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The Kernel of the Reciprocity Map of Simple Normal Crossing Varieties over Finite Fields

by

Patrick Forré

Abstract

For a smooth and proper variety Y over a finite field k the reciprocity map $\rho^Y : \mathrm{CH}_0(Y) \to \pi_1^{\mathrm{ab}}(Y)$ is injective with dense image. For a proper simple normal crossing variety this is no longer true in general. In this paper we give a description of the kernel and cokernel of the reciprocity map in terms of homology groups of a complex filled with descent data using an algebraic Seifert–van Kampen theorem. Furthermore, we give a new criterion for the injectivity of the reciprocity map for proper simple normal crossing varieties over finite fields.

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§1. Introduction

The class field theory of smooth, proper varieties Y over finite fields k was developed by S. Lang, S. Bloch, K. Kato and S. Saito. To determine the structure of the abelianized étale fundamental group $\pi_1^{\mathrm{ab}}(Y)$ classifying the finite abelian étale coverings of Y, the following reciprocity map was considered.

Definition 1.1 (The reciprocity map). Let Y be a scheme of finite type over $\mathbb{Z}$ and $\pi_1^{\mathrm{ab}}(Y) = \bigoplus_{Y' \in \pi_0(Y)} \pi_1^{\mathrm{ab}}(Y')$ the abelianized étale fundamental group of Y. Let $y \in Y$ be a closed point. Then the residue field $\kappa(y)$ is a finite field. Therefore we can consider the image of the Frobenius automorphism $\varphi_y \in G_{\kappa(y)} = \pi_1^{\mathrm{ab}}(\mathrm{Spec}(\kappa(y)))$ in $\pi_1^{\mathrm{ab}}(Y)$ via the push-forward of the natural map

$$i_y : \mathrm{Spec}(\kappa(y)) \to Y.$$

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P. Forré: Fakultät für Mathematik, Universität Regensburg,
Universitätsstr. 31, 93053 Regensburg, Germany;
e-mail: `patrick.forre@mathematik.uni-regensburg.de`

This mapping extends linearly to the group of zero-cycles of Y:

$$\rho' : Z_0(Y) = \bigoplus_{y \in Y_{(0)}} \mathbb{Z} \cdot y \to \pi_1^{\mathrm{ab}}(Y), \qquad \sum_y n_y \cdot y \mapsto \sum_y n_y \cdot (i_y)_*(\varphi_y).$$

ρ' is called the *reciprocity map* of Y.

Theorem 1.2. 1. *Let Y be a proper scheme over a finite field k. The reciprocity map ρ' factors through rational equivalence to give a map from the Chow group of zero-cycles* (cf. [Ful98, §1.3]):

$$\rho : \mathrm{CH}_0(Y) \to \pi_1^{\mathrm{ab}}(Y),$$

which is also called the reciprocity map *of Y.*

2. *Let Y be a smooth and proper variety over a finite field k. Then the reciprocity map $\rho : \mathrm{CH}_0(Y) \to \pi_1^{\mathrm{ab}}(Y)$ is injective and the cokernel of ρ is isomorphic to $(\hat{\mathbb{Z}}/\mathbb{Z})^{\pi_0(Y)}$, which is a uniquely divisible group. Moreover, we have a commutative diagram of exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & A_0(Y) \arrow[r] \arrow[d, hook, "\rho_0", "\sim"'] & \mathrm{CH}_0(Y) \arrow[r, "\deg"] \arrow[d, hook, "\rho"] & \mathbb{Z}^{\pi_0(Y)} \arrow[d, hook, "\mathrm{incl}"] \\
0 \arrow[r] & \pi_1^{\mathrm{geo}}(Y) \arrow[r] & \pi_1^{\mathrm{ab}}(Y) \arrow[r, "\deg"] & \hat{\mathbb{Z}}^{\pi_0(Y)}
\end{tikzcd}$$

where the restriction ρ_0 of the reciprocity map induces an isomorphism of finite groups between the kernels $A_0(Y)$ and $\pi_1^{\mathrm{geo}}(Y)$ of the degree maps.

Note that the degree maps for connected schemes Y are defined as follows: $\deg : \pi_1^{\mathrm{ab}}(Y) \to \hat{\mathbb{Z}}$ is just the push-forward map $\pi_1^{\mathrm{ab}}(Y) \to \pi_1^{\mathrm{ab}}(\mathrm{Spec}(k))$ composed with the isomorphism $\pi_1^{\mathrm{ab}}(\mathrm{Spec}(k)) \cong \hat{\mathbb{Z}}$, sending the Frobenius automorphism of k to 1. The map $\deg : \mathrm{CH}_0(Y) \to \mathbb{Z}$ is given by sending a cycle $\sum_i a_i \cdot [x_i]$ to $\sum_i a_i \cdot [\kappa(x) : k]$, which is well-defined for proper schemes Y over k and coincides in this case with the proper push-forward map $\mathrm{CH}_0(Y) \to \mathrm{CH}_0(\mathrm{Spec}(k))$ (cf. [Ful98, §1.4]).

Proof. See [Lan56], [Blo81], [KS83], and [Ras95, §5] for a summary. We also mention [Wie06], [Wie07], [JS03], [KS09], [KS10]. □

In the study of varieties over local fields (cf. e.g. [Sat05], [JS03]) one is confronted with the reciprocity map of simple normal crossing varieties over finite fields.

Definition 1.3 (Simple normal crossing varieties). Let k be a field and Y an equidimensional and separated scheme of finite type over k. Then Y is called a *normal crossing variety* over k if Y is everywhere étale locally isomorphic to

$$\mathrm{Spec}(k[T_0, \dots, T_d]/(T_0T_1 \cdots T_r)),$$

with $d = \dim Y$ and some $0 \le r \le d$.

A normal crossing variety Y is called *simple* if any irreducible component of Y is smooth over k.

The main results in this paper concerning the reciprocity map of simple normal crossing varieties over finite fields are summarized in the following theorem. In the proof, an algebraic Seifert–van Kampen theorem [Sti06] is used, leading to a more explicit understanding of the kernel of the reciprocity map, in contrast to [JS03] using étale homology theory and cohomological Hasse principles. Also here the characteristic of k poses no problem.

