{N} = (N, \beta)$ consisting of an $R[K]$-module N and a $(K \times \{\pm\})$-map $\beta : \Theta_K^{(0)} \to N$, we define $\boldsymbol{\varphi}^\# \boldsymbol{N} = (\varphi^\# N, \psi^\# \beta)$, where $\varphi^\# N$ is an $R[H]$-module and $\psi^\# \beta : \Theta_H^{(0)} \to \varphi^\# N$, so that the underlying R-module of $\varphi^\# N$ is the same as N but the H-action on $\varphi^\# N$ is given by $(h, x) \mapsto \varphi(h)x$ for $h \in H$ and $x \in \varphi^\# N$, and $\psi^\# \beta(t) = \beta(\psi(t))$ for $t \in \Theta_H^{(0)}$.

For a pair $\boldsymbol{M} = (M, \alpha)$ consisting of an $R[H]$-module M and an $(H \times \{\pm 1\})$-map $\alpha : \Theta_H^{(0)} \to M$, we define $\boldsymbol{\varphi}_\# \boldsymbol{M} = (\varphi_\# M, \psi_\# \alpha)$, where $\varphi_\# M$ is an $R[K]$-module and $\psi_\# \alpha : \Theta_K^{(0)} \to \varphi_\# M$, by $\varphi_\# M = R[K] \otimes_{R[H], \varphi} M$ and

$$\psi_\# \alpha(t) = \sum_{[g, t']} \{g \otimes \alpha(t') \mid [g, t'] \in K \times_{H, \varphi} \Theta_H^{(0)} \text{ such that } g\psi(t') = t\}$$

for $t \in \Theta_K^{(0)}$.

These $\boldsymbol{\varphi}^\# \boldsymbol{N}$ and $\boldsymbol{\varphi}_\# \boldsymbol{M}$ are simple analogies of those in [10, p. 2347]. Thus the conclusion of the lemma above follows from the same arguments used in the proof of Theorem 12.10 of [10].

Let $\rho : \Theta \to \mathcal{S}(G)$ be a G-map and $f : \mathcal{S}(G) \to \mathfrak{P}(\Theta)$; $H \mapsto \Theta_H$, an intersection preserving, ρ-compatible G-map such that $f(G) = \Theta$. Let $\gamma : \Theta^{(2)} \to \Theta$ be a G-map such that the diagram (4.2) commutes and

$$\gamma(\Theta_H^{(2)}) = \Theta_H \quad (H \in \mathcal{S}(G)). \tag{4.4}$$

Lemma 4.8. *If $\rho : \Theta \to \mathcal{S}(G)$ is S-bijective, $f : \mathcal{S}(G) \to \mathfrak{P}(\Theta)$ is (ρ, S)-saturated and $f_{\Theta^{(2)}} : \mathcal{S}(G) \to \mathfrak{P}(\Theta^{(2)})$ is $(\rho^{(2)}, S)$-saturated, then the correspondence*

$$H \mapsto W_n(\boldsymbol{A}_H, \boldsymbol{\Theta}_H)_{\text{free}} \quad (H \in \mathcal{S}(G))$$

is a module over the Green functor

$$H \mapsto \mathrm{SGW}_0(R, H, S_H, \Theta_H) \qquad (H \in \mathcal{S}(G)).$$

Proof. We can argue in the same way as in the proof of [10, Theorem 12.10] with a modification using Lemma 4.6.

§5. A deleting-inserting theorem

Deleting (resp. inserting) G-fixed submanifolds from (resp. to) given ambient G-manifolds is useful for the study of fixed point data of G-manifolds. For example, it has been applied to the study of the Smith problem on tangential representations at fixed points on spheres. In this section we prove Theorem 5.1 below. Let $\mathcal{G}^1_p(G)$ denote the set of all subgroups H of G possessing normal subgroups $P \trianglelefteq H$ such that P has p-power order and H/P is cyclic, where P is possibly the trivial group. An element H of $\mathcal{G}^1_p(G)$ is called a *mod-$\mathcal{P}_p$ cyclic group.* We set

$$\mathcal{G}^1(G) = \bigcup_{p\,\mathrm{prime}} \mathcal{G}^1_p(G).$$

If H lies in $\mathcal{G}^1(G)$ then H is referred to as a *mod-$\mathcal{P}$ cyclic group.*

