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Generalized Whittaker Functions for Degenerate Principal Series of $\mathrm{GL}(4, \mathbb{R})$

by

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Abstract

For degenerate principal series representations of $\mathrm{GL}(n, \mathbb{R})$, we show that the spaces of corresponding class one generalized Whittaker functions are characterized by explicit systems of differential operators. By using this characterization, we give detailed calculations on $\mathrm{GL}(4, \mathbb{R})$. We examine the dimensions of the spaces of generalized Whittaker functions and give their generators in terms of hypergeometric functions of one and two variables. We show that generalized Whittaker functions have multiplicity one by using the theory of hypergeometric functions.

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§1. Introduction

In this paper we shall investigate generalized Whittaker functions of degenerate principal series representations of $\mathrm{GL}(n, \mathbb{R})$ and give a detailed computation for the $\mathrm{GL}(4, \mathbb{R})$ case.

It is known that real analytic Eisenstein series are constructed from these representations (see [18]). Then generalized Whittaker functions appear as Fourier coefficients of the Eisenstein series and play important roles in establishing certain analytic properties. For example, in [29] A. Terras gives a Fourier expansion of Epstein zeta functions which are Eisenstein series associated with degenerate principal series induced from characters of the maximal parabolic subgroup $P_{1,n}$ fixing the unit vector $e_n = (0, \ldots, 0, 1)$ (cf. [12]). Here the generalized Whittaker functions appearing as Fourier coefficients are given by modified Bessel functions.

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However it seems that there has not been much study of generalized Whittaker functions of degenerate series induced from other parabolic subgroups. Thus in this paper we study generalized Whittaker functions of degenerate principal series of $\mathrm{GL}(4,\mathbb{R})$ induced from characters of $P_{1,4}$ and $P_{2,4}$; the latter case will give a generalization of Terras's case. We note that for $\mathrm{GL}(n,\mathbb{R})$ $(n \leq 3)$, all maximal parabolic subgroups are reduced to $P_{1,n}$ by conjugations.

In contrast to the method of Terras, our method relies on the representation theory of degenerate principal series of $\mathrm{GL}(n,\mathbb{R})$. Let us explain this precisely. Let $G = \mathrm{GL}(n,\mathbb{R})$ and consider an Iwasawa decomposition $G = KAN$. Take an increasing sequence of positive integers stopped at n, i.e., $\Theta = \{n_1, \dots, n_L\}$ with $0 < n_1 < \cdots < n_L = n$. Then let P_Θ be the parabolic subgroup corresponding to Θ and take the Langlands decomposition $P_\Theta = M_\Theta A_\Theta N_\Theta$. For a linear mapping $\lambda \in \mathrm{Hom}_{\mathbb{R}}(\mathrm{Lie}(A_\Theta), \mathbb{C})$, we can consider the induced representation $\pi_{\Theta,\lambda} = C^\infty\text{-}\mathrm{Ind}_{P_\Theta}^G(1_{M_\Theta} \otimes e^\lambda \otimes 1_{N_\Theta})$, called a degenerate principal series representation. Then we consider the annihilator ideal $\mathrm{Ann}_{U(\mathfrak{g})}(\pi_{\Theta,\lambda})$ of this representation in $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}_\mathbb{C} = \mathfrak{gl}(n,\mathbb{C})$. Also we consider $I_\Theta(\lambda) = \iota(\mathrm{Ann}_{U(\mathfrak{g})}(\pi_{\Theta,\lambda}))$ where ι is the antiautomorphism of $U(\mathfrak{g})$ such that $\iota(XY) = (-Y)(-X)$ for $X, Y \in \mathfrak{g}$. For a closed subgroup U of N, let (η, V_η) be an irreducible unitary representation of U and consider the space $C_\eta^\infty(U\backslash G) = \{f\colon G \to V_\eta^\infty \text{ smooth} \mid f(ug) = \eta(u)f(g),\ u \in U,\ g \in G\}$. Let $X_{\Theta,\lambda}$ be the Harish-Chandra module of $\pi_{\Theta,\lambda}$ and $X_{\Theta,\lambda}^*$ its dual Harish-Chandra module. *Generalized Whittaker models* are images of $X_{\Theta,\lambda}$ under elements of $\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda}, C_\eta^\infty(U\backslash G))$. We will prove the following characterization theorem for generalized Whittaker models.

Theorem 1.1 (see Theorem 3.5). *Suppose that $\lambda \in \mathfrak{a}_\mathbb{C}^*$ is regular and dominant. Take a nonzero K-fixed vector f_0 in $X_{\Theta,\lambda}^*$. Then the mapping*

$$\tilde{\Phi}\colon \mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda}^*, C_\eta^\infty(U\backslash G)) \xrightarrow{\sim} C_\eta^\infty(U\backslash G/K, I_\Theta(\lambda)), \qquad W \mapsto W(f_0)(g),$$

is a linear isomorphism. Here

$$\begin{aligned} &C_\eta^\infty(U\backslash G/K, I_\Theta(\lambda)) \\ &\quad = \{f\colon G \to V_\eta^\infty \text{ smooth} \mid f(ngk) = \eta(n)f(g),\ g \in G,\ n \in U,\ k \in K \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{and } R_X f(g) = 0,\ X \in I_\Theta(\lambda)\} \end{aligned}$$

and R_X is right translation by $X \in U(\mathfrak{g})$.

The elements in $C_\eta^\infty(U\backslash G/K, I_\Theta(\lambda))$ are called *class one generalized Whittaker functions.* Here we note that this theorem can be obtained as a corollary of Yamashita's result (Corollary 1.8 in [36]) where the irreducibility of $X_{\Theta,\lambda}^*$ is assumed. We give a new proof without using this assumption.

After the general theory for $\mathrm{GL}(n,\mathbb{R})$, we study the particular case of $\mathrm{GL}(4,\mathbb{R})$. We consider degenerate principal series representations induced from characters of the maximal parabolic subgroups $P_{1,4}$ and $P_{2,4}$. Then we examine the dimensions and find generators of $C^\infty_\eta(U\backslash G/K, I_\Theta(\lambda))$. Here for the closed subgroup $U \subset N$ and its unitary character χ, we assume that $L^2\text{-}\mathrm{Ind}_U^N \chi$ is an irreducible unitary representation of N. Under this assumption, we shall give elements in $C^\infty_\eta(U\backslash G/K, I_\Theta(\lambda))$ explicitly by using hypergeometric functions of several variables, namely Horn's hypergeometric functions H_{10} and modified Bessel functions. Using the theory of hypergeometric functions we shall examine the dimension of $C^\infty_\eta(U\backslash G/K, I_\Theta(\lambda)) \cong \mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X^*_{\Theta,\lambda}, C^\infty_\eta(U\backslash G))$. Also in the Appendix, we give some facts about Horn's hypergeometric functions.

Our results lead to the following observation. For degenerate principal series representations induced from characters of $P_{1,4}$, the multiplicity one theorem for generalized Whittaker models is true. On the other hand, for representations induced from characters of $P_{2,4}$, the multiplicity one theorem is no longer true. This fact seems to correspond to the result of Terras [30] who could determine only the nonsingular terms in the Fourier expansion of Eisenstein series corresponding to the degenerate principal series representations induced from characters of $P_{2,4}$ (Theorem 1 in [30]). Here the multiplicity one theorem is valid from our result. However degenerate terms are not computed because the corresponding generalized Whittaker models have multiplicities. Since our result gives a base of the space of generalized Whittaker functions explicitly, it makes it possible to overcome this difficulty.

Finally we mention some previous related work. Ishii and Oda [14] give an explicit calculation of generalized Whittaker functions of degenerate principal series of $\mathrm{SL}(3,\mathbb{R})$. For general studies of generalized Whittaker models of representations of general reductive Lie groups, see for example [7], [19], [32]–[34]. These generalized Whittaker models are associated with nondegenerate (admissible) characters of subgroups $U \subset N$. Thus they do not cover our results completely. The recent work of Oshima and Shimeno [25] gives Whittaker functions associated with a nondegenerate character of a maximal unipotent subgroup N as confluent hypergeometric functions obtained from Heckman–Opdam hypergeometric functions. Also there are various explicit presentations of Whittaker functions as hypergeometric functions of several variables given by Hirano, Ishii, Oda and other researchers (see [11] for the references).

§2. Spherical degenerate principal series representations of $\mathrm{GL}(n,\mathbb{R})$

In this section we review the results of T. Oshima on degenerate principal series representations of $\mathrm{GL}(n,\mathbb{R})$ and their annihilators. T. Oshima shows that the image of a degenerate principal series representation under the Poisson transform is characterized by the kernel of the annihilator [21] and moreover gives explicit generators of this annihilator [22], [24].

§2.1. Spherical degenerate principal series representations of $\mathrm{GL}(n,\mathbb{R})$

Let $G = \mathrm{GL}(n,\mathbb{R})$ with Lie algebra $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{R}) = M(n,\mathbb{R})$, the space of $n\times n$ matrices with real entries. The Iwasawa decomposition of G is $G = KAN$, where $K = O(n)$, A is the group of $n\times n$ diagonal matrices with positive real entries and N is the group of lower triangular matrices with 1s on the diagonal. Let E_{ij} be the matrix with 1 in the (i,j)-entry and 0 elsewhere. A nondegenerate bilinear form on $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(n,\mathbb{C}) = M(n,\mathbb{C})$ is defined by $\langle X,Y\rangle = \mathrm{tr}(XY)$ for $X,Y\in\mathfrak{g}_{\mathbb{C}}$. Via this bilinear form, we identify $\mathfrak{g}_{\mathbb{C}}$ with its dual space $\mathfrak{g}_{\mathbb{C}}^*$. The dual basis $\{E_{ij}^*\}$ of $\{E_{ij}\}$ is given by $E_{ij}^* = E_{ji}$. For simplicity we write $e_i = E_{ii}^*$.

Consider the Lie algebra

$$\mathfrak{a} = \Big\{\sum_{i=1}^{n} a_i E_{ii} \,\Big|\, a_i\in\mathbb{R},\ i=1,\dots,n\Big\}$$

of A. Then the root system of $(\mathfrak{g},\mathfrak{a})$ is $\triangle(\mathfrak{g},\mathfrak{a}) = \{e_i - e_j \mid 1\le i\neq j\le n\}$. Put $\alpha_i = e_{i+1} - e_i$ for $i = 1,\dots,n-1$ and fix a simple system of $\triangle(\mathfrak{g},\mathfrak{a})$ to be $\Pi(\mathfrak{g},\mathfrak{a}) = \{\alpha_1,\dots,\alpha_{n-1}\}$. Then the positive system of $\triangle(\mathfrak{g},\mathfrak{a})$ associated with $\Pi(\mathfrak{g},\mathfrak{a})$ is $\triangle^+(\mathfrak{g},\mathfrak{a}) = \{e_i - e_j \mid 1\le j< i\le n\}$. The Lie algebra $\mathfrak{n}$ of N is

$$\mathfrak{n} = \sum_{\alpha\in\triangle^+(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_\alpha = \sum_{i>j}\mathbb{R}E_{ij}$$

where $\mathfrak{g}_\alpha = \{X\in\mathfrak{g}\mid \mathrm{ad}(H)X = \alpha(H)X \text{ for } H\in\mathfrak{a}\}$. Similarly, let $\overline{N}$ be the group of upper triangular matrices with 1s on the diagonal. Then the Lie algebra $\overline{\mathfrak{n}}$ of $\overline{N}$ is

$$\overline{\mathfrak{n}} = \sum_{\alpha\in\triangle^+(\mathfrak{g},\mathfrak{a})} \mathfrak{g}_{-\alpha} = \sum_{i<j}\mathbb{R}E_{ij}.$$

Let $\Theta = \{n_1,\dots,n_L\}$ be a sequence of strictly increasing positive integers stopped at n, i.e., $(n_0 =)0 < n_1 < \cdots < n_L(= n)$. For this Θ, the associated standard parabolic subgroup P_Θ is defined as follows. Let

$$\mathfrak{a}_\Theta = \Big\{\sum_{k=1}^{L} a_k \sum_{i=n_{k-1}+1}^{n_k} E_{ii} \,\Big|\, a_k\in\mathbb{R},\ k=1,\dots,L\Big\}.$$

Let L_Θ be the centralizer of $\mathfrak{a}_\Theta$ in G, i.e.,

$$L_\Theta = \left\{ l = \begin{pmatrix} l_1 & & \\ & \ddots & \\ & & l_L \end{pmatrix} \,\middle|\, l_i\in\mathrm{GL}(n_i - n_{i-1},\mathbb{R}) \right\}$$

and $\mathfrak{l}_\Theta$ its Lie algebra, which is the centralizer of $\mathfrak{a}_\Theta$ in $\mathfrak{g}$. We put

$$\mathfrak{n}_\Theta = \sum_{\iota_\Theta(i) > \iota_\Theta(j)} \mathbb{R}E_{ij}$$

where $\iota_\Theta(\nu) = i$ if $n_{i-1} < \nu \leq n_i$ for $i = 1, \dots, L$. The corresponding analytic subgroup of G is $N_\Theta = \exp \mathfrak{n}_\Theta$, i.e.,

$$N_\Theta = \left\{ n = \begin{pmatrix} I_{n'_1} & & & & \\ N_{21} & I_{n'_2} & & & \\ N_{31} & N_{32} & I_{n'_3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ N_{L1} & N_{L2} & N_{L3} & \cdots & I_{n'_L} \end{pmatrix} \middle| \, N_{ij} \in M(n'_i, n'_j; \mathbb{R}),\ n'_i = n_i - n_{i-1} \right\}.$$

Here I_m denotes the identity matrix of size m and $M(k, l; \mathbb{R})$ denotes the space of $k \times l$ matrices with entries in $\mathbb{R}$. We also define $\overline{\mathfrak{n}}_\Theta = \sum_{\iota_\Theta(i) < \iota_\Theta(j)} \mathbb{R}E_{ij}$ and $\overline{N}_\Theta = \exp \overline{\mathfrak{n}}_\Theta$.

Then we define the parabolic subgroup P_Θ to be $L_\Theta N_\Theta$, i.e.,

$$P_\Theta = \left\{ p = \begin{pmatrix} g_1 & & \\ * & \ddots & \\ \vdots & & \\ * & \cdots & g_L \end{pmatrix} \in \mathrm{GL}(n, \mathbb{R}) \, \middle| \, g_i \in \mathrm{GL}(n_i - n_{i-1}, \mathbb{R}) \right\}.$$

Its Lie algebra is $\mathfrak{p}_\Theta = \mathfrak{l}_\Theta \oplus \mathfrak{n}_\Theta$.

For $(\lambda_1, \dots, \lambda_L) \in \mathbb{C}^L$, define a 1-dimensional representation of P_Θ, $\lambda \colon P_\Theta \to \mathbb{C}^\times$, by

$$\lambda(p) = |\det(g_1)|^{\lambda_1} \cdots |\det(g_L)|^{\lambda_L} \qquad \text{for } p \in P_\Theta.$$

Then the *spherical degenerate principal series representation* of G, denoted by $\pi_{\Theta,\lambda} = C^\infty\text{-}\mathrm{ind}_{P_\Theta}^G(\lambda)$, is defined as follows. The underlying representation space is

$$C^\infty(G/P_\Theta; \lambda) = \{\phi \in C^\infty(G) \mid \phi(gp) = \lambda(p)\phi(g),\ g \in G,\ p \in P_\Theta\}$$

where $C^\infty(G)$ is the space of C^∞-functions on G. The action of G on this space is by left translation, $\pi_{\Theta,\lambda}(g)\phi(x) = \phi(g^{-1}x)$ for $g \in G$ and $\phi \in C^\infty(G/P_\Theta; \lambda)$.

We consider the annihilator of $C^\infty(G/P_\Theta; \lambda)$ in $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. Recall that $U(\mathfrak{g})$ can be seen as the ring of left G-invariant differential operators on $C^\infty(G)$ by the natural extension of the differential of right translation,

$$R_X(f)(g) = \frac{d}{dt} f(g \exp(tX)) \bigg|_{t=0}$$

for $X \in \mathfrak{g}, f \in C^\infty(G)$. The representation of $U(\mathfrak{g})$ on $C^\infty(G/P_\Theta;\lambda)$ is defined by the differential of $\pi_{\Theta,\lambda}$, i.e., for $X \in \mathfrak{g}$, $\phi \in C^\infty(G/P_\Theta;\lambda)$, $\pi_{\Theta,\lambda}(X)\phi(x) = \frac{d}{dt}\phi(\exp(-tX)x)|_{t=0}$.

Let L_g and R_g be left and right translations by $g \in G$ respectively, i.e., $L_g f(x) = f(g^{-1}x)$ and $R_g f(x) = f(xg)$ for $f \in C^\infty(G)$.

Definition 2.1. The *annihilator* of $C^\infty(G/P_\Theta;\lambda)$ in $U(\mathfrak{g})$ is

$$\mathrm{Ann}_{U(\mathfrak{g})}(\pi_{\Theta,\lambda}) = \{X \in U(\mathfrak{g}) \mid \pi_{\Theta,\lambda}(X)\phi(x) = 0 \text{ for all } \phi \in C^\infty(G/P_\Theta;\lambda)\}.$$

We define an antiautomorphism ι of $U(\mathfrak{g})$ by $\iota(XY) = (-Y)(-X)$ for $X, Y \in \mathfrak{g}_\mathbb{C}$. Let us denote the differential of λ by $d\lambda\colon \mathfrak{p}_\Theta \to \mathbb{C}$.

Proposition 2.2. *We have*

$$\iota(\mathrm{Ann}_{U(\mathfrak{g})}(\pi_{\Theta,\lambda})) = \bigcap_{g\in G} \mathrm{Ad}(g) J_\Theta(d\lambda).$$

Here

$$J_\Theta(d\lambda) = \sum_{X\in\mathfrak{p}_\Theta} U(\mathfrak{g})(X - d\lambda(X))$$

is a left ideal of $U(\mathfrak{g})$.

Proof. For $X \in \mathfrak{p}_\Theta$ and $f \in C^\infty(G/P_\Theta;\lambda)$, we have

$$R_X f(g) = \frac{d}{dt} f(g\cdot\exp tX)\bigg|_{t=0} = \frac{d}{dt}\lambda(\exp tX)\bigg|_{t=0} f(g). \tag{2.1}$$

This implies $R_X f = 0$ for $X \in J_\Theta(d\lambda)$.

Recall the equation $L_X f(g) = R_{\mathrm{Ad}(g^{-1})\iota(X)} f(g)$ for $X \in U(\mathfrak{g})$. Since $X \in \bigcap_{g\in G}\mathrm{Ad}(g)J_\Theta(d\lambda)$ implies $\mathrm{Ad}(g)X \in J_\Theta(d\lambda)$, we have

$$L_{\iota(X)} f(g) = R_{\mathrm{Ad}(g^{-1})X} f(g) = 0$$

for $X \in \bigcap_{g\in G}\mathrm{Ad}(g)J_\Theta(d\lambda)$. Hence $\bigcap_{g\in G}\mathrm{Ad}(g)J_\Theta(d\lambda) \subset \iota(\mathrm{Ann}_{U(\mathfrak{g})}(\pi_{\Theta,\lambda}))$.

Conversely, take $X \in \mathrm{Ann}_{U(\mathfrak{g})}(\pi_{\Theta,\lambda})$ and put $X_{g_0} = \mathrm{Ad}(g_0^{-1})\iota(X)$ for $g_0 \in G$. Then

$$\begin{aligned} R_{X_{g_0}} f(g) &= L_{g_0 g^{-1}}(R_{X_{g_0}} f)(g_0) = R_{X_{g_0}}(L_{g_0 g^{-1}} f)(g_0) \\ &= L_X(L_{g_0 g^{-1}} f)(g_0) = L_X(\pi_{\Theta,\lambda}(g_0 g^{-1})f)(g_0) = 0 \end{aligned} \tag{2.2}$$

for $f \in C^\infty(G/P_\Theta;\lambda)$. By the decomposition $\mathfrak{g} = \bar{\mathfrak{n}}_\Theta \oplus \mathfrak{p}_\Theta$ and the Poincaré–Birkhoff–Witt theorem, we have

$$U(\mathfrak{g}) = U(\bar{\mathfrak{n}}_\Theta) \oplus J_\Theta(d\lambda)$$

where $U(\overline{\mathfrak{n}}_\Theta)$ is the universal enveloping algebra of $\overline{\mathfrak{n}}_\Theta \otimes_{\mathbb{R}} \mathbb{C}$. Take $Y \in U(\overline{\mathfrak{n}}_\Theta)$ and $Z \in J_\Theta(d\lambda)$ such that $X_{g_0} = Y + Z$. By (2.1), we have $R_Z f(g) = 0$ for $g \in G$ and $f \in C^\infty(G/P_\Theta; \lambda)$. Therefore (2.2) tells us that $0 = R_{X_{g_0}} f(g) = R_Y f(g)$.

We shall show $Y = 0$, which yields $X_{g_0} \in J_\Theta(d\lambda)$, so $\iota(\mathrm{Ann}_{U(\mathfrak{g})}(\pi_{\Theta,\lambda})) \subset \bigcap_{g\in G} \mathrm{Ad}(g) J_\Theta(d\lambda)$.