Theorem 1.4. *Let $Y = \bigcup_{i=1}^m Y_i$ be a proper simple normal crossing variety over a finite field k with irreducible components $Y_i \hookrightarrow Y$. Let*

$$Y^{[k]} := \coprod_{i_0 < i_1 < \cdots < i_k} Y_{i_0} \times_Y Y_{i_1} \times_Y \cdots \times_Y Y_{i_k}$$

be the disjoint union of the k-fold intersections, $k \ge 0$, and Γ_Y be the corresponding dual complex. Consider the complex

$$\pi_1^{\mathrm{ab}}(Y^{[\bullet]}): \quad \cdots \xrightarrow{d_2} \pi_1^{\mathrm{ab}}(Y^{[1]}) \xrightarrow{d_1} \pi_1^{\mathrm{ab}}(Y^{[0]}) \xrightarrow{d_0} \pi_1^{\mathrm{ab}}(Y)$$

with $d_k := \sum_{j=0}^k (-1)^j (\delta_j^k)_*$.

1. *The cokernel and kernel of the reciprocity map*

   $$\rho^Y : \mathrm{CH}_0(Y) \to \pi_1^{\mathrm{ab}}(Y)$$

   are given by

   $$\mathrm{coker}(\rho^Y) \cong \mathrm{H}_{-1}(\pi_1^{\mathrm{ab}}(Y^{[\bullet]})) \oplus (\hat{\mathbb{Z}}/\mathbb{Z})^{\pi_0(Y)}, \qquad \ker(\rho^Y) \cong \mathrm{H}_0(\pi_1^{\mathrm{ab}}(Y^{[\bullet]})),$$

   where the last group is finite.
2. *There is also an exact sequence*

   $$\mathrm{H}_2(\Gamma_Y, \hat{\mathbb{Z}}) \to \mathrm{CH}_0(Y) \xrightarrow{\rho^Y} \pi_1^{\mathrm{ab}}(Y) \to (\hat{\mathbb{Z}}/\mathbb{Z})^{\pi_0(Y)} \oplus \mathrm{H}_1(\Gamma_Y, \hat{\mathbb{Z}}) \to 0.$$

3. *If, furthermore, every component of $Y^{[0]}$ and of $Y^{[1]}$ is geometrically connected over k and $\mathrm{CH}_0(Y^{[0]})$ is torsion-free, then ρ^Y is injective.*

The last point gives a new criterion for the injectivity of the reciprocity map for simple normal crossing varieties which does not use the vanishing of the second homology group $\mathrm{H}_2(\Gamma_Y, \hat{\mathbb{Z}})$ of the dual complex.

The next sections will lead to the proof of the theorem above.

§2. An algebraic Seifert–van Kampen theorem

In this section we will introduce the dual complex of a variety carrying information about how the components intersect. Furthermore, we will cite an algebraic Seifert–van Kampen theorem and compute the abelianized version. The missing information is then controlled by the first and second homology groups of the dual complex.

Definition 2.1 (The dual complex). Let $(I, <)$ be a totally ordered set. Let $Y = \bigcup_{i \in I} Y_i$ be a locally noetherian scheme with closed subschemes $Y_i \hookrightarrow Y$. For $k \geq 0$ we let

$$Y^{[k]} := \coprod_{i_0 < i_1 < \cdots < i_k} Y_{i_0, i_1, \ldots, i_k}$$

be the disjoint union of

$$Y_{i_0, i_1, \ldots, i_k} := Y_{i_0} \times_Y Y_{i_1} \times_Y \ldots \times_Y Y_{i_k},$$

the k-fold scheme-theoretic intersection of the Y_i, and $Y^{[-1]} := Y$. For instance we have

$$Y^{[0]} = \coprod_{i_0 \in I} Y_{i_0} \quad \text{and} \quad Y^{[1]} = \coprod_{i_0 < i_1 \in I} Y_{i_0} \cap Y_{i_1}.$$

For every integer $k \geq 1$ there are $k + 1$ morphisms

$$\delta_j^k : Y^{[k]} \to Y^{[k-1]} \qquad \text{for } j = 0, \ldots, k$$

given by the closed immersions

$$\delta_j^k : Y_{i_0, \ldots, i_k} \hookrightarrow Y_{i_0, \ldots, \hat{i_j}, \ldots, i_k},$$

where the index $i_j \in I$ is omitted on the right side. For $j = 0, \ldots, k$ we get induced maps

$$\partial_j^k : \pi_0(Y^{[k]}) \to \pi_0(Y^{[k-1]})$$

on the connected components. Notice that we also have a canonical map $h = \delta_0^0 : Y^{[0]} \to Y = Y^{[-1]}$ and the corresponding induced maps.

Therefore $\Gamma := (\pi_0(Y^{[\bullet]}), (\partial_j^\bullet)_j)$ is a semi-simplicial complex, called the *dual complex* to $(Y, (Y_i)_{i \in I}, (I, <))$. The elements of $\pi_0(Y^{[k]})$ are called *k-simplices.*

The homology groups of the complex

$$C(\Gamma, A): \quad \cdots \xrightarrow{d_3} A^{(\pi_0(Y^{[k]}))} \xrightarrow{d_2} A^{(\pi_0(Y^{[1]}))} \xrightarrow{d_1} A^{(\pi_0(Y^{[0]}))}$$

with $d_k := \sum_{j=0}^{k}(-1)^j \partial_j^k$ will be denoted by $\mathrm{H}_k(\Gamma, A) := \ker(d_k)/\mathrm{im}(d_{k+1})$. Notice that $A^{(\pi_0(Y^{[k]}))}$ is placed in degree k.

Theorem 2.2 (Algebraic Seifert–van Kampen Theorem). *Let $(I,<)$ be a finite and totally ordered set and $Y = \bigcup_{v\in I} Y_v$ be a locally noetherian and connected scheme with closed and connected subschemes $Y_v \hookrightarrow Y$. Let $h : Y^{[0]} = \coprod_{v\in I} Y_v \to Y$ be the canonical map.*

Let $\overline{s}$ be a geometric point of Y and for every $k = 0, 1, 2$ and every $t \in \pi_0(Y^{[k]})$ let $\overline{s}(t)$ be a geometric point of t. Let Γ be the dual complex to $(Y, (Y_i)_{i\in I}, (I,<))$. Fix a maximal subtree T of Γ and for every boundary map $\partial : t \to t'$ in $\Gamma_{\leq 2}$ let $\gamma_{t,t'} : \overline{s}(t') \rightsquigarrow \Gamma(\partial)\overline{s}(t)$ be a fixed path in the sense of algebraic paths between base points, i.e. a fixed isomorphism between the corresponding fibre functors. Then canonically with respect to all these choices we have an isomorphism

$$\pi_1(Y, \overline{s}) \cong \Big(\Big(\mathop{\ast}_{v\in I} \pi_1(Y_v, \overline{s}(v))\Big) * \hat{\pi}_1(\Gamma, T)\Big)/H,$$

where H is the closed normal subgroup generated by the edge and cocycle relations:

$$\begin{aligned} \mathrm{edg}(e, g_e) &:= \overrightarrow{e} \cdot \pi_1(\delta_0^1)(g_e) \cdot (\overrightarrow{e})^{-1} \cdot \pi_1(\delta_1^1)(g_e)^{-1}, \\ \mathrm{coc}(f) &:= (\overrightarrow{\partial_2^2 f}) \cdot \alpha_{102}^{(f)}(\alpha_{120}^{(f)})^{-1} \cdot (\overrightarrow{\partial_0^2 f}) \cdot \alpha_{210}^{(f)}(\alpha_{201}^{(f)})^{-1} \cdot (\overrightarrow{\partial_1^2 f})^{-1} \cdot \alpha_{021}^{(f)}(\alpha_{012}^{(f)})^{-1}, \end{aligned}$$

for all parameter values $e \in \Gamma_1$, $g_e \in \pi_1(e, \overline{s}(e))$ and $f \in \Gamma_2$. Here $\overrightarrow{e}$ is defined to be the corresponding topological generators of the profinite group

$$\hat{\pi}_1(\Gamma, T) := \Big(\mathop{\ast}_{e\in\Gamma_1} \hat{\mathbb{Z}}\Big)/(\overrightarrow{e'} \mid e' \in T_1) \cong \mathop{\ast}_{e\in\Gamma_1\setminus T_1} \hat{\mathbb{Z}}.$$

The map $\pi_1(\delta_i^1)$ uses the fixed path $\gamma_{\delta_i^1(e),e}$. Finally $\alpha_{ijk}^{(f)}$ is defined using the i-th vertex $v = v_i(f) \in \Gamma_0$ of f and the edge $e = e_{ij}(f) \in \Gamma_1$ with vertices $\{v_i(f), v_j(f)\} = \{\partial_0^1(e), \partial_1^1(e)\}$, as

$$\alpha_{ijk}^{(f)} := \gamma_{v_i(f),\, e_{ij}(f)} \circ \gamma_{e_{ij}(f),\, f} \circ (\gamma_{v_i(f),\, f})^{-1} \in \pi_1(Y_{v_i}, \overline{s}(v_i)).$$

Proof. This is a special case of [Sti06, Corollary 5.4] noting that closed immersions are monomorphisms ([Sti06, Definition 4.2]) and using [Sti06, Theorem 5.2(1)] to see that h—a proper, surjective morphism of finite presentation—is a universal effective descent morphism for finite étale covers ([Sti06, Definition 5.1]). □

Corollary 2.3 (The abelianized fundamental group). *Let the notation be as in 2.2. The abelianized fundamental group of Y is then given by*

$$\pi_1^{\mathrm{ab}}(Y) = \big(\pi_1^{\mathrm{ab}}(Y^{[0]}) \oplus \hat{\pi}_1^{\mathrm{ab}}(\Gamma, T)\big)/\overline{H},$$

where

$$\pi_1^{\mathrm{ab}}(Y^{[k]}) := \bigoplus_{Z \in \pi_0(Y^{[k]})} \pi_1^{\mathrm{ab}}(Z),$$
$$\hat{\pi}_1^{\mathrm{ab}}(\Gamma, T) := \Big(\bigoplus_{e \in \Gamma_1} \hat{\mathbb{Z}}\Big)/(\overrightarrow{e'} \mid e' \in T_1) \cong \bigoplus_{e \in \Gamma_1 \setminus T_1} \hat{\mathbb{Z}},$$

and $\overline{H}$ is topologically generated by the relations

$$d_1(g) \quad \text{and} \quad \overline{\mathrm{coc}}(f) = \beta(f) + \overline{d}_2(f),$$

where

$$d_1 : \pi_1^{\mathrm{ab}}(Y^{[1]}) \to \pi_1^{\mathrm{ab}}(Y^{[0]}), \qquad g \mapsto \pi_1^{\mathrm{ab}}(\delta_0^1)(g) - \pi_1^{\mathrm{ab}}(\delta_1^1)(g),$$
$$\overline{d}_2 : \bigoplus_{f \in \Gamma_2} \hat{\mathbb{Z}} \to \hat{\pi}_1^{\mathrm{ab}}(\Gamma, T), \qquad \sum_f n_f \cdot f \mapsto \sum_f n_f \cdot (\overrightarrow{\partial_0^2 f} - \overrightarrow{\partial_1^2 f} + \overrightarrow{\partial_2^2 f}),$$
$$\beta : \bigoplus_{f \in \Gamma_2} \hat{\mathbb{Z}} \to \pi_1^{\mathrm{ab}}(Y^{[0]}),$$
$$\sum_f n_f \cdot f \mapsto \sum_f n_f \cdot (\overline{\alpha}_{102}^{(f)} - \overline{\alpha}_{120}^{(f)} + \overline{\alpha}_{210}^{(f)} - \overline{\alpha}_{201}^{(f)} + \overline{\alpha}_{021}^{(f)} - \overline{\alpha}_{012}^{(f)}),$$
$$\overline{\mathrm{coc}} : \bigoplus_{f \in \Gamma_2} \hat{\mathbb{Z}} \to \pi_1^{\mathrm{ab}}(Y^{[0]}) \oplus \hat{\pi}_1^{\mathrm{ab}}(\Gamma, T), \qquad \sum_f n_f \cdot f \mapsto \sum_f n_f \cdot \big(\beta(f) + \overline{d}_2(f)\big),$$

with all the abelian groups written additively. Furthermore, for an abelian group A consider the complex

$$\pi_1^{\mathrm{ab}}(Y^{[\bullet]}) \otimes_{\mathbb{Z}} A : \quad \pi_1^{\mathrm{ab}}(Y^{[1]}) \otimes_{\mathbb{Z}} A \xrightarrow{d_1} \pi_1^{\mathrm{ab}}(Y^{[0]}) \otimes_{\mathbb{Z}} A \xrightarrow{d_0} \pi_1^{\mathrm{ab}}(Y) \otimes_{\mathbb{Z}} A$$

with homology groups $\mathrm{H}_i(\pi_1^{\mathrm{ab}}(Y^{[\bullet]}), A)$. We then have an isomorphism

$$\mathrm{H}_1(\Gamma, A) \cong \mathrm{H}_{-1}(\pi_1^{\mathrm{ab}}(Y^{[\bullet]}), A) := \mathrm{coker}(d_0),$$

and a surjection

$$\mathrm{H}_2(\Gamma, A) \overset{\beta_|}{\twoheadrightarrow} \mathrm{H}_0(\pi_1^{\mathrm{ab}}(Y^{[\bullet]}), A) := \ker(d_0)/\mathrm{im}(d_1).$$

