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Existence and Uniqueness Theorem for a Class of Singular Nonlinear Partial Differential Equations

by

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Abstract

This paper deals with singular nonlinear partial differential equations of the form $t\partial u/\partial t = F(t, x, u, \partial u/\partial x)$, with independent variables $(t, x) \in \mathbb{R} \times \mathbb{C}$, and where $F(t, x, u, v)$ is a function continuous in t and holomorphic in the other variables. Using the Banach fixed point theorem, we show that a unique solution $u(t, x)$ exists under the condition that $F(0, x, 0, 0) = 0$, $F_u(0, x, 0, 0) = 0$ and $F_v(0, x, 0, 0) = x\,\gamma(x)$ with $\mathrm{Re}\,\gamma(0) < 0$.

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§1. Introduction

Consider the first order singular nonlinear partial differential equation

$$t\frac{\partial u}{\partial t} = F\left(t, x, u, \frac{\partial u}{\partial x}\right). \tag{1.1}$$

Suppose $F(t, x, u, v)$ is a function holomorphic in a neighborhood of the origin $(0, 0, 0, 0) \in \mathbb{C}^4$ and $F(0, x, 0, 0) \equiv 0$ near $x = 0$. Then we can write F as

$$F\left(t, x, u, \frac{\partial u}{\partial x}\right) = a(x)t + \lambda(x)u + b(x)\frac{\partial u}{\partial x} + \sum_{i+j+\alpha \geq 2} a_{i,j,\alpha}(x)t^i u^j \left(\frac{\partial u}{\partial x}\right)^{\alpha}.$$

In this situation, solving (1.1) can be divided into three cases:

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(C_1) $b(x) \equiv 0$;

(C_2) $b(0) \neq 0$;

(C_3) $b(x) = x^p \gamma(x)$ where $\gamma(0) \neq 0$ and $p \in \mathbb{N}^* := \{1, 2, \dots\}$.

In the case (C_1), the equation (1.1) is called a Briot-Bouquet type partial differential equation with respect to t. Gérard–Tahara [5] proved the existence and uniqueness of holomorphic solution of this equation when $\lambda(0) \notin \mathbb{N}^*$; Yamazawa [12] then solved the case $\lambda(0) \in \mathbb{N}^*$. For the second case (C_2), by the implicit function theorem we can rewrite (1.1) in the form

$$\frac{\partial u}{\partial x} = G\left(t, x, u, t\frac{\partial u}{\partial t}\right)$$

and so we can apply the Cauchy–Kowalewski theorem to this equation with data on $x = 0$. The equation (1.1) is said to be a totally characteristic type partial differential equation if it satisfies (C_3). In this case we have the following results: for $p = 1$, Chen–Tahara [2] and Tahara [10] established the solvability of the equation when $\gamma(0) \in \mathbb{C} \setminus [0, \infty)$, whereas for $p \geq 2$, Chen–Luo–Tahara [3] studied Gevrey type estimates of formal solutions, and Luo–Chen–Zhang [8] showed the solvability in a sectorial domain by using summability theory.

On the other hand, assuming $F(t, x, u, v)$ is holomorphic with respect to the variables (x, u, v) but only continuous in t, Baouendi–Goulaouic [1] formulated existence and uniqueness theorems for some nonlinear partial differential equations. Their results were then extended by Lope–Roque–Tahara [7] for a wider class of equations using the concept of weight functions. The equations in [1] and [7] correspond to the case (C_1) (the case of Briot–Bouquet type partial differential equations).

This paper aims to answer the following problem:

Problem 1.1. Solve the equation (1.1) in the case (C_3) (the case of totally characteristic type), where $p = 1$, under the assumption that $F(t, x, u, v)$ is holomorphic with respect to the variables (x, u, v) but only continuous in t.

§2. Main result

Let $(t, x) \in \mathbb{R} \times \mathbb{C}$, $T_0 > 0$, $R_0 > 0$ and $\rho_0 > 0$. For any $s > 0$, we denote by D_s the open disk $\{x \in \mathbb{C} : |x| < s\}$. We study (1.1) under the following assumptions:

(A_1) $F(t, x, u, v)$ is continuous on $\Delta_0 = [0, T_0] \times D_{R_0} \times D_{\rho_0} \times D_{\rho_0}$ and holomorphic in the variables (x, u, v) for any fixed t;

(A_2) $F(0, x, 0, 0) = 0$ on D_{R_0};

(A_3) $F_v(0, x, 0, 0) = x\gamma(x)$ with $\gamma(0) \neq 0$.

Set $a(t,x) = F(t,x,0,0)$, $\lambda(t,x) = F_u(t,x,0,0)$, $b(t) = F_v(t,0,0,0)$, and $c(t,x) = (F_v(t,x,0,0) - F_v(t,0,0,0))/x$. Then, using the Taylor expansion of $F(t,x,u,v)$ with respect to the variables (u,v), (1.1) can be rewritten as

$$t\frac{\partial u}{\partial t} = a(t,x) + \lambda(t,x)u + (b(t) + xc(t,x))\frac{\partial u}{\partial x} + R_2\left(t,x,u,\frac{\partial u}{\partial x}\right), \tag{2.1}$$

where $R_2(t,x,u,v)$ is the sum of all the terms in the Taylor expansion whose degrees with respect to (u,v) are at least 2. Our assumptions imply that $a(t,x)$, $\lambda(t,x)$ and $c(t,x)$ are continuous functions on $[0,T_0] \times D_{R_0}$ and holomorphic in x for any fixed t, and $b(t)$ is a continuous function on $[0,T_0]$. Moreover, we have $a(0,x) \equiv 0$, $b(0) = 0$, and $c(0,x) = \gamma(x)$, and hence $c(0,0) \neq 0$.

In order to describe the decreasing order of $a(t,x) = o(1)$ (as $t \to 0$) and $b(t) = o(1)$ (as $t \to 0$), we introduce a concept of a weight function. We say that a real-valued function $\mu(t)$ is a *weight function* on $(0,T_0]$ if it satisfies the following conditions:

(i) $\mu(t)$ is continuous on $(0,T_0]$;
(ii) $\mu(t) > 0$ and increasing on $(0,T_0]$;
(iii) $\int_0^{T_0} (\mu(s)/s)\,ds < +\infty$.

The first two conditions imply that $\lim_{t\to 0}\mu(t) = 0$, while condition (iii) allows us to define the function

$$\varphi(t) = \int_0^t \frac{\mu(s)}{s}\,ds, \qquad 0 \le t \le T_0. \tag{2.2}$$

Examples of such weight functions are t^η and $1/(-\log t)^{\eta+1}$ for any $\eta > 0$.

We suppose that there is a weight function $\mu(t)$ such that

$$a(t,x) = O(\mu(t)) \quad \text{uniformly on } D_{R_0} \text{ (as } t \to 0\text{), and} \tag{2.3}$$

$$b(t) = O(\mu(t)) \quad (\text{as } t \to 0). \tag{2.4}$$

For any $r > 0$, $T > 0$ and $R > 0$, we define the region $W_{r,R}$ by

$$W_{r,R} = \{(t,x) : 0 \le t \le T \text{ and } |x| + \varphi(t)/r < R\}.$$

We also define two function spaces on $W = W_{r,R}$ or $[0,T] \times D_R$:

$$X_0(W) = \{w(t,x) \in C^0(W)\colon w \text{ is holomorphic in } x \text{ for any fixed } t\},$$
$$X_1(W) = X_0(W) \cap C^1(W \cap \{t > 0\}).$$

The following is our main result.