Theorem 5.1 (Deleting-inserting theorem). *Let G be a finite Oliver group and Y a smooth G-manifold such that the underlying manifold of Y is diffeomorphic to the disk of dimension $n \geq 5$ and $Y^G \neq \emptyset$. Let $F_1, \ldots, F_t$ denote all the underlying spaces of connected components of Y^G, and let $n_1, \ldots, n_t$ be non-negative integers. Suppose the following:*

(5.1.1) *Y satisfies the weak gap condition on $\mathcal{PH}(G)$.*

(5.1.2) $\dim Y^{=H} \geq 3$ *for any $H \in \mathcal{G}^1(G)$.*

(5.1.3) $\dim Y^P \geq 5$ *for any $P \in \mathcal{P}(G)$.*

(5.1.4) *$\pi_1(Y^P)$ is finite and of order prime to $|P|$ for any $P \in \mathcal{P}(G)$.*

(5.1.5) *For $1 \leq i, j \leq t$, n_i coincides with n_j if some connected component Y^H_α of Y^H, $H \in \mathcal{L}(G)$, contains both F_i and F_j.*

(5.1.6) *For $1 \leq i \leq t$, n_i is equal to 1 if some connected component Y^H_α of Y^H, $H \in \mathcal{L}(G)$, contains F_i and $\partial Y^H_\alpha \neq \emptyset$.*

(5.1.7) *If* $\dim Y^P = 2\dim Y^H$ *for $(P, H) \in \mathcal{PH}(G)$ then $(P, H) \in \mathcal{PH}_2(G)$ and* $\dim Y^{>H} \leq \dim Y^H - 2$.

Then there exists a smooth G-action on the disk D of dimension n such that

(i) *∂D is G-diffeomorphic to ∂Y,*

(ii) D^G *has the form of the disjoint union of copies of* F_i*'s:*

$$D^G = \coprod_{i=1}^{t} \coprod_{j=1}^{n_i} F_{i,j} \quad (\textit{each } F_{i,j} \textit{ is diffeomorphic to } F_i), \textit{ and}$$

(iii) *the normal bundle* $\nu(F_{i,j}, D)$ *is* G*-isomorphic to* $\nu(F_i, Y)$*.*

Furthermore if Y^H (*resp.* Y^P) *is connected* (*resp. simply connected*) *for an element* $H \in \mathcal{G}^1(G)$ (*resp.* $P \in \mathcal{P}(G)$)*, then one can choose the* G*-action so that* D^H (*resp.* D^P) *is connected* (*resp. simply connected*) *for the subgroup* H (*resp.* P)*.*

Proof. The procedure is the same as that of proving Theorem 1.3 of [11, §5]. Let $\boldsymbol{f} = (f, b)$, $f : (X, \partial X) \to (Y, \partial Y)$ and $b : T(X) \oplus \varepsilon_X(\mathbb{R}^u) \to f^*T(Y) \oplus \varepsilon_X(\mathbb{R}^u)$, be the degree-one G-framed map obtained in Section 4 of [11]. Note that for $P \in \mathcal{P}(G)$, Y^P is orientable and the map $f^P : (X^P, \partial X^P) \to (Y^P, \partial Y^P)$ has degree one.

The details of the proof differ in some points from the proof of Theorem 1.3 of [11, §5]. The differences occur in Steps A and B below.

Step A. The step converting $f^P : X^P \to Y^P$ to a mod p homology equivalence, where $P \in \mathcal{P}(G)$ possesses $H \in \mathcal{S}(G)$ such that $2 \dim X^H = \dim X^P$ and p is the prime dividing $|P|$.

Step B. The step converting $f : X \to Y$ to a homotopy equivalence, when there is (at least one) $H \in \mathcal{S}(G)$ such that $2 \dim X^H = \dim X$.

In these steps, the condition (5.1.7) is used to get rid of technical difficulties.