Proof. The first statement immediately follows from 2.2 by abelianization. For clarity in the description of the homology groups we suppress the terms $\otimes_{\mathbb{Z}} A$ in every line. Now d_0 is the canonical map

$$d_0 : \pi_1^{\mathrm{ab}}(Y^{[0]}) \to \pi_1^{\mathrm{ab}}(Y) = \big(\pi_1^{\mathrm{ab}}(Y^{[0]}) \;\oplus\; \hat{\pi}_1^{\mathrm{ab}}(\Gamma, T)\big)/\overline{H}.$$

So we get

$$\begin{aligned}
\mathrm{H}_{-1}(\pi_1^{\mathrm{ab}}(Y^{[\bullet]}), A) &= \operatorname{coker}(d_0) \\
&= \big(\pi_1^{\mathrm{ab}}(Y^{[0]}) \oplus \hat{\pi}_1^{\mathrm{ab}}(\Gamma, T)\big) / \big(\pi_1^{\mathrm{ab}}(Y^{[0]}) + \overline{H}\big) \\
&= \big(\pi_1^{\mathrm{ab}}(Y^{[0]}) \oplus \hat{\pi}_1^{\mathrm{ab}}(\Gamma, T)\big) / \big(\pi_1^{\mathrm{ab}}(Y^{[0]}) \oplus \operatorname{im}(\overline{d}_2)\big) \\
&\cong \hat{\pi}_1^{\mathrm{ab}}(\Gamma, T)) / \operatorname{im}(\overline{d}_2) \\
&= \mathrm{H}_1(\Gamma, T; A) \cong \mathrm{H}_1(\Gamma, A).
\end{aligned}$$

The last isomorphism holds because T is a maximal subtree of Γ. Moreover,

$$\mathrm{H}_0(\pi_1^{\mathrm{ab}}(Y^{[\bullet]}), A) = \ker(d_0)/\operatorname{im}(d_1) = \big(\pi_1^{\mathrm{ab}}(Y^{[0]}) \cap \overline{H}\big)/\operatorname{im}(d_1).$$

Now consider the exact sequence

$$0 \to \big(\pi_1^{\mathrm{ab}}(Y^{[0]}) \cap \overline{H}\big)/\operatorname{im}(d_1) \to \overline{H}/\operatorname{im}(d_1) \overset{\omega}{\twoheadrightarrow} \big(\pi_1^{\mathrm{ab}}(Y^{[0]}) + \overline{H}\big)/\pi_1^{\mathrm{ab}}(Y^{[0]}) \to 0.$$

Because $\overline{H} = \langle \operatorname{im}(d_1), \operatorname{im}(\overline{\mathrm{coc}})\rangle$ we have a surjection

$$\overline{\mathrm{coc}} : \bigoplus_{f \in \Gamma_2} \hat{\mathbb{Z}} \twoheadrightarrow \overline{H}/\operatorname{im}(d_1).$$

Further $\overline{d}_2 : \bigoplus_{f \in \Gamma_2} \hat{\mathbb{Z}} \to \hat{\pi}_1^{\mathrm{ab}}(\Gamma, T)$ factors through ω:

$$\overline{d}_2 : \bigoplus_{f \in \Gamma_2} \hat{\mathbb{Z}} \xrightarrow{\overline{\mathrm{coc}}} \overline{H}/\operatorname{im}(d_1) \overset{\omega}{\twoheadrightarrow} \big(\pi_1^{\mathrm{ab}}(Y^{[0]}) + \overline{H}\big)/\pi_1^{\mathrm{ab}}(Y^{[0]}) \xrightarrow{\mathrm{pr}_2} \hat{\pi}_1^{\mathrm{ab}}(\Gamma, T).$$

Therefore the restriction of $\overline{\mathrm{coc}}$ to $\ker(\overline{d}_2)$, which by definition coincides with the restriction of β to $\ker(\overline{d}_2)$, induces a well-defined and surjective map

$$\beta_{|} = \overline{\mathrm{coc}}_{|} : \ker(\overline{d}_2) \twoheadrightarrow \big(\pi_1^{\mathrm{ab}}(Y^{[0]}) \cap \overline{H}\big)/\operatorname{im}(d_1).$$

Because T is a maximal subtree of Γ, we have $\ker(d_2) = \ker(\overline{d}_2)$. Moreover

$$\begin{aligned}
&\beta : \bigoplus_{f \in \Gamma_2} \hat{\mathbb{Z}} \to \pi_1^{\mathrm{ab}}(Y^{[0]}), \\
&\sum_f n_f \cdot f \mapsto \sum_f n_f \cdot (\overline{\alpha}_{102}^{(f)} - \overline{\alpha}_{120}^{(f)} + \overline{\alpha}_{210}^{(f)} - \overline{\alpha}_{201}^{(f)} + \overline{\alpha}_{021}^{(f)} - \overline{\alpha}_{012}^{(f)}),
\end{aligned}$$

vanishes on the image of

$$d_3 : \bigoplus_{Z \in \Gamma_3} \hat{\mathbb{Z}} \to \bigoplus_{f \in \Gamma_2} \hat{\mathbb{Z}}$$

by definition of the $\overline{\alpha}_{ijk}^{(f)}$ and alternating signs. Therefore we also get a surjection

$$\mathrm{H}_2(\Gamma, A) = \ker(d_2)/\operatorname{im}(d_3) \overset{\beta_|}{\twoheadrightarrow} \mathrm{H}_0(\pi_1^{\mathrm{ab}}(Y^{[\bullet]}), A),$$

which finishes the proof. $\square$

§3. The reciprocity map for simple normal crossing varieties

In this section we will determine the kernel and cokernel of the reciprocity map of simple normal crossing varieties in terms of homology groups of a complex filled with descent data. From this description we will deduce a criterion for the injectivity of the reciprocity map without using the vanishing of the second homology group $\mathrm{H}_2(\Gamma)$.

Lemma 3.1. 1. *Let $f : Y \to \mathrm{Spec}(k)$ be a normal scheme which is separated, of finite type and geometrically connected over a finite field k. The degree map*

$$\deg : Z_0(Y) \to \mathbb{Z}, \qquad \sum_{y\in Y_0} a_y \cdot y \mapsto \sum_{y\in Y_0} a_y \cdot [\kappa(y) : k],$$

is then surjective.