Theorem 2.1 (Main Theorem). *Suppose* (A_1)–(A_3), (2.3) *and* (2.4) *hold, and*

$$\operatorname{Re}\lambda(0,0) < 0 \quad \textit{and} \quad \operatorname{Re} c(0,0) < 0. \tag{2.5}$$

Then there exist $R > 0$, $r > 0$, $M > 0$ *and* $T > 0$ *with* $M\mu(T) < \rho_0$ *such that* (2.1) *has a unique solution* $u(t,x)$ *in* $X_1(W_{r,R})$ *that satisfies*

$$|u(t,x)| \le M\mu(t) \quad \textit{and} \quad \left|\frac{\partial u}{\partial x}(t,x)\right| \le M\mu(t) \quad \textit{on } W_{r,R}. \tag{2.6}$$

For simplicity, we set

$$\mathcal{P} = t\frac{\partial}{\partial t} - \lambda(t,x) - xc(t,x)\frac{\partial}{\partial x}$$

and

$$\Phi[u] = b(t)\frac{\partial u}{\partial x} + R_2\Big(t,x,u,\frac{\partial u}{\partial x}\Big).$$

So the equation (2.1) may be written as

$$\mathcal{P}u = a(t,x) + \Phi[u].$$

The remaining part of this paper is organized as follows. In Section 3, we investigate the equation $\mathcal{P}w = g(t,x)$ on $[0,T]\times D_R$. Next, we examine the same equation $\mathcal{P}w = g(t,x)$ on $W_{r,R}$. Then, in the last section, we solve (2.1) by using the Banach fixed point theorem as in Walter [11].

§3. On the equation $\mathcal{P}w = g$ on $[0,T]\times D_R$

Let $0 < T < T_1 < T_0$ and $0 < R < R_1 < R_0$. Consider the equation

$$t\frac{\partial w}{\partial t} - \lambda(t,x)w - xc(t,x)\frac{\partial w}{\partial x} = g(t,x) \tag{3.1}$$

on $[0,T]\times D_R$. Since we know that $\lambda(t,x)$ and $c(t,x)$ belong to $X_0([0,T_0]\times D_{R_0})$, we can choose $T_1 > 0$ and $R_1 > 0$ sufficiently small so that

(B_1) $\operatorname{Re}\lambda(t,x) \le -L$ on $[0,T_1]\times D_{R_1}$ for some $L > 0$;

(B_2) $\operatorname{Re} c(t,x) \le -\delta$ on $[0,T_1]\times D_{R_1}$ for some $\delta \ge 0$.

We admit the case $\delta = 0$ in this section, and so (B_2) is weaker than the condition posed in (2.5). Since $0 < R < R_1$, it also follows that $|\lambda_x(t,x)| \le \Lambda$ on $[0,T_1]\times D_R$ for some $\Lambda > 0$.

The purpose of this section is to show the following:

Proposition 3.1. *Suppose* (B_1) *and* (B_2) *hold. For any given* $g(t,x) \in X_0([0,T]\times D_R)$*, the equation* (3.1) *has a unique solution* $w(t,x)$ *in* $X_1([0,T]\times D_R)$*. Moreover, if* $|g(t,x)| \le K$ *and* $|g_x(t,x)| \le K_1$ *on* $[0,T]\times D_R$*, then*

$$(3.2)\qquad |w(t,x)| \le \frac{K}{L} \quad \text{and} \quad \left|\frac{\partial w}{\partial x}(t,x)\right| \le \left(\frac{K_1}{L} + \frac{\Lambda K}{L^2}\right) H \quad \text{on } [0,T]\times D_R,$$

where $H > 0$ *is a constant independent of* $g(t,x)$.

Before proving the above proposition, let us first investigate the integral curves of the vector field

$$\tau = t\frac{\partial}{\partial t} - xc(t,x)\frac{\partial}{\partial x}.$$

The integral curve of τ passing through the point $(t_0, x_0) \in (0, T_1] \times D_{R_1}$ is given by the solution of the initial value problem

$$(3.3)\qquad \begin{cases} t\dfrac{dx}{dt} = -xc(t,x), \\ x(t_0) = x_0. \end{cases}$$

Lemma 3.2. *For any* $(t_0, x_0) \in (0, T_1] \times D_{R_1}$*, the initial value problem* (3.3) *has a unique solution* $x(t)$ *on* $(0, t_0]$ *satisfying* $|x(t)| \le |x_0|(t/t_0)^\delta$ *on* $(0, t_0]$.

Proof. Since $c(t,x)$ satisfies the Lipschitz condition on D_{R_1}, (3.3) has a unique local solution $x(t)$ on $(t_1, t_0]$ for some $0 < t_1 < t_0$. Moreover, the solution satisfies

$$x(t) = x_0 \exp\left[\int_t^{t_0} c(s, x(s))\, \frac{ds}{s}\right] \quad \text{on } (t_1, t_0],$$

and thus we have

$$\begin{aligned} |x(t)| &= |x_0| \exp\left[\int_t^{t_0} \frac{\operatorname{Re} c(s, x(s))}{s}\, ds\right] \\ &\le |x_0| \exp\left[\int_t^{t_0} \frac{-\delta}{s}\, ds\right] = |x_0|(t/t_0)^\delta \quad \text{on } (t_1, t_0]. \end{aligned}$$

We show that the solution can be continued to $(0, t_0]$. Suppose it can only be extended to $(\epsilon, t_0]$ for some $\epsilon > 0$. By the above estimate, $x(t) \in K_0 = \{x \in D_{R_1} : |x| \le |x_0|\}$ for any $\epsilon < t \le t_0$. As a consequence, since K_0 is a compact subset of D_{R_1}, the solution may be continued to the left of ϵ (by Theorem 4.1 in [4]), a contradiction to our original supposition. Therefore, $\epsilon = 0$ and we have a unique solution on $(0, t_0]$, which is the continuation of the local solution $x(t)$ to $(0, t_0]$. □

Denote by $\chi(t; t_0, x_0)$ the unique solution of (3.3); $\chi(t; t_0, x_0)$ is regarded as a function on

$$\Omega_1 = \{(t, t_0, x_0) : 0 < t \le t_0 \text{ and } (t_0, x_0) \in (0, T_1] \times D_{R_1}\}.$$

The fact that $\chi(t;t_0,x_0)$ belongs to $C^1(\Omega_1)$ follows from a result concerning the dependence on initial data of solutions of ordinary differential equations (see Theorem 7.2 in [4]). Since $c(t,x)$ is holomorphic in $x \in D_{R_1}$, it is easy to see that $\chi(t;t_0,x_0)$ is holomorphic in $x_0 \in D_{R_1}$. Moreover, $|\chi(t;t_0,x_0)| \le |x_0|(t/t_0)^\delta$ on Ω_1.