Step A. In this step, we set $n_P = \dim X^P$, $k_P = n_P/2$, $\lambda = (-1)^{k_P}$, $T = N_G(P)/P$, $w = w_{X^P} : T \to \{\pm 1\}$, and furthermore

$$\begin{aligned}
&R = \mathbb{Z}_{(p)},\\
&S = \{g \in T(2) \mid \dim(X^P)^g = k_P\} \ (= S(X^P)),\\
&Q = \{g \in T(2) \mid \dim(X^P)^g = k_P - 1\} \ (= Q(X^P)),\\
&\mathfrak{S} = \{(X^P)^g \mid g \in S\} \ (= \mathfrak{S}(X^P)),\\
&\Theta^{(0)} = \Theta^{(0)}(X^P), \quad \Theta^{(2)} = \Theta^{(2)}(X^P),\\
&\rho = \rho^{(2)}_{X^P} : \Theta^{(2)} \to \mathcal{S}(T), \quad \boldsymbol{\Theta} = (\Theta^{(0)}, \Theta^{(2)}, p_{\Theta^{(0)}}, \rho),
\end{aligned}$$

where $p_{\Theta^{(0)}} : \Theta^{(0)} \to \Theta^{(2)}$ is the canonical map. Without any loss of generality we can suppose that $f^P : X^P \to Y^P$ is k_P-connected. Then by Theorem 3.5 the T-surgery obstruction $\sigma(f^P, b^P)$ to the (T, R)-surgery map (f^P, b^P) being a $\mathbb{Z}_{(p)}$-homology equivalence lies in $W_{n_P}(R, T, S, Q, \boldsymbol{\Theta})_{\text{free}}$.

For a subgroup K of T, set $S_K = S(\operatorname{res}^T_K X^P)$, $Q_K = Q(\operatorname{res}^T_K X^P)$, $\mathfrak{S}_K = \mathfrak{S}(\operatorname{res}^T_K X^P)$, $\Theta^{(0)}_K = \Theta^{(0)}(\operatorname{res}^T_K X^P)$, $\Theta^{(2)}_K = \Theta^{(2)}(\operatorname{res}^T_K X^P)$, $\rho_K = \rho^{(2)}_{\operatorname{res}^T_K X^P} : \Theta^{(2)}_K \to \mathcal{S}(K)$, and

$$\boldsymbol{\Theta}_K = (\Theta^{(0)}_K, \Theta^{(2)}_K, p_{\Theta^{(0)}_K}, \rho_K).$$

By Lemmas 2.4, 4.7 and 4.8, the correspondence

$$K \mapsto W_{n_P}(R, K, S_K, Q_K, \boldsymbol{\Theta}_K)_{\text{free}} \quad (K \in \mathcal{S}(T))$$

affords a Mackey functor, and moreover a module over the Green functor

$$K \mapsto \mathrm{SGW}_0(\mathbb{Z}, K, S_K, \Theta^{(2)}_K/S_K) \quad (K \in \mathcal{S}(T)).$$

Thus the argument in [11, §5, Case 2] using the relation between the equivariant connected sum operation and the $\Omega(T)$-action on the surgery obstruction group (cf. [11, (5.2)]), works in the present situation. This ensures that by using equivariant connected sum and G-surgery of isotropy type (P), we can convert $f^P : X^P \to Y^P$ to a $\mathbb{Z}_{(p)}$-homology equivalence.

Step B. In this case, Y is 1-connected and $n = \dim Y = \dim X$. We set $k = n/2$, $\lambda = (-1)^k$, $w = w_X : G \to \{\pm 1\}$, $R = \mathbb{Z}$, $S = S(X)$, $Q = Q(X)$, $\mathfrak{S} = \mathfrak{S}(X)$, $\Theta^{(0)} = \Theta^{(0)}(X)$, $\Theta^{(2)} = \Theta^{(2)}(X)$, $\rho = \rho^{(2)}_X : \Theta^{(2)} \to \mathcal{S}(G)$, and

$$\boldsymbol{\Theta} = (\Theta^{(0)}, \Theta^{(2)}, p_{\Theta^{(0)}}, \rho),$$

where $p_{\Theta^{(0)}} : \Theta^{(0)} \to \Theta^{(2)}$ is the canonical map. Without loss of generality we can suppose that $f : X \to Y$ is k-connected. Since