Consider the space $C_0^\infty(\overline{N}_\Theta)$ of compactly supported C^∞-functions on $\overline{N}_\Theta$. For $g \in \overline{N}_\Theta P_\Theta$, we take $\bar{n}(g) \in \overline{N}_\Theta$ and $p(g) \in P_\Theta$ so that $g = \bar{n}(g)p(g)$. Then we have an injection

$$C_0^\infty(\overline{N}_\Theta) \to C^\infty(G/P_\Theta; \lambda), \qquad f \mapsto \begin{cases} \lambda(p(g)) f(\bar{n}(g)) & \text{if } g \in \overline{N}_\Theta P_\Theta, \\ 0 & \text{otherwise.} \end{cases}$$

Via this injection, we can regard $C_0^\infty(\overline{N}_\Theta)$ as a subset of $C^\infty(G/P_\Theta; \lambda)$. Therefore recalling that $R_Y f(g) = 0$ for $g \in G$ and $f \in C^\infty(G/P_\Theta; \lambda)$, we have

$$R_Y f(\bar{n}) = 0 \qquad \text{for } \bar{n} \in \overline{N}_\Theta,\ f \in C_0^\infty(\overline{N}_\Theta).$$

For any $\psi \in C^\infty(\overline{N}_\Theta)$ and $\bar{n} \in \overline{N}_\Theta$, there exists $f \in C_0^\infty(\overline{N}_\Theta)$ such that $\psi = f$ on some neighbourhood of $\bar{n}$ in $\overline{N}_\Theta$. This implies

$$R_Y \psi(\bar{n}) = 0 \qquad \text{for } \bar{n} \in \overline{N}_\Theta,\ \psi \in C^\infty(\overline{N}_\Theta).$$

Therefore $Y \in U(\overline{\mathfrak{n}}_\Theta)$ must be 0, because $U(\overline{\mathfrak{n}}_\Theta)$ is identified with the ring of all left invariant differential operators on $\overline{N}_\Theta$. Hence $X_{g_0} \in J_\Theta(d\lambda)$ for any $g_0 \in G$, as desired. □

§2.2. Poisson transform of degenerate principal series representations

For simplicity we denote $I_\Theta(\lambda) = \bigcap_{g\in G} \mathrm{Ad}(g) J_\Theta(d\lambda)$. We shall see that $I_\Theta(\lambda)$ characterizes the image of the Poisson transform of the degenerate principal series. To explain this fact, we extend the representation space to the space of hyperfunctions on G. The space $\mathcal{B}(G)$ of hyperfunctions on G is a left G-module under left translation $G \times \mathcal{B}(G) \ni (g, f(x)) \mapsto f(g^{-1}x)$. Take a parabolic subgroup P_Θ of G and a character $\lambda\colon P_\Theta \to \mathbb{C}^\times$ for $(\lambda_1, \dots, \lambda_L) \in \mathbb{C}^L$. Then we can define a G-submodule

$$\mathcal{B}(G/P_\Theta; \lambda) = \{f \in \mathcal{B}(G) \mid f(xp) = \lambda(p) f(x) \text{ for } p \in P_\Theta\}$$

as in §2.1. Let $M = \{k \in K \mid kak^{-1} = a,\ a \in A\}$ and define the minimal parabolic subgroup $P_0 = P_{\{1,\dots,n\}} = MAN$. A character of P_0 is defined by

$$\lambda_\Theta\colon P_0 \to \mathbb{C}^\times, \qquad man \mapsto \prod_{i=1}^{L} \prod_{j=n_i+1}^{n_{i+1}} a_j^{\lambda_i},$$

for $m \in M$, $a \in A$, $n \in N$. Now we introduce the Poisson transform of $\mathcal{B}(G/P_0; \lambda_\Theta)$.

Definition 2.3. The *Poisson transform* is the G-homomorphism

$$\mathcal{P}^{\lambda}\colon \mathcal{B}(G/P_0;\lambda_\Theta)\to\mathcal{B}(G/K),\qquad f\mapsto F(x)=\int_K f(xk)\,dk,\ x\in G.$$

Here dk is the normalized Haar measure on K so that $\int_K dk=1$.

Let us recall a character of the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$, the so-called infinitesimal character of $\pi_{\Theta,\lambda}$. Let $d\lambda_\Theta\colon \mathrm{Lie}(P_0)\to\mathbb{C}$ be the differential of λ_Θ. By restriction to $\mathfrak{a}\subset\mathrm{Lie}(P_0)$, we can regard $d\lambda_\Theta\in\mathfrak{a}^*_{\mathbb{C}}$. Let ω be the projection map from $U(\mathfrak{g})$ to the symmetric algebra $S(\mathfrak{a})$ of $\mathfrak{a}_{\mathbb{C}}=\mathfrak{a}\otimes_{\mathbb{R}}\mathbb{C}$ along the decomposition $U(\mathfrak{g})=S(\mathfrak{a})\oplus(\overline{\mathfrak{n}}U(\mathfrak{g})+U(\mathfrak{g})\mathfrak{n})$. It is known that ω is an algebra homomorphism from $Z(\mathfrak{g})$ into $S(\mathfrak{a})$. We can identify the symmetric algebra $S(\mathfrak{a})$ with the algebra of polynomials on $\mathfrak{a}^*_{\mathbb{C}}$. Hence if we consider the evaluation of $\omega(\cdot)\in S(\mathfrak{a})$ at $d\lambda_\Theta$, we obtain a character of $Z(\mathfrak{g})$,

$$\chi_\lambda\colon Z(\mathfrak{g})\ni X\mapsto\omega(X)(d\lambda_\Theta)\in\mathbb{C}.$$

Define a subspace of $C^\infty(G/K)$ by

$$C^\infty(G/K;\mathcal{M}_\lambda)=\{f\in C^\infty(G/K)\mid R_Xf=\chi_\lambda(X)f\ \text{for}\ X\in Z(\mathfrak{g})\}$$

and put

$$e(\lambda_\Theta)=\prod_{\alpha\in\triangle^+(\mathfrak{g},\mathfrak{a})}\Gamma\biggl(\frac14\biggl(3+\frac{2\langle\lambda_\Theta,\alpha\rangle}{\langle\alpha,\alpha\rangle}\biggr)\biggr)^{-1}\Gamma\biggl(\frac14\biggl(1+\frac{2\langle\lambda_\Theta,\alpha\rangle}{\langle\alpha,\alpha\rangle}\biggr)\biggr)^{-1}.$$

The following theorem is known as *Helgason's conjecture* [9].

Theorem 2.4 ([16]). *The Poisson transform $\mathcal{P}^\lambda$ is a G-isomorphism*

$$\mathcal{B}(G/P_0;\lambda_\Theta)\cong C^\infty(G/K;\mathcal{M}_\lambda)$$

if and only if $e(\lambda_\Theta)\neq0$.

We can also define the Poisson transform on the subspace $\mathcal{B}(G/P_\Theta;\lambda)$ of $\mathcal{B}(G/P_0;\lambda_\Theta)$. Thus next we shall discuss a characterization of the image of $\mathcal{B}(G/P_\Theta;\lambda)$. Consider the subspace $C^\infty(G/K;I_\Theta(\lambda))=\{f\in C^\infty(G/K)\mid R_Xf=0\ \text{for}\ X\in I_\Theta(\lambda)\}$ of $C^\infty(G/K;\mathcal{M}_\lambda)$.

Remark 2.5. We can easily show that

$$I_\Theta(\lambda)\supset\sum_{D\in Z(\mathfrak{g})}U(\mathfrak{g})(D-\chi_\lambda(D))$$

(see Remark 4.3 in [24]). Hence $C^\infty(G/K;I_\Theta(\lambda))$ is a subspace of $C^\infty(G/K;\mathcal{M}_\lambda)$. Moreover $C^\infty(G/K;I_\Theta(\lambda))\subset C^\infty(G/K;\mathcal{M}_\lambda)\subset C^\omega(G)$. Here $C^\omega(G)$ is the space of real analytic functions on G.

Assume that $\lambda_\Theta + \rho \in \mathfrak{a}_{\mathbb{C}}^*$ is regular and dominant where $\rho = \frac{1}{2}\mathrm{tr}(\mathrm{ad}|_{\mathfrak{n}}) \in \mathfrak{a}_{\mathbb{C}}^*$, i.e., $\rho = \frac{1}{2}\sum_{1\le i<j\le n}(e_j - e_i) = \sum_{i=1}^n (i - \frac{n+1}{2})e_i$. This is equivalent to

$$\frac{2\langle \lambda_\Theta + \rho, \alpha\rangle}{\langle \alpha, \alpha\rangle} \notin \{0, -1, -2, \dots\} \qquad \text{for } \alpha \in \triangle^+(\mathfrak{g}, \mathfrak{a}).$$

We keep this assumption throughout the remainder of this paper.

Theorem 2.6 (T. Oshima, Theorem 5.1 in [24]). *Under the above assumption, the Poisson transform*

$$\mathcal{P}_\Theta^\lambda \colon \mathcal{B}(G/P_\Theta; \lambda) \to C^\infty(G/K; I_\Theta(\lambda)), \qquad f \mapsto F(x) = \int_K f(xk)\, dk,\ x \in G,$$

is a G-isomorphism.

In [21], [22] and [24], T. Oshima obtained several good generator systems of $I_\Theta(\lambda)$. We introduce one of them here.

Denote the space of $n \times n$ matrices with entries in $U(\mathfrak{g})$ by $M(n, U(\mathfrak{g}))$. For $\mathbb{E} = (E_{ij})_{ij} \in M(n, U(\mathfrak{g}))$, we define elements in $Z(\mathfrak{g})$ by

$$\Delta_k = \mathrm{tr}(\mathbb{E}^k) \qquad \text{for } k = 1, \dots, n.$$

Then it is known that $Z(\mathfrak{g}) \cong \mathbb{C}[\Delta_1, \dots, \Delta_n]$ as $\mathbb{C}$-algebras.

Theorem 2.7 (T. Oshima, Corollary 4.6 in [24]). *Assume that $\lambda_\Theta + \rho \in \mathfrak{a}_{\mathbb{C}}^*$ is regular and dominant. Then*

$$I_\Theta(\lambda) = \sum_{i=1}^n \sum_{j=1}^n U(\mathfrak{g}) \Big(\prod_{k=1}^L (\mathbb{E} - \lambda_k - n_{k-1}) \Big)_{ij} + \sum_{k=1}^{L-1} U(\mathfrak{g})(\Delta_k - \chi_\lambda(\Delta_k)).$$

§3. Generalized Whittaker functions

Generalized Whittaker functions are the main object of study in this paper. We shall give a characterization of the space of generalized Whittaker functions of a degenerate principal series $\pi_{\Theta,\lambda}$ as a function space whose elements are killed by $I_\Theta(\lambda)$. This is an analogy of Yamashita's method in the case of irreducible highest weight modules [36]. The substantial part of his method is that the maximal globalization (in the sense of W. Schmid [28]) of highest weight modules is given by the kernel of a certain differential operator. The corresponding theorem for degenerate principal series is Theorem 2.6 in §2.2. Moreover thanks to Theorem 2.7, we know explicit structures of these differential operators. Hence we can carry out explicit calculations on the space of generalized Whittaker functions.

For a complete Hausdorff locally convex space V with a continuous G-action, the space of K-finite vectors of V is denoted by V_K. Let $X_{\Theta,\lambda}$ be $C^\infty(G/P_\Theta;\lambda)_K$, which is a $(\mathfrak{g}_{\mathbb{C}},K)$-module where the $\mathfrak{g}_{\mathbb{C}}$-action is the differential of $\pi_{\Theta,\lambda}$ and the K-action is the restriction of $\pi_{\Theta,\lambda}$; furthermore the actions of $\mathfrak{g}_{\mathbb{C}}$ and K are compatible. Also $X_{\Theta,\lambda}$ is a *Harish-Chandra module*, i.e., finitely generated as a $U(\mathfrak{g})$-module and with finite K-multiplicities.

§3.1. Maximal globalization

For the Harish-Chandara module $X_{\Theta,\lambda}$, consider its dual Harish-Chandra module X_{Θ,λ^*}. Here the character λ^* of P_Θ is

$$\lambda^* = -\bar{\lambda} - 2\rho_\Theta = (n - n_0 - n_1 - \bar{\lambda}_1, \ldots, n - n_{L-1} - n_L - \bar{\lambda}_L),$$

where $\rho_\Theta = \frac{1}{2}\mathrm{tr}(\mathrm{ad}|_{\mathfrak{n}_\Theta}) \in \mathfrak{a}_{\mathbb{C}}^*$, i.e.,

$$\rho_\Theta = \sum_{i=1}^{L} \frac{n_{i-1} + n_i - n}{2} \sum_{j=n_{i-1}}^{n_i} e_j.$$

In fact the pairing $\langle\ ,\ \rangle_{\lambda,\lambda^*} \colon C^\infty(G/P_\Theta;\lambda) \times C^\infty(G/P_\Theta;\lambda^*) \to \mathbb{C}$ defined by

$$\langle f, g\rangle_{\lambda,\lambda^*} = \int_K f(k)\overline{g(k)}\,dk$$

for $(f,g) \in C^\infty(G/P_\Theta;\lambda) \times C^\infty(G/P_\Theta;\lambda^*)$ is a G-equivariant nondegenerate sesquilinear pairing. Via this pairing, $X_{\Theta,\lambda^*} = C^\infty(G/P_\Theta;\lambda^*)_K$ can be identified with the dual Harish-Chandra module $(X_{\Theta,\lambda})^*$.

We can consider the natural $(\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}, K \times K)$-bimodule structures on $X_{\Theta,\lambda} \otimes X_{\Theta,\lambda^*}$ and $C^\infty(G)$. For $X_1, X_2 \in \mathfrak{g}_{\mathbb{C}}$ and $k_1, k_2 \in K$, put

$$\begin{aligned}(X_1,X_2)(f\otimes f^*) &= \pi_{\Theta,\lambda^*}(X_1)f \otimes f^* + f \otimes \pi_{\Theta,\lambda^*}(X_2)f^*,\\ (k_1,k_2)(f\otimes f^*) &= \pi_{\Theta,\lambda}(k_1)f \otimes \pi_{\Theta,\lambda^*}(k_2)f^*\end{aligned}$$

for $f \in X_{\Theta,\lambda}$ and $f^* \in X_{\Theta,\lambda^*}$. Also define

$$(X_1,X_2)h = L_{X_1}f + R_{X_2}h, \qquad (k_1,k_2)h = L_{k_1}R_{k_2}h$$

for $h \in C^\infty(G)$. Let us introduce the matrix coefficient map $c \colon X_{\Theta,\lambda} \otimes X_{\Theta,\lambda^*} \to C^\infty(G)$ (cf. [4]) satisfying

1. c is a $(\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}, K \times K)$-bimodule homomorphism,
2. for any $f \in X_{\Theta,\lambda}$ and $f^* \in X_{\Theta,\lambda^*}$, the evaluation $c(f \otimes f^*)(e)$ at the origin $e \in G$ equals $\langle f, f^*\rangle_{\lambda,\lambda^*}$.

It is known that this matrix coefficient map is uniquely determined (see Theorem 8.7 in [4]).

Theorem 2.6 tells us that the restriction of the Poisson transform $\mathcal{P}_\Theta^\lambda$ to $X_{\Theta,\lambda}$ gives us a $(\mathfrak{g}_\mathbb{C}, K)$-isomorphism $\mathcal{P}_\Theta^\lambda \colon X_{\Theta,\lambda} \xrightarrow{\sim} C^\infty(G/K; I_\Theta(\lambda))_K$.

Take a K-fixed vector $f_0 \in X_{\Theta,\lambda^*}$ such that $f_0|_K \equiv 1$. Then the restriction of the Poisson transform to $X_{\Theta,\lambda}$ is the matrix coefficient of an element of $X_{\Theta,\lambda}$ with $f_0 \in X_{\Theta,\lambda^*}$, i.e.,

$$\mathcal{P}_\Theta^\lambda(f) = c(f \otimes f_0). \tag{3.1}$$

Lemma 3.1. *The dual Harish-Chandra module X_{Θ,λ^*} is a cyclic $U(\mathfrak{g})$-module with a cyclic vector $f_0 \in X_{\Theta,\lambda^*}$ such that $f_0|_K \equiv 1$.*

Proof. Put $W = \{\pi_{\Theta,\lambda^*}(X)f_0 \mid X \in U(\mathfrak{g})\}$. This is a $(\mathfrak{g}_\mathbb{C}, K)$-module. We restrict the pairing $\langle\ ,\ \rangle_{\lambda,\lambda^*}$ to $X_{\Theta,\lambda} \times W$. Take an element $f \in X_{\Theta,\lambda}$ so that $\langle f, w\rangle_{\lambda,\lambda^*} = 0$ for any $w \in W$. Since $\mathcal{P}_\lambda(f)$ is K-finite and $Z(\mathfrak{g})$-finite, it is a real analytic function on G. Let C be a sufficiently small open neighbourhood of 0 in $\mathfrak{g}$. Then we have the Taylor expansion at the origin $e \in G$,

$$\begin{aligned}\mathcal{P}_\lambda(f)(\exp X) &= \sum_{\nu=0}^\infty \frac{1}{\nu!} R_{X^\nu}(\mathcal{P}_\lambda(f))(e) = \sum_{\nu=0}^\infty \frac{1}{n!} R_{X^\nu}(c(f \otimes f_0))(e)\\ &= \sum_{\nu=0}^\infty \frac{1}{\nu!} \langle f, \pi_{\Theta,\lambda^*}(X^\nu) f_0\rangle_{\lambda,\lambda^*} = 0\end{aligned}$$

for $X \in C$. Here we have used the equation (3.1). We can extend this equality to the identity component G^0 of G because both functions are real analytic. Also we can extend it to G by the equation $G = KG^0$. The injectivity of the Poisson transform $\mathcal{P}_\lambda$ implies $f = 0$. Hence the bilinear form on $X_{\Theta,\lambda} \times W$ is nondegenerate. Therefore $W = X_{\Theta,\lambda^*}$ by Lemma 2 in Section 5.2 of [31]. ☐

Now we consider $\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C^\infty(G))$ and recall that this space inherits a Fréchet topology and a continuous G-action. This continuous G-module is called the *maximal globalization* of the Harish-Chandra module $X_{\Theta,\lambda}$ (cf. [28] and [17]). The space $C^\infty(G)$ is a Fréchet space with the topology of uniform convergence on compact sets for functions on G and their derivatives. Let $\{|\cdot|_\alpha\}_{\alpha\in\Lambda}$ be a countable family of seminorms on $C^\infty(G)$ which defines the Fréchet topology on $C^\infty(G)$, where Λ is an index set. Take some $\alpha \in \Lambda$ and $v \in X_{\Theta,\lambda^*}$. Then we define the real-valued function $|\cdot|_{\alpha,v} \colon \mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C^\infty(G)) \to \mathbb{R}_{\geq 0}$ by $|I|_{\alpha,v} = |I(v)|_\alpha$ for $I \in \mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda_*}, C^\infty(G))$. Let $\{v_m\}$ be a countable vector space basis of the Harish-Chandra module X_{Θ,λ^*}. Then the family of seminorms $\{|\cdot|_{\alpha,v_k}\}_{\alpha\in\Lambda,\, v_k\in\{v_m\}}$

defines a Fréchet topology on $\mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}},K}(X_{\Theta,\lambda_*}, C^\infty(G))$, and a continuous G-action on this space is defined by left translation on $C^\infty(G)$.

Lemma 3.2. *Take a K-fixed vector $f_0 \in X_{\Theta,\lambda^*}$ such that $f_0|_K \equiv 1$. Then*

$$\Phi\colon \mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}},K}(X_{\Theta,\lambda^*}, C^\infty(G)) \to C^\infty(G), \qquad I \mapsto I(f_0)(g)\ (g \in G),$$

is a continuous mapping. Moreover for any seminorm $|\cdot|_{\alpha,v_m}$ on the space $\mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}},K}(X_{\Theta,\lambda^*}, C^\infty(G))$, there exists a continuous seminorm μ_{α,v_m} on $C^\infty(G)$ such that

$$\mu_{\alpha,v_m}(\Phi(I)) = |I|_{\alpha,v_m}$$

for $I \in \mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}},K}(X_{\Theta,\lambda^*}, C^\infty(G))$. Thus Φ is injective.

Proof. The first and the second assertions are well-known. The final one immediately follows from them. □

The maximal globalization of $X_{\Theta,\lambda}$ is isomorphic to a subspace of $C^\infty(G/K)$, as shown by the following proposition (see [28], [17]). We give a proof for completeness.

Proposition 3.3. *Take a K-fixed vector $f_0 \in X_{\Theta,\lambda^*}$ such that $f_0|_K \equiv 1$. Then we have a topological G-isomorphism*

$$\Phi\colon \mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}},K}(X_{\Theta,\lambda^*}, C^\infty(G)) \xrightarrow{\sim} C^\infty(G/K; I_\Theta(\lambda)), \qquad I \mapsto I(f_0).$$

Here $C^\infty(G/K; I_\Theta(\lambda))$ has Fréchet topology as a closed subspace of $C^\infty(G)$.

Proof. We can immediately see that Φ preserves the action of G. First we show that Φ is well-defined. Take $I \in \mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}},K}(X_{\Theta,\lambda^*}, C^\infty(G))_K$. Then by evaluation at the origin $e \in G$, we can regard $I(\cdot)(e)$ as an element of $X_{\Theta,\lambda} \cong (X_{\Theta,\lambda^*})^*$. As we see in Remark 2.5, $I(f_0) \in C^\infty(G/K; I_\Theta(\lambda)) \subset C^\omega(G)$. Thus the same argument as in the proof of Lemma 3.1 shows that we have the Taylor expansion at $e \in G$,

$$\begin{aligned} I(f_0)(\exp X) &= \sum_{\nu=0}^{\infty} \frac{1}{\nu!} R_{X^\nu}(I(f_0))(e) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} \langle I(\cdot)(e), \pi_{\Theta,\lambda^*}(X^\nu) f_0 \rangle_{\lambda,\lambda_*} \\ &= \sum_{\nu=0}^{\infty} \frac{1}{\nu!} c(I(\cdot)(e) \otimes \pi_{\Theta,\lambda^*}(X^\nu) f_0)(e) = \sum_{\nu=0}^{\infty} \frac{1}{\nu!} R_{X^\nu} c(I(\cdot)(e) \otimes f_0)(e) \\ &= c(I(\cdot)(e) \otimes f_0)(\exp X) \end{aligned}$$

for $X \in C$ where C is a sufficiently small neighbourhood of 0 in $\mathfrak{g}$. We can extend this equality to all $g \in G$ as in Lemma 3.1. Hence by Theorem 2.6 and (3.1), we have

$$\Phi(\mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}},K}(X_{\Theta,\lambda^*}, C^\infty(G))_K) \subset C^\infty(G/K; I_\Theta(\lambda)).$$

We recall that for a continuous representation of G on a locally convex complete space V, the space V_K of K-finite vectors is dense in V (see, for example, Lemma 1.9 in Ch. IV of [10]). Moreover we know that Φ is a continuous mapping by Lemma 3.2. Hence $\Phi(\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C^\infty(G))) \subset C^\infty(G/K; I_\Theta(\lambda))$, which proves that Φ is well-defined.