Set

$$\phi(s,t,x) = \chi(s;t,x) \quad \text{on } \Omega_1, \tag{3.4}$$

where $\Omega_1 = \{(s,t,x) : 0 < s \le t \text{ and } (t,x) \in (0,T_1] \times D_{R_1}\}$. Then $\phi(s,t,x)$ is a C^1 function on Ω_1 that is holomorphic in $x \in D_{R_1}$ for any fixed (s,t), and $|\phi(s,t,x)| \le |x|(s/t)^\delta$ on Ω_1. Furthermore, we have the following lemma:

Lemma 3.3. *The above $\phi(s,t,x)$ is the unique solution of*

$$\begin{cases} t\dfrac{\partial \phi}{\partial t} - xc(t,x)\dfrac{\partial \phi}{\partial x} = 0 \quad \textit{on } \Omega_1, \\ \phi(t,t,x) = x \quad \textit{on } (0,T_1] \times D_{R_1} \end{cases} \tag{3.5}$$

that is differentiable in s and t, holomorphic in x, and $|\phi(s,t,x)| \le |x|(s/t)^\delta$ on Ω_1.

Proof. Take any $(s,t_0,x_0) \in \Omega_1$ and set $\xi_0 = \chi(s;t_0,x_0)$. Consider the solution $\chi(t;s,\xi_0)$ of (3.3) with initial point (s,ξ_0). Since $\chi(t;t_0,x_0)$ is defined on $(0,t_0]$, $\chi(t;s,\xi_0)$ can be continued to $(0,t_0]$, and we have $\chi(t;s,\xi_0) = \chi(t;t_0,x_0)$ on $(0,t_0]$. In particular, $\phi(t_0;s,\xi_0) = x_0$.

Let $t \in (0,t_0]$ and set $x = \chi(t;t_0,x_0)$. Then we also have $x = \chi(t;s,\xi_0)$. This means that $\xi_0 = \chi(s;t,x) = \phi(s,t,x)$ and so

$$\xi_0 = \phi(s,t,\chi(t;s,\xi_0)).$$

Applying $t\partial/\partial t$ on both sides of this equation and using the fact that $\chi(t;s,\xi)$ satisfies (3.3) gives

$$\begin{aligned} 0 &= t\frac{\partial \phi}{\partial t}(s,t,\chi(t;s,\xi_0)) + \frac{\partial \phi}{\partial x}(s,t,\chi(t;s,\xi_0)) \cdot t\frac{d\chi}{dt}(t;s,\xi_0) \\ &= t\frac{\partial \phi}{\partial t}(s,t,\chi(t;s,\xi_0)) - \chi(t;s,\xi_0)c(t,\chi(t;s,\xi_0))\frac{\partial \phi}{\partial x}(s,t,\chi(t;s,\xi_0)) \\ &= t\frac{\partial \phi}{\partial t}(s,t,x) - xc(t,x)\frac{\partial \phi}{\partial x}(s,t,x). \end{aligned}$$

In particular, the last equation is true for $(t,x) = (t_0,x_0)$. Since (s,t_0,x_0) is arbitrarily chosen from Ω_1, we conclude that $\phi(s,t,x)$ is a solution to (3.5).

We now proceed to the uniqueness proof. Let $\psi(s,t,x)$ be another solution of (3.5) defined on Ω_1. Our claim is that $\psi(s,t_0,x_0) = \phi(s,t_0,x_0)$ for any $(s,t_0,x_0) \in \Omega_1$. Let us prove this claim.

Similar to the arguments above, we set $x = \chi(t; t_0, x_0)$ and $\xi_0 = \chi(s; t_0, x_0) = \phi(s, t_0, x_0)$. Then again we have $x = \chi(t; s, \xi_0)$. By setting $f(t) = \psi(s, t, \chi(t; s, \xi_0))$ on $(0, t_0]$ we have $f(t_0) = \psi(s, t_0, x_0)$ and $f(s) = \psi(s, s, \xi_0) = \xi_0$. Taking the derivative of $f(t)$ with respect to t and again using the fact that $\chi(t; s, \xi_0)$ satisfies (3.3) yields

$$\begin{aligned} f'(t) &= \frac{\partial \psi}{\partial t}(s, t, \chi(t; s, \xi_0)) + \frac{\partial \psi}{\partial x}(s, t, \chi(t; s, \xi_0)) \frac{d\chi}{dt}(t; s, \xi_0) \\ &= \frac{\partial \psi}{\partial t}(s, t, x) - \frac{xc(t, x)}{t} \frac{\partial \psi}{\partial x}(s, t, x) = 0. \end{aligned}$$

Thus, $f(t)$ is constant, and consequently we have $\psi(s, t_0, x_0) = f(t_0) = f(s) = \xi_0 = \phi(s, t_0, x_0)$. □

Let us now prove Proposition 3.1.

Proof of Proposition 3.1. We set

$$w(t, x) = \int_0^t \exp\left[\int_s^t \lambda(\tau, \phi(\tau, t, x)) \frac{d\tau}{\tau}\right] g(s, \phi(s, t, x)) \frac{ds}{s} \tag{3.6}$$

where $\phi(s, t, x)$ is the unique solution of (3.5). Since we are considering the equation (3.1) where $0 < T < T_1$ and $0 < R < R_1$, we may suppose that $|(\partial\phi/\partial x)(s, t, x)| \leq H$ on $\Omega = \{(s, t, x) : 0 < s \leq t \text{ and } (t, x) \in (0, T] \times D_R\}$ for some $H > 0$. We recall that $|\lambda_x(t, x)| \leq \Lambda$ on $[0, T] \times D_R$. Then, if $|g(t, x)| \leq K$ on $[0, T] \times D_R$, we have

$$\begin{aligned} |w(t, x)| &\leq \int_0^t \exp\left[\int_s^t \mathrm{Re}\, \lambda(\tau, \phi(\tau, t, x)) \frac{d\tau}{\tau}\right] |g(s, \phi(s, t, x))| \frac{ds}{s} \\ &\leq \int_0^t \exp\left[\int_s^t -L \frac{d\tau}{\tau}\right] K \frac{ds}{s} = \int_0^t \left(\frac{s}{t}\right)^L K \frac{ds}{s} = \frac{K}{L} \quad \text{on } [0, T] \times D_R. \end{aligned}$$

From (3.6), we get

$$\begin{aligned} \frac{\partial w}{\partial x}(t, x) = &\int_0^t \exp\left[\int_s^t \lambda(\tau, \phi(\tau, t, x)) \frac{d\tau}{\tau}\right] \frac{\partial g}{\partial x}(s, \phi(s, t, x)) \frac{\partial \phi}{\partial x}(s, t, x) \frac{ds}{s} \\ &+ \int_0^t \exp\left[\int_s^t \lambda(\tau, \phi(\tau, t, x)) \frac{d\tau}{\tau}\right] \left(\int_s^t \frac{\partial \lambda}{\partial x}(\tau, \phi(\tau, t, x)) \frac{\partial \phi}{\partial x}(\tau, t, x) \frac{d\tau}{\tau}\right) \\ &\times g(s, \phi(s, t, x)) \frac{ds}{s}. \end{aligned}$$

Therefore, if $|g_x(t,x)| \le K_1$ on $[0,T]\times D_R$, we have

$$\begin{aligned}(3.7)\qquad \left|\frac{\partial w}{\partial x}(t,x)\right| &\le \int_0^t \left(\frac{s}{t}\right)^L K_1 H \frac{ds}{s} + \int_0^t \left(\frac{s}{t}\right)^L \left(\int_s^t \Lambda H \frac{d\tau}{\tau}\right) K \frac{ds}{s} \\ &\le \frac{K_1 H}{L} + \Lambda H K \int_0^t \left(\frac{s}{t}\right)^L \log\left(\frac{t}{s}\right) \frac{ds}{s} \\ &= \left(\frac{K_1}{L} + \frac{\Lambda K}{L^2}\right) H \quad \text{on } [0,T]\times D_R.\end{aligned}$$

Here, we have used the fact that $\int_0^1 x^L \log(1/x)\, dx/x = 1/L^2$ if $L > 0$.