$$K_k(f; R) = \operatorname{Ker}[f_* : H_k(X; R) \to H_k(Y; R)]$$

is a projective $R[G]$-module but not necessarily a stably free $R[G]$-module, Theorem 6.3 in [2] says that the G-surgery obstruction $\sigma(f, b)$ to the (G, R)-surgery map (f, b) being a homotopy equivalence lies in the obstruction group $W_n(R, G, S, Q, \boldsymbol{\Theta})_{\text{proj}}$. But by employing the relation

$$(1 + (-\beta)^{\%})\widetilde{K}_0(R[G]) = 0$$

described in [11, §5, Case 3] and by taking a suitable equivariant connected sum, we may assume that $K_k(f; R)$ is a stably free $R[G]$-module. Then $\sigma(f, b)$ lies in the obstruction group $W_n(R, G, S, Q, \boldsymbol{\Theta})_{\text{free}}$.

For a subgroup K of G, set $S_K = S(\operatorname{res}^G_K X)$, $Q_K = Q(\operatorname{res}^G_K X)$, $\mathfrak{S}_K = \mathfrak{S}(\operatorname{res}^G_K X)$, $\Theta^{(0)}_K = \Theta^{(0)}(\operatorname{res}^G_K X)$, $\Theta^{(2)}_K = \Theta^{(2)}(\operatorname{res}^G_K X)$, $\rho_K = \rho^{(2)}_{\operatorname{res}^G_K X} : \Theta^{(2)}_K \to \mathcal{S}(K)$, and

$$\boldsymbol{\Theta}_K = (\Theta^{(0)}_K, \Theta^{(2)}_K, p_{\Theta^{(0)}_K}, \rho_K).$$

By Lemmas 2.4, 4.7 and 4.8, the correspondence

$$K \mapsto W_n(R, K, S_K, Q_K, \mathbf{\Theta}_K)_{\text{free}} \quad (K \in \mathcal{S}(G))$$

affords a Mackey functor, and a module over the Green functor

$$K \mapsto \mathrm{SGW}_0(\mathbb{Z}, K, S_K, \Theta_K^{(2)}/S_K) \quad (K \in \mathcal{S}(G)).$$

Thus the argument in [11, §5, Case 3] works in the present situation. Hence, by using equivariant connected sum and G-surgery of isotropy type $(\{e\})$, we can convert $f : X \to Y$ to a homotopy equivalence.

Putting all this together, we have proved the theorem above.

§6. Applications of the deleting-inserting theorem

Let G be a finite group. One may conjecture that if V and W are $\mathcal{P}$-matched $\mathcal{L}$-free real G-modules then V and W are stably Smith equivalent, with which the following is concerned.

Definition 6.1. We call a real G-module V *admissible* if it satisfies the following conditions.

(6.1.1) V satisfies the weak gap condition on $\mathcal{PH}(G)$.

(6.1.2) $\dim V^{=H} \geq 3$ for any $H \in \mathcal{G}^1(G)$.

(6.1.3) $\dim V^P \geq 5$ for any $P \in \mathcal{P}(G)$.

(6.1.4) If $\dim V^P = 2 \dim V^H$ for $(P, H) \in \mathcal{PH}(G)$ then (P, H) belongs to $\mathcal{PH}_2(G)$ and $\dim V^{>H} \leq \dim V^H - 2$.

The next lemma is an elaboration of [6, Theorem B]. In [6], we worked with real G-modules V such that all transformations $g : V^H \to V^{gHg^{-1}}$ are orientation preserving for $g \in G$ and $H \in \mathcal{S}(G)$ (cf. [6, p. 491 (3.3.6)]).

Lemma 6.2. *Let G be an Oliver group, m a positive integer, and V an admissible real G-module. Then there exists a smooth G-action on the standard sphere S_V such that S_V^G consists of m points $x_1, \ldots, x_m$ and each $T_{x_i}(S_V)$, $1 \leq i \leq m$, is isomorphic to V.*