Next we prove Φ is a bijective map. By Lemma 3.2, Φ is injective. We need to prove that it is surjective. For any $F \in C^\infty(G/K; I_\Theta(\lambda))_K$, there exists $h \in X_{\Theta,\lambda}$ such that $F = c(h \otimes f_0)$ by Theorem 2.6 and (3.1). We choose $I_h \in \mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda}, C^\infty(G))$ so that $I_h(v) = c(h \otimes v)$ for $v \in X_{\Theta,\lambda^*}$. Then $\Phi(I_h) = I_h(f_0) = c(h \otimes f_0) = F$. Hence we have the inclusion $C^\infty(G/K; I_\Theta(\lambda))_K \subset \Phi(\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda}, C^\infty(G)))$. Because $C^\infty(G/K; I_\Theta(\lambda))_K$ is a dense subspace of $C^\infty(G/K; I_\Theta(\lambda))$, for any $f \in C^\infty(G/K; I_\Theta(\lambda))$ we can choose a convergent sequence $f_\nu \to f$ $(\nu \to \infty)$ where $f_\nu \in C^\infty(G/K; I_\Theta(\lambda))_K$ for $\nu \in \mathbb{N}$. The above inclusion shows that there exist $I_\nu \in \mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda}, C^\infty(G))$ such that $\Phi(I_\nu) = f_\nu$. From the second assertion in Lemma 3.2, $\{I_\nu\}$ is a Cauchy sequence in $\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda}, C^\infty(G))$. Since $\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda}, C^\infty(G))$ is a Fréchet space, i.e., complete space, there exists $I \in \mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda}, C^\infty(G))$ such that $I_\nu \to I$ $(\nu \to \infty)$. Thus $\Phi(I) = f$ by the continuity of Φ. This shows that Φ is surjective. The open mapping theorem implies that Φ is a homeomorphism. ☐

§3.2. Generalized Whittaker functions

We shall define generalized Whittaker models and functions for $X_{\Theta,\lambda}$. Fix a closed subgroup U of N and its irreducible unitary representation η on a Hilbert space V_η. Let V_η^∞ be the space of C^∞-vectors in V_η. Define

$$C_\eta^\infty(U\backslash G) = \{f\colon G \to V_\eta^\infty \text{ smooth} \mid f(ng) = \eta(n)f(g),\, g \in G,\, n \in U\},$$

which is a G-module with respect to right translation.

Definition 3.4. The image $W(X_{\Theta,\lambda^*})$ of $W \in \mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C_\eta^\infty(U\backslash G))$ is called a *generalized Whittaker model* of X_{Θ,λ^*}. Elements in $W(X_{\Theta,\lambda^*}) \subset C_\eta^\infty(U\backslash G)$ are called *generalized Whittaker functions*. In particular, the image $W(f_K)$ of a K-fixed vector $f_K \in X_{\Theta,\lambda^*}$ is called a *class one generalized Whittaker function*.

The following theorem gives a characterization of the space of class one generalized Whittaker functions as a function space whose elements are killed by $I_\Theta(\lambda)$. Also this shows that the space of class one generalized Whittaker functions is isomorphic to $\mathrm{Hom}_{\mathfrak{g}_\mathbb{C},K}(X_{\Theta,\lambda^*}, C_\eta^\infty(U\backslash G))$ as a linear space.

Theorem 3.5. *Take a K-fixed vector f_0 in X_{Θ,λ^*} such that $f_0|_K \equiv 1$. Then*

$$\tilde{\Phi}\colon \operatorname{Hom}_{\mathfrak{g}_{\mathbb{C}},K}(X_{\Theta,\lambda^*}, C_\eta^\infty(U\backslash G)) \xrightarrow{\sim} C_\eta^\infty(U\backslash G/K; I_\Theta(\lambda)), \qquad W \mapsto W(f_0)(g),$$

is a linear isomorphism. Here

$$\begin{aligned} &C_\eta^\infty(U\backslash G/K; I_\Theta(\lambda)) \\ &\quad = \{f\colon G \to V_\eta^\infty \text{ smooth} \mid f(ngk) = \eta(n)f(g),\ g\in G,\ n\in U,\ k\in K \\ &\qquad\qquad\qquad\qquad \text{and } R_X f(g) = 0,\ X \in I_\Theta(\lambda)\}. \end{aligned}$$

Proof. Fix a nonzero element $\xi \in V_\eta$ and define a linear mapping

$$T\colon C_\eta^\infty(U\backslash G) \ni f \mapsto \langle \xi, f(g)\rangle_\eta \in C^\infty(G),$$

which commutes with G and $\mathfrak{g}_{\mathbb{C}}$ actions from the right; here $\langle\ ,\ \rangle_\eta$ is the inner product on the Hilbert space V_η. Since (η, V_η) is an irreducible unitary representation of U, the mapping T is injective (see Theorem 2.4 in [35]). It also yields an injective map

$$\tilde{T}\colon \operatorname{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(X_{\Theta,\lambda^*}, C_\eta^\infty(U\backslash G)) \to \operatorname{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(X_{\Theta,\lambda^*}, C^\infty(G)), \qquad W \mapsto T\circ W.$$

For any $W \in \operatorname{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(X_{\Theta,\lambda^*}, C_\eta^\infty(U\backslash G))$, we have $T(\tilde{\Phi}(W)) = T(W(f_0)) = T\circ W(f_0) = \tilde{T}(W)(f_0) = \Phi(\tilde{T}(W))$. Hence we have the commutative diagram

$$\begin{tikzcd} \operatorname{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(X_{\Theta,\lambda^*}, C_\eta^\infty(U\backslash G)) \arrow[r, "\tilde{\Phi}"] \arrow[d, "\tilde{T}"'] & C_\eta^\infty(U\backslash G/K) \arrow[d, "T"] \\ \operatorname{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(X_{\Theta,\lambda^*}, C^\infty(G)) \arrow[r, "\Phi"] & C^\infty(G/K) \end{tikzcd}$$

Since Φ, T and $\tilde{T}$ are injective, $\tilde{\Phi}$ is also injective.

Next we show that $\operatorname{Im}\tilde{\Phi} \subset C_\eta^\infty(U\backslash G/K; I_\Theta(\lambda))$. Take an element $W \in \operatorname{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(X_{\Theta,\lambda^*}, C_\eta^\infty(U\backslash G))$. Then $T(\tilde{\Phi}(W)) = \langle \xi, W(f_0)\rangle \in C^\infty(G/K; I_\Theta(\lambda))$. Hence $0 \equiv R_X T(\tilde{\Phi}(W)) = T(R_X\tilde{\Phi}(W))$ for $X \in I_\Theta(\lambda)$. Since T is injective, we have $R_X W(f_0) \equiv 0$ for $X \in I_\Theta(\lambda)$, i.e., $\operatorname{Im}\tilde{\Phi} \subset C_\eta^\infty(U\backslash G/K; I_\Theta(\lambda))$.

Finally, we show that $\tilde{\Phi}$ is surjective. Let $f \in C_\eta^\infty(U\backslash G/K; I_\Theta(\lambda))$. For $v \in X_{\Theta,\lambda^*}$ there exists $X_v \in U(\mathfrak{g})$ such that $v = \pi_{\Theta,\lambda^*}(X_v)f_0$ by Lemma 3.1. Then we define a mapping $W_f\colon X_{\Theta,\lambda^*} \ni v = \pi_{\Theta,\lambda^*}(X_v)f_0 \mapsto R_{X_v}f(g) \in C_\eta^\infty(U\backslash G)$. We need to check that it is well-defined. If for $X_v, X_v' \in \mathfrak{g}$ we have $v = \pi_{\Theta,\lambda^*}(X_v)f_0 = \pi_{\Theta,\lambda^*}(X_v')f_0$, then $\pi_{\Theta,\lambda^*}(X_v - X_v')f_0 = 0$. Since $T(f) \in C^\infty(G/K; I_\Theta(\lambda))$, there exists $I_f \in \operatorname{Hom}_{\mathfrak{g}_{\mathbb{C}},K}(X_{\Theta,\lambda^*}, C^\infty(G))$ such that $T(f) = \Phi(I_f)$ by Proposition 3.3. Put $Z = X_v - X_v'$. Then $T(R_Z f) = R_Z T(f) = \Phi(R_Z I_f) = R_Z I_f(f_0)(g) = I_f(\pi_{\Theta,\lambda^*}(Z)f_0)(g) = 0$. Hence by the injectivity of T, we have $R_Z f(g) = 0$, i.e., $R_{X_v}f = R_{X_v'}f$. This implies that W_f is well-defined.

Also we can check that W_f is compatible with the $\mathfrak{g}_{\mathbb{C}}$ and K actions. Hence $W_f \in \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(X_{\Theta,\lambda^*}, C_\eta^\infty(U\backslash G))$ and $\tilde{\Phi}(W_f) = W_f(f_0) = f$. Thus $\tilde{\Phi}$ is surjective. □

§4. Calculus in the case of $\mathrm{GL}(4,\mathbb{R})$

In the previous section, we gave a characterization of the space of class one generalized Whittaker functions as the kernel of an explicit differential operator. Now by using this characterization, we study the particular case of $\mathrm{GL}(4,\mathbb{R})$. In the cases of $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{GL}(2,\mathbb{R})$, Whittaker functions are well understood. Also for $\mathrm{SL}(3,\mathbb{R})$ Ishii and Oda computed generalized Whittaker functions of degenerate principal series [14]. We shall consider the spherical degenerate principal series representations induced from the maximal parabolic subgroups $P_{1,4}$, $P_{2,4}$, examine the dimensions of the spaces of class one generalized Whittaker functions, and find their bases.

Now $G = \mathrm{GL}(4,\mathbb{R})$, $K = O(4)$, A is the group of 4×4 diagonal matrices with positive real entries and N is the group of 4×4 strictly lower triangular matrices with 1s on the diagonal. We put $P_k = P_{k,4}$, $k = 1,2$. For $(\lambda_1,\lambda_2) \in \mathbb{C}^2$, we define the character $\lambda\colon P_k \to \mathbb{C}^\times$ and degenerate principal series representations induced from λ as before. Let $X_{k,\lambda}$ be the Harish-Chandra modules of these degenerate principal series representations. Then by Theorem 2.7, their annihilator ideals in $U(\mathfrak{g})$ are

$$
\begin{aligned}
(4.1)\qquad I_k(\lambda) &= I_{\{k,4\}}(\lambda)\\
&= \sum_{i=1}^{4}\sum_{j=1}^{4} U(\mathfrak{g})((\mathbb{E}-\lambda_1)(\mathbb{E}-\lambda_2-k))_{ij} + U(\mathfrak{g})\Big(\sum_{i=1}^{4} E_{ii} - k\lambda_1 - (4-k)\lambda_2\Big)
\end{aligned}
$$

for $k = 1,2$. Throughout this section, we assume $\lambda_1 - \lambda_2 \notin \mathbb{Z}$.

§4.1. Equivalence classes of $C_\eta^\infty(U\backslash G)$

Generalized Whittaker models are images of embeddings of X_{Θ,λ^*} into $C_\eta^\infty(U\backslash G)$ where U is a closed subgroup of N and η is its irreducible unitary representation. In this paper, we only consider the space $C_\eta^\infty(U\backslash G)$ where the closed subgroup $U \subset N$ and its unitary character η are chosen to satisfy that

$$L^2\text{-}\mathrm{Ind}_U^N\, \eta \text{ is an irreducible unitary representation of } N.$$

Therefore we first review the classification of irreducible unitary representations of N.

4.1.1. Classification of the unitary dual of N. First we give the classification of the unitary dual of the maximal unipotent subgroup N of G using Kirillov's method of coadjoint orbits. The material of this subsection is standard; the details can be found in [3] for example.

We denote the dual $\mathbb{R}$-vector space of $\mathfrak{n}$ by $\mathfrak{n}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{n}, \mathbb{R})$ and identify it with the subspace of $M(4,\mathbb{R})$ consisting of the matrices

$$\alpha_{21}E_{21}^* + \alpha_{31}E_{31}^* + \alpha_{32}E_{32}^* + \alpha_{41}E_{41}^* + \alpha_{42}E_{42}^* + \alpha_{43}E_{43}^* = \begin{pmatrix} 0 & \alpha_{21} & \alpha_{31} & \alpha_{41} \\ & 0 & \alpha_{32} & \alpha_{42} \\ & & 0 & \alpha_{43} \\ & & & 0 \end{pmatrix}$$

for any $\alpha_{ij} \in \mathbb{R}$; here $E_{ij}^*(E_{i'j'}) = \mathrm{tr}(E_{ij}^* E_{i'j'})$. Let us put

$$\begin{aligned} &n(x_{21}, x_{31}, x_{32}, x_{41}, x_{42}, x_{43}) \\ &\qquad = x_{21}E_{21} + x_{31}E_{31} + x_{32}E_{32} + x_{41}E_{41} + x_{42}E_{42} + x_{43}E_{43} \in \mathfrak{n}, \\ &l(\alpha_{21}, \alpha_{31}, \alpha_{32}, \alpha_{41}, \alpha_{42}, \alpha_{43}) \\ &\qquad = \alpha_{21}E_{21}^* + \alpha_{31}E_{31}^* + \alpha_{32}E_{32}^* + \alpha_{41}E_{41}^* + \alpha_{42}E_{42}^* + \alpha_{43}E_{43}^* \in \mathfrak{n}^*. \end{aligned}$$

Then the coadjoint action of N on $\mathfrak{n}^*$ is

$$\begin{aligned} (4.2) \qquad &(\mathrm{Ad}^* \exp(n(x_{21}, \ldots, x_{43})))(l(\alpha_{21}, \ldots, \alpha_{43})) \\ &= \alpha_{41}E_{41}^* + (\alpha_{31} + x_{43}\alpha_{41})E_{31}^* + (\alpha_{42} - x_{21}\alpha_{41})E_{42}^* \\ &\quad + (\alpha_{21} + x_{32}\alpha_{31} + \alpha_{41}(x_{42} + x_{32}x_{43}/2))E_{21}^* \\ &\quad + (\alpha_{32} + x_{43}\alpha_{42} - x_{21}\alpha_{31} - x_{21}x_{43}\alpha_{41})E_{32}^* \\ &\quad + (\alpha_{43} - x_{32}\alpha_{42} - \alpha_{41}(x_{31} - x_{21}x_{32}/2))E_{43}^*. \end{aligned}$$

Proposition 4.1. *The coadjoint orbits of $\mathfrak{n}^*$ under the action of N are classified as follows:*

(1) *For $\alpha_{41} \in \mathbb{R} \setminus \{0\}$ and $\alpha_{32} \in \mathbb{R}$,*

$$\begin{aligned} &\mathrm{Ad}^* N(\alpha_{41}E_{41}^* + \alpha_{32}E_{32}^*) \\ &\quad = \{\alpha_{41}E_{41}^* + t_1E_{31}^* + t_2E_{42}^* + s_1E_{21}^* + (\alpha_{32} + t_1t_2/\alpha_{41})E_{32}^* + s_2E_{43}^* \mid \\ &\qquad\qquad t_1, t_2, s_1, s_2 \in \mathbb{R}\} \\ &\quad = \Big\{ \sum_{1 \le j < i \le 4} \beta_{ij}E_{ij}^* \in \mathfrak{n}^* \ \Big|\ \beta_{41} = \alpha_{41},\ \alpha_{41}\beta_{32} = \alpha_{32}\alpha_{41} + \beta_{31}\beta_{42} \Big\}. \end{aligned}$$

Here $\dim \mathrm{Ad}^* N(\alpha_{41}E_{41}^* + \alpha_{32}E_{32}^*) = 4$.

(2) *For* $\alpha_{21}, \alpha_{31}, \alpha_{42}, \alpha_{43} \in \mathbb{R}$ *such that* $\alpha_{31}\alpha_{42} \neq 0$,

$$\begin{aligned}&\mathrm{Ad}^* N(\alpha_{21}E_{21}^* + \alpha_{31}E_{31}^* + \alpha_{42}E_{42}^* + \alpha_{43}E_{43}^*)\\ &\quad = \{\alpha_{31}E_{31}^* + \alpha_{42}E_{42}^* + (\alpha_{31}t_1 + \alpha_{21})E_{21}^* + t_2E_{32}^* + (\alpha_{43} - \alpha_{42}t_1)E_{43}^* \mid \\ &\qquad\qquad t_1, t_2 \in \mathbb{R}\}\\ &\quad = \Big\{ \sum_{1\le j<i\le 4} \beta_{ij}E_{ij}^* \in \mathfrak{n}^* \;\Big|\; \beta_{41} = 0,\ \beta_{31} = \alpha_{31},\ \beta_{42} = \alpha_{42},\\ &\qquad\qquad \alpha_{31}\beta_{43} + \alpha_{42}\beta_{41} = \alpha_{42}\alpha_{21} + \alpha_{31}\alpha_{43}\Big\}.\end{aligned}$$

Here $\dim \mathrm{Ad}^* N(\alpha_{21}E_{21}^* + \alpha_{31}E_{31}^* + \alpha_{42}E_{42}^* + \alpha_{43}E_{43}^*) = 2$.

(3) *For* $\alpha_{21}, \alpha_{32}, \alpha_{43} \in \mathbb{R}$,

$$\mathrm{Ad}^* N(\alpha_{21}E_{21}^* + \alpha_{32}E_{32}^* + \alpha_{43}E_{43}^*) = \alpha_{21}E_{21}^* + \alpha_{32}E_{32}^* + \alpha_{43}E_{43}^*.$$

Here $\dim \mathrm{Ad}^* N(\alpha_{21}E_{21}^* + \alpha_{32}E_{32}^* + \alpha_{43}E_{43}^*) = 0$.

Proof. This follows by direct computation using (4.2). ☐

To construct irreducible unitary representations of N from the coadjoint orbit of $l \in \mathfrak{n}^*$, we should determine its radical $\mathfrak{r}_l$ and choose a maximal subordinate subalgebra $\mathfrak{s}_l$. We define the coadjoint action of the Lie algebra $\mathfrak{n}$ on $l \in \mathfrak{n}^*$ by $((\mathrm{ad}^* X)l)(Y) = l([Y, X])$ for $X, Y \in \mathfrak{n}$.

Definition 4.2. For $l \in \mathfrak{n}^*$, the *radical* of l is the subalgebra of $\mathfrak{n}$ defined by

$$\mathfrak{r}_l = \{X \in \mathfrak{n} \mid (\mathrm{ad}^* X)l = 0\}.$$

Here we note that $\exp \mathfrak{r}_l = \{x \in N \mid (\mathrm{Ad}^*\, x)l = l\}$ (see Lemma 1.3.1 in [3]).

Definition 4.3. For $l \in \mathfrak{n}^*$, we can regard $l([X, Y])$ as a bilinear form for $(X, Y) \in \mathfrak{n} \times \mathfrak{n}$. By the antisymmetry of the Lie bracket $[X, Y] = -[Y, X]$ $(X, Y \in \mathfrak{n})$, this is an alternating form on $\mathfrak{n} \times \mathfrak{n}$. Any subalgebra $\mathfrak{s}_l \subset \mathfrak{n}$ which is isotropic for l, i.e., $l([X, Y]) = 0$ for $X, Y \in \mathfrak{s}_l$, and has codimension $\frac{1}{2}\dim_{\mathbb{R}}(\mathfrak{n}/\mathfrak{r}_l)$ is called a *maximal subordinate subalgebra* of $\mathfrak{n}$ for l.

Let us construct radicals and choose maximal subordinate subalgebras for coadjoint orbits (1), (2), (3) which are classified in Proposition 4.1.

Case (1). Equation (4.2) implies

$$\begin{aligned}&(\mathrm{Ad}^* \exp(n(x_{21}, \ldots, x_{43})))(\alpha_{41}E_{41}^* + \alpha_{32}E_{32}^*)\\ &\quad = \alpha_{41}E_{41}^* + x_{31}E_{31}^* + x_{42}E_{42}^* + x_{21}E_{21}^* + (\alpha_{32} + x_{31}x_{42}/\alpha_{41})E_{32}^* + x_{43}E_{43}^*.\end{aligned}$$

Thus $\mathfrak{r}_{\alpha_{41}E_{41}^* + \alpha_{32}E_{32}^*} = \mathbb{R}E_{41} + \mathbb{R}E_{32}$.

Although the radical is uniquely determined from $l \in \mathfrak{n}^*$, there are several maximal subordinate subalgebras to choose. Among these, we choose

$$\mathfrak{s}_{\alpha_{41}E_{41}^*+\alpha_{32}E_{32}^*} = \mathbb{R}E_{32} + \mathbb{R}E_{31} + \mathbb{R}E_{42} + \mathbb{R}E_{43} = \mathfrak{n}_{2,4}.$$

Here $\mathfrak{n}_{2,4} = \mathfrak{n}_\Theta$ with $\Theta = \{2,4\}$.

Case (2). As in case (1), we can see that the radical for $\alpha_{21}E_{21}^*+\alpha_{31}E_{31}^*+\alpha_{42}E_{42}^*+\alpha_{43}E_{43}^*$ is given by

$$\mathfrak{r}_{\alpha_{21}E_{21}^*+\alpha_{31}E_{31}^*+\alpha_{42}E_{42}^*+\alpha_{43}E_{43}^*} = \mathbb{R}(\alpha_{31}E_{43} + \alpha_{42}E_{21}) + \mathbb{R}E_{31} + \mathbb{R}E_{42} + \mathbb{R}E_{41}.$$

Also we can choose a maximal subordinate subalgebra

$$\mathfrak{s}_{\alpha_{21}E_{21}^*+\alpha_{31}E_{31}^*+\alpha_{42}E_{42}^*+\alpha_{43}E_{43}^*} = \mathbb{R}E_{21} + \mathbb{R}E_{43} + \mathbb{R}E_{31} + \mathbb{R}E_{42} + \mathbb{R}E_{41} = \mathfrak{n}_{1,3,4}.$$

Here $\mathfrak{n}_{1,3,4} = \mathfrak{n}_\Theta$ with $\Theta = \{1,3,4\}$.

Case (3). As in (1) and (2), the radical for $\alpha_{21}E_{21}^* + \alpha_{32}E_{32}^* + \alpha_{43}E_{43}^*$ is

$$\mathfrak{r}_{\alpha_{21}E_{21}^*+\alpha_{32}E_{32}^*+\alpha_{43}E_{43}^*} = \mathfrak{n}.$$

Also we can choose a maximal subordinate subalgebra

$$\mathfrak{s}_{\alpha_{21}E_{21}^*+\alpha_{32}E_{32}^*+\alpha_{43}E_{43}^*} = \mathfrak{n}.$$

Let us recall Kirillov's orbit method. Let $\mathfrak{s}_l$ be a maximal subordinate subalgebra for $l \in \mathfrak{n}^*$ and let $S_l = \exp \mathfrak{s}_l$. We can extend $l|_{\mathfrak{s}_l} \colon \mathfrak{s}_l \to \mathbb{R}$ to a map $\chi_l \colon S_l \to \mathbb{C}^1$ by

$$\chi_l(\exp X) = e^{2\pi\sqrt{-1}\, l(X)}, \qquad X \in \mathfrak{s}_l.$$

This is a group homomorphism, i.e., a unitary character of S_l because $\mathfrak{s}_l$ is an isotropic subspace for l. Define a Hilbert space by

$$\mathcal{H}_{\chi_l} = \left\{ f \colon N \to \mathbb{C} \text{ measurable} \;\middle|\; \begin{array}{l} f(sx) = \chi_l(s)f(x) \text{ for } s \in S_l,\ x \in N, \\ \qquad\qquad \text{and } \displaystyle\int_{S_l\backslash N} |f(x)|^2 \, d\dot{x} < \infty \end{array} \right\},$$

where $d\dot{x}$ is the right-invariant measure on $S_l\backslash N$, with the inner product defined by

$$\langle f, f' \rangle = \int_{S_l\backslash N} f(x)\overline{f'(x)}\, d\dot{x}.$$

It can be shown that $\mathcal{H}_{\chi_l}$ is complete with this inner product. The action of N on $\mathcal{H}_{\chi_l}$ is by right translation. From the right-invariance of $d\dot{x}$ this action on $\mathcal{H}_{\chi_l}$ gives a unitary representation of N, which is said to be induced from χ_l and denoted by $L^2\text{-}\mathrm{Ind}_{S_l}^N \chi_l$.