In a similar way, we can verify that $w(t,x)$ given by the integral in (3.6) is a well-defined function belonging to $X_1([0,T]\times D_R)$. A straightforward calculation also shows that it is a solution to the equation (3.1).

To show the uniqueness of solution, we prove that

$$(3.8)\qquad \left(t\frac{\partial}{\partial t} - \lambda(t,x) - xc(t,x)\frac{\partial}{\partial x}\right) w(t,x) = 0$$

only when $w \equiv 0$ in $X_1([0,T]\times D_R)$.

Suppose $w(t,x) \in X_1([0,T]\times D_R)$ satisfies (3.8). It suffices to show that $w \equiv 0$ on $(0,T]\times D_R$. Let $(t_0,x_0) \in (0,T]\times D_R$ and set $w_0(t) = w(t,\chi(t;t_0,x_0))$ and $\lambda_0(t) = \lambda(t,\chi(t;t_0,x_0))$ on $(0,t_0]$. Then we have $w_0(t) \in C^1((0,t_0])$, $w_0(t) = O(1)$ (as $t \to 0$), $\lambda_0(t) \in C^0((0,t_0])$, $\operatorname{Re}\lambda_0(t) \le -L$ and

$$\begin{aligned}&t\frac{dw_0}{dt}(t) - \lambda_0(t)w_0(t) \\ &\quad = t\frac{\partial w}{\partial t}(t,\chi(t;t_0,x_0)) + \frac{\partial w}{\partial x}(t,\chi(t;t_0,x_0))\cdot t\frac{d\chi}{dt}(t;t_0,x_0) \\ &\qquad - \lambda(t,\chi(t;t_0,x_0))w(t,\chi(t;t_0,x_0)) \\ &\quad = \left(t\frac{\partial w}{\partial t}(t,x) - xc(t,x)\frac{\partial w}{\partial x}(t,x) - \lambda(t,x)w(t,x)\right)\Big|_{x=\chi(t;t_0,x_0)} = 0.\end{aligned}$$

This implies that

$$\frac{d}{dt}\left(\exp\left[\int_t^{t_0} \lambda_0(\tau)\frac{d\tau}{\tau}\right] w_0(t)\right) = 0,$$

and integrating this from t to t_0 yields

$$w(t_0) - \exp\left[\int_t^{t_0} \lambda_0(\tau)\frac{d\tau}{\tau}\right] w_0(t) = 0.$$

Since $w_0(t_0) = w(t_0, x_0)$, we have

$$|w(t_0, x_0)| \le \exp\left[\int_t^{t_0} \operatorname{Re}\lambda_0(\tau)\,\frac{d\tau}{\tau}\right]|w_0(t)| \le \exp\left[\int_t^{t_0} -L\,\frac{d\tau}{\tau}\right]|w_0(t)|$$
$$= (t/t_0)^L|w_0(t)| \to 0 \quad \text{as } t \to 0,$$

which shows that $w(t_0, x_0) = 0$. Since (t_0, x_0) is taken arbitrarily from $(0, T] \times D_R$, we then have $w \equiv 0$ on $(0, T] \times D_R$. □

§4. On the equation $\mathcal{P}w = g$ on $W_{r,R}$

Let Λ, H, $0 < T < T_1 < T_0$ and $0 < R < R_1 < R_0$ be as in Section 3. For simplicity, we assume that $0 < R \le 1$. In this section, we consider the following equation, which is the same as (3.1), on $W_{r,R}$:

$$t\frac{\partial w}{\partial t} - \lambda(t, x)w - xc(t, x)\frac{\partial w}{\partial x} = g(t, x). \tag{4.1}$$

Let Λ_2 and H_2 be constants satisfying $|(\partial/\partial x)^2\lambda(t, x)| \le \Lambda_2$ on $[0, T] \times D_R$ and $|(\partial/\partial x)^2\phi(s, t, x)| \le H_2$ on Ω. Then we have a result which is analogous to Proposition 3.1.

Proposition 4.1. *Suppose (B_1) and (B_2) hold. For any given $g(t, x) \in X_0(W_{r,R})$, the equation (4.1) has a unique solution $w(t, x)$ in $X_1(W_{r,R})$, and it is given by*

$$w(t, x) = \int_0^t \exp\left[\int_s^t \lambda(\tau, \phi(\tau, t, x))\,\frac{d\tau}{\tau}\right]g(s, \phi(s, t, x))\,\frac{ds}{s}, \tag{4.2}$$

where $\phi(s, t, x)$ is the unique solution of (3.5). Moreover, the following are true on $W_{r,R}$ given any nondecreasing, nonnegative functions $\psi(t)$ and $\psi_1(t)$:

(a) *If $|g(t, x)| \le K\psi(t)$, then $|w(t, x)| \le (K/L)\psi(t)$.*
(b) *In addition, if $|g_x(t, x)| \le A_1\psi(t)$ and $|g_{xx}(t, x)| \le A_2\psi(t)$, then*

$$\left|\frac{\partial w}{\partial x}(t, x)\right| \le \frac{\Lambda H}{L^2}K\psi(t) + \frac{H}{L}A_1\psi(t), \tag{4.3}$$

$$\left|\frac{\partial^2 w}{\partial x^2}(t, x)\right| \le \left(\frac{2(\Lambda H)^2}{L^3} + \frac{\Lambda_2 H^2 + \Lambda H_2}{L^2}\right)K\psi(t) + \left(\frac{2\Lambda H^2}{L^2} + \frac{H_2}{L}\right)A_1\psi(t) + \frac{H^2}{L}A_2\psi(t). \tag{4.4}$$

(c) *If $|g(t, x)| \le K\psi(t)$ and $|g_x(t, x)| \le \dfrac{K_1\psi_1(t)\mu(t)}{(R - |x| - \varphi(t)/r)^{1/2}}$, then*

$$\left|\frac{\partial w}{\partial x}(t, x)\right| \le \frac{\Lambda H}{L^2}K\psi(t) + 2\sqrt{R}HrK_1\,\psi_1(t), \tag{4.5}$$

$$(4.6)\quad \left|\frac{\partial^2 w}{\partial x^2}(t,x)\right| \le \left(\frac{2(\Lambda H)^2}{L^3}+\frac{\Lambda_2 H^2+\Lambda H_2}{L^2}\right)K\psi(t) + \left(\frac{4\sqrt{R}\Lambda H^2}{Le}+2\sqrt{R}H_2\right)rK_1\psi_1(t)+\frac{(3\sqrt{3})H^2 rK_1\psi_1(t)}{(R-|x|-\varphi(t)/r)^{1/2}}.$$

Proof. By the same arguments as in Section 3, we can easily verify that the function $w(t,x)$ defined by (4.2) is the unique solution of (4.1) belonging to $X_1(W_{r,R})$.