Proof. Let Y be the unit disk $D(V)$ of V with respect to some G-invariant inner product. Then Y satisfies the conditions (5.1.1)–(5.1.7). By Theorem 5.1, we obtain a smooth G-action on a disk D_0 such that D_0 does not have G-fixed points and ∂D_0 is G-diffeomorphic to $S(V) = \partial D(V)$. On the other hand, by Theorem 5.1 there exists a smooth G-action on a disk D_m such that D_m^G consists of m points $x_1, \ldots, x_m$, ∂D_0 is G-diffeomorphic to $S(V) = \partial D(V)$, and the tangential representations $T_{x_i}(D_m)$ are all isomorphic to V. Then glue D_0 and D_m along the boundary and obtain a smooth G-action on a homotopy sphere Σ_V such that Σ_V^G consists of m points $x_1, \dots, x_m$ and $T_{x_i}(\Sigma_V)$ are isomorphic to V. Taking the equivariant connected sum of copies of Σ_V (cf. [7, Proposition 1.3, Example 1.2]), we can obtain a smooth G-action on the standard sphere as desired.

Lemma 1.1 implies that $\mathbb{R}[G]_{\mathcal{L}}^{\oplus 3}$ is an admissible real G-module. Hence Theorems 1.3 and 1.4 immediately follow from the lemma above.

Theorem 6.3. *Let G be an Oliver group. Let $V_1, \dots, V_m$ be $\mathcal{L}$-free real G-modules any two of which are $\mathcal{P}$-matched. Then there exists an integer N_1 such that for any integer $\ell \geq N_1$, there exists a smooth G-action on the disk D with exactly m G-fixed points $x_1, \dots, x_m$ for which the tangential representation $T_{x_i}(D)$ is isomorphic to $V_i \oplus \mathbb{R}[G]_{\mathcal{L}}^{\oplus \ell}$ for $1 \leq i \leq m$.*

Proof. Consider the space $F = \{x_1, \dots, x_m\}$ with the trivial G-action. We have the $\mathcal{L}$-free real G-vector bundle $\nu = \varepsilon_{\{x_1\}}(V_1) \amalg \cdots \amalg \varepsilon_{\{x_m\}}(V_m)$ over F. Clearly $\mathrm{res}^G_{\{e\}} \nu$ is isomorphic to $\varepsilon_F(\mathbb{R}^n)$ for $n = \dim V_1$ and $\mathrm{res}^G_P \nu$ is isomorphic to $\varepsilon_F(\mathrm{res}^G_P V_1)$ for any $P \in \mathcal{P}(G)$. By [14, Theorem 21], there exists an integer N_1 as desired.

Proof of Theorem 1.5. Let N_1 be the non-negative integer obtained in Theorem 6.3 for the G-modules $V_1, \dots, V_m$. There exists an integer $N \geq N_1$ such that the real G-modules $V_i \oplus \mathbb{R}[G]_{\mathcal{L}}^{\oplus N}$, $1 \leq i \leq m$, are all admissible. Then for all $\ell \geq N$, the real G-modules

$$W_i = V_i \oplus \mathbb{R}[G]_{\mathcal{L}}^{\oplus \ell}, \qquad 1 \leq i \leq m,$$

are also admissible. Again by Theorem 6.3, there exists a smooth G-action on the disk Y such that $Y^G = \{x_1, \dots, x_m\}$ and $T_{x_i}(Y) \cong W_i$ for $1 \leq i \leq m$. Let Z denote the double $Y \cup_{\partial Y} Y$ of Y. Then Z is a sphere having the G-fixed points $x_1, \dots, x_m$, $x_1', \dots, x_m'$ such that $T_{x_i}(Z) \cong T_{x_i'}(Z) \cong W_i$ for $1 \leq i \leq m$. By Lemma 6.2, there exist smooth G-actions on spheres S_i, $1 \leq i \leq m$, such that $S_i^G = \{x_i''\}$ and $T_{x_i''}(S_i) \cong W_i$. Let S denote the G-manifold obtained as the G-connected sum of Z and S_i, $1 \leq i \leq m$, at pairs $(x_i', x_i'') \in Z \times S_i$. Then the underlying manifold of S is diffeomorphic to the standard sphere and moreover S possesses the properties required in Theorem 1.5.