Theorem 4.4 (Kirillov [15]). *Take* $l \in \mathfrak{n}^*$ *and let* $\mathfrak{s}_l$ *be a maximal subordinate subalgebra of* $\mathfrak{n}$ *for* l*.*

(1) *The induced representation* $L^2\text{-}\mathrm{Ind}_{S_l}^N \chi_l$ *is an irreducible representation of* N*.*

(2) *Let* $\mathfrak{s}_l'$ *be another maximal subordinate subalgebra of* $\mathfrak{n}$ *for* l *and* $S_l' = \exp \mathfrak{s}_l'$*. Then* $L^2\text{-}\mathrm{Ind}_{S_l'}^N \chi_l$ *is unitarily equivalent to* $L^2\text{-}\mathrm{Ind}_{S_l}^N \chi_l$*. Hence we may write* π_l *for* $L^2\text{-}\mathrm{Ind}_{S_l}^N \chi_l$*.*

(3) *Let* $l' \in \mathfrak{n}^*$*. Then* $\pi_{l'}$ *is unitarily equivalent to* π_l *if and only if* $l' \in (\mathrm{Ad}^* N)l$*.*

(4) *Let* π *be an irreducible unitary representation of* N*. Then there exists an* $l \in \mathfrak{n}^*$ *such that* π *is unitarily equivalent to* π_l*.*

By this theorem, we can obtain all equivalence classes of irreducible unitary representations of N.

Proposition 4.5. *We retain the above notation. Then every irreducible unitary representation of* N *is unitarily equivalent to one of the following representations.*

(1) *For* $\alpha_{41}E_{41}^* + \alpha_{32}E_{32}^* \in \mathfrak{n}^*$ *and its maximal subordinate subalgebra* $\mathfrak{n}_{2,4}$*, we define the representation*

$$L^2\text{-}\mathrm{Ind}_{N_{2,4}}^N \chi_{\alpha_{41}E_{41}^* + \alpha_{32}E_{32}^*}.$$

Here $N_{2,4} = N_\Theta$ *with* $\Theta = \{2, 4\}$ *and* $\alpha_{41} \in \mathbb{R}\backslash\{0\}$*,* $\alpha_{32} \in \mathbb{R}$*.*

(2) *For* $\alpha_{21}E_{21}^* + \alpha_{31}E_{31}^* + \alpha_{42}E_{42}^* + \alpha_{43}E_{43}^* \in \mathfrak{n}^*$ *and its maximal subordinate subalgebra* $\mathfrak{n}_{1,3,4}$*, we define the representation*

$$L^2\text{-}\mathrm{Ind}_{N_{1,3,4}}^N \chi_{\alpha_{21}E_{21}^* + \alpha_{31}E_{31}^* + \alpha_{42}E_{42}^* + \alpha_{43}E_{43}^*}.$$

Here $N_{1,3,4} = N_\Theta$ *with* $\Theta = \{1, 3, 4\}$ *and* $\alpha_{21}, \alpha_{31}, \alpha_{42}, \alpha_{43} \in \mathbb{R}$ $(\alpha_{31}\alpha_{42} \neq 0)$*.*

(3) *For* $\alpha_{21}E_{21}^* + \alpha_{32}E_{32}^* + \alpha_{43}E_{43}^* \in \mathfrak{n}^*$*, we define the unitary character of* N*,*

$$\chi_{\alpha_{21}E_{21}^* + \alpha_{32}E_{32}^* + \alpha_{43}E_{43}^*}.$$

Here $\alpha_{21}, \alpha_{32}, \alpha_{43} \in \mathbb{R}$*.*

4.1.2. Conjugacy classes of $C_\eta^\infty(U\backslash G)$**.** Next we investigate G-equivalence of the following spaces:

(1) $C^\infty_{\chi_{\alpha_{41}E_{41}^* + \alpha_{32}E_{32}^*}}(N_{2,4}\backslash G)$, $\alpha_{41} \in \mathbb{R}\backslash\{0\}$, $\alpha_{32} \in \mathbb{R}$,

(2) $C^\infty_{\chi_{\alpha_{21}E_{21}^* + \alpha_{31}E_{31}^* + \alpha_{42}E_{42}^* + \alpha_{43}E_{43}^*}}(N_{1,3,4}\backslash G)$, $\alpha_{21}, \alpha_{31}, \alpha_{42}, \alpha_{43} \in \mathbb{R}$, $\alpha_{31}\alpha_{42} \neq 0$,

(3) $C^\infty_{\chi_{\alpha_{21}E_{21}^* + \alpha_{32}E_{32}^* + \alpha_{43}E_{43}^*}}(N\backslash G)$, $\alpha_{21}, \alpha_{32}, \alpha_{43} \in \mathbb{R}$.

Put $g^x = xgx^{-1}$ for $g, x \in G$. Let H be a closed subgroup of G and π a continuous representation of H on a complete locally convex space E. Then for

$x \in N_G(H) = \{g \in G \,|\, h^g \in H$ for any $h \in H\}$, define conjugation of π by $\pi^x(h) = \pi(h^x)$. Then we have the following fact about the induced representation $C^\infty_\pi(H\backslash G) = \{f\colon G \to E\,\text{smooth} \mid f(hg) = \pi(h)f(g),\ g \in G,\ h \in H\}$ on which G acts by right translation.

Lemma 4.6. *We retain the above notation. The map*

$$C^\infty_\pi(H\backslash G) \xrightarrow{\sim} C^\infty_{\pi^x}(H\backslash G), \qquad f(g) \mapsto F(g) = f(xg),$$

is an isomorphism of G-modules.

Proof. Obvious. □

Lemma 4.7. *Fix a maximal subordinate subalgebra $\mathfrak{s}_l \subset \mathfrak{n}$ for $l \in \mathfrak{n}^*$ and put $S_l = \exp \mathfrak{s}_l$. Define a character $\chi_l\colon S_l \to \mathbb{C}^1$ so that $\chi_l(\exp X) = e^{2\pi\sqrt{-1}\,l(X)}$ for $X \in \mathfrak{s}_l$. Then the character χ_l is invariant under conjugation by S_l, i.e., $\chi_l^x(s) = \chi_l(s)$ for $s, x \in S_l$.*

Proof. Obvious. □

Using these lemmas, we obtain the following.

Proposition 4.8. *Case* (1). *For $\alpha_{41} \in \mathbb{R}\backslash\{0\}$ and $\alpha_{32} \in \mathbb{R}$, we have*

$$C^\infty_{\chi_{\alpha_{41}E^*_{41}+\alpha_{32}E^*_{32}}}(N_{2,4}\backslash G) \cong \begin{cases} C^\infty_{\chi_{E^*_{31}+E^*_{42}}}(N_{2,4}\backslash G) & \text{if } \alpha_{32} \neq 0, \qquad (1_a)\\ C^\infty_{\chi_{E^*_{32}}}(N_{2,4}\backslash G) & \text{if } \alpha_{32} = 0. \qquad (1_b)\end{cases}$$

Case (2). *Choose $\alpha_{21}, \alpha_{31}, \alpha_{42}, \alpha_{43} \in \mathbb{R}$ so that $\alpha_{31}\alpha_{42} \neq 0$. Then*

$$\begin{aligned} &C^\infty_{\chi_{\alpha_{21}E^*_{21}+\alpha_{31}E^*_{31}+\alpha_{42}E^*_{42}+\alpha_{43}E^*_{43}}}(N_{1,3,4}\backslash G)\\ &\cong \begin{cases} C^\infty_{\chi_{E^*_{21}+E^*_{42}}}(N_{1,3,4}\backslash G) & \text{if } (\alpha_{21},\alpha_{31})\cdot(\alpha_{43},\alpha_{42}) \neq 0, \qquad (2_a)\\ C^\infty_{\chi_{E^*_{21}+E^*_{43}}}(N_{1,3,4}\backslash G) & \text{if } (\alpha_{21},\alpha_{31})\cdot(\alpha_{43},\alpha_{42}) = 0\\ & \quad \text{and } \alpha_{31} \neq 0,\ \alpha_{42} \neq 0, \qquad (2_b)\\ C^\infty_{\chi_{E^*_{21}}}(N_{1,3,4}\backslash G) & \text{if } (\alpha_{21},\alpha_{31})\cdot(\alpha_{43},\alpha_{42}) = 0\\ & \quad \text{and } \alpha_{31} \neq 0,\ \alpha_{42} = 0, \qquad (2_c)\\ C^\infty_{\chi_{E^*_{43}}}(N_{1,3,4}\backslash G) & \text{if } (\alpha_{21},\alpha_{31})\cdot(\alpha_{43},\alpha_{42}) = 0\\ & \quad \text{and } \alpha_{31} = 0,\ \alpha_{42} \neq 0. \qquad (2_d)\end{cases}\end{aligned}$$

Here $(a,b)\cdot(c,d) = ac + bd$ for $a,b,c,d \in \mathbb{R}$ is a natural inner product in $\mathbb{R}^2$ which is induced from the structure of Heisenberg Lie algebra, namely

$$(\alpha_{21}\alpha_{42} + \alpha_{31}\alpha_{43})E_{14} = [\alpha_{21}E_{12} + \alpha_{31}E_{13}, \alpha_{42}E_{24} + \alpha_{43}E_{34}].$$

Case (3). *For* $\alpha_{21}, \alpha_{32}, \alpha_{43} \in \mathbb{R}$, *we have*

$$C^\infty_{\chi_{\alpha_{21}E^*_{21}+\alpha_{32}E^*_{32}+\alpha_{43}E^*_{43}}}(N\backslash G) \cong$$

$$\begin{cases} C^\infty_{\chi_{E^*_{21}+E^*_{32}+E^*_{43}}}(N\backslash G) & \textit{if } \alpha_{21} \neq 0,\ \alpha_{32} \neq 0,\ \alpha_{43} \neq 0, \quad (3_a)\\ C^\infty_{\chi_{E^*_{21}+E^*_{32}}}(N\backslash G) & \textit{if } \alpha_{21} \neq 0,\ \alpha_{32} \neq 0,\ \alpha_{43} = 0, \quad (3_b)\\ C^\infty_{\chi_{E^*_{21}+E^*_{43}}}(N\backslash G) & \textit{if } \alpha_{21} \neq 0,\ \alpha_{32} = 0,\ \alpha_{43} \neq 0, \quad (3_c)\\ C^\infty_{\chi_{E^*_{32}+E^*_{43}}}(N\backslash G) & \textit{if } \alpha_{21} = 0,\ \alpha_{32} \neq 0,\ \alpha_{43} \neq 0, \quad (3_d)\\ C^\infty_{\chi_{E^*_{21}}}(N\backslash G) & \textit{if } \alpha_{21} \neq 0,\ \alpha_{32} = 0,\ \alpha_{43} = 0, \quad (3_e)\\ C^\infty_{\chi_{E^*_{32}}}(N\backslash G) & \textit{if } \alpha_{21} = 0,\ \alpha_{32} \neq 0,\ \alpha_{43} = 0, \quad (3_f)\\ C^\infty_{\chi_{E^*_{43}}}(N\backslash G) & \textit{if } \alpha_{21} = 0,\ \alpha_{32} = 0,\ \alpha_{43} \neq 0, \quad (3_g)\\ C^\infty(N\backslash G) & \textit{if } \alpha_{21} = 0,\ \alpha_{32} = 0,\ \alpha_{43} = 0. \quad (3_h) \end{cases}$$

Proof. (1) The normalizer $N_G(N_{2,4})$ of $N_{2,4}$ in G is the semidirect product $L_{2,4} \ltimes N_{2,4}$ where

$$L_{2,4} = \left\{ \left(\begin{array}{c|c} A & 0_2 \\ \hline 0_2 & B \end{array} \right) \middle| A, B \in \mathrm{GL}(2,\mathbb{R}) \right\}.$$

Here $0_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M(2,\mathbb{R})$. Define the action of $N_G(N_{2,4})$ on $\widehat{N_{2,4}}$, the set of unitary characters of $N_{2,4}$, as follows. For $x \in N_G(N_{2,4})$, $\chi \in \widehat{N_{2,4}}$ and $s \in N_{2,4}$, we define $(x \cdot \chi)(s) = \chi(s^{x^{-1}})$. Then by Lemma 4.6, if $\chi, \chi' \in \widehat{N_{2,4}}$ are in the same $N_G(N_{2,4})$-orbit, the spaces $C^\infty_\chi(N_{2,4}\backslash G)$ and $C^\infty_{\chi'}(N_{2,4}\backslash G)$ are G-equivalent. Also by Lemma 4.7, it suffices to consider the action of $L_{2,4}$ on $\widehat{N_{2,4}}$.

Now we see that $\widehat{N_{2,4}}$ has three orbits $\mathrm{Ad}^*(N_G(N_{2,4}))(\chi_{\varepsilon_1 E^*_{31}+\varepsilon_2 E^*_{32}+\varepsilon_3 E^*_{42}})$ for $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1,0,1), (0,1,0), (0,0,0)$. It is easy to see that

$$\begin{aligned} &\chi_{\alpha_{41}E^*_{41}+\alpha_{32}E^*_{32}} \in \mathrm{Ad}^*(N_G(N_{2,4}))(\chi_{E^*_{31}+E^*_{42}}) && \text{if } \alpha_{32} \neq 0,\\ &\chi_{\alpha_{41}E^*_{41}+\alpha_{32}E^*_{32}} \in \mathrm{Ad}^*(N_G(N_{2,4}))(\chi_{E^*_{32}}) && \text{if } \alpha_{32} = 0. \end{aligned}$$

(2) The normalizer of $N_{1,3,4}$ in G is the semidirect product $L_{1,3,4} \ltimes N_{1,3,4}$ where

$$L_{1,3,4} = \left\{ n_{(a,b,A)} = \begin{pmatrix} a & \mathbf{0}_2 & 0 \\ {}^t\mathbf{0}_2 & A & {}^t\mathbf{0}_2 \\ 0 & \mathbf{0}_2 & b \end{pmatrix} \in G \,\middle|\, a, b \in \mathbb{R}^\times,\ A \in \mathrm{GL}(2,\mathbb{R}) \right\}.$$

Here $\mathbf{0}_2 = (0,0)$ and ${}^t\mathbf{0}_2 = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. As in (1), let us consider the $N_G(N_{1,3,4})$-action on $\widehat{N_{1,3,4}}$, the set of unitary characters of $N_{1,3,4}$. This action has the following orbits:

$$\begin{aligned}
&\{\chi_{v_1E^*_{31}+v_2E^*_{42}+w_1E^*_{21}+w_2E^*_{43}} \mid v_1w_1+v_2w_2\neq 0\},\\
&\{\chi_{v_1E^*_{31}+v_2E^*_{42}+w_1E^*_{21}+w_2E^*_{43}} \mid (v_1,v_2)\neq(0,0),\ (w_1,w_2)\neq(0,0)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and } v_1w_1+v_2w_2=0\},\\
&\{\chi_{v_1E^*_{31}+v_2E^*_{42}+w_1E^*_{21}+w_2E^*_{43}} \mid (v_1,v_2)\neq(0,0),\ (w_1,w_2)=(0,0)\},\\
&\{\chi_{v_1E^*_{31}+v_2E^*_{42}+w_1E^*_{21}+w_2E^*_{43}} \mid (v_1,v_2)=(0,0),\ (w_1,w_2)\neq(0,0)\},\\
&\{\chi_{v_1E^*_{31}+v_2E^*_{42}+w_1E^*_{21}+w_2E^*_{43}} \mid (v_1,v_2)=(0,0),\ (w_1,w_2)=(0,0)\}.
\end{aligned}$$

(3) The normalizer of N in G is the semidirect product $L\ltimes N$ where

$$L=\left\{\left.\begin{pmatrix} a_1 &&& \\ & a_2 && \\ && a_3 & \\ &&& a_4\end{pmatrix}\right| a_1,\dots,a_4\in\mathbb{R}^\times\right\}.$$

Then the lemma easily follows. □

Remark 4.9. In the above list, the characters

$$\begin{aligned}
&\chi_{E^*_{31}+E^*_{42}}\colon N_{2,4}\to\mathbb{C}^1 && (1_a),\\
&\chi_{E^*_{21}+E^*_{42}}\colon N_{1,3,4}\to\mathbb{C}^1 && (2_a),\\
&\chi_{E^*_{21}+E^*_{32}+E^*_{43}}\colon N\to\mathbb{C}^1 && (3_a)
\end{aligned}$$

are nondegenerate (also called admissible) (cf. [19], [33]).

§4.2. Spaces of class one generalized Whittaker functions and their vanishing

In Proposition 4.8, a list of $C^\infty_\chi(U\backslash G)$ with labels $(1_a),\dots,(3_h)$ is given. The purpose of this paper is to study the spaces of class one generalized Whittaker functions $C^\infty_\chi(U\backslash G/K;I_k(\lambda))$ for pairs (U,χ) corresponding to these labels $(1_a),\dots(3_h)$. In this section we shall find equalities and isomorphisms between these spaces with different labels and also show their vanishing.

For a closed subgroup U of N, there is a smooth cross section $\theta\colon U\backslash N\to N$ with a smooth splitting of $n\in N$ so that $n=u(n)s(n)$ for $u(n)\in U$ and $s(n)\in\theta(U\backslash N)$ (cf. Theorem 1.2.12 in [3]). Set $\tilde U=\theta(U\backslash N)$. Then we have a diffeomorphism $N\cong U\times\tilde U$. Recalling the Iwasawa decomposition $G=NAK$, we have the linear isomorphism

$$\Xi\colon C^\infty_\chi(U\backslash G/K)\xrightarrow{\sim}C^\infty(\tilde U\times A),\qquad f\mapsto\Xi(f)(x,a)=f(xa), \tag{4.3}$$

for $x\in\tilde U$ and $a\in A$. Here $C^\infty_\chi(U\backslash G/K)=\{f\in C^\infty(G)\mid f(ugk)=\chi(u)f(g)$ for $u\in U$, $g\in G$, $k\in K\}$ for a character χ of U.

Let us denote $X \cdot f = \Xi(R_X \Xi^{-1}(f))$ for $f \in C^\infty(\tilde{U} \times A)$ and $X \in \mathfrak{g}$. We sometimes omit the dot and write simply Xf. Then for an ideal $I \subset U(\mathfrak{g})$, we have

$$\begin{aligned} C^\infty_\chi(U\backslash G/K; I) &= \{f \in C^\infty_\chi(U\backslash G/K) \mid R_X f = 0 \text{ for all } X \in I\} \\ &\xrightarrow{\sim} C^\infty(\tilde{U} \times A; I) = \{f \in C^\infty(\tilde{U} \times A) \mid X \cdot f = 0 \text{ for all } X \in I\}. \end{aligned}$$

The Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$ and the P-B-W theorem induce a decomposition $U(\mathfrak{g}) = U(\mathfrak{g})\mathfrak{k} \oplus U(\mathfrak{a} \oplus \mathfrak{n})$. We shall see how elements in $U(\mathfrak{g})\mathfrak{k}$ and $U(\mathfrak{a} \oplus \mathfrak{n})$ are realized as differential operators on $C^\infty(\tilde{U} \times A)$.

We note that $E_{ii} \in \mathfrak{a}$, $i = 1, \ldots, 4$, can be realized on $C^\infty(\tilde{U} \times A)$ as $\vartheta_{a_i} = a_i \frac{\partial}{\partial a_i}$, $i = 1, \ldots, 4$, where we denote the elements of A by $a = \mathrm{diag}(a_1, \ldots, a_4)$.

We have the following symmetric relation among the generators of the annihilator ideal $I_k(\lambda)$.

Lemma 4.10. *We have* $((\mathbb{E}-\lambda_1)(\mathbb{E}-\lambda_2-k))_{ij} \equiv ((\mathbb{E}-\lambda_1)(\mathbb{E}-\lambda_2-k))_{ji}$ *modulo* $U(\mathfrak{g})\mathfrak{k}$ *for* $1 \le i, j \le 4$, *and* $k = 1, 2$.

Proof. Note that $E_{ij} - E_{ji}$ $(1 \le i < j \le 4)$ generate $\mathfrak{k}$. Then we have

$$\begin{aligned} &((\mathbb{E}-\lambda_1)(\mathbb{E}-\lambda_2-k))_{ij} - ((\mathbb{E}-\lambda_1)(\mathbb{E}-\lambda_2-k))_{ji} \\ &\quad = \Big(\sum_{l=1}^{4} E_{il}E_{lj} - (\lambda_1+\lambda_2+k)E_{ij} + \lambda_1(\lambda_2+k)\delta_{ij}\Big) \\ &\qquad - \Big(\sum_{l=1}^{4} E_{jl}E_{li} - (\lambda_1+\lambda_2+k)E_{ji} + \lambda_1(\lambda_2+k)\delta_{ji}\Big) \\ &\quad = \sum_{l=1}^{4}(E_{il}(E_{lj}-E_{jl}) + E_{jl}(E_{il}-E_{li})) - (\lambda_1+\lambda_2+k-1)(E_{ij}-E_{ji}) \in U(\mathfrak{g})\mathfrak{k}. \end{aligned}$$

□

Let us find the projections of $((\mathbb{E}-\lambda_1)(\mathbb{E}-\lambda_2-k))_{ij}$ to $U(\mathfrak{a} \oplus \mathfrak{n})$ along the decomposition $U(\mathfrak{g}) = U(\mathfrak{g})\mathfrak{k} \oplus U(\mathfrak{a} \oplus \mathfrak{n})$.