Let us show the estimates in (a)–(c).

Statement (a) follows immediately from (4.2):

$$|w(t,x)| \le \int_0^t \left(\frac{s}{t}\right)^L K\psi(s)\,\frac{ds}{s} \le K\psi(t)\int_0^t \left(\frac{s}{t}\right)^L \frac{ds}{s} = \frac{K}{L}\psi(t) \quad \text{on } W_{r,R}.$$

Computations similar to those in (3.2) give the first estimate (4.3) in (b). Similarly, we can obtain (4.4) using the fact that $\int_0^1 x^L(\log x)^2\,dx/x = 2/L^3$.

The next lemma is essential to estimating some integral expressions that we encounter in proving (c).

Lemma 4.2. *For a weight function $\mu(t)$ and $\varphi(t)$ given by (2.2), we have:*

(i) $$\int_0^t \frac{\mu(s)}{(R-|x|-\varphi(s)/r)^{1/2}}\,\frac{ds}{s} \le 2r\sqrt{R},$$

(ii) $$\int_0^t \frac{\mu(s)}{(R-|x|-\varphi(s)/r)^{3/2}}\,\frac{ds}{s} \le \frac{2r}{(R-|x|-\varphi(t)/r)^{1/2}}.$$

Proof. The first inequality is verified as follows:

$$\begin{aligned}\int_0^t \frac{\mu(s)}{(R-|x|-\varphi(s)/r)^{1/2}}\,\frac{ds}{s} &= \int_0^t \frac{\varphi'(s)}{(R-|x|-\varphi(s)/r)^{1/2}}\,ds\\ &= \left[-2r(R-|x|-\varphi(s)/r)^{1/2}\right]_0^t\\ &= -2r(R-|x|-\varphi(t)/r)^{1/2}+2r(R-|x|)^{1/2}\\ &\le 2r(R-|x|)^{1/2} \le 2r\sqrt{R}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\int_0^t \frac{\mu(s)}{(R-|x|-\varphi(s)/r)^{3/2}}\,\frac{ds}{s} &= \int_0^t \frac{\varphi'(s)}{(R-|x|-\varphi(s)/r)^{3/2}}\,ds\\ &= \left[\frac{2r}{(R-|x|-\varphi(s)/r)^{1/2}}\right]_0^t\\ &= \frac{2r}{(R-|x|-\varphi(t)/r)^{1/2}}-\frac{2r}{(R-|x|)^{1/2}}\\ &\le \frac{2r}{(R-|x|-\varphi(t)/r)^{1/2}}.\end{aligned}$$

□

By the preceding lemma and similar arguments to those in (3.2), we establish the first estimate (4.5) in (c):

$$\begin{aligned}\left|\frac{\partial w}{\partial x}(t,x)\right| &\le \int_0^t \left(\frac{s}{t}\right)^L \frac{K_1\psi_1(s)\mu(s)}{(R-|\phi(s,t,x)|-\varphi(s)/r)^{1/2}}\cdot H\,\frac{ds}{s}\\ &\quad + \int_0^t \left(\frac{s}{t}\right)^L \left(\int_s^t \Lambda H\frac{d\tau}{\tau}\right)\cdot K\psi(s)\,\frac{ds}{s}\\ &\le K_1H\psi_1(t)\int_0^t \frac{\mu(s)}{(R-|x|-\varphi(s)/r)^{1/2}}\,\frac{ds}{s}\\ &\quad + \Lambda HK\psi(t)\int_0^t \left(\frac{s}{t}\right)^L \log\left(\frac{t}{s}\right)\frac{ds}{s}\\ &\le K_1H\psi_1(t)\cdot 2r\sqrt{R} + (\Lambda HK/L^2)\psi(t).\end{aligned}$$

Finally, to prove (4.6), we recall Nagumo's lemma which provides a bound for the derivative of a holomorphic function.

Lemma 4.3. *Let $f(x)$ be a holomorphic function on D_R. If*

$$|f(x)| \le \frac{C}{(R-|x|)^a} \qquad \text{on } D_R$$

for some $C \ge 0$ and $a \ge 0$, then

$$\left|\frac{\partial f}{\partial x}(x)\right| \le \frac{\gamma_a C}{(R-|x|)^{a+1}} \qquad \text{on } D_R,$$

where $\gamma_0 = 1$ and $\gamma_a = (1+a)(1+1/a)^a$ for $a > 0$.

For the proof, see [6, Lemma 5.1.3] or [9]. The above lemma gives the following estimate for the second derivative of $g(t,x)$:

$$\left|\frac{\partial^2 g}{\partial x^2}(t,x)\right| \le \frac{(3\sqrt{3}/2)K_1\psi_1(t)\mu(t)}{(R-|x|-\varphi(t)/r)^{3/2}} \qquad \text{on } W_{r,R}.$$

By using this estimate, Lemma 4.2, and the fact that $\int_0^1 x^L(\log x)^2\,dx/x = 2/L^3$, we can verify (4.6) in a way similar to our previous calculations. □

§5. Proof of Main Theorem

Consider the equation

$$t\frac{\partial u}{\partial t} - \lambda(t,x)u - xc(t,x)\frac{\partial u}{\partial x} = a(t,x) + b(t,x)\frac{\partial u}{\partial x} + R_2\left(t,x,u,\frac{\partial u}{\partial x}\right). \tag{5.1}$$

Let $T_1 > 0$, $R_1 > 0$, $\rho_1 > 0$ and $\Delta_1 = [0, T_1] \times D_{R_1} \times D_{\rho_1} \times D_{\rho_1}$. Suppose that $\lambda(t, x)$, $c(t, x)$, $a(t, x)$ and $b(t, x)$ are functions belonging to $X_0([0, T_1] \times D_{R_1})$, and $R_2(t, x, u, v)$ is a continuous function on Δ_1 that is holomorphic in (x, u, v) with Taylor expansion in (u, v) of the form

$$R_2(t, x, u, v) = \sum_{i+j \geq 2} a_{i,j}(t, x) u^i v^j.$$

In addition to (B_1) and (B_2), we suppose:

(B_3) $|a(t, x)| \leq A\mu(t)$ on $[0, T_1] \times D_{R_1}$ for some $A \geq 0$;

(B_4) $|b(t, x)| \leq B\mu(t)$ on $[0, T_1] \times D_{R_1}$ for some $B \geq 0$.

For simplicity, we assume again that $0 < R_1 \leq 1$.

In this section, we prove the following theorem, stronger than Theorem 2.1.

Theorem 5.1. *Suppose (B_1)–(B_4) hold. Then, for any $0 < R < R_1$ and $0 < \rho < \rho_1$, there exist $T > 0$, $r > 0$ and $M > 0$ with $M\mu(T) \leq \rho$ such that the equation (5.1) has a unique solution $u(t, x)$ in $X_1(W_{r,R})$ that satisfies*

$$|u(t, x)| \leq M\mu(t) \quad \textit{and} \quad \left| \frac{\partial u}{\partial x}(t, x) \right| \leq M\mu(t) \quad \textit{on } W_{r,R}. \tag{5.2}$$

Remark 5.2. Comparing (2.1) with (5.1), we see that the coefficient $b(t)$ in (2.1) is generalized to $b(t, x)$ in (5.1). Since the case $\delta = 0$ is admitted in (B_2), we can also apply Theorem 5.1 to the case $c(t, x) \equiv 0$, which is just the equation discussed in [7].