2. *Let $Y = \bigcup_{v\in I} Y_v$ be a connected scheme which is proper over a finite field k such that Y_v are closed normal connected subschemes of Y. Let $k_v := \mathcal{O}_{Y_v}(Y_v)$. The image of the degree map $\mathrm{CH}_0(Y) \to \mathrm{CH}_0(k) \cong \mathbb{Z}$ is then given by*

$$\operatorname{im}\deg = \gcd_{v\in I}([k_v : k]) \cdot \mathbb{Z}.$$

Proof. 1. Consider the commutative diagram

$$\begin{tikzcd} Z_0(Y) \arrow[r, "\rho'"] \arrow[d, "\deg"] & \pi_1^{\mathrm{ab}}(Y) \arrow[d, "f_*"] \\ \mathbb{Z} \arrow[r, hook] & \mathrm{Gal}_k \end{tikzcd}$$

where $\mathrm{Gal}_k \cong \hat{\mathbb{Z}}$ and f_* is surjective by [Sza09, Prop. 5.5.4] because Y is geometrically connected. Now let n be the natural number given by $n\mathbb{Z} = \operatorname{im}\deg$. By Lang's theorem (see [Sza09, Thm. 5.8.16], [Mil80, Section VI.12]) ρ' has dense image and therefore so does $f_* \circ \rho'$. But $n\mathbb{Z}$ is only dense in $\hat{\mathbb{Z}}$ if and only if $n = 1$.

2. By the Stein factorisation every Y_v is geometrically connected over the finite field k_v. Now consider the commutative diagram

$$\begin{tikzcd} \bigoplus_{v\in I} \mathrm{CH}_0(Y_v) \arrow[r, two heads] \arrow[d, "\oplus_v \deg_v"] & \mathrm{CH}_0(Y) \arrow[d, "\deg"] \\ \bigoplus_{v\in I} \mathrm{CH}_0(k_v) \arrow[r] & \mathrm{CH}_0(k) \end{tikzcd}$$

where $\oplus_v \deg_v$ is surjective by the first point and the bottom map is componentwise on $\mathbb{Z}$ given by multiplying with the degrees $[k_v : k]$ and summing up. So the image of deg equals the image of the bottom map, which is $\gcd_{v\in I}([k_v : k]) \cdot \mathbb{Z}$. □

Proposition 3.2. *Let $Y = \bigcup_{v\in I} Y_v$ be a proper scheme over a finite field k with closed and connected subschemes $Y_v \hookrightarrow Y$ which are smooth over k such that $Y_{v_0} \times_Y Y_{v_1}$ are also smooth over k for all $v_0, v_1 \in I$. Let n be an arbitrary integer and consider the complex*

$$\pi_1^{\mathrm{ab}}(Y^{[\bullet]})/n : \quad \cdots \to \pi_1^{\mathrm{ab}}(Y^{[1]})/n \xrightarrow{d_1} \pi_1^{\mathrm{ab}}(Y^{[0]})/n \xrightarrow{d_0} \pi_1^{\mathrm{ab}}(Y)/n$$

with $d_k := \sum_{j=0}^{k}(-1)^j(\delta_j^k)_*$. Then the kernel and the cokernel of the reciprocity map modulo n

$$\rho_n : \mathrm{CH}_0(Y)/n \to \pi_1^{\mathrm{ab}}(Y)/n$$

are given by the homology groups of $\pi_1^{\mathrm{ab}}(Y^{[\bullet]})/n$:

$$\ker(\rho_n) \cong \mathrm{H}_0(\pi_1^{\mathrm{ab}}(Y^{[\bullet]})/n), \qquad \operatorname{coker}(\rho_n) \cong \mathrm{H}_{-1}(\pi_1^{\mathrm{ab}}(Y^{[\bullet]})/n).$$

Furthermore, we have an exact sequence of finite abelian groups

$$\mathrm{H}_2(\Gamma, \mathbb{Z}/n) \to \mathrm{CH}_0(Y)/n \to \pi_1^{\mathrm{ab}}(Y)/n \to \mathrm{H}_1(\Gamma, \mathbb{Z}/n) \to 0,$$

where Γ is the dual complex to $(Y, (Y_i)_{i\in I}, (I, <))$.

Proof. We have a commutative diagram of complexes

$$\begin{tikzcd}
\mathrm{CH}_0(Y^{[1]})/n \arrow[r, "d_1'"] \arrow[d, "\rho_n^1", "\wr"'] & \mathrm{CH}_0(Y^{[0]})/n \arrow[r, "d_0'"] \arrow[d, "\rho_n^0", "\wr"'] & \mathrm{CH}_0(Y)/n \arrow[r] \arrow[d, "\rho_n"] & 0 \\
\pi_1^{\mathrm{ab}}(Y^{[1]})/n \arrow[r, "d_1"] & \pi_1^{\mathrm{ab}}(Y^{[0]})/n \arrow[r, "d_0"] & \pi_1^{\mathrm{ab}}(Y)/n &
\end{tikzcd}$$

where the first row is exact in analogy to [Ful98, Ex. 1.3.1, 1.8.1] and the first two vertical maps are isomorphisms by 1.2, since $Y^{[0]}$ and $Y^{[1]}$ are smooth and proper by assumption and $(\hat{\mathbb{Z}}/\mathbb{Z})^{\pi_0(Y^{[k]})}$ is uniquely divisible. By the isomorphisms ρ_n^0 and ρ_n^1 we have $\operatorname{coker}(d_1) \cong \mathrm{CH}_0(Y)/n$, and

$$\overline{d}_0 : \operatorname{coker}(d_1) \to \pi_1^{\mathrm{ab}}(Y)/n$$

coincides with ρ_n. Therefore we get

$$\ker(\rho_n) \cong \ker(\overline{d}_0) = \mathrm{H}_0(\pi_1^{\mathrm{ab}}(Y^{[\bullet]})/n),$$
$$\operatorname{coker}(\rho_n) = \operatorname{coker}(\overline{d}_0) = \mathrm{H}_{-1}(\pi_1^{\mathrm{ab}}(Y^{[\bullet]})/n).$$

The statement now follows from the abelianized Seifert–van Kampen theorem 2.3 with $A = \mathbb{Z}/n$:

$$\mathrm{H}_0(\pi_1^{\mathrm{ab}}(Y^{[\bullet]})/n) \cong \mathrm{H}_1(\Gamma, \mathbb{Z}/n), \qquad \mathrm{H}_{-1}(\pi_1^{\mathrm{ab}}(Y^{[\bullet]})/n) \twoheadleftarrow \mathrm{H}_2(\Gamma, \mathbb{Z}/n). \qquad \square$$

Remark 3.3. The proofs of 3.2 and 2.3 show that

$$\ker(\rho_n) \cong \operatorname{im}\bigl(\beta_| : \ker(d_2) \to (\pi_1^{\mathrm{ab}}(Y^{[0]})/n)/\operatorname{im}(d_1)\bigr).$$

Therefore non-vanishing of $\beta_|$ results in a non-trivial kernel of the reciprocity map modulo n.