Lemma 4.11. *Representatives of* $((\mathbb{E}-\lambda_1)(\mathbb{E}-\lambda_2-k))_{ij}$ *modulo* $U(\mathfrak{g})\mathfrak{k}$, *for* $k = 1, 2$ *and* $1 \le i < j \le 4$, *are*

$$((i,j) = (1,1)) \qquad E_{11}^2 + E_{21}^2 + E_{31}^2 + E_{41}^2 - (\lambda_1+\lambda_2+k-3)E_{11} - (E_{22}+E_{33}+E_{44}) + \lambda_1(\lambda_2+k),$$

$$((i,j) = (1,2)) \qquad E_{21}(E_{11}+E_{22}-(\lambda_1+\lambda_2+k-3)) + E_{32}E_{31} + E_{42}E_{41},$$

$$((i,j) = (1,3)) \qquad E_{31}(E_{11}+E_{33}-(\lambda_1+\lambda_2+k-2)) + E_{32}E_{21} + E_{43}E_{41},$$

$$
\begin{aligned}
&((i,j)=(1,4)) && E_{41}(E_{11}+E_{44}-(\lambda_1+\lambda_2+k-2))+E_{42}E_{21}+E_{31}E_{43},\\
&((i,j)=(2,2)) && E_{22}^2-(\lambda_1+\lambda_2+k-2)E_{22}+E_{21}^2+E_{32}^2+E_{42}^2\\
& && \qquad -(E_{33}+E_{44})+\lambda_1(\lambda_2+k),\\
&((i,j)=(2,3)) && E_{32}(E_{22}+E_{33}-(\lambda_1+\lambda_2+k-2))+E_{21}E_{31}+E_{43}E_{42},\\
&((i,j)=(2,4)) && E_{42}(E_{22}+E_{44}-(\lambda_1+\lambda_2+k-2))+E_{21}E_{41}+E_{32}E_{43},\\
&((i,j)=(3,3)) && E_{33}^2-(\lambda_1+\lambda_2+k-1)E_{33}+E_{31}^2+E_{32}^2+E_{43}^2\\
& && \qquad -E_{44}+\lambda_1(\lambda_2+k),\\
&((i,j)=(3,4)) && E_{43}(E_{33}+E_{44}-(\lambda_1+\lambda_2+k-1))+E_{31}E_{41}+E_{32}E_{42},\\
&((i,j)=(4,4)) && E_{44}^2-(\lambda_1+\lambda_2+k)E_{44}+E_{41}^2+E_{42}^2+E_{43}^2+\lambda_1(\lambda_2+k).
\end{aligned}
$$

Proof. If we note that $E_{ij}-E_{ji}$ $(1\le i<j\le 4)$ are the generators of $\mathfrak{k}$, this lemma can be obtained by direct computations. $\square$

Let us define an automorphism of G by $j\colon G\ni g\mapsto J\,{}^t g^{-1}J^{-1}\in G$ where $J\in G$ has 1s in all the antidiagonal entries and 0s in the other entries. Also define $j\colon U(\mathfrak{g})\to U(\mathfrak{g})$ as the extension of the Lie algebra automorphism $j\colon \mathfrak{g}\ni X\mapsto \mathrm{Ad}\,(J)(-{}^tX)\in\mathfrak{g}$. Here we notice that $j\circ j=\mathrm{id}$.

Lemma 4.12. *We have $j(I_k(\lambda))\equiv I_k(\lambda')$ modulo $U(\mathfrak{g})\mathfrak{k}$ where $\lambda'=(\lambda_1',\lambda_2')=(-\lambda_1+4-k,-\lambda_2-k)$.*

Proof. It follows from Lemma 4.11 and a little computation that a set of generators of $I_k(\lambda)$ modulo $U(\mathfrak{g})\mathfrak{k}$ consists of

$$
\begin{aligned}
&E_{11}^2+E_{21}^2+E_{31}^2+E_{41}^2-(\lambda_1+\lambda_2+k-4)E_{11}+\lambda_2(\lambda_1-4+k),\\
&E_{21}(E_{11}+E_{22}-(\lambda_1+\lambda_2+k-3))+E_{32}E_{31}+E_{42}E_{41},\\
&E_{31}(E_{11}+E_{33}-(\lambda_1+\lambda_2+k-2))+E_{32}E_{21}+E_{43}E_{41},\\
&E_{41}(E_{11}+E_{44}-(\lambda_1+\lambda_2+k-2))+E_{42}E_{21}+E_{31}E_{43},\\
&E_{22}^2-(\lambda_1+\lambda_2+k-3)E_{22}+E_{21}^2+E_{32}^2+E_{42}^2+E_{11}+\lambda_2(\lambda_1-4+k),\\
&E_{32}(E_{22}+E_{33}-(\lambda_1+\lambda_2+k-2))+E_{21}E_{31}+E_{43}E_{42},\\
&E_{42}(E_{22}+E_{44}-(\lambda_1+\lambda_2+k-2))+E_{21}E_{41}+E_{32}E_{43},\\
&E_{33}^2-(\lambda_1+\lambda_2+k-1)E_{33}+E_{31}^2+E_{32}^2+E_{43}^2-E_{44}+\lambda_1(\lambda_2+k),\\
&E_{43}(E_{33}+E_{44}-(\lambda_1+\lambda_2+k-1))+E_{31}E_{41}+E_{32}E_{42},\\
&E_{44}^2-(\lambda_1+\lambda_2+k)E_{44}+E_{41}^2+E_{42}^2+E_{43}^2+\lambda_1(\lambda_2+k),\\
&\sum_{i=1}^4 E_{ii}-k\lambda_1-(4-k)\lambda_2.
\end{aligned}
$$

Direct computation shows that $j\colon U(\mathfrak{g}) \to U(\mathfrak{g})$ carries this set to the set whose elements are obtained by changing (λ_1, λ_2) to (λ_1', λ_2') except the final element which is $-\sum_{i=1}^4 E_{ii} + k\lambda_1' + (4-k)\lambda_2'$.

Notice that $U(\mathfrak{g})\mathfrak{k}$ is invariant under the map j. This shows the lemma. □

From this lemma we have the following isomorphism between spaces of class one generalized Whittaker functions.

Proposition 4.13. *Fix a closed subgroup* $U \subset N$ *and its character* χ. *Define* $U^\dagger = \{j(u) \mid u \in U\} \subset N$ *and its character* $\chi^\dagger(u') = \chi(j(u'))$ $(u' \in U^\dagger)$. *Then*

$$C_\chi^\infty(U\backslash G/K; I_k(\lambda)) \to C_{\chi^\dagger}^\infty(U^\dagger\backslash G/K; I_k(\lambda')), \qquad f \mapsto f \circ j,$$

is a linear isomorphism. Here $\lambda' = (-\lambda_1 + 4 - k, -\lambda_2 - k)$.

Proof. Let us note that K is invariant under j. Thus the proposition follows from Lemma 4.12. □

For each $C_\chi^\infty(U\backslash G)$ listed in Proposition 4.8, we shall give realizations of elements in $\mathfrak{n}$ as differential and scalar operators on $C^\infty(\tilde{U} \times A) \cong C_\chi^\infty(U\backslash G/K)$.

Case (1). We consider the space $C^\infty_{\chi_{\varepsilon_1 E_{31}^* + \varepsilon_2 E_{32}^* + \varepsilon_3 E_{42}^*}}(N_{2,4}\backslash G/K)$, where

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \begin{cases} (1,0,1) & \text{for case } (1_a), \\ (0,1,0) & \text{for case } (1_b), \end{cases} \tag{4.4}$$

as in Proposition 4.8. If we notice that $\mathfrak{n}_{2,4} = \mathbb{R}E_{31} + \mathbb{R}E_{32} + \mathbb{R}E_{41} + \mathbb{R}E_{43}$ is not only a subalgebra of $\mathfrak{n}$ but also an ideal of $\mathfrak{n}$, then

$$\mathfrak{n}_{2,4}\backslash\mathfrak{n} \cong \tilde{\mathfrak{n}}_{2,4} = \mathbb{R}E_{21} + \mathbb{R}E_{43}$$

is a subalgebra of $\mathfrak{n}$. Hence $N_{2,4}\backslash N$ is isomorphic to the subgroup

$$\tilde{N}_{2,4} = \exp(\tilde{\mathfrak{n}}_{2,4}) = \{\exp(uE_{21} + vE_{43}) \mid u, v \in \mathbb{R}\}$$

of N. Then we have a diffeomorphism

$$N \cong N_{2,4} \rtimes \tilde{N}_{2,4}$$

and a linear isomorphism

$$\Xi_{(1)}\colon C^\infty_{\chi_{\varepsilon_1 E_{31}^* + \varepsilon_2 E_{32} + \varepsilon_3 E_{43}}}(N_{2,4}\backslash G/K) \xrightarrow{\sim} C^\infty(\tilde{N}_{2,4} \times A)$$

as in (4.3). We introduce a coordinate system on $\tilde{N}_{2,4} \times A$,

$$\begin{gathered} \mathbb{R}^2 \times (\mathbb{R}_{>0})^4 \xrightarrow{\sim} \tilde{N}_{2,4} \times A, \\ ((u,v),(a_1,a_2,a_3,a_4)) \mapsto (\exp(uE_{21} + vE_{43}), \operatorname{diag}(a_1,a_2,a_3,a_4)). \end{gathered}$$

Proposition 4.14. *We regard the space* $C^\infty(\tilde{N}_{2,4}\times A)$ *as the image of the space* $C^\infty_{\chi_{\varepsilon_1 E^*_{31}+\varepsilon_2 E^*_{32}+\varepsilon_3 E^*_{42}}}(N_{2,4}\backslash G/K)$ *under the mapping* $\Xi_{(1)}$*. Then for elements in* $\mathfrak{n}$*, we have*

$$\begin{aligned} E_{21}F &= \frac{a_2}{a_1}\frac{\partial}{\partial u}F, & E_{31}F &= 2\pi\sqrt{-1}\,\frac{a_3}{a_1}\varepsilon_1 F,\\ E_{41}F &= 0, & E_{32}F &= 2\pi\sqrt{-1}\,\frac{a_3}{a_2}(-u\varepsilon_1+\varepsilon_2+v\varepsilon_3)F,\\ E_{42}F &= 2\pi\sqrt{-1}\,\frac{a_4}{a_2}\varepsilon_3 F, & E_{43}F &= \frac{a_4}{a_3}\frac{\partial}{\partial v}F. \end{aligned}$$

Here $F\in C^\infty(\tilde{N}_{2,4}\times A)$ *and* $(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ *are chosen as in* (4.4)*.*

Proof. For $F\in C^\infty(\tilde{N}_{2,4}\times A)$, there exists $f\in C^\infty_{\chi_{\varepsilon_1 E^*_{31}+\varepsilon_2 E^*_{32}+\varepsilon_3 E^*_{42}}}(N_{2,4}\backslash G/K)$ such that $F(u,v;a)=\Xi_{(1)}(f)=f(\exp(uE_{21}+vE_{43})a)$ for $u,v\in\mathbb{R}$ and $a\in A$. Hence for E_{ij} $(1\le j<i\le 4)$ we have $E_{ij}F=\Xi_{(1)}(R_{E_{ij}}f)$ and

$$\begin{aligned} E_{ij}F(u,v;a) &= \Xi_{(1)}(R_{E_{ij}}f)=\frac{d}{dt}f(\exp(uE_{21}+vE_{43})a\exp(tE_{ij}))\Big|_{t=0}\\ &= \frac{d}{dt}f(\exp(uE_{21}+vE_{43})\exp(t\mathrm{Ad}(a)E_{ij})a)\Big|_{t=0}\\ &= \frac{a_i}{a_j}\frac{d}{dt}f(\exp(uE_{21}+vE_{43})\exp(tE_{ij})a)\Big|_{t=0}. \end{aligned}\tag{4.5}$$

Direct computation shows

$$\begin{aligned} &\exp(uE_{21}+vE_{43})\cdot\exp n(z,\dots,x_3)\\ &\qquad=\exp n(z',y'_1,y'_2,0,x_2,0)\cdot\exp((u+x_1)E_{21}+(v+x_3)E_{43}), \end{aligned}$$

where

$$\begin{aligned} z' &= z+vy_1-uy_2+\tfrac{1}{2}x_3y_1-\tfrac{1}{2}x_1y_2-uvx_2-\tfrac{1}{2}vx_1x_2-\tfrac{1}{2}ux_2x_3-\tfrac{1}{3}x_1x_2x_3,\\ y'_1 &= y_1-ux_2-x_1x_2/2,\\ y'_2 &= y_2+vx_2+x_2x_3/2. \end{aligned}$$

Here we put $n(z,\dots,x_3)=zE_{41}+y_1E_{31}+y_2E_{32}+x_1E_{21}+x_2E_{32}+x_3E_{43}$.

Hence we have

$$\begin{aligned} &f(\exp(uE_{21}+vE_{43})\exp n(z,\dots,x_3)a)\\ &=f(\exp n(z',y'_1,y'_2,0,x_2,0)\exp((u+x_1)E_{21}+(v+x_3)E_{43})a)\\ &=\chi_{\varepsilon_1 E^*_{31}+\varepsilon_2 E^*_{32}+\varepsilon_3 E^*_{42}}(\exp n(z',y'_1,y'_2,0,x_2,0))f(\exp((u+x_1)E_{21}+(v+x_3)E_{43})a)\\ &=e^{2\pi\sqrt{-1}\,(\varepsilon_1 y'_1+\varepsilon_2 x_2+\varepsilon_3 y'_2)}f(\exp((u+x_1)E_{21}+(v+x_3)E_{43})a)\\ &=e^{2\pi\sqrt{-1}\,(\varepsilon_1 y'_1+\varepsilon_2 x_2+\varepsilon_3 y'_2)}F(u+x_1,v+x_3;a). \end{aligned}\tag{4.6}$$

Combining formulas (4.5) and (4.6), we obtain the proposition. ☐

Case (2). We consider the space $C^\infty_{\chi_{\varepsilon_1 E^*_{21}+\varepsilon_2 E^*_{42}+\varepsilon_3 E^*_{43}}}(N_{1,3,4}\backslash G/K)$. Here each $(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ corresponds to cases (2_i) $(i=a,b,c,d)$ in Proposition 4.8 as follows:

$$(\varepsilon_1,\varepsilon_2,\varepsilon_3)=\begin{cases}(1,1,0) & \text{for case } (2_a),\\ (1,0,1) & \text{for case } (2_b),\\ (1,0,0) & \text{for case } (2_c),\\ (0,0,1) & \text{for case } (2_d).\end{cases}\tag{4.7}$$

From the same argument as in case (1), the homogeneous space $N_{1,3,4}\backslash N$ is isomorphic to the subgroup $\tilde{N}_{1,3,4}=\{\exp(uE_{32})\mid u\in\mathbb{R}\}$ of N. This isomorphism gives a smooth section $\theta_{(2)}\colon N_{1,3,4}\backslash N\to N$ and we have a linear bijection

$$\Xi_{(2)}\colon C^\infty_{\chi_{\varepsilon_1 E^*_{21}+\varepsilon_2 E^*_{42}+\varepsilon_3 E^*_{43}}}(N_{1,3,4}\backslash G/K)\xrightarrow{\sim} C^\infty(\tilde{N}_{1,3,4}\times A).$$

Let us introduce a coordinate system on $\tilde{N}_{1,3,4}\times A$,

$$\begin{gathered}\mathbb{R}\times(\mathbb{R}_{>0})^4\xrightarrow{\sim}\tilde{N}_{1,3,4}\times A,\\ (u,(a_1,a_2,a_3,a_4))\mapsto(\exp(uE_{32}),\,\mathrm{diag}(a_1,a_2,a_3,a_4)).\end{gathered}$$

Then we can write down the action of $\mathfrak{n}$ on $C^\infty(\tilde{N}_{1,3,4}\times A)$.

Proposition 4.15. *We regard the space $C^\infty(\tilde{N}_{1,3,4}\times A)$ as the image of the space $C^\infty_{\chi_{\varepsilon_1 E^*_{21}+\varepsilon_2 E^*_{42}+\varepsilon_3 E^*_{43}}}(N_{1,3,4}\backslash G/K)$ under the mapping $\Xi_{(2)}$. Then elements in $\mathfrak{n}$ can be realized as follows:*

$$\begin{aligned}E_{21}F&=2\pi\sqrt{-1}\,\frac{a_2}{a_1}\varepsilon_1F, & E_{31}F&=0,\\ E_{41}F&=0, & E_{32}F&=\frac{a_3}{a_2}\frac{\partial}{\partial u}F,\\ E_{42}F&=2\pi\sqrt{-1}\,\frac{a_4}{a_2}\varepsilon_2F, & E_{43}F&=2\pi\sqrt{-1}\,\frac{a_4}{a_3}(\varepsilon_3-\varepsilon_2u)F.\end{aligned}$$

Here $F\in C^\infty(\tilde{N}_{1,3,4}\times A)$ and $(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ are chosen as in (4.7).

Proof. The proposition can be obtained in the same way as in case (1) via

$$\exp(uE_{32})\cdot\exp n(z,\dots,x_3)=\exp(n(z',y_1',y_2',x_1,0,x_3))\cdot\exp((u+x_2)E_{32}),$$

where

$$\begin{aligned}z'&=z+\tfrac{1}{6}x_1x_2x_3,\\ y_1'&=y_1+x_1u+\tfrac{1}{2}x_1x_2,\\ y_2'&=y_2-x_3u-\tfrac{1}{2}x_2x_3.\end{aligned}$$

□

Case (3). We consider the space $C^\infty_{\chi_{\varepsilon_1 E^*_{21}+\varepsilon_2 E^*_{32}+\varepsilon_3 E^*_{43}}}(N\backslash G/K)$ where

$$(\varepsilon_1,\varepsilon_2,\varepsilon_3)=\begin{cases}(1,1,1) & \text{for case } (3_a),\\ (1,1,0) & \text{for case } (3_b),\\ (1,0,1) & \text{for case } (3_c),\\ (0,1,1) & \text{for case } (3_d),\\ (1,0,0) & \text{for case } (3_e),\\ (0,1,0) & \text{for case } (3_f),\\ (0,0,1) & \text{for case } (3_g),\\ (0,0,0) & \text{for case } (3_h).\end{cases} \tag{4.8}$$

By the Iwasawa decomposition, we have the linear bijection

$$\Xi_{(3)}\colon C^\infty_{\chi_{\varepsilon_1 E^*_{21}+\varepsilon_2 E^*_{32}+\varepsilon_3 E^*_{43}}}(N\backslash G/K)\ni f\mapsto f|_A\in C^\infty(A).$$

Proposition 4.16. *Let us consider the space $C^\infty(A)$ as the image of the space $C^\infty_{\chi_{\varepsilon_1 E^*_{21}+\varepsilon_2 E^*_{32}+\varepsilon_3 E^*_{43}}}(N\backslash G/K)$ under the mapping $\Xi_{(3)}$. Then*

$$\begin{aligned} &E_{21}F=2\pi\sqrt{-1}\,\frac{a_2}{a_1}\varepsilon_1 F, && E_{31}F=0,\\ &E_{41}F=0, && E_{32}F=2\pi\sqrt{-1}\,\frac{a_3}{a_2}\varepsilon_2 F,\\ &E_{42}F=0, && E_{43}F=2\pi\sqrt{-1}\,\frac{a_4}{a_3}\varepsilon_3 F.\end{aligned}$$

Here $F\in C^\infty(A)$ and $(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ are chosen as in (4.8).

Proof. This is obvious from the formula

$$(E_{ij}F)(a)=\frac{d}{dt}f(a\exp(tE_{ij}))\Big|_{t=0}=\frac{d}{dt}f(\exp(t\mathrm{Ad}(a)E_{ij})a)\Big|_{t=0}$$

for $1\leq i\neq j\leq 4$. Here $f=\Xi_{(3)}^{-1}(F)$. □

Before studying in detail the spaces $C^\infty_\chi(U\backslash G/K;I_k(\lambda))$ corresponding to $(1_a),\dots,(3_h)$ respectively, we record the following relations among them.

Proposition 4.17. (1) *We have*

$$C^\infty_{\chi_{\mu_1 E^*_{21}+\mu_2 E^*_{42}+\mu_3 E^*_{43}}}(N_{1,3,4}\backslash G/K;I_k(\lambda))=C^\infty_{\chi_{\nu_1 E^*_{21}+\nu_2 E^*_{32}+\nu_3 E^*_{43}}}(N\backslash G/K;I_k(\lambda))$$

for $k=1,2$ if one of the following is satisfied:

(i) $(\mu_1,\mu_2,\mu_3)=(1,0,1)$ *and* $(\nu_1,\nu_2,\nu_3)=(1,0,1)$,
(ii) $(\mu_1,\mu_2,\mu_3)=(1,0,0)$ *and* $(\nu_1,\nu_2,\nu_3)=(1,0,0)$,
(iii) $(\mu_1,\mu_2,\mu_3)=(0,0,1)$ *and* $(\nu_1,\nu_2,\nu_3)=(0,0,1)$.

(2) *We have*

$$C^\infty_{\chi_{\mu_1 E^*_{31}+\mu_2 E^*_{32}+\mu_3 E^*_{42}}}(N_{2,4}\backslash G/K; I_k(\lambda)) = C^\infty_{\chi_{\nu_1 E^*_{21}+\nu_2 E^*_{32}+\nu_3 E^*_{43}}}(N\backslash G/K; I_k(\lambda))$$

for $k = 1, 2$ *if* $(\mu_1, \mu_2, \mu_3) = (0, 1, 0)$ *and* $(\nu_1, \nu_2, \nu_3) = (0, 1, 0)$.

Remark 4.18. Each of the three conditions in (1) above implies

$$\mu_1 E^*_{21} + \mu_2 E^*_{42} + \mu_3 E^*_{43} = \nu_1 E^*_{21} + \nu_2 E^*_{32} + \nu_3 E^*_{43}.$$

Hence

$$C^\infty_{\chi_{\mu_1 E^*_{21}+\mu_2 E^*_{42}+\mu_3 E^*_{43}}}(N_{1,3,4}\backslash G) \supset C^\infty_{\chi_{\nu_1 E^*_{21}+\nu_2 E^*_{32}+\nu_3 E^*_{43}}}(N\backslash G)$$

under these conditions. Also in (2), we can see that

$$\mu_1 E^*_{31} + \mu_2 E^*_{32} + \mu_3 E^*_{42} = \nu_1 E^*_{21} + \nu_2 E^*_{32} + \nu_3 E^*_{43}$$

and

$$C^\infty_{\chi_{\mu_1 E^*_{31}+\mu_2 E^*_{32}+\mu_3 E^*_{42}}}(N_{2,4}\backslash G) \supset C^\infty_{\chi_{\nu_1 E^*_{21}+\nu_2 E^*_{32}+\nu_3 E^*_{43}}}(N\backslash G)$$

under the given condition.

To show the proposition, we prepare the following lemma.