To prove Theorem 5.1, we use the Banach fixed point theorem as in Walter [11]. Set

$$a = a(t, x) \quad \text{and} \quad \Phi[u] = b(t, x) \frac{\partial u}{\partial x} + R_2\Big(t, x, u, \frac{\partial u}{\partial x}\Big).$$

Proposition 4.1 tells us that equation (5.1) is equivalent to the integral equation

$$u(t, x) = \int_0^t \exp\Big[\int_s^t \lambda(\tau, \phi(\tau, t, x)) \frac{d\tau}{\tau}\Big] (a + \Phi[u])(s, \phi(s, t, x)) \frac{ds}{s}. \tag{5.3}$$

Therefore, if the operator $\mathcal{R}$ defined by

$$\mathcal{R}[u](t, x) = \int_0^t \exp\Big[\int_s^t \lambda(\tau, \phi(\tau, t, x)) \frac{d\tau}{\tau}\Big] (a + \Phi[u])(s, \phi(s, t, x)) \frac{ds}{s}$$

is a contraction mapping from a suitable function space E (which is a complete metric space) into itself, we have a unique solution of

$$u = \mathcal{R}[u] \quad \text{in } E. \tag{5.4}$$

To define E, fix any $0 < R < R_1$ and $0 < \rho < \rho_1$. For $r > 0$ and $T > 0$, we denote by $\mathscr{X}(W_{r,R})$ the set of all functions $u(t,x) \in X_0(W_{r,R})$ satisfying the following estimates on $W_{r,R}$ for some $C > 0$:

$$|u(t,x)| \le C\mu(t), \qquad \left|\frac{\partial u}{\partial x}(t,x)\right| \le C\mu(t),$$
$$\left|\frac{\partial^2 u}{\partial x^2}(t,x)\right| \le \frac{C\mu(t)}{(R-|x|-\varphi(t)/r)^{1/2}}.$$

We define a norm $\|u\|_{\mathscr{X}}$ of $u \in \mathscr{X}(W_{r,R})$ by

$$\|u\|_{\mathscr{X}} = \max\{\|u\|_0, \|u\|_1, \|u\|_2\},$$

where

$$\|u\|_0 = \sup_{(t,x)\in W_{r,R}, t>0} \frac{|u(t,x)|}{\mu(t)}, \qquad \|u\|_1 = \sup_{(t,x)\in W_{r,R},\, t>0} \frac{|(\partial u/\partial x)(t,x)|}{\mu(t)},$$
$$\|u\|_2 = \sup_{(t,x)\in W_{r,R},\, t>0} \frac{(R-|x|-\varphi(t)/r)^{1/2}|(\partial^2 u/\partial x^2)(t,x)|}{\mu(t)}.$$

It is clear that $(\mathscr{X}(W_{r,R}), \|\cdot\|_{\mathscr{X}})$ is a Banach space.

For $M > 0$, we set $E_M = \{u \in \mathscr{X}(W_{r,R}) : \|u\|_{\mathscr{X}} \le M\}$. This is a closed subset of $\mathscr{X}(W_{r,R})$ and so it is a complete metric space.

Proposition 5.3. *For any sufficiently large M, we can choose $r > 0$ and $T > 0$ so small that $M\mu(T) \le \rho$ and the mapping $\mathcal{R} : E_M \to E_M$ is a contraction map.*

Let us prove this proposition. The following lemma implies that the mapping $\mathcal{R} : E_M \to E_M$ is well-defined.

Lemma 5.4. *Suppose $M\mu(T) \le \rho$. If $u \in E_M$, then the following hold on $W_{r,R}$:*

(i) $$|\mathcal{R}[u](t,x)| \le K_{0,0}\mu(t) + K_{0,1}M\mu(t)^2 + K_{0,2}M^2\mu(t)^2,$$

(ii) $$\left|\frac{\partial \mathcal{R}[u]}{\partial x}(t,x)\right| \le (K_{1,0}\mu(t) + K_{1,1}M\mu(t)^2 + K_{1,2}M^2\mu(t)^2) + (K_{1,3}M\mu(t) + K_{1,4}M^2)r\mu(t),$$

(iii) $$\left|\frac{\partial^2 \mathcal{R}[u]}{\partial x^2}(t,x)\right| \le (K_{2,0}\mu(t) + K_{2,1}M\mu(t)^2 + K_{2,2}M^2\mu(t)^2) + (K_{2,3}M + K_{2,4}M^2)r\mu(t) + \frac{(K_{2,5}M + K_{2,6}M^2)r\mu(t)}{(R-|x|-\varphi(t)/r)^{1/2}},$$

where $K_{i,j} > 0$ are constants depending only on R, ρ and the estimates of $a(t,x)$, $b(t,x)$ and $R_2(t,x,u,v)$ on $\Delta = [0,T_1] \times D_R \times D_\rho \times D_\rho$. Moreover, the $K_{i,j}$ are independent of $T > 0$, $r > 0$ and $M > 0$.

Proof. Take any $u \in E_M$. Then on $W_{r,R}$ we have

$$|u(t,x)| \leq M\mu(t), \qquad \left|\frac{\partial u}{\partial x}(t,x)\right| \leq M\mu(t),$$
$$\left|\frac{\partial^2 u}{\partial x^2}(t,x)\right| \leq \frac{M\mu(t)}{(R-|x|-\varphi(t)/r)^{1/2}},$$

and $\Phi[u] \in X_0(W_{r,R})$. Set $w(t,x) = \mathcal{R}[u]$. Then $w(t,x) \in X_1(W_{r,R})$ and

$$t\frac{\partial w}{\partial t} - \lambda(t,x)w - xc(t,x)\frac{\partial w}{\partial x} = a(t,x) + \Phi[u] \quad \text{on } W_{r,R}.$$

Let $w_1(t,x), w_2(t,x) \in X_1(W_{r,R})$ be the unique solutions of the equations

$$t\frac{\partial w_1}{\partial t} - \lambda(t,x)w_1 - xc(t,x)\frac{\partial w_1}{\partial x} = a(t,x), \tag{5.5}$$
$$t\frac{\partial w_2}{\partial t} - \lambda(t,x)w_2 - xc(t,x)\frac{\partial w_2}{\partial x} = \Phi[u], \tag{5.6}$$

respectively. Then we have $w(t,x) = w_1(t,x) + w_2(t,x)$, and hence $\mathcal{R}[u] = w_1(t,x) + w_2(t,x)$. To estimate the function $\mathcal{R}[u]$, we estimate $w_1(t,x)$ and $w_2(t,x)$ by applying Proposition 4.1 to the equations (5.5) and (5.6).