Let $\mathrm{WP}(G)$ denote the set consisting of $[V] - [W] \in \mathrm{RO}(G)^{\mathcal{L}}$ such that V and W both are $\mathcal{L}$-free and satisfy the weak gap condition on $\mathcal{PH}_2(G)$. Note that G is a weak gap group if and only if $\mathrm{WP}(G)_{\mathcal{P}} = \mathrm{RO}(G)^{\mathcal{L}}_{\mathcal{P}}$. Since the set

$$-\mathrm{WP}(G) = \{-x \in \mathrm{RO}(G) \mid x \in \mathrm{WP}(G)\}$$

coincides with $\mathrm{WP}(G)$, we can prove the next proposition without difficulties.

Proposition 6.4. *The set* $\mathrm{WP}(G)$ *is a subgroup of* $\mathrm{RO}(G)$.

Theorem 1.9 can be reformulated as follows:

Theorem 6.5. *If* H *is a subgroup of an Oliver group* G *then*

$$\mathrm{ind}_H^G(\mathrm{WP}(H)_{\mathcal{P}}) \subset \mathrm{Sm}(G)_{\mathcal{P}}^{\mathcal{L}}.$$

For a pair $(P,H) \in \mathcal{PH}(G)$, define a $\mathbb{Z}$-linear map $f_{P,H} : \mathrm{RO}(G) \to \mathbb{Z}$ by

$$f_{P,H}([V]) = \dim V^P - 2\dim V^H.$$

Next define

$$\mathrm{P}_+(\mathcal{PH}_2(G)) = \{x \in \mathrm{RO}(G)^{\mathcal{L}} \mid f_{P,H}(x) \geq 0 \text{ for all } (P,H) \in \mathcal{PH}_2(G)\},$$
$$\mathrm{P}_-(\mathcal{PH}_2(G)) = \{x \in \mathrm{RO}(G)^{\mathcal{L}} \mid f_{P,H}(x) \leq 0 \text{ for all } (P,H) \in \mathcal{PH}_2(G)\}.$$

It is clear that $\mathrm{P}_-(\mathcal{PH}_2(G)) = -\mathrm{P}_+(\mathcal{PH}_2(G))$.

Lemma 6.6. *For an arbitrary finite group* G*, we have*

$$\mathrm{P}_+(\mathcal{PH}_2(G)) \cup \mathrm{P}_-(\mathcal{PH}_2(G)) \subset \mathrm{WP}(G).$$

Proof. Let $x = [V] - [W] \in \mathrm{P}_+(\mathcal{PH}_2(G))$, where V and W are $\mathcal{L}$-free real G-modules. By [13, Proposition 2.3], W is isomorphic to a G-submodule of $\mathbb{R}[G]_{\mathcal{L}}^{\oplus m}$, where $m = \dim W$. Thus we can assume $W = \mathbb{R}[G]_{\mathcal{L}}^{\oplus m}$ without any loss of generality. Then the inequality

$$f_{P,H}(x) = (\dim V^P - 2\dim V^H) - m(\dim(\mathbb{R}[G]_{\mathcal{L}})^P - 2\dim(\mathbb{R}[G]_{\mathcal{L}})^H) \geq 0$$

for $(P,H) \in \mathcal{PH}_2(G)$ reads

$$\dim V^P - 2\dim V^H \geq m(\dim(\mathbb{R}[G]_{\mathcal{L}})^P - 2\dim(\mathbb{R}[G]_{\mathcal{L}})^H).$$

Since the right-hand side above is non-negative, V satisfies the weak gap condition on $\mathcal{PH}_2(G)$ as also does $W = \mathbb{R}[G]_{\mathcal{L}}^{\oplus m}$, which ensures that the element $x = [V] - [W]$ belongs to $\mathrm{WP}(G)$, hence $\mathrm{P}_+(\mathcal{PH}_2(G)) \subset \mathrm{WP}(G)$.

In addition, we have

$$\mathrm{P}_-(\mathcal{PH}_2(G)) = -\mathrm{P}_+(\mathcal{PH}_2(G)) \subset -\mathrm{WP}(G) = \mathrm{WP}(G).$$

This completes the proof.

The next claim immediately follows from Theorem 6.5 and Lemma 6.6.