Lemma 4.19. *Suppose the assumption in Proposition 4.17 is satisfied and*

$$F \in C^\infty_{\chi_{\mu_1 E^*_{31}+\mu_2 E^*_{32}+\mu_3 E^*_{42}}}(N_{2,4}\backslash G/K; I_k(\lambda))$$
$$(\textit{resp. } F \in C^\infty_{\chi_{\mu_1 E^*_{21}+\mu_2 E^*_{42}+\mu_3 E^*_{43}}}(N_{1,3,4}\backslash G/K; I_k(\lambda))).$$

Then $R_X F = 0$ *for* $X \in \mathfrak{n}_{1,3,4}$ (*resp.* $X \in \mathfrak{n}_{2,4}$).

Proof. Put $X = E_{31}(E_{11} + E_{33} - (\lambda_1 + \lambda_2 + k - 2)) + E_{32}E_{21} + E_{43}E_{41}$ and $X' = E_{42}(E_{22} + E_{44} - (\lambda_1 + \lambda_2 + k - 2)) + E_{21}E_{41} + E_{32}E_{43}$. By the proof of Lemma 4.12 we see that $R_X F = R_{X'} F = 0$ for $F \in C^\infty(G/K; I_k(\lambda))$.

First we take $F \in C^\infty_{\chi_{\mu_1 E^*_{31}+\mu_2 E^*_{32}+\mu_3 E^*_{42}}}(N_{2,4}\backslash G/K; I_k(\lambda))$. Then Lemma 4.14 shows that $0 = R_X F = R_{E_{32}} \circ R_{E_{21}} F$ and $R_{E_{21}}$ is a scalar operator and commutes with $R_{E_{32}}$. Thus $R_{E_{32}} F = 0$. Similarly we have $0 = R_{X'} F = R_{E_{32}} \circ R_{E_{43}} F$, which implies $R_{E_{43}} F = 0$.

Also for $F \in C^\infty_{\chi_{\mu_1 E^*_{21}+\mu_2 E^*_{42}+\mu_3 E^*_{43}}}(N_{1,3,4}\backslash G/K; I_k(\lambda))$, we can show $R_{E_{32}} F = 0$ similarly. Then the lemma easily follows from Lemmas 4.14 and 4.15. ☐

Proof of Proposition 4.17. First we show (1). Suppose the assumption is satisfied. Then we have

$$C^\infty_{\chi_{\mu_1 E^*_{21}+\mu_2 E^*_{42}+\mu_3 E^*_{43}}}(N_{1,3,4}\backslash G/K; I_k(\lambda)) \supset C^\infty_{\chi_{\nu_1 E^*_{21}+\nu_2 E^*_{32}+\nu_3 E^*_{43}}}(N\backslash G/K; I_k(\lambda)).$$

We shall show the converse inclusion.

Since $N \cong N_{1,3,4} \rtimes \tilde{N}_{1,3,4}$ and $\chi_{\nu_1 E_{21}^* + \nu_2 E_{32}^* + \nu_3 E_{43}^*}(n) = 1$ for $n \in \tilde{N}_{1,3,4}$, it suffices to see that

$$f(ng) = \chi_{\nu_1 E_{21}^* + \nu_2 E_{32}^* + \nu_3 E_{43}^*}(n) f(g) = f(g)$$

for all $n \in \tilde{N}_{1,3,4}$ and $f \in C^\infty_{\chi_{\mu_1 E_{21}^* + \mu_2 E_{42}^* + \mu_3 E_{43}^*}}(N_{1,3,4} \backslash G/K; I_k(\lambda))$. This is equivalent to showing that $L_X f = 0$ for all $X \in \tilde{\mathfrak{n}}_{1,3,4}$ since $\tilde{N}_{1,3,4} = \exp \tilde{\mathfrak{n}}_{1,3,4}$.

Since $\tilde{N}_{1,3,4}$ is a commutative group and normalized by $N_{1,3,4}$ and A, for any $\tilde{n} \in \tilde{N}_{1,3,4}$ there exists $\tilde{n}' \in \tilde{N}_{1,3,4}$ such that $f(\tilde{n}nak) = f(\tilde{n}na) = f(na\tilde{n}')$ where $f \in C^\infty(G/K)$, $(n, a, k) \in N \times A \times K$. Then Lemma 4.19 shows that for $f \in C^\infty_{\chi_{\mu_1 E_{21}^* + \mu_2 E_{42}^* + \mu_3 E_{43}^*}}(N_{1,3,4} \backslash G/K; I_k(\lambda))$ and $X \in \tilde{\mathfrak{n}}_{1,3,4}$ we have $R_X f = 0$. This yields $L_X f = 0$.

The second assertion follows from the same argument as the first one. □

By this proposition it suffices to consider $C^\infty_\chi(U \backslash G/K; I_k(\lambda))$ for the pairs (U, χ) with the labels

$$(1_a), (1_b) = (3_f), (2_a), (2_b) = (3_c), (2_c) = (3_e), (2_d) = (3_g), (3_a), (3_b), (3_d), (3_h).$$

Moreover by Proposition 4.13 we have an isomorphism

$$C^\infty_\chi(U \backslash G/K; I_k(\lambda)) \cong C^\infty_{\chi'}(U' \backslash G/K; I_k(\lambda'))$$

when (U, χ) corresponds to either (3_b), (2_c) or (3_e) and (U', χ') to (3_d), (2_d) or (3_g) respectively. Here $\lambda' = (-\lambda_1 - 4 + k, -\lambda_2 - k)$.

Now let us see that some spaces of class one generalized Whittaker functions vanish.

Proposition 4.20. (1) *If* $k = 1$, *then*

$$C^\infty_{\chi_{E_{31}^* + E_{42}^*}}(N_{2,4} \backslash G/K; I_1(\lambda)) = \{0\}. \quad (1_a)$$

(2) *For* $k = 1, 2$,

$$C^\infty_{\chi_{E_{21}^* + E_{42}^*}}(N_{1,3,4} \backslash G/K; I_k(\lambda)) = \{0\}. \quad (2_a)$$

(3) *If* $k = 1$, *then*

$$C^\infty_{\chi_{E_{21}^* + E_{43}^*}}(N_{1,3,4} \backslash G/K; I_k(\lambda)) = C^\infty_{\chi_{E_{21}^* + E_{43}^*}}(N \backslash G/K; I_k(\lambda)) = \{0\}. \quad (2_b) = (3_c)$$

(4) *For* $k = 1, 2$,

$$\begin{aligned} C^\infty_{\chi_{E_{21}^* + E_{32}^* + E_{43}^*}}(N \backslash G/K; I_k(\lambda)) &= \{0\}, \quad (3_a) \\ C^\infty_{\chi_{E_{21}^* + E_{32}^*}}(N \backslash G/K; I_k(\lambda)) &= \{0\}, \quad (3_b) \\ C^\infty_{\chi_{E_{32}^* + E_{43}^*}}(N \backslash G/K; I_k(\lambda)) &= \{0\}. \quad (3_d) \end{aligned}$$

Proof. First we show (1). All $F \in C^\infty_{\chi_{E^*_{31}+E^*_{42}}}(N_{2,4}\backslash G/K; I_1(\lambda))$ are killed by $I_k(\lambda)$ and $U(\mathfrak{g})\mathfrak{k}$. Let us consider representatives of $((\mathbb{E}-\lambda_1)(\mathbb{E}-\lambda_2-k))_{ij}$ modulo $U(\mathfrak{g})\mathfrak{k}$ given in Lemma 4.11 and in particular focus on elements labelled by $((i,j)=(1,3))$ and $((i,j)=(2,4))$. Then recalling Proposition 4.14, we see that F is killed by right translation by $E_{31}E_{42}(\sum_{i=1}^4 E_{ii}-2\lambda_1-2\lambda_2+2)+E_{32}E_{42}E_{21}+E_{32}E_{31}E_{43}$. Here we notice that by Proposition 4.14 right translations by E_{31}, E_{32} and E_{42} on $C^\infty_{\chi_{E^*_{31}+E^*_{42}}}(N_{2,4}\backslash G/K; I_1(\lambda))$ are nonzero scalar operators, thus they are mutually commutative. Also let us consider the representative labelled by $((i,j)=(1,4))$ in Lemma 4.11. Then it follows that F is killed by right translation by $E_{42}E_{21}+E_{31}E_{43}$. Thus we see that F is killed by $\sum_{i=1}^4 E_{ii}-2\lambda_1-2\lambda_2+2$ since right translation by $E_{31}E_{42}$ is a scalar operator on $C^\infty_{\chi_{E^*_{31}+E^*_{42}}}(N_{2,4}\backslash G/K; I_1(\lambda))$. On the other hand, F is killed by $\sum_{i=1}^4 E_{ii}-\lambda_1-3\lambda_2$ as well. Thus $F=0$.

Let us show (3). Consider the elements in Lemma 4.11 labelled by $((i,j)=(1,2))$ and $((i,j)=(3,4))$. Then from Propositions 4.15 and 4.16, it follows that all elements in $C^\infty_{\chi_{E^*_{21}+E^*_{43}}}(N_{1,3,4}\backslash G/K; I_k(\lambda))=C^\infty_{\chi_{E^*_{21}+E^*_{43}}}(N\backslash G/K; I_k(\lambda))$ are killed by $\sum_{i=1}^4 E_{ii}-2\lambda_1-2\lambda_2+2$. Since they are also killed by $\sum_{i=1}^4 E_{ii}-\lambda_1-3\lambda_2$, they must be 0.

To show (2), consider the element in Lemma 4.11 labelled by $((i,j)=(1,4))$. Then from Proposition 4.15, all elements in $C^\infty_{\chi_{E^*_{21}+E^*_{42}}}(N_{1,3,4}\backslash G/K; I_k(\lambda))$ are killed by $E_{42}E_{21}$ which induces a nonzero scalar operator. Thus they are all 0.

We can show (4) in the same way as (2) by using the elements in Lemma 4.11 labelled by $((i,j)=(1,3))$ and $((i,j)=(2,4))$. Indeed at least one of these induces a nonzero scalar operator by Proposition 4.16. □

§4.3. Spaces of class one generalized Whittaker functions

We shall examine the dimensions and find generators of spaces of class one generalized Whittaker functions. From the results in §4.2, it suffices to consider $C^\infty_\chi(U\backslash G/K; I_k(\lambda))$ where pairs (U,χ) correspond to the labels $(1_a),(1_b)=(3_f)$, $(2_b)=(3_c),(2_c)=(3_e),(2_d)=(3_g),(3_h)$.

First we consider differential equations satisfied by class one generalized Whittaker functions.

Proposition 4.21. *The space $C^\infty_{\chi_{E^*_{31}+E^*_{42}}}(N_{2,4}\backslash G/K; I_k(\lambda))$ corresponding to (1_a) can be identified via the isomorphism $\Xi_{(1)}$ with the solution space of the following system of differential equations on $C^\infty(\tilde{N}_{2,4}\times A)$:*

$$\begin{aligned}(4.9)\qquad \Bigg[\vartheta_{a_1}^2-(\lambda_1+\lambda_2+k-3)\vartheta_{a_1}+\left(\frac{a_2}{a_1}\right)^2\frac{\partial^2}{\partial u^2}+\left(\frac{a_3}{a_1}\right)^2(2\pi\sqrt{-1})^2 \\ -(\vartheta_{a_2}+\vartheta_{a_3}+\vartheta_{a_4})+\lambda_1(\lambda_2+k)\Bigg]\phi=0,\end{aligned}$$

$$\left[\frac{\partial}{\partial u}(\vartheta_{a_1}+\vartheta_{a_2}-(\lambda_1+\lambda_2+k-3))+\left(\frac{a_3}{a_2}\right)^2(2\pi\sqrt{-1}\,)^2(v-u)\right]\phi=0, \tag{4.10}$$

$$\left[(\vartheta_{a_1}+\vartheta_{a_3}-(\lambda_1+\lambda_2+k-2))+(v-u)\frac{\partial}{\partial u}\right]\phi=0, \tag{4.11}$$

$$\left[\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right]\phi=0, \tag{4.12}$$

$$\begin{aligned}\Bigg[\vartheta_{a_2}^2-(\lambda_1+\lambda_2+k-2)\vartheta_{a_2}+\left(\frac{a_2}{a_1}\right)^2\frac{\partial^2}{\partial u^2}+\left(\frac{a_3}{a_2}\right)^2(2\pi\sqrt{-1}\,)^2(v-u)^2\\ +\left(\frac{a_4}{a_2}\right)^2(2\pi\sqrt{-1}\,)^2-(\vartheta_{a_3}+\vartheta_{a_4})+\lambda_1(\lambda_2+k)\Bigg]\phi=0,\end{aligned} \tag{4.13}$$

$$\left[(v-u)(\vartheta_{a_2}+\vartheta_{a_3}-(\lambda_1+\lambda_2+k-2))+\left(\frac{a_2}{a_1}\right)^2\frac{\partial}{\partial u}+\left(\frac{a_4}{a_3}\right)^2\frac{\partial}{\partial v}\right]\phi=0, \tag{4.14}$$

$$\left[(\vartheta_{a_2}+\vartheta_{a_4}-(\lambda_1+\lambda_2+k-2))+(v-u)\frac{\partial}{\partial v}\right]\phi=0, \tag{4.15}$$

$$\begin{aligned}\Bigg[\vartheta_{a_3}^2-(\lambda_1+\lambda_2+k-1)\vartheta_{a_3}+\left(\frac{a_3}{a_1}\right)^2(2\pi\sqrt{-1}\,)^2\\ +\left(\frac{a_3}{a_2}\right)^2(2\pi\sqrt{-1}\,)^2(v-u)^2+\left(\frac{a_4}{a_3}\right)^2\frac{\partial^2}{\partial v^2}-\vartheta_{a_4}+\lambda_1(\lambda_2+k)\Bigg]\phi=0,\end{aligned} \tag{4.16}$$

$$\left[\frac{\partial}{\partial v}(\vartheta_{a_3}+\vartheta_{a_4}-(\lambda_1+\lambda_2+k-1))+\left(\frac{a_3}{a_2}\right)^2(2\pi\sqrt{-1}\,)^2(v-u)\right]\phi=0, \tag{4.17}$$

$$\left[\vartheta_{a_4}^2-(\lambda_1+\lambda_2+k)\vartheta_{a_4}+\left(\frac{a_4}{a_2}\right)^2(2\pi\sqrt{-1}\,)^2+\left(\frac{a_4}{a_3}\right)^2\frac{\partial^2}{\partial v^2}+\lambda_1(\lambda_2+k)\right]\phi=0, \tag{4.18}$$

$$[(\vartheta_{a_1}+\vartheta_{a_2}+\vartheta_{a_3}+\vartheta_{a_4}-k\lambda_1-(4-k)\lambda_2)]\phi=0. \tag{4.19}$$

Here $\phi\in C^\infty(\tilde{N}_{2,4}\times A)$.

Proof. Recall that $I_k(\lambda)$ is written as in the proof of Lemma 4.12 modulo $U(\mathfrak{g})\mathfrak{k}$. Then these differential equations immediately follow from Proposition 4.14. □

Proposition 4.22. *Each space* $C^\infty_{\chi_{\varepsilon_1E_{21}^*+\varepsilon_2E_{32}^*+\varepsilon_3E_{43}^*}}(N\backslash G/K;I_k(\lambda))$ *for* $k=1,2$ *and* $(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ *chosen as in* (4.8) *can be identified via the isomorphism* $\Xi_{(3)}$ *with the solution space of the following system of differential equations on* $C^\infty(A)$*:*

$$\begin{aligned}\Bigg[\vartheta_{a_1}^2-(\lambda_1+\lambda_2+k-3)\vartheta_{a_1}+\left(2\pi\sqrt{-1}\,\frac{a_2}{a_1}\right)^2\varepsilon_1\\ -(\vartheta_{a_2}+\vartheta_{a_3}+\vartheta_{a_4})+\lambda_1(\lambda_2+k)\Bigg]\phi=0,\end{aligned} \tag{4.20}$$

$$\varepsilon_1 2\pi\sqrt{-1}\,\frac{a_2}{a_1}(\vartheta_{a_1}+\vartheta_{a_2}-(\lambda_1+\lambda_2+k-3))\phi=0, \tag{4.21}$$

$$\varepsilon_1\varepsilon_2\phi = 0, \tag{4.22}$$

$$\begin{aligned}\Bigg[\vartheta_{a_2}^2 - (\lambda_1+\lambda_2+k-2)\vartheta_{a_2} + \left(2\pi\sqrt{-1}\,\frac{a_2}{a_1}\right)^2\varepsilon_1 + \left(2\pi\sqrt{-1}\,\frac{a_3}{a_2}\right)^2\varepsilon_2 \\ - (\vartheta_{a_3}+\vartheta_{a_4}) + \lambda_1(\lambda_2+k)\Bigg]\phi = 0,\end{aligned} \tag{4.23}$$

$$\varepsilon_2 2\pi\sqrt{-1}\,\frac{a_3}{a_2}(\vartheta_{a_2}+\vartheta_{a_3} - (\lambda_1+\lambda_2+k-2))\phi = 0, \tag{4.24}$$

$$\varepsilon_2\varepsilon_3\phi = 0, \tag{4.25}$$

$$\begin{aligned}\Bigg[\vartheta_{a_3}^2 - (\lambda_1+\lambda_2+k-1)\vartheta_{a_3} + \left(2\pi\sqrt{-1}\,\frac{a_3}{a_2}\right)^2\varepsilon_2 \\ + \left(2\pi\sqrt{-1}\,\frac{a_4}{a_3}\right)^2\varepsilon_3 - \vartheta_{a_4} + \lambda_1(\lambda_2+k)\Bigg]\phi = 0,\end{aligned} \tag{4.26}$$

$$\varepsilon_3 2\pi\sqrt{-1}\,\frac{a_4}{a_3}(\vartheta_{a_3}+\vartheta_{a_4} - (\lambda_1+\lambda_2+k-1))\phi = 0, \tag{4.27}$$

$$\left[\vartheta_{a_4}^2 - (\lambda_1+\lambda_2+k)\vartheta_{a_4} + \left(2\pi\sqrt{-1}\,\frac{a_4}{a_3}\right)^2\varepsilon_3 + \lambda_1(\lambda_2+k)\right]\phi = 0, \tag{4.28}$$

$$[\vartheta_{a_1}+\vartheta_{a_2}+\vartheta_{a_3}+\vartheta_{a_4} - k\lambda_1 - (4-k)\lambda_2]\phi = 0. \tag{4.29}$$

Here $\phi \in C^\infty(A)$*.*

Proof. Just as in Proposition 4.21, this system of differential equations is obtained by direct computation from Lemma 4.10 and Propositions 4.11 and 4.16. ☐

Cases (1_a) **and** (1_b)**.** Let us study the spaces $C^\infty_{\chi_{E^*_{31}+E^*_{42}}}(N_{2,4}\backslash G/K; I_k(\lambda))$ and $C^\infty_{\chi_{E^*_{32}}}(N_{2,4}\backslash G/K; I_k(\lambda))$ corresponding to (1_a) and (1_b) respectively.

Case (1_a). First we investigate the space $C^\infty_{\chi_{E^*_{31}+E^*_{42}}}(N_{2,4}\backslash G/K; I_k(\lambda))$. We have already handled the case $k=1$ in Proposition 4.20. Thus we consider the case $k=2$. We introduce a new coordinate system:

$$\begin{aligned}
&x_1 = a_1a_2a_3a_4, \\
&x_2 = (\pi\sqrt{-1}\,)^2\left(\left(\frac{a_3}{a_2}\right)^2(v-u)^2 + \left(\frac{a_4}{a_2}\right)^2 + \left(\frac{a_3}{a_1}\right)^2\right), \\
&x_3 = \left(\frac{a_1a_3}{a_2a_4}(v-u)^2 + \frac{a_2a_3}{a_1a_4} + \frac{a_1a_4}{a_2a_3}\right)^{-2}, \\
&x_4 = \frac{a_1a_3}{a_2a_4}, \qquad x_5 = \frac{a_1a_4}{a_2a_3}, \qquad x_6 = u.
\end{aligned} \tag{4.30}$$

Proposition 4.23. *Consider the system of differential equations in Proposition 4.21. By addition, substitution and multiplying by some rational functions, the* *system of differential equations in the new coordinate system* $x_1, \dots, x_6$ *can be written as follows:*

$$\left(\vartheta_{x_1} - \frac{\lambda_1 + \lambda_2}{2}\right)\phi = 0, \tag{4.31}$$

$$[x_2 - (\vartheta_{x_2} - \tfrac{1}{2})(2\vartheta_{x_3} - \vartheta_{x_2})]\phi = 0, \tag{4.32}$$

$$\begin{aligned}[x_3(\vartheta_{x_2} - 2\vartheta_{x_3})(\vartheta_{x_2} - 2\vartheta_{x_3} - 1)& \\ - (\vartheta_{x_3} - \tfrac{1}{4}(\lambda_2 - \lambda_1) - 1)(\vartheta_{x_3} + \tfrac{1}{4}(\lambda_2 - \lambda_1))]\phi = 0,&\end{aligned} \tag{4.33}$$

$$\frac{\partial}{\partial x_4}\phi = 0, \tag{4.34}$$

$$\frac{\partial}{\partial x_5}\phi = 0, \tag{4.35}$$

$$\frac{\partial}{\partial x_6}\phi = 0. \tag{4.36}$$

Proof. First, we put

$$\begin{aligned} \alpha_1 &= a_1 a_2, \quad & \alpha_2 &= a_1 a_2^{-1}, \quad & \alpha_3 &= a_3 a_4, \\ \alpha_4 &= a_3 a_4^{-1}, \quad & u' &= u, \quad & v' &= v - u. \end{aligned}$$

Then the differential equation (4.12) becomes

$$\frac{\partial}{\partial u'}\phi = 0. \tag{4.37}$$

Furthermore we change the variables α_2, α_4, v' to

$$w = \alpha_2 \alpha_4 v'^2 + \alpha_2 \alpha_4^{-1} + \alpha_2^{-1}\alpha_4, \qquad \beta_2 = \alpha_2 \alpha_4, \qquad \beta_4 = \alpha_2 \alpha_4^{-1}.$$

Then equations (4.14) and (4.15) become

$$\vartheta_{\beta_4}\phi = 0, \tag{4.38}$$

$$\vartheta_{\beta_2}\phi = 0, \tag{4.39}$$

respectively. On setting

$$\beta_1 = \alpha_1 \alpha_3, \qquad \beta_3 = \alpha_1 \alpha_3^{-1},$$

equation (4.19) becomes

$$(2\vartheta_{\beta_1} - (\lambda_1 + \lambda_2))\phi = 0. \tag{4.40}$$

Also we can see that equation (4.10) can be written as

$$\left[2w\frac{\partial}{\partial w}(2(\vartheta_{\beta_1} + \vartheta_{\beta_3}) - (\lambda_1 + \lambda_2 - 1)) - (2\pi\sqrt{-1})^2 \beta_3^{-1} w\right]\phi = 0. \tag{4.41}$$