By (B_3), we have $|a(t,x)| \leq A\mu(t)$, $|a_x(t,x)| \leq A_1\mu(t)$ and $|a_{xx}(t,x)| \leq A_2\mu(t)$ on $W_{r,R}$ for some constants $A_j > 0$ $(j = 1,2)$. On the other hand, we have $|\Phi[u](t,x)| \leq B\mu(t)M\mu(t) + B_1(M\mu(t))^2$ for some $B_1 > 0$. Since

$$\begin{aligned}\frac{\partial \Phi[u]}{\partial x}(t,x) = {}& \frac{\partial b}{\partial x}(t,x)\frac{\partial u}{\partial x} + b(t,x)\frac{\partial^2 u}{\partial x^2} + \frac{\partial R_2}{\partial x}(t,x,u,\partial u/\partial x) \\ &+ \frac{\partial R_2}{\partial u}(t,x,u,\partial u/\partial x)\frac{\partial u}{\partial x} + \frac{\partial R_2}{\partial v}(t,x,u,\partial u/\partial x)\frac{\partial^2 u}{\partial x^2},\end{aligned}$$

we also have

$$\begin{aligned}\left|\frac{\partial \Phi[u]}{\partial x}(t,x)\right| \leq {}& \mathcal{K}_1\mu(t)M\mu(t) + \mathcal{K}_2\mu(t)\frac{M\mu(t)}{(R-|x|-\varphi(t)/r)^{1/2}} \\ &+ \mathcal{K}_3(M\mu(t))^2 + \frac{\mathcal{K}_4(M\mu(t))^2}{(R-|x|-\varphi(t)/r)^{1/2}} \\ \leq {}& \frac{(\mathcal{K}_5 M + \mathcal{K}_6 M^2)\mu(t)^2}{(R-|x|-\varphi(t)/r)^{1/2}} \quad \text{on } W_{r,R}\end{aligned}$$

for some constants $\mathcal{K}_j > 0$ $(1 \leq j \leq 6)$. Lastly, we apply Proposition 4.1 to obtain the assertions of this lemma. □

Note that since we restrict $0 < R < R_1 \leq 1$, we have $1/(R-|x|-\varphi(t)/r)^{1/2} > 1$ on $W_{r,R}$.

Now, we choose $M > 0$, $T > 0$ and $r > 0$ satisfying the system

$$(5.7)\qquad \begin{cases} M\mu(T) \le \rho, \\ K_{0,0} + (K_{0,1}M + K_{0,2}M^2)\mu(T) \le M, \\ K_{1,0} + (K_{1,1}M + K_{1,2}M^2)\mu(T) + (K_{1,3}M + K_{1,4}M^2)r \le M, \\ K_{2,0} + (K_{2,1}M + K_{2,2}M^2)\mu(T) + (K_{2,3}M + K_{2,4}M^2)r \\ \qquad\qquad + (K_{2,5}cM + K_{2,6}M^2)r \le M. \end{cases}$$

This can be done by first taking $M > \max\{K_{0,0}, K_{1,0}, K_{2,0}\}$, and then choosing T and r small enough so that (5.7) holds. These values ensure that on $W_{r,R}$ we have

$$|\mathcal{R}[u](t,x)| \le M\mu(t), \qquad \left|\frac{\partial \mathcal{R}[u]}{\partial x}(t,x)\right| \le M\mu(t),$$
$$\left|\frac{\partial^2 \mathcal{R}[u]}{\partial x^2}(t,x)\right| \le \frac{M\mu(t)}{(R - |x| - \varphi(t)/r)^{1/2}},$$

and so $\mathcal{R}[u] \in E_M$, which implies that $\mathcal{R} : E_M \to E_M$ is well-defined.

Next, we show that $\mathcal{R} : E_M \to E_M$ is a contraction mapping.

Lemma 5.5. *Suppose $M\mu(T) \le \rho$. If $u, v \in E_M$ and $\|u - v\|_{\mathscr{X}} = C$, then the following hold on $W_{r,R}$:*

(i) $$|\mathcal{R}[u](t,x) - \mathcal{R}[v](t,x)| \le (K^*_{0,1}\mu(t) + K^*_{0,2}M\mu(t))C\mu(t),$$

(ii) $$\left|\frac{\partial \mathcal{R}[u]}{\partial x}(t,x) - \frac{\partial \mathcal{R}[v]}{\partial x}(t,x)\right| \le (K^*_{1,1}\mu(t) + K^*_{1,2}M\mu(t))C\mu(t) + (K^*_{1,3} + K^*_{1,4}M)rC\mu(t),$$

(iii) $$\left|\frac{\partial^2 \mathcal{R}[u]}{\partial x^2}(t,x) - \frac{\partial^2 \mathcal{R}[v]}{\partial x^2}(t,x)\right| \le (K^*_{2,1}\mu(t) + K^*_{2,2}M\mu(t))C\mu(t) + (K^*_{2,3} + K^*_{2,4}M)rC\mu(t) + \frac{(K^*_{2,5} + K^*_{2,6}M)rC\mu(t)}{(R - |x| - \varphi(t)/r)^{1/2}},$$

where $K^*_{i,j} > 0$ are constants depending only on R, ρ and the estimates of $a(t,x)$, $b(t,x)$ and $R_2(t,x,u,v)$ on $\Delta = [0,T_1] \times D_R \times D_\rho \times D_\rho$. Moreover, the $K^*_{i,j}$ are independent of $T > 0$, $r > 0$ and $M > 0$.

Proof. Set $W(t,x) = \mathcal{R}[u] - \mathcal{R}[v]$ and $G(t,x) = \Phi[u] - \Phi[v]$. Then $W(t,x) \in X_1(W_{r,R})$, $G(t,x) \in X_0(W_{r,R})$ and

$$(5.8)\qquad t\frac{\partial W}{\partial t} - \lambda(t,x)W - xc(t,x)\frac{\partial W}{\partial x} = G(t,x) \quad \text{on } W_{r,R}.$$

Since

$$G(t,x) = b(t,x)\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\right) + R_2\left(t,x,u,\frac{\partial u}{\partial x}\right) - R_2\left(t,x,v,\frac{\partial v}{\partial x}\right)$$

and by hypothesis, $\|u-v\|_{\mathscr{X}} = C$, which implies that on $W_{r,R}$,

$$|u(t,x)-v(t,x)| \le C\mu(t), \qquad \left|\frac{\partial u}{\partial x}(t,x) - \frac{\partial v}{\partial x}(t,x)\right| \le C\mu(t),$$
$$\left|\frac{\partial^2 u}{\partial x^2}(t,x) - \frac{\partial^2 v}{\partial x^2}(t,x)\right| \le \frac{C\mu(t)}{(R-|x|-\varphi(t)/r)^{1/2}},$$

it follows that $|G(t,x)| \le (B\mu(t) + B_2 M\mu(t))C\mu(t)$ and

$$|G_x(t,x)| \le \frac{(B_3\mu(t) + B_4 M\mu(t))C\mu(t)}{(R-|x|-\varphi(t)/r)^{1/2}} \quad \text{on } W_{r,R}$$

for some constants $B_j > 0$ $(j = 2,3,4)$. Again, we use Proposition 4.1 to (5.8) to obtain the desired estimates. □

Now, we take a $0 < \delta_0 < 1$. Besides (5.7), we require M, T and r to satisfy

$$\text{(5.9)} \qquad \begin{cases} (K^*_{0,1} + K^*_{0,2}M)\mu(T) \le \delta_0, \\ (K^*_{1,1} + K^*_{1,2}M)\mu(T) + (K^*_{1,3} + K^*_{1,4}M)r \le \delta_0, \\ (K^*_{2,1} + K^*_{2,2}M)\mu(T) + (K^*_{2,3} + K^*_{2,4}M)r + (K^*_{2,5} + K^*_{2,6}M)r \le \delta_0. \end{cases}$$