If we could choose the geometric points of 2.2 such that the paths $\gamma_{t,t'}$ generate trivial $\alpha_{ijk}^{(f)}$ for all parameters, then we get a vanishing $\beta_|$ and therefore a trivial kernel of the reciprocity map modulo n for every integer n.

Notation 3.4. For an abelian group A and a set $\mathbb{L}$ of primes let $\mathbb{N}(\mathbb{L})$ be the monoid of all natural numbers which have prime divisors only in $\mathbb{L}$. We define $A_{\mathbb{L}}$ to be the $\mathbb{L}$-completion

$$A_{\mathbb{L}} := \varprojlim_{n\in\mathbb{N}(\mathbb{L})} A/n,$$

and $\hat{A}$ to be the $\hat{\mathbb{Z}}$-completion

$$\hat{A} := \varprojlim_{n\in\mathbb{N}} A/n.$$

Theorem 3.5 (The reciprocity map and its $\mathbb{L}$-completion). *Let $\mathbb{L}$ be a set of prime numbers and let $Y = \bigcup_{v\in I} Y_v$ be a proper scheme over a finite field k with a finite number of closed and connected subschemes $Y_v \hookrightarrow Y$ which are smooth over k such that $Y_{v_0} \times_Y Y_{v_1}$ are also smooth over k for all $v_0, v_1 \in I$. Let Γ be the dual complex to $(Y, (Y_i)_{i\in I}, (I, <))$. Consider the complex*

$$\pi_1^{\mathrm{ab}}(Y^{[\bullet]})_{\mathbb{L}}: \quad \cdots \xrightarrow{d_2} \pi_1^{\mathrm{ab}}(Y^{[1]})_{\mathbb{L}} \xrightarrow{d_1} \pi_1^{\mathrm{ab}}(Y^{[0]})_{\mathbb{L}} \xrightarrow{d_0} \pi_1^{\mathrm{ab}}(Y)_{\mathbb{L}}$$

with $d_k := \sum_{j=0}^{k} (-1)^j (\delta_j^k)_*$ and the reciprocity maps

$$\rho : \mathrm{CH}_0(Y) \to \pi_1^{\mathrm{ab}}(Y), \qquad \rho_0 : A_0(Y) \to \pi_1^{\mathrm{geo}}(Y),$$
$$\rho_{\mathbb{L}} : \mathrm{CH}_0(Y)_{\mathbb{L}} \to \pi_1^{\mathrm{ab}}(Y)_{\mathbb{L}}, \qquad \hat{\rho} : \mathrm{CH}_0\hat{}(Y) \to \pi_1^{\mathrm{ab}}\hat{}(Y) = \pi_1^{\mathrm{ab}}(Y),$$

where $A_0(Y)$ and $\pi_1^{\mathrm{geo}}(Y)$ are the kernels of the corresponding degree maps.

1. *Then the kernel of $\rho_{\mathbb{L}}$ is a finite abelian group and a factor group of $\mathrm{H}_2(\Gamma, \mathbb{Z}_{\mathbb{L}})$, and satisfies*

$$\ker(\rho_{\mathbb{L}}) \cong \mathrm{H}_0(\pi_1^{\mathrm{ab}}(Y^{[\bullet]})_{\mathbb{L}}).$$

The cokernel of $\rho_{\mathbb{L}}$ satisfies

$$\operatorname{coker}(\rho_{\mathbb{L}}) \cong \mathrm{H}_{-1}(\pi_1^{\mathrm{ab}}(Y^{[\bullet]})_{\mathbb{L}}) \cong \mathrm{H}_1(\Gamma, \mathbb{Z}_{\mathbb{L}}).$$

Therefore, we have an exact sequence of finitely generated $\mathbb{Z}_{\mathbb{L}}$-modules:

$$\mathrm{H}_2(\Gamma, \mathbb{Z}_{\mathbb{L}}) \to \mathrm{CH}_0(Y)_{\mathbb{L}} \xrightarrow{\rho_{\mathbb{L}}} \pi_1^{\mathrm{ab}}(Y)_{\mathbb{L}} \to \mathrm{H}_1(\Gamma, \mathbb{Z}_{\mathbb{L}}) \to 0.$$

2. *For every set* $\mathbb{L}$ *of primes with* $\#A_0(Y) \in \mathbb{N}(\mathbb{L})$ *we have*

$$\ker(\rho_{\mathbb{L}}) = \ker(\rho_0) = \ker(\rho) = \ker(\hat{\rho}) \cong \mathrm{H}_0(\pi_1^{\mathrm{ab}}(Y^{[\bullet]})).$$

3. *The cokernels of* ρ *and* $\hat{\rho}$ *satisfy*

$$\operatorname{coker}(\hat{\rho}) \cong \mathrm{H}_{-1}(\pi_1^{\mathrm{ab}}(Y^{[\bullet]})) \cong \mathrm{H}_1(\Gamma, \hat{\mathbb{Z}}), \qquad \operatorname{coker}(\rho) \cong (\hat{\mathbb{Z}}/\mathbb{Z})^{\pi_0(Y)} \oplus \mathrm{H}_1(\Gamma, \hat{\mathbb{Z}}).$$

Therefore, we have an exact sequence of abelian groups

$$\mathrm{H}_2(\Gamma, \hat{\mathbb{Z}}) \to \mathrm{CH}_0(Y) \xrightarrow{\rho} \pi_1^{\mathrm{ab}}(Y) \to (\hat{\mathbb{Z}}/\mathbb{Z})^{\pi_0(Y)} \oplus \mathrm{H}_1(\Gamma, \hat{\mathbb{Z}}) \to 0.$$

Proof. For the $\mathbb{L}$-completion we have the analogous results from 3.2 by taking the inverse limit with the additional information that for a finitely generated abelian group A we have $A_{\mathbb{L}} = A \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathbb{L}}$, so that the universal coefficient theorem

$$0 \to \mathrm{H}_i(\Gamma, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathbb{L}} \to \mathrm{H}_i(\Gamma, \mathbb{Z}_{\mathbb{L}}) \to \mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{H}_{i-1}(\Gamma, \mathbb{Z}), \mathbb{Z}_{\mathbb{L}}) \to 0$$