If we eliminate ϑ_{β_1} from (4.41) by using (4.40), we can write (4.41) as

$$\left[2w\frac{\partial}{\partial w}(2\vartheta_{\beta_3}+1)-(2\pi\sqrt{-1}\,)^2\beta_3^{-1}w\right]\phi=0. \tag{4.42}$$

We note that (4.17) can be reduced to the same equation. Taking into account (4.40) and (4.42), equation (4.9) can be reduced to

$$\left[(\vartheta_{\beta_3}+\vartheta_w-\tfrac{1}{2}(\lambda_1-\lambda_2-4))(\vartheta_{\beta_3}+\vartheta_w+\tfrac{1}{2}(\lambda_1-\lambda_2))-4\frac{\partial^2}{\partial w^2}\right]\phi=0. \tag{4.43}$$

We can also see that (4.13), (4.16) and (4.18) can be written as the same equation (4.43). Finally, we put

$$\gamma_1=(\pi\sqrt{-1}\,)^2\beta_3^{-1}w, \qquad \gamma_2=w^{-2}.$$

Then (4.42) is equivalent to

$$[(\vartheta_{\gamma_1}-2\vartheta_{\gamma_2})(\tfrac{1}{2}-\vartheta_{\gamma_1})-\gamma_1]\phi=0. \tag{4.44}$$

Also (4.43) can be written as

$$[(\vartheta_{\gamma_2}-\tfrac{1}{4}(\lambda_2-\lambda_1)-1)(\vartheta_{\gamma_2}+\tfrac{1}{4}(\lambda_2-\lambda_1))-\gamma_2(\vartheta_{\gamma_1}-2\vartheta_{\gamma_2})(\vartheta_{\gamma_1}-2\vartheta_{\gamma_2}-1)]\phi=0. \tag{4.45}$$

If we put

$$\begin{aligned} x_1&=\beta_1, & x_2&=\gamma_1, & x_3&=\gamma_2,\\ x_4&=\beta_2, & x_5&=\beta_4, & x_6&=u', \end{aligned}$$

then the theorem follows from (4.37), (4.38), (4.39), (4.40), (4.44) and (4.45). ☐

Let us look at the differential equations (4.32) and (4.33). If $f(x_2,x_3)$ is a solution of them, we consider the function $F(x_2,x_3)$ such that

$$f=x_2^{1/2}x_3^{(\lambda_1-\lambda_2)/4}F.$$

Then $F(x_2,x_3)$ satisfies

$$[x_2-\vartheta_{x_2}(2\vartheta_{x_3}-\vartheta_{x_2}+\tfrac{1}{2}(\lambda_1-\lambda_2-1))]F(x_2,x_3)=0, \tag{4.46}$$

$$\begin{aligned}[x_3(2\vartheta_{x_3}-\vartheta_{x_2}+\tfrac{1}{2}(\lambda_1-\lambda_2-1))(2\vartheta_{x_3}-\vartheta_{x_2}+\tfrac{1}{2}(\lambda_1-\lambda_2-1)+1)&\\ -\vartheta_{x_3}(\vartheta_{x_3}+\tfrac{1}{2}(\lambda_1-\lambda_2)-1)]F(x_2,x_3)=0.&\end{aligned} \tag{4.47}$$

These are the differential equations for Horn's hypergeometric function $\mathrm{H}_{10}(\frac{1}{2}(\lambda_1-\lambda_2-1),\frac{1}{2}(\lambda_1-\lambda_2);x_2,x_3)$ (cf. [13]). Let $\mathfrak{H}_{10}(a,d;x,y)$ be the solution space of the system of partial differential equations for Horn's hypergeometric function $\mathrm{H}_{10}(a,d;x,y)$, i.e.,

$$
\begin{aligned}
&[x(2\vartheta_x-\vartheta_y+a)(2\vartheta_x-\vartheta_y+a+1)-\vartheta_x(\vartheta_x+d-1)]f(x,y)=0,\\
&\qquad\qquad\qquad\qquad\qquad [y-\vartheta_y(2\vartheta_x-\vartheta_y+a)]f(x,y)=0.
\end{aligned}
$$

It is known that the dimension of the solution space is 4 on a nonsingular domain (cf. [5]). We can find more detailed properties of $\mathfrak{H}_{10}(a,d;x,y)$ in the Appendix.

Theorem 4.24. *Set*

$$
\begin{aligned}
x_1 &= a_1a_2a_3a_4,\\
x_2 &= (\pi\sqrt{-1}\,)^2\left(\left(\frac{a_3}{a_2}\right)^2(v-u)^2+\left(\frac{a_4}{a_2}\right)^2+\left(\frac{a_3}{a_1}\right)^2\right),\\
x_3 &= \left(\frac{a_1a_3}{a_2a_4}(v-u)^2+\frac{a_2a_3}{a_1a_4}+\frac{a_1a_4}{a_2a_3}\right)^{-2},\\
x_4 &= \frac{a_1a_3}{a_2a_4},\qquad x_5=\frac{a_1a_4}{a_2a_3},\qquad x_6=u.
\end{aligned}
$$

Then:

1. *For any* $F\in\Xi_{(1)}(C^\infty_{\chi_{E^*_{21}+E^*_{42}}}(N_{2,4}\backslash G/K;I_2(\lambda)))\subset C^\infty(\tilde{N}_{2,4}\times A)$, *there exists* $f(x,y)\in\mathfrak{H}_{10}(\frac{1}{2}(\lambda_1-\lambda_2-1),\frac{1}{2}(\lambda_1-\lambda_2);x,y)$ *such that*

$$F(x_1,\dots,x_6)=x_1^{(\lambda_1+\lambda_2)/2}x_2^{1/2}x_3^{(\lambda_1-\lambda_2)/4}f(x_2,x_3).$$

2. $\dim C^\infty_{\chi_{E^*_{21}+E^*_{42}}}(N_{2,4}\backslash G/K;I_2(\lambda))\le 4$.
3. *Suppose that there exists* $F\in\Xi_{(1)}(C^\infty_{\chi_{E^*_{21}+E^*_{42}}}(N_{2,4}\backslash G/K;I_2(\lambda)))$ *such that*

$$\sup_{(a,(u,v))\in A\times\tilde{N}_{2,4}}|x_1^{-(\lambda_1+\lambda_2)/2}x_2^{\alpha_1}x_3^{\alpha_2}F(x_1,\dots,x_6)|<\infty$$

for all sufficiently large positive integers α_1 *and* α_2. *Then there exists a constant* C *such that*

$$
\begin{aligned}
&x_1^{-(\lambda_1+\lambda_2)/2}x_2^{-1/2}x_3^{-(\lambda_1-\lambda_2)/4}F=C\times\\
&\int_{\sigma_1-\sqrt{-1}\,\infty}^{\sigma_1+\sqrt{-1}\,\infty}\int_{\sigma_2-\sqrt{-1}\,\infty}^{\sigma_2+\sqrt{-1}\,\infty}\Gamma(s_1)\Gamma(s_1-2s_2-a)\Gamma(s_2)\Gamma(s_2-d+1)x_2^{-s_1}x_3^{-s_2}\,ds_1\,ds_2.
\end{aligned}
$$

Here σ_1 *and* σ_2 *are sufficiently large positive integers and* $a=\frac{1}{2}(\lambda_1-\lambda_2-1)$ *and* $b=\frac{1}{2}(\lambda_1-\lambda_2)$.

Proof. The first statement is already shown. Let us note that for a domain $\mathcal{O} \subset G$, the restriction map $\mathrm{Res}\colon C^\omega(G) \to C^\omega(\mathcal{O})$ is injective. We also recall that $C^\infty_{\chi_{E^*_{21}+E^*_{42}}}(N_{2,4}\backslash G/K; I_2(\lambda)) \subset C^\omega(G)$ as we have seen in Remark 2.5. Then the second statement follows from the first one.

Regarding $x_1, \dots, x_6$ as functions on $\tilde{N}_{2,4} \times A$, we can see that $\{(-x_2, x_3) \mid ((u,v),a) \in \tilde{N}_{2,4} \times A\} = (\mathbb{R}_{>0})^2$. Thus the third statement follows from Theorem C.1. $\square$

Case $(1_b) = (3_f)$. Let us now investigate the spaces $C^\infty_{\chi_{E^*_{32}}}(N_{2,4}\backslash G/K; I_k(\lambda)) = C^\infty_{\chi_{E^*_{32}}}(N\backslash G/K; I_2(\lambda))$.

If we put $x_1 = a_1,\ x_2 = a_3/a_2,\ x_3 = a_2a_3,\ x_4 = a_4$, the differential equations for the case $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (0,1,0)$ in Proposition 4.22 can be written as follows:

$$\begin{aligned}
&(\vartheta_{x_1} - (\lambda_1 + (k-4)))(\vartheta_{x_1} - \lambda_2)\phi = 0,\\
&[\vartheta_{x_2}^2 - \vartheta_{x_2} - 1/4(\lambda_1 + \lambda_2 + k - 2)(\lambda_1 + \lambda_2 + k)\\
&\qquad\qquad\qquad\qquad + (2\pi\sqrt{-1}\,x_2)^2 - \vartheta_{x_4} + \lambda_1(\lambda_2 + k)]\phi = 0,\\
&(2\vartheta_{x_3} - (\lambda_1 + \lambda_2 + k - 2))\phi = 0,\\
&(\vartheta_{x_4} - \lambda_1)(\vartheta_{x_4} - (\lambda_2 + k))\phi = 0,\\
&[\vartheta_{x_1} + 2\vartheta_{x_3} + \vartheta_{x_4} - k\lambda_1 - (4-k)\lambda_2]\phi = 0.
\end{aligned}$$

To solve this system of differential equations, let us recall the differential equation of modified Bessel functions. Let $\mathfrak{MB}(\nu; x)$ $(\nu \in \mathbb{C})$ be the solution space of the differential equation

$$\left[\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \left(1 + \frac{\nu^2}{x^2}\right)\right] f(x) = 0,$$

the modified Bessel equation. This differential equation has $x = 0$ as the unique singular point in $\mathbb{C}$. We have $\dim \mathfrak{MB}(\nu; x) = 2$. In $\mathfrak{MB}(\nu; x)$, there is a series solution

$$I_\nu(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{\nu+m}}{m!\Gamma(\nu+m+1)}.$$

Also there is a solution which is a slowly increasing function on $\mathbb{R}_{>0}$ defined by

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi},$$

and any slowly increasing function in $\mathfrak{MB}(\nu; x)$ is a constant multiple of $K_\nu(x)$. Here slowly increasing functions are defined as follows.

Definition 4.25. Let $U \subset \mathbb{R}^n$ be a domain. A function $f(x)$ on U is called *slowly increasing* on U if there exists $N \in \mathbb{N}$ such that

$$\sup_{x \in U} (1 + |x|)^{-N} |f(x)| < \infty$$

where $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ for $x \in U$.

In terms of modified Bessel functions, we can solve the above differential equations as shown below.

Theorem 4.26. *Set* $x_1 = a_1$, $x_2 = a_3/a_2$, $x_3 = a_2a_3$, $x_4 = a_4$. *Then the space* $\Xi_{(1)}(C^\infty_{\chi_{E^*_{32}}}(N_{2,4}\backslash G/K; I_k(\lambda)))$ *is spanned by*

$$x_1^{\lambda_2} x_2^{1/2} x_3^{(\lambda_1+\lambda_2-1)/2} x_4^{\lambda_2+1} f(2\pi x_2) \quad \text{if } k = 1$$

for $f(x) \in \mathfrak{MB}((\lambda_1 - \lambda_2 - 2)/2; x)$ *and*

$$C x_1^{\lambda_1-2} x_2^{1/2} x_3^{(\lambda_1+\lambda_2+k-2)/2} x_4^{\lambda_2+2} g_1(2\pi x_2) + C' x_1^{\lambda_2} x_2^{1/2} x_3^{(\lambda_1+\lambda_2)/2} x_4^{\lambda_1} g_2(2\pi x_2) \qquad \text{if } k = 2$$

for $C, C' \in \mathbb{C}$, $g_1 \in \mathfrak{MB}((\lambda_1 - \lambda_2 - 3)/2; x)$ *and* $g_2 \in \mathfrak{MB}((\lambda_1 - \lambda_2)/2; x)$. *Thus*

$$\dim C^\infty_{\chi_{E^*_{32}}}(N_{2,4}\backslash G/K; I_k(\lambda)) = \begin{cases} 2 & \text{if } k = 1, \\ 4 & \text{if } k = 2. \end{cases}$$

Moreover $\Xi_{(1)}(C^\infty_{\chi_{E^*_{32}}}(N_{2,4}\backslash G/K; I_k(\lambda)))$ *for* $k = 1$ (*resp.* $k = 2$) *contains a* 1*-dimensional* (*resp.* 2*-dimensional*) *subspace of slowly increasing functions on* $\{(x_1, \ldots, x_4) \mid x_i \in \mathbb{R}_{>0},\ i = 1, \ldots, 4\}$.

Proof. Consider the system of differential equations

$$\begin{aligned} (\vartheta_{x_1} - (\lambda_1 + (k-4)))(\vartheta_{x_1} - \lambda_2)\phi = 0, \\ (\vartheta_{x_4} - \lambda_1)(\vartheta_{x_4} - (\lambda_2 + k))\phi = 0, \\ (\vartheta_{x_1} + 2\vartheta_{x_3} + \vartheta_4 - k\lambda_1 - (4-k)\lambda_2)\phi = 0. \end{aligned}$$

The general solution of the system is $C x_1^{\lambda_2} x_3^{(\lambda_1+\lambda_2-1)/2} x_4^{\lambda_2+1}$ if $k = 1$ and $C_1 x_1^{\lambda_1-2} x_3^{(\lambda_1+\lambda_2)/2} x_4^{\lambda_2+2} + C_2 x_1^{\lambda_2} x_3^{(\lambda_1+\lambda_2)/2} x_4^{\lambda_1}$ if $k = 2$, for constants C, C_1 and C_2. The remaining differential equations can be reduced to the equation of modified Bessel functions, proving the theorem. □

Cases (2_b)**,** (2_c) **and** (2_d)

Case $(2_b) = (3_c)$ *and* $k = 2$

Theorem 4.27. *If* $k = 2$*, the space* $\Xi_{(2)}(C^\infty_{\chi_{E^*_{21}+E^*_{43}}}(N_{1,3,4}\backslash G/K; I_2(\lambda)))$ *consists of*

$$x_1^{(\lambda_1+\lambda_2-1)/2} x_2^{1/2} x_3^{(\lambda_1+\lambda_2+1)/2} x_4^{1/2} f(2\pi x_2) g(2\pi x_3)$$

for $f(x), g(x) \in \mathfrak{MB}((\lambda_1-\lambda_2-2)/2; x)$*. Here we put* $x_1 = a_1a_2,\ x_2 = a_1^{-1}a_2,\ x_3 = a_3a_4,\ x_4 = a_3^{-1}a_4$*. Thus* $\dim C^\infty_{\chi_{E^*_{21}+E^*_{43}}}(N_{1,3,4}\backslash G/K; I_2(\lambda)) = 4$.

Moreover, $\Xi_{(2)}(C^\infty_{\chi_{E^*_{21}+E^*_{43}}}(N_{1,3,4}\backslash G/K; I_2(\lambda)))$ *contains a slowly increasing function on* $\{(x_1, \ldots, x_4, u) \mid x_i \in \mathbb{R}_{>0},\ u \in \mathbb{R}\}$ *and it is unique up to a constant.*

Proof. Since $C^\infty_{\chi_{E^*_{21}+E^*_{43}}}(N_{1,3,4}\backslash G/K; I_2(\lambda)) = C^\infty_{\chi_{E^*_{21}+E^*_{43}}}(N\backslash G/K; I_2(\lambda))$, it suffices to solve the system of differential equations in the case $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, 0, 1)$ in Proposition 4.22. Set $x_1 = a_1a_2,\ x_2 = a_1^{-1}a_2,\ x_3 = a_3a_4,\ x_4 = a_3^{-1}a_4$. Then we can rewrite the differential equations as follows:

$$\begin{aligned}
&[2\vartheta_{x_1} - (\lambda_1 + \lambda_2 - 1)]\phi = 0,\\
&[2\vartheta_{x_3} - (\lambda_1 + \lambda_2 + 1)]\phi = 0,\\
&\left[\vartheta_{x_2}^2 - \vartheta_{x_2} + (2\pi\sqrt{-1}\,x_2)^2 - \left(\frac{\lambda_1 - \lambda_2 - 2}{2}\right)^2 - \frac{1}{4}\right]\phi = 0,\\
&\left[\vartheta_{x_4}^2 - \vartheta_{x_4} + (2\pi\sqrt{-1}\,x_4)^2 - \left(\frac{\lambda_1 - \lambda_2 - 2}{2}\right)^2 - \frac{1}{4}\right]\phi = 0.
\end{aligned}$$

We take ϕ' such that $\phi = x_2^{1/2}x_4^{1/2}\phi'$. Then ϕ' satisfies the equations

$$\begin{aligned}
&[2\vartheta_{x_1} - (\lambda_1 + \lambda_2 - 1)]\phi' = 0,\\
&[2\vartheta_{x_3} - (\lambda_1 + \lambda_2 + 1)]\phi' = 0,\\
&\left[\vartheta_{x_2}^2 - \left((2\pi\sqrt{-1}\,x_2)^2 + \left(\frac{\lambda_1 - \lambda_2 - 2}{2}\right)^2\right)\right]\phi' = 0,\\
&\left[\vartheta_{x_4}^2 - \left((2\pi\sqrt{-1}\,x_4)^2 + \left(\frac{\lambda_1 - \lambda_2 - 2}{2}\right)^2\right)\right]\phi = 0.
\end{aligned}$$

We can conclude that

$$\phi(x_1, x_2, x_3, x_4) = x_1^{(\lambda_1+\lambda_2-1)/2} x_2^{1/2} x_3^{(\lambda_1+\lambda_2+1)/2} x_4^{1/2} f(2\pi x_2) g(2\pi x_3),$$

where $f, g \in \mathfrak{MB}((\lambda_1 - \lambda_2 - 2)/2; x)$. Hence the conclusion follows. ☐

Case $(2_c) = (3_e)$

Theorem 4.28. *Set* $x_1 = a_1a_2$*,* $x_2 = a_1^{-1}a_2$*,* $x_3 = a_3$*,* $x_4 = a_4$*. Then the space* $\Xi_{(2)}(C^{\infty}_{\chi_{E^*_{21}}}(N_{1,3,4}\backslash G/K; I_k(\lambda)))$ $(k = 1, 2)$ *is spanned by*

$$x_1^{(\lambda_1+\lambda_2-2)/2}x_2^{1/2}(x_3x_4)^{\lambda_2+1}f(2\pi x_2) \quad \textit{if } k = 1$$

for some $f(x) \in \mathfrak{MB}((\lambda_1 - \lambda_2 - 3)/2; x)$ *and*

$$(C_1x_3^{\lambda_2}x_4^{\lambda_1} + C_2x_3^{\lambda_1-1}x_4^{\lambda_2+2}) \times x_1^{(\lambda_1+\lambda_2-1)/2}x_2^{1/2}f(2\pi x_2) \quad \textit{if } k = 2$$

for some $f(x) \in \mathfrak{MB}((\lambda_1 - \lambda_2 - 2)/2; x)$ *and* $C_1, C_2 \in \mathbb{C}$*. Thus*

$$\dim C^{\infty}_{\chi_{E^*_{21}}}(N_{1,3,4}\backslash G/K; I_1(\lambda)) = \begin{cases} 2 & \textit{if } k = 1, \\ 4 & \textit{if } k = 2. \end{cases}$$

Moreover, $\Xi_{(2)}(C^{\infty}_{\chi_{E^*_{21}}}(N_{1,3,4}\backslash G/K; I_k(\lambda)))$ *for* $k = 1$ (*resp.* $k = 2$) *contains a* 1*-dimensional* (*resp.* 2*-dimensional*) *subspace of slowly increasing functions on* $\{(x_1, \ldots, x_4, u) \mid x_i \in \mathbb{R}_{>0}, u \in \mathbb{R}\}$.

Proof. Since $C^{\infty}_{\chi_{E^*_{21}}}(N_{1,3,4}\backslash G/K; I_k(\lambda)) = C^{\infty}_{\chi_{E^*_{21}}}(N\backslash G/K; I_k(\lambda))$, it suffices to solve the system of differential equations in the case of $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, 0, 0)$ in Proposition 4.22. Set $x_1 = a_1a_2$, $x_2 = a_1^{-1}a_2$. For a solution ϕ of the system in Proposition 4.22, we take ϕ' with $\phi = x_2^{1/2}\phi'$. Then

$$\left[\vartheta_{x_1} - \frac{\lambda_1 + \lambda_2 - 3 + k}{2}\right]\phi' = 0, \tag{4.48}$$

$$\left[\vartheta_{x_2}^2 - \left((2\pi x_2)^2 + \left(\frac{\lambda_1 - \lambda_2 - (4 - k)}{2}\right)^2\right)\right]\phi' = 0, \tag{4.49}$$

$$[\vartheta_{a_3}^2 - (\lambda_1 + \lambda_2 - 1 + k)\vartheta_{a_3} - \vartheta_{a_4} + \lambda_1(\lambda_2 + k)]\phi' = 0, \tag{4.50}$$

$$(\vartheta_{a_4} - \lambda_1)(\vartheta_{a_4} - (\lambda_2 + k))\phi' = 0. \tag{4.51}$$

The solution of (4.48) and (4.49) is

$$\phi'(x_1, x_2, a_3, a_4) = c(a_3, a_4)x_1^{(\lambda_1+\lambda_2-3+k)/2}f(2\pi x_2)$$

for an arbitrary function $c(a_3, a_4)$ and $f(x) \in \mathfrak{MB}((\lambda_1 - \lambda_2 - (4 - k))/2; x)$. We solve the equations (4.50) and (4.51) to determine $c(a_3, c_4)$. We find

$$c(a_3, a_4) = \begin{cases} (a_3a_4)^{\lambda_2+1} & \text{for } k = 1, \\ C_1a_3^{\lambda_2+1}a_4^{\lambda_1} + C_2a_3^{\lambda_1-1}a_4^{\lambda_2+2} & \text{for } k = 2, \end{cases}$$

for some constants $C_1, C_2 \in \mathbb{C}$. This concludes the proof. □

Case $(2_d) = (3_g)$

Theorem 4.29. *Set* $x_1 = a_1,\ x_2 = a_2,\ x_3 = a_3^{-1}a_4,\ x_4 = a_3a_4$. *Then the space* $\Xi_{(2)}(C^\infty_{\chi_{E^*_{43}}}(N_{1,3,4}\backslash G/K; I_k(\lambda)))$ $(k = 1, 2)$ *is spanned by*

$$x_4^{(\lambda_1+\lambda_2)/2}x_3^{1/2}(x_3x_4)^{\lambda_2}f(2\pi x_2) \qquad \textit{if } k = 1$$

for some $f(x) \in \mathfrak{MB}((\lambda_2 - \lambda_1 - 1)/2; x)$ *and*

$$(C_1x_1^{\lambda_1-2}x_2^{\lambda_2-2} + C_2x_1^{\lambda_2-4}x_2^{\lambda_1-1}) \times x_4^{(\lambda_1+\lambda_2+1)/2}x_3^{1/2}f(2\pi x_3) \qquad \textit{if } k = 2$$

for some $f(x) \in \mathfrak{MB}((\lambda_2 - \lambda_1 + 2)/2; x)$ *and* $C_1, C_2 \in \mathbb{C}$. *Thus*

$$\dim_{\mathbb{C}} C^\infty_{\chi_{E^*_{43}}}(N_{1,3,4}\backslash G/K; I_k(\lambda)) = \begin{cases} 2 & \textit{if } k = 1, \\ 4 & \textit{if } k = 2. \end{cases}$$

Moreover, $\Xi_{(2)}(C^\infty_{\chi_{E^*_{43}}}(N_{1,3,4}\backslash G/K; I_k(\lambda)))$ *for* $k = 1$ (*resp.* $k = 2$) *contains a* 1*-dimensional* (*resp.* 2*-dimensional*) *subspace of slowly increasing functions on* $\{(x_1, \dots, x_4, u) \mid x_i \in \mathbb{R}_{>0}, u \in \mathbb{R}\}$.