We can guarantee that (5.9) holds by choosing sufficiently small T and r. Consequently, on $W_{r,R}$ we have

$$|\mathcal{R}[u](t,x) - \mathcal{R}[v](t,x)| \le \delta_0 C\mu(t), \qquad \left|\frac{\partial \mathcal{R}[u]}{\partial x}(t,x) - \frac{\partial \mathcal{R}[v]}{\partial x}(t,x)\right| \le \delta_0 C\mu(t),$$
$$\left|\frac{\partial^2 \mathcal{R}[u]}{\partial x^2}(t,x) - \frac{\partial^2 \mathcal{R}[v]}{\partial x^2}(t,x)\right| \le \frac{\delta_0 C\mu(t)}{(R-|x|-\varphi(t)/r)^{1/2}}.$$

Therefore, $\|\mathcal{R}[u] - \mathcal{R}[v]\|_{\mathscr{X}} \le \delta_0\|u-v\|_{\mathscr{X}}$, which confirms that $\mathcal{R} : E_M \to E_M$ is indeed a contraction mapping. This completes the proof of Proposition 5.3.

Now, let us complete the proof of Theorem 5.1. We have seen in Proposition 5.3 that the mapping $\mathcal{R} : E_M \to E_M$ is a contraction map, which means that the equation (5.4) has a unique solution $u(t,x) \in X_0(W_{r,R})$ satisfying the estimates in (5.2). Also, by the expression (5.3), we can see that this $u(t,x)$ is in $X_1(W_{r,R})$. Since (5.3) solves the equation (5.1), we have obtained a solution $u(t,x)$ of (5.1). This concludes the existence part of the proof of Theorem 5.1.

To show the uniqueness of solution, we suppose that there is another solution $u_2(t,x)$ in $X_1(W_{r,R})$ such that $|u_2(t,x)| \le M\mu(t)$ and $|(\partial u_2/\partial x)(t,x)| \le M\mu(t)$ on the set $W_{r,R}$. Set $w(t,x) = u_2(t,x) - u(t,x)$. Then $w(t,x) \in X_1(W_{r,R})$, and both $w(t,x)$ and $(\partial w/\partial x)(t,x)$ are bounded by $2M\mu(t)$ on $W_{r,R}$. Furthermore, $w(t,x)$ satisfies the linear partial differential equation

$$t\frac{\partial w}{\partial t} - \lambda(t,x)w - xc(t,x)\frac{\partial w}{\partial x} = a_1(t,x)w + b_1(t,x)\frac{\partial w}{\partial x} \tag{5.10}$$

on $W_{r,R} \cap \{t > 0\}$, where

$$a_1(t,x) = \int_0^1 \frac{\partial R_2}{\partial u}\left(t, x, u(t,x) + \theta w(t,x), \frac{\partial u}{\partial x}(t,x) + \theta\frac{\partial w}{\partial x}(t,x)\right)d\theta$$

and

$$b_1(t,x) = b(t,x) + \int_0^1 \frac{\partial R_2}{\partial v}\left(t, x, u(t,x) + \theta w(t,x), \frac{\partial u}{\partial x}(t,x) + \theta\frac{\partial w}{\partial x}(t,x)\right)d\theta.$$

Note that $a_1(t,x)$, $b_1(t,x) \in X_0(W_{r,R})$, and $|a_1(t,x)| \le K_1M\mu(t)$ and $|b_1(t,x)| \le K_2M\mu(t)$ on $W_{r,R}$ for some constants $K_i > 0$ $(i = 1,2)$ depending only on R, ρ and the estimates of $b(t,x)$ and $R_2(t,x,u,v)$ on $\Delta = [0,T_1] \times D_R \times D_\rho \times D_\rho$.

Set

$$\Psi[w] = a_1(t,x)w + b_1(t,x)\frac{\partial w}{\partial x}.$$

Then we have

$$w(t,x) = \int_0^t \exp\left[\int_s^t \lambda(\tau, \phi(\tau,t,x))\,\frac{d\tau}{\tau}\right]\Psi[w](s,\phi(s,t,x))\,\frac{ds}{s}. \tag{5.11}$$

We also set $\alpha = (K_1 + (3\sqrt{3}/2)K_2)M \times 2r$ and $\beta = 2(K_1 + K_2)M^2$. Observe that $\alpha < 1$ for sufficiently small r.

The following lemma completes the proof of uniqueness.

Lemma 5.6. *If $\alpha < 1$ then $w \equiv 0$ on $W_{r,R}$.*

Proof. Let us show by induction that the following estimate holds for any $k = 0,1,2,\dots$:

$$|\Psi[w](t,x)| \le \frac{\alpha^k\beta\mu(t)^2}{(R - |x| - \varphi(t)/r)^{3/2}} \quad \text{on } W_{r,R}. \tag{5.12}$$

The case $k = 0$ is clear due to the fact that $R < 1$ and

$$\begin{aligned}|\Psi[w](t,x)| &\le K_1M\mu(t)(2M\mu(t)) + K_2M\mu(t)(2M\mu(t)) \\ &= 2(K_1 + K_2)M^2\mu(t)^2 \le \frac{2(K_1 + K_2)M^2\mu(t)^2}{(R - |x| - \varphi(t)/r)^{3/2}} \quad \text{on } W_{r,R}.\end{aligned}$$

Assume now that (5.12) holds for $k = n$. Then, by (5.11) and Lemma 4.2,

$$|w(t,x)| \le \int_0^t \left(\frac{s}{t}\right)^L \frac{\alpha^n \beta \mu(s)^2}{(R - |\phi(s,t,x)| - \varphi(s)/r)^{3/2}} \frac{ds}{s} \le \frac{2r\alpha^n \beta \mu(t)}{(R - |x| - \varphi(t)/r)^{1/2}} \quad \text{on } W_{r,R}, \tag{5.13}$$

and applying Lemma 4.3 gives

$$\left|\frac{\partial w}{\partial x}(t,x)\right| \le \frac{(3\sqrt{3}/2)2r\alpha^n \beta \mu(t)}{(R - |x| - \varphi(t)/r)^{3/2}} \quad \text{on } W_{r,R}. \tag{5.14}$$

Using the estimates (5.13) and (5.14), we get

$$\begin{aligned} |\Psi[w](t,x)| &\le K_1 M\mu(t) \cdot \frac{2r\alpha^n \beta \mu(t)}{(R - |x| - \varphi(t)/r)^{1/2}} \\ &\quad + K_2 M\mu(t) \cdot \frac{(3\sqrt{3}/2)2r\alpha^k \beta \mu(t)}{(R - |x| - \varphi(t)/r)^{3/2}} \\ &\le (K_1 + (3\sqrt{3}/2)K_2)M \cdot 2r \cdot \frac{\alpha^n \beta \mu(t)^2}{(R - |x| - \varphi(t)/r)^{3/2}} \\ &= \frac{\alpha^{n+1} \beta \mu(t)^2}{(R - |x| - \varphi(t)/r)^{3/2}} \quad \text{on } W_{r,R}, \end{aligned}$$

which is the case $k = n + 1$. Therefore, (5.12) is true for all $k = 0, 1, 2, \ldots$.

Finally, we obtain $w \equiv 0$ on $W_{r,R}$ by letting k approach $+\infty$, since $\alpha < 1$. □

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