gives that $\mathrm{H}_i(\Gamma, \mathbb{Z}_{\mathbb{L}}) = \mathrm{H}_i(\Gamma, \mathbb{Z})_{\mathbb{L}}$, because $\mathbb{Z}_{\mathbb{L}}$ is torsion-free and therefore we get $\mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{H}_{i-1}(\Gamma, \mathbb{Z}), \mathbb{Z}_{\mathbb{L}}) = 0$. $\ker(\rho_{\mathbb{L}})$ is finite because it lies in the kernel of the degree map (see diagram below), which is finite (cf. [Blo81, Thm. 4.2] and [KS86, Thm. 6.1]). With 3.1 we have the commutative and exact diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker(\rho_0) \arrow[r, "\sim"] \arrow[d, hook] & \ker(\rho) \arrow[rr] \arrow[d, hook] & & 0 \arrow[d] & & \mathbb{Z}^{\pi_0(Y)} \arrow[d, hook, "\mathrm{incl}"] \\
0 \arrow[r] & A_0(Y) \arrow[r] \arrow[d, "\rho_0"] & \mathrm{CH}_0(Y) \arrow[rr, "\deg"] \arrow[d, "\rho"] & & \operatorname{im}\deg \arrow[r] \arrow[d, hook] \arrow[urr, hook] & 0 & \hat{\mathbb{Z}}^{\pi_0(Y)} \\
0 \arrow[r] & \pi_1^{\mathrm{geo}}(Y) \arrow[r] \arrow[d, two heads] & \pi_1^{\mathrm{ab}}(Y) \arrow[rr, "\deg'"] \arrow[d, two heads] & & \operatorname{im}\deg' \arrow[r] \arrow[d, two heads] \arrow[urr, hook] & 0 & \\
0 \arrow[r] & \operatorname{coker}(\rho_0) \arrow[r] & \operatorname{coker}(\rho) \arrow[rr] & & \bigoplus_{W \in \pi_0(Y)} \hat{\mathbb{Z}}/m_W\mathbb{Z} \arrow[r] & 0 &
\end{tikzcd}$$

where $m_W := \gcd_{Y_v \subseteq W}([k_v : k_W])$ and $k_v := \mathcal{O}_{Y_v}(Y_v)$ and $k_W := \mathcal{O}_W(W)$.

Taking the $\hat{\mathbb{Z}}$-completion of this diagram, we see that the two middle horizontal lines stay exact and that $\ker(\rho_0) \cong \ker(\hat{\rho})$. Note that $A_0(Y)$ and $\pi_1^{\mathrm{geo}}(Y)$ are profinite groups and do not change under $\hat{\mathbb{Z}}$-completion, i.e. we have $\hat{\rho}_0 = \rho_0$. Comparing with the original bottom line sequence we get a commutative diagram of exact sequences

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \ker(f) \arrow[r] \arrow[d, hook] & \ker(g) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \operatorname{coker}(\rho_0) \arrow[r] \arrow[d, equal] & \operatorname{coker}(\rho) \arrow[r] \arrow[d, "f"] & \bigoplus_{W\in\pi_0(Y)} \hat{\mathbb{Z}}/m_W\mathbb{Z} \arrow[r] \arrow[d, two heads, "g"] & 0 \\
0 \arrow[r] & \operatorname{coker}(\rho_0) \arrow[r] & \operatorname{coker}(\hat{\rho}) \arrow[r] & \bigoplus_{W\in\pi_0(Y)} \hat{\mathbb{Z}}/m_W\hat{\mathbb{Z}} \arrow[r] & 0
\end{tikzcd}$$

from which it follows that f is surjective, and $\ker(f) \cong \ker(g) \cong (\hat{\mathbb{Z}}/\mathbb{Z})^{\pi_0(Y)}$, which is a divisible group. We therefore get an isomorphism

$$\operatorname{coker}(\rho) \cong (\hat{\mathbb{Z}}/\mathbb{Z})^{\pi_0(Y)} \oplus \operatorname{coker}(\hat{\rho}).$$

Now let $m \in \mathbb{N}(\mathbb{L})$ be an integer with $m \cdot A_0(Y) = 0$ and $m \cdot \pi_1^{\mathrm{geo}}(Y)_{\mathbb{L}\text{-tors}} = 0$, which exists by assumption and since $\pi_1^{\mathrm{geo}}(Y)$ is a finitely generated abelian profinite group. Then ρ_0 factors as

$$\rho_0 : A_0(Y) \to \pi_1^{\mathrm{geo}}(Y)_{\mathbb{L}\text{-tors}} \hookrightarrow \pi_1^{\mathrm{geo}}(Y).$$

Since $\pi_1^{\mathrm{geo}}(Y)/\pi_1^{\mathrm{geo}}(Y)_{\mathbb{L}\text{-tors}}$ is $\mathbb{L}$-torsion-free and $m \in \mathbb{N}(\mathbb{L})$ we get an injection

$$\pi_1^{\mathrm{geo}}(Y)_{\mathbb{L}\text{-tors}}/m \hookrightarrow \pi_1^{\mathrm{geo}}(Y)/m,$$

and therefore a factorisation modulo m:

$$\rho_{0,m} : A_0(Y)/m \to \pi_1^{\mathrm{geo}}(Y)_{\mathbb{L}\text{-tors}}/m \hookrightarrow \pi_1^{\mathrm{geo}}(Y)/m.$$

Because $A_0(Y)/m = A_0(Y)$ and $\pi_1^{\mathrm{geo}}(Y)_{\mathbb{L}\text{-tors}}/m = \pi_1^{\mathrm{geo}}(Y)_{\mathbb{L}\text{-tors}}$ we have

$$\ker(\rho_{0,m}) = \ker(\rho_0) = \ker(\rho).$$

Now taking the limit over $\mathbb{N}(\mathbb{L})$ shows

$$\ker(\rho_{\mathbb{L}}) = \ker(\rho_0) = \ker(\rho)$$

for such sets $\mathbb{L}$ and also for the set of all prime numbers, which gives the results for $\hat{\rho}$. □

Corollary 3.6. *Let the setting be as in 3.5. Furthermore, assume that all components of Y, $Y^{[1]}$ and $Y^{[0]}$ are geometrically connected over k and $Y^{[0]}$ consists of "geometrically simply connected" components, i.e.*

$$\pi_1^{\mathrm{geo}}(Y^{[0]}) := \ker\big(\pi_1^{\mathrm{ab}}(Y^{[0]}) \xrightarrow{\deg} \mathrm{Gal}_k^{\pi_0(Y^{[0]})} \cong \hat{\mathbb{Z}}^{\pi_0(Y^{[0]})}\big)$$

vanishes. (*Note that by* 1.2 *this assumption is equivalent to saying that* $\mathrm{CH}_0(Y^{[0]})$ *is torsion-free.*) *The kernel of the reciprocity map then vanishes and also modulo* n *for every integer* n.