Theorem 6.7. *If* H *is a subgroup of an Oliver group* G *then*

$$\mathrm{ind}_H^G\left(\mathrm{P}_+(\mathcal{PH}_2(H))_{\mathcal{P}} \cup \mathrm{P}_-(\mathcal{PH}_2(H))_{\mathcal{P}}\right) \subset \mathrm{Sm}(G)_{\mathcal{P}}^{\mathcal{L}}.$$

Proof of Theorem 1.10. It is clear that

$$\mathrm{RO}(H)^{\mathcal{L}}_{\mathcal{H}} \subset \mathrm{P}_+(\mathcal{PH}_2(H))_{\mathcal{H}} \subset \mathrm{P}_+(\mathcal{PH}_2(H))_{\mathcal{P}}$$

and

$$\mathrm{ind}_H^G(\mathrm{RO}(H)_{\mathcal{H}}) \subset \mathrm{RO}(G)_{\mathcal{H}}.$$

Thus Theorem 1.10 follows from Theorem 6.7.

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References

[1] A. Bak, *K-theory of forms*, Ann. of Math. Stud. 98, Princeton Univ. Press, Princeton, 1981. Zbl 0465.10013 MR 0632404

[2] A. Bak and M. Morimoto, Equivariant intersection theory and surgery theory for manifolds with middle dimensional singular sets, J. K-Theory **2** (2008), Special Issue 03, 507–600. Zbl 1156.57028 MR 2465449

[3] S. E. Cappell and J. L. Shaneson, The codimension two placement problem and homology equivalent manifolds, Ann. of Math. **99** (1974), 277–348. Zbl 0279.57011 MR 0339216

[4] X. M. Ju, The Smith set of the group $S_5 \times C_2 \times \cdots \times C_2$, Osaka J. Math. **47** (2010), 215–236. Zbl 1228.55004 MR 2666133

[5] X. M. Ju, K. Matsuzaki and M. Morimoto, Mackey and Frobenius structures on odd dimensional surgery obstruction groups, *K*-Theory **29** (2003), 285–312. Zbl 1044.19003 MR 2029081

[6] E. Laitinen and M. Morimoto, Finite groups with smooth one fixed point actions on spheres, Forum Math. **10** (1998), 479–520. Zbl 0905.57023 MR 1631012

[7] E. Laitinen, M. Morimoto and K. Pawałowski, Deleting-inserting theorem for smooth actions of finite nonsolvable groups on spheres, Comment. Math. Helv. **70** (1995), 10–38. Zbl 0843.57034 MR 1314939

[8] M. Morimoto, Bak groups and equivariant surgery, *K*-Theory **2** (1989), 465–483. Zbl 0669.57018 MR 0990572

[9] ———, Bak groups and equivariant surgery II, *K*-Theory **3** (1990), 505–521. Zbl 0712.57013 MR 1071893

[10] ———, Induction theorems of equivariant surgery obstruction groups, Trans. Amer. Math. Soc. **355** (2003), 2341–2384. Zbl 1019.19002 MR 1973993

[11] ———, Fixed-point sets of smooth actions on spheres, J. K-Theory **1** (2008), 95–128. Zbl 1154.57037 MR 2424568

[12] M. Morimoto and K. Iizuka, Extendibility of *G*-maps to pseudoequivalences to finite *G*-CW-complexes whose fundamental groups are finite, Osaka J. Math. **21** (1984), 59–69. Zbl 0542.57031 MR 0736970

[13] M. Morimoto and K. Pawałowski, The equivariant bundle subtraction theorem and its applications, Fund. Math. **161** (1999), 279–303. Zbl 0947.57035 MR 1716021

[14] ______, Smooth actions of finite Oliver groups on spheres, Topology **42** (2003), 395-421. Zbl 1019.57020 MR 1941442

[15] B. Oliver, Fixed point sets and tangent bundles of actions on disks and euclidean spaces, Topology **35** (1996), 583–615. Zbl 0861.57047 MR 1396768

[16] ______, Fixed-point sets of groups on finite acyclic complexes, Comment. Math. Helv. **50** (1975), 155–177. Zbl 0304.57020 MR 0375361

[17] K. Pawałowski and R. Solomon, Smith equivalence and finite Oliver groups with Laitinen number 0 or 1, Algebr. Geom. Topol. **2** (2002), 843–895. Zbl 1022.57019 MR 1936973

[18] C. T. C. Wall, *Surgery on compact manifolds*, Academic Press, London, 1970. Zbl 0219.57024 MR 0431216