Proof. This follows from the isomorphism given in Proposition 4.13 and Theorem 4.28. ☐

Case (3_h)

Theorem 4.30. *For* $k = 1, 2$, *the space* $\Xi_{(3)}(C^\infty(N\backslash G/K; I_k(\lambda)))$ *consists of the following functions: if* $k = 1$,

$$\begin{aligned} C_1a_1^{\lambda_2}a_2^{\lambda_2}a_3^{\lambda_2}a_4^{\lambda_1} &+ C_2a_1^{\lambda_2}a_2^{\lambda_1-2}a_3^{\lambda_2+1}a_4^{\lambda_2+1} \\ &+ C_3a_1^{\lambda_1-3}a_2^{\lambda_2+1}a_3^{\lambda_2+1}a_4^{\lambda_2+1} + C_4a_1^{\lambda_2}a_2^{\lambda_2}a_3^{\lambda_1-1}a_4^{\lambda_2+1} \end{aligned}$$

for $C_i \in \mathbb{C}$, $i = 1, \dots, 4$; *and if* $k = 2$,

$$\begin{aligned} C_1a_1^{\lambda_2}a_2^{\lambda_2}a_3^{\lambda_1}a_4^{\lambda_1} &+ C_2a_1^{\lambda_1-2}a_2^{\lambda_1-1}a_3^{\lambda_2+1}a_4^{\lambda_1} \\ &+ C_3a_1^{\lambda_1-2}a_2^{\lambda_2+1}a_3^{\lambda_2+1}a_4^{\lambda_1} + C_4a_1^{\lambda_2}a_2^{\lambda_1-1}a_3^{\lambda_1-1}a_4^{\lambda_2+2} \\ &+ C_5a_1^{\lambda_1-2}a_2^{\lambda_2+1}a_3^{\lambda_1-1}a_4^{\lambda_2+2} + C_6a_1^{\lambda_1-2}a_2^{\lambda_1-2}a_3^{\lambda_2+2}a_4^{\lambda_2+2} \end{aligned}$$

for $C_i \in \mathbb{C}$, $i = 1, \dots, 6$.

Proof. The relevant differential equations are

$$\begin{aligned} &(\vartheta_{a_1} - (\lambda_1 - (k-4)))(\vartheta_{a_1} - \lambda_2)\phi = 0, \\ &[\vartheta_{a_2}^2 - (\lambda_1 + \lambda_2 + k - 2)\vartheta_{a_2} - (\vartheta_{a_3} + \vartheta_{a_4}) + \lambda_1(\lambda_2 + k)]\phi = 0, \end{aligned}$$

$$
\begin{aligned}
&[\vartheta_{a_3}^2 - (\lambda_1 + \lambda_2 + k - 1)\vartheta_{a_3} - \vartheta_{a_4} + \lambda_1(\lambda_2 + k)]\phi = 0,\\
&(\vartheta_{a_4} - \lambda_1)(\vartheta_{a_4} - (\lambda_2 + k))\phi = 0,\\
&[\vartheta_{a_1} + \vartheta_{a_2} + \vartheta_{a_3} + \vartheta_{a_4} - k\lambda_1 - (4 - k)\lambda_2]\phi = 0.
\end{aligned}
$$

The result follows by solving this system of differential equations. □

Appendix A. The table of dimensions of generalized Whittaker functions of $\mathrm{GL}(4,\mathbb{R})$

We summarize the dimensions of generalized Whittaker functions of $\mathrm{GL}(4,\mathbb{R})$ in the table below. The notation is as in §4. The first row of the table describes the basis of the space of generalized Whittaker functions. The second row describes the dimensions and the third row the dimensions of the spaces of functions which satisfy the growth conditions. For detailed conditions, see §4.

Generalized Whittaker functions for $X_{1,\lambda}$

	(1_a)	$(1_b) = (3_f)$	(2_a)	$(2_b) = (3_c)$	$(2_c) = (3_e)$	$(2_d) = (3_g)$	(3_a)	(3_b)	(3_d)	(3_h)
basis	0	$\mathfrak{MB}$	0	0	$\mathfrak{MB}$	$\mathfrak{MB}$	0	0	0	x^α
dim	0	2	0	0	2	2	0	0	0	4
$\dim^{\text{growth}}$	0	1	0	0	1	1	0	0	0	4

Generalized Whittaker functions for $X_{2,\lambda}$

(1_a)	$(1_b) = (3_f)$	(2_a)	$(2_b) = (3_c)$	$(2_c) = (3_e)$	$(2_d) = (3_g)$	(3_a)	(3_b)	(3_d)	(3_h)
$\mathfrak{H}_{10}$	$\mathfrak{MB} + \mathfrak{MB}$	0	$\mathfrak{MB} \times \mathfrak{MB}$	$(x^\alpha + x^\beta)\mathfrak{MB}$	$(x^\alpha + x^\beta)\mathfrak{MB}$	0	0	0	x^α
≤ 4	4	0	4	4	4	0	0	0	6
≤ 1	2	0	1	2	2	0	0	0	6

B. The multiplicity one theorem for Horn's hypergeometric functions

We consider the asymptotic behaviour at infinity of Horn's hypergeometric functions, to apply it in the multiplicity theorem for generalized Whittaker functions.

Let $P_i(x)$ and $Q_i(x)$ be nonzero polynomials of $x = (x_1, \dots, x_n)$ for $i = 1, \dots, n$. Then Horn's hypergeometric functions are defined as solutions of the system of linear partial differential equations

$$[x_i P_i(\vartheta) - Q_i(\vartheta)] f(x) = 0, \quad i = 1, \dots, n. \tag{B.1}$$

Here $\vartheta_i = x_i \frac{\partial}{\partial x_i}$ and $\vartheta = (\vartheta_1, \dots, \vartheta_n)$. We assume that P_i and Q_i can be decomposed into products of linear factors, i.e.,

$$P_i(s) = \prod_{k=1}^{p} (\langle A_k, s\rangle - c_k), \qquad Q_i(s) = \prod_{l=1}^{q} (\langle B_l, s\rangle - d_l)$$

for $s \in \mathbb{R}^n$, $A_k, B_l \in \mathbb{R}^n$, $c_k, d_l \in \mathbb{C}$ and $\langle \ , \ \rangle$ denotes the natural inner product in $\mathbb{R}^n$. We also assume that $P_i(s)$, $Q_i(s + e_i)$ are relatively prime for $i = 1, \dots, n$. Here $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ (1 in the ith position).

We consider the following system of difference equations associated with (B.1):

$$P_i(-(s + e_i))\phi(s + e_i) = Q_i(-s)\phi(s), \qquad i = 1, \dots, n. \tag{B.2}$$

Remark B.1. Let ϕ be a solution of (B.2). We consider the integral

$$f(x) = \int_{\mathcal{C}} \phi(s) x^{-s}\, ds.$$

Then under the assumptions below, $f(x)$ is a solution of (B.1).

1. For any $i = 1, \dots, n$, the translation of the contour $\mathcal{C}$ with respect to the basis e_i is homologically equivalent to $\mathcal{C}$ in the complement of the set of singularities of the integrand $\phi(s)$ in $\mathbb{C}^n$.
2. The integral converges absolutely and it can be differentiated with respect to x sufficiently many times.

We put

$$R_i(s) = \frac{Q_i(-s)}{P_i(-(s + e_i))}, \qquad i = 1, \dots, n.$$

Theorem B.2 (Ore [20], Sato [27], Sadykov [26])**.** 1. *The system of difference equations* (B.2) *is solvable if and only if*

$$R_i(s + e_j) R_j(s) = R_j(s + e_i) R_i(s), \quad i, j = 1, \dots, n. \tag{B.3}$$

2. *If* (B.2) *is solvable, then its solution is unique up to an arbitrary periodic function $\psi(s)$ with respect to e_i, i.e.,*

$$\psi(s + e_i) = \psi(s)$$

for $i = 1, \dots, n$. Furthermore, there exist $p', q' \in \mathbb{N}$, $A'_k, B'_l \in \mathbb{R}^n$ $(1 \leq k \leq p'$,

$1 \leq l \leq q'$), $c'_k, d'_l \in \mathbb{C}$ ($1 \leq k \leq p', 1 \leq l \leq q'$) and $t_i \in \mathbb{R}$ ($i = 1, \ldots, n$) such that the general solution of (B.2) is

$$\phi(s) = t^{-s} \frac{\prod_{l=1}^{q'} \Gamma(\langle B'_l, s\rangle - d'_l)}{\prod_{k=1}^{p'} \Gamma(\langle A'_k, s\rangle - c'_k)} \psi(s),$$

where $t^{-s} = t_1^{-s_1} \cdots t_n^{-s_n}$ and $\psi(s)$ is an arbitrary periodic function satisfying $\psi(s + e_i) = \psi(s)$.

We make the following assumption for the multiplicity theorem.

(A) The system of difference equations (B.2) is solvable, i.e., the condition (B.3) is satisfied, and we can choose a solution

$$\phi(s) = t^{-s} \frac{\prod_{l=1}^{q'} \Gamma(\langle B'_l, s\rangle - d'_l)}{\prod_{k=1}^{p'} \Gamma(\langle A'_k, s\rangle - c'_k)}$$

which satisfies the following conditions:

(i) We have

$$\sum_{l=1}^{q'} |\langle B'_l, s\rangle| - \sum_{k=1}^{p'} |\langle A'_k, s\rangle| \geq \sum_{i=1}^{n} |s_i| \quad \text{for } s \in \mathbb{R}^n.$$

(ii) The function $\phi(s)$ has no zero if each $\mathrm{Re}(s_i)$ is sufficiently large for $i = 1, \ldots, n$.

Remark B.3. Consider the integral

$$f(x) = \int_{\sigma_1 - \sqrt{-1}\infty}^{\sigma_1 + \sqrt{-1}\infty} \cdots \int_{\sigma_n - \sqrt{-1}\infty}^{\sigma_n + \sqrt{-1}\infty} \phi(s) x^{-s}\, ds$$

for appropriate $\sigma_i \in \mathbb{R}$, $i = 1, \ldots, n$. Under assumption (A)(i), this integral is absolutely convergent in $\{x \in \mathbb{R}^n \mid (t_1 x_1, \ldots, t_n x_n) \in (\mathbb{R}_{\geq 0})^n\}$.

The following theorem is a generalization of the theorem of Diaconu and Goldfeld (Theorem 6.1.6 in [6]).

Theorem B.4 (Multiplicity one). *Suppose that the system of difference equations (B.2) associated with the system of differential equations (B.1) satisfies assumption (A). Let $f(x)$ be a solution of (B.1) which satisfies the growth condition*

$$\sup_{x \in (\mathbb{R}_{\geq 0})^n} |x^\alpha f(tx)| < \infty$$

for sufficiently large integers $\alpha_i \in \mathbb{N}$, $i = 1, \ldots, n$. Then it is unique up to constant multiples. Here $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $tx = (t_1 x_1, \ldots, t_n x_n)$.

Proof. We consider the Mellin transform of $f(tx)$ as a function of x,

$$\mathcal{M}[f,s] = \int_0^\infty \cdots \int_0^\infty f(tx) x^{s-1}\, dx.$$

This integral converges absolutely and $\mathcal{M}[f,s]$ is an analytic function of s if each $\mathrm{Re}(s_i)$ is sufficiently large by the assumption on $f(x)$. Changing the variable x to $tx = (t_1x_1, \dots, t_nx_n)$, we have

$$\mathcal{M}[f,s] = t^{-s} \int_0^{t_1^{-1}\infty} \cdots \int_0^{t_n^{-1}\infty} f(x) x^{s-1}\, dx.$$

By the growth condition on $f(x)$, we have

$$\int_0^{t_1^{-1}\infty} \cdots \int_0^{t_n^{-1}\infty} \frac{\partial^k}{\partial x_i^k} f(x) x^{s-1}\, dx = (-1)^k \int_0^{t_1^{-1}\infty} \cdots \int_0^{t_n^{-1}\infty} f(x) \frac{\partial^k}{\partial x_i^k} x^{s-1}\, dx$$

by integration by parts for $i = 1, \dots, n$. Recall that $f(x)$ satisfies the system of partial differential equations (B.1); then we have the system of difference equations for $\mathcal{M}[f,s]$,

$$P_i(-(s+e_i))\mathcal{M}[f, s+e_i] = Q_i(-s)\mathcal{M}[f,s], \qquad i = 1, \dots, n.$$

Hence by Theorem B.2, there is a periodic function $\psi(s)$ such that

$$\text{(B.4)} \qquad \frac{\prod_{l=1}^{q'} \Gamma(\langle B'_l, s\rangle - d'_l)}{\prod_{k=1}^{p'} \Gamma(\langle A'_k, s\rangle - c'_i)} \psi(s) = \int_0^{t_1^{-1}\infty} \cdots \int_0^{t_n^{-1}\infty} f(x) x^{s-1}\, dx.$$

By Stirling's formula and assumption (A)(i), we obtain for $\mathrm{Re}(s_i) > 0$ $(i = 1, \dots, n)$ the estimate

$$\frac{\prod_{l=1}^{q} \Gamma(\langle B_l, s\rangle - d_l)}{\prod_{k=1}^{p} \Gamma(\langle A_k, s\rangle - c_i)} = O\Biggl(\exp\Biggl(-\frac{1}{2}\pi \sum_{i=1}^n |\mathrm{Im}(s_i)|\Biggr)\Biggr) \quad \text{as} \quad \sum_{i=1}^n |\mathrm{Im}(s_i)| \to \infty.$$

Also by the Riemann–Lebesgue theorem, we have

$$\mathcal{M}[f,s] \to 0 \quad \text{as} \quad \sum_{i=1}^n |\mathrm{Im}(s_i)| \to \infty.$$

Combining these estimates, we obtain the asymptotic behaviour of the periodic function

$$\text{(B.5)} \qquad \psi(s) = O(\exp(\tfrac{1}{2}\pi|\mathrm{Im}(s_i)|))$$

as $\mathrm{Im}(s_i) \to \infty$ and the other s_j $(i \neq j)$ are fixed. The right hand side of (B.4) is an analytic function of s when $\mathrm{Re}(s_i)$ $(i = 1, \dots, n)$ are sufficiently large. Thus if we recall assumption (A)(ii) and the periodicity of $\psi(s)$, we can see that $\psi(s)$ is an entire function. We put $z_i = \exp 2\pi\sqrt{-1}\, s_i$ for $i = 1, \dots, n$. Now consider the

Laurent expansion of $\phi(s)$ with respect to z_1,

$$\psi(s) = \sum_{k=-\infty}^{\infty} c_k^{(1)}(s_2, \ldots, s_n) z_1^k.$$

Here $c_k^{(1)}(s_2, \ldots, s_n)$ are periodic and entire functions for $(s_2, \ldots, s_n) \in \mathbb{C}^{n-1}$. We write $s_i = \sigma_i + \sqrt{-1}\,\tau_i$ for $\sigma_i, \tau_i \in \mathbb{R}$, $i = 1, \ldots, n$. We consider the integral

$$\begin{aligned}\int_0^1 |\psi(s)|^2 \, d\sigma_i &= \sum_{k=-\infty}^{\infty} |c_k^{(1)}(s_2, \ldots, s_n)|^2 \exp(-4\pi k \tau_i) \\ &\geq |c_t^{(1)}(s_2, \ldots, s_n)|^2 \exp(-4\pi t \tau_i)\end{aligned}$$

for every $t = 0, \pm 1, \pm 2, \ldots$. However (B.5) tells us that there exist constants $M_i \in \mathbb{R}_{>0}$ such that

$$\exp(\pi|\tau_i|) > M_i \int_0^1 |\psi(s)|^2 \, d\sigma_i$$

for sufficiently large τ_i. Thus we have $c_t^{(1)}(s_2, \ldots, s_n) = 0$ for $t = \pm 1, \pm 2, \ldots$. The remaining coefficient $c_0^{(1)}(s_2, \ldots, s_n)$ is also a periodic and entire function for $(s_2, \ldots, s_n) \in \mathbb{C}^{n-1}$. Hence we can apply the same argument for $c_0^{(1)}(s_2, \ldots, s_n)$ with respect to s_2. Also we can proceed inductively for $i = 3, \ldots, n$. Thus we conclude that $\psi(s)$ must be a constant. This completes the proof of the theorem. □

C. Horn's hypergeometric function H_{10}

We give some facts about Horn's two-variable hypergeometric function H_{10}. This function is the hypergeometric series defined by

$$\mathrm{H}_{10}(a, d; x, y) = \sum_{m=0, n=0}^{\infty} \frac{(a)_{2m-n}}{(d)_m m! n!} x^m y^n.$$

Here $(a)_m$ means the Pochhammer symbol, i.e., $(a)_m = a(a+1)\cdots(a+(m-1))$ for $a \in \mathbb{C}$ and $m \in \mathbb{N}$. It is not hard to see that this power series satisfies the system of hypergeometric partial differential equations

$$\begin{aligned}\{x(2\vartheta_x - \vartheta_y + a)(2\vartheta_x - \vartheta_y + a + 1) - \vartheta_x(\vartheta_x + d - 1)\}\phi(x, y) = 0, \\ \{y - \vartheta_y(2\vartheta_x - \vartheta_y + a)\}\phi(x, y) = 0.\end{aligned} \tag{C.1}$$

It is known that the dimension of the solution space is 4 (cf. [2]). We define another convergent series

$$\tilde{\mathbb{H}}_{10}(a, d; x, y) = \sum_{m=0, n=0}^{\infty} \frac{(-1)^{m+2n}}{(a+1)_{m+2n} (d)_n m! n!} x^m y^n.$$

Then a basis of the solution space is given by the power series

$$\begin{aligned}&\mathrm{H}_{10}(a,d;x,y), \quad y^{-d+1}\mathrm{H}_{10}(a-2d+2,-d+2;x,y),\\ &x^a\tilde{\mathbb{H}}_{10}(a,d;x,x^2y), \quad x^a y^{-d+1}\tilde{\mathbb{H}}_{10}(a-2d+3,-d+2;x,x^2y).\end{aligned}$$

The system of hypergeometric differential equations (C.1) has a solution which has the Mellin–Barnes integral representation

$$\phi(x,y)=\int_{\sigma_1-\sqrt{-1}\,\infty}^{\sigma_1+\sqrt{-1}\,\infty}\int_{\sigma_2-\sqrt{-1}\,\infty}^{\sigma_2+\sqrt{-1}\,\infty}\Gamma(s_1)\Gamma(s_1-2s_2-a)\Gamma(s_2)\Gamma(s_2-d+1)(-x)^{-s_1}y^{-s_2}\,ds_1\,ds_2.$$

Here $\sigma_1\in\mathbb{R}$ and $\sigma_2\in\mathbb{R}$ satisfy the conditions $\sigma_1>0$, $\sigma_2>\max\{0,\mathrm{Re}(d-1)\}$ and $\sigma_1-2\sigma_2>\mathrm{Re}(a)$. This integral converges absolutely for $x\in\mathbb{R}_{\leq 0}$ and $y\in\mathbb{R}_{\geq 0}$.

Theorem C.1. *If $f(x,y)$ is a solution of the system (C.1) which satisfies*

$$\sup_{x,y\in\mathbb{R}_{\geq 0}}|x^{\alpha_1}y^{\alpha_2}f(-x,y)|<\infty$$

for sufficiently large $\alpha_1,\alpha_2\in\mathbb{N}$, then $f(x,y)=C\phi(x,y)$ for some constant C.

Proof. It is easy to see that ϕ satisfies the assumptions of Theorem B.4. Hence we only need to check that ϕ satisfies the growth condition. If we write a complex number $s=\sigma+\sqrt{-1}\,\tau$, we have $\left||x|^{-s}\right|=|x|^{-\sigma}$. Thus we obtain the inequality $|\phi(x,y)|\leq M|x|^{-\sigma_1}|y|^{-\sigma_2}$ for $x\in\mathbb{R}_{\leq 0}$ and $y\in\mathbb{R}_{\geq 0}$. Here the constant M is

$$M=\left|\int_{\sigma_1-\sqrt{-1}\,\infty}^{\sigma_1+\sqrt{-1}\,\infty}\int_{\sigma_2-\sqrt{-1}\,\infty}^{\sigma_2+\sqrt{-1}\,\infty}\Gamma(s_1)\Gamma(s_1-2s_2-a)\Gamma(s_2)\Gamma(s_2-d+1)\,ds_1\,ds_2\right|.$$

We can choose σ_1 and σ_2 with $\sigma_1>0$, $\sigma_2>\max\{0,\mathrm{Re}(d-1)\}$ and $\sigma_1-2\sigma_2>\mathrm{Re}(a)$. Thus $\phi(x,y)$ satisfies the growth condition. ☐

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