Proof. We have a commutative diagram of complexes

$$\begin{tikzcd}
\pi_1^{\mathrm{geo}}(Y^{[1]}) \arrow[r] \arrow[d, hook] & \pi_1^{\mathrm{geo}}(Y^{[0]}) \arrow[r] \arrow[d, hook] & \pi_1^{\mathrm{geo}}(Y) \arrow[d, hook] \\
\pi_1^{\mathrm{ab}}(Y^{[1]}) \arrow[r] \arrow[d, "\deg"] & \pi_1^{\mathrm{ab}}(Y^{[0]}) \arrow[r] \arrow[d, "\deg"] & \pi_1^{\mathrm{ab}}(Y) \arrow[d, "\deg"] \\
\mathrm{Gal}_k^{\pi_0(Y^{[1]})} \arrow[r, "\alpha"] & \mathrm{Gal}_k^{\pi_0(Y^{[0]})} \arrow[r] & \mathrm{Gal}_k^{\pi_0(Y)}
\end{tikzcd}$$

By geometrical connectedness the degree maps are surjective. Therefore we have a short exact sequence of complexes

$$0 \to \pi_1^{\mathrm{geo}}(Y^{[\bullet]}) \to \pi_1^{\mathrm{ab}}(Y^{[\bullet]}) \to \mathrm{Gal}_k^{\pi_0(Y^{[\bullet]})} \to 0.$$

And because $\mathrm{Gal}_k \cong \hat{\mathbb{Z}}$ is torsion-free, we get short exact sequences of complexes for every integer n:

$$0 \to \pi_1^{\mathrm{geo}}(Y^{[\bullet]})/n \to \pi_1^{\mathrm{ab}}(Y^{[\bullet]})/n \to (\mathbb{Z}/n)^{\pi_0(Y^{[\bullet]})} \to 0.$$

From the long exact sequence follows the exact sequence

$$\mathrm{H}_0(\pi_1^{\mathrm{geo}}(Y^{[\bullet]})/n) \to \mathrm{H}_0(\pi_1^{\mathrm{ab}}(Y^{[\bullet]})/n) \to \mathrm{H}_0((\mathbb{Z}/n)^{\pi_0(Y^{[\bullet]})}),$$

where the first term vanishes by assumption. For the last term we mention that (cf. [Liu02, §2.4, Ex. 4.4])

$$\mathrm{coker}(\alpha) \cong \mathrm{H}_0(\Gamma, \hat{\mathbb{Z}}) \cong \hat{\mathbb{Z}}^{\pi_0(\Gamma)} \xrightarrow{\sim} \hat{\mathbb{Z}}^{\pi_0(Y)}.$$

The isomorphism above also holds with $\mathbb{Z}/n$-coefficients. Hence $\mathrm{H}_0((\mathbb{Z}/n)^{\pi_0(Y^{[\bullet]})}) = 0$. So $\mathrm{H}_0(\pi_1^{\mathrm{ab}}(Y^{[\bullet]})/n)$ vanishes. By 3.2 the last term is isomorphic to $\ker(\rho_n)$. So the claim follows. The same way one shows that $\ker(\rho) = 0$. □

§4. Examples

Here we will give some examples of commonly used varieties and make some essential observations about the interaction of $\mathrm{H}_2(\Gamma_Y)$ with the kernel of the reciprocity map and their torsion parts with $\mathrm{CH}_0(Y)$.

Example 4.1. *Let k be a finite field and $Y = V_+(T_0 \cdot T_1 \cdot T_2 \cdot T_3) \subseteq \mathbb{P}^3_k = \mathrm{Proj}(k[T_0, \ldots, T_3])$ be the surface of the projective tetrahedron. Then the reciprocity map*

$$\rho : \mathrm{CH}_0(Y) \to \pi_1^{\mathrm{ab}}(Y)$$

has trivial kernel and so also do the reciprocity maps modulo n for every integer n. But

$$\mathrm{H}_2(\Gamma_Y, \mathbb{Z}/n) \cong \mathbb{Z}/n.$$

So $\mathrm{H}_2(\Gamma_Y, \mathbb{Z}/n)$ surjects onto $\ker(\rho_n)$, but e.g. does not inject into $\mathrm{CH}_0(Y)/n$.

Proof. The calculation of $\mathrm{H}_2(\Gamma_Y, \mathbb{Z}/n)$ is clear. The rest follows from 3.6 and the fact that $\pi_1(\mathbb{P}^m_k, \overline{x}_i) \cong \mathrm{Gal}_k$, i.e. $\mathbb{P}^m_k$ is "geometrically simply connected". Note that every intersection of irreducible components is isomorphic to a $\mathbb{P}^m_k$. □

This example can be generalized to the following:

Lemma 4.2. *Let k be a field and let $i : W \hookrightarrow Z$ be a closed immersion between proper smooth geometrically connected k-varieties such that $d := \dim Z \geq 3$, and the complement $U = Z \setminus W$ is affine. Further assume one of the following properties:*

- *The natural map $\pi_1(Z, \overline{z}) \to \mathrm{Gal}_k$ has a section for a geometric point $\overline{z}$ on Z (which is the case if Z has a k-rational point).*
- *The cohomology group $\mathrm{H}^2(\mathrm{Gal}_k, \mathbb{Q}/\mathbb{Z})$ vanishes.*

Then the push-forward map $i_* : \pi_1^{\mathrm{ab}}(W) \to \pi_1^{\mathrm{ab}}(Z)$ is an isomorphism.

Proof. For a separably closed field k this is due to [Sat05]. The proof there uses Poincaré duality [Mil80, VI.11.1], the affine Lefschetz theorem [Mil80, VI.7.2] and the assumption dim ≥ 3 to show that $\mathrm{H}^i_c(U, \mathbb{Q}/\mathbb{Z})[\ell]$ vanishes for $i = 1, 2$ and $\ell \neq p = \mathrm{char}(k)$. For $\ell = p$ one needs duality results from [JSS09, Thm. 1.6, 1.7] (cf. [Mos99], [Mil86, §1]), and the corresponding vanishing results from [Suw95, 2.1] for the cohomology of the logarithmic part of de Rham–Witt sheaves (cf. [Ill79]). For an arbitrary field k, one base changes with a separable closure $\overline{k}$ of k and uses the homotopy exact sequence from [Gro71, IX, Thm. 6.1] to get a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \pi_1(\overline{W}) \arrow[r] \arrow[d] & \pi_1(W) \arrow[r] \arrow[d] & \mathrm{Gal}_k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \pi_1(\overline{Z}) \arrow[r] & \pi_1(Z) \arrow[r] & \mathrm{Gal}_k \arrow[r] \arrow[l, dashed, bend right] & 0
\end{tikzcd}$$

suppressing the geometric points. This induces a commutative diagram of exact sequences

$$\begin{tikzcd}
 & \pi_1^{\mathrm{ab}}(\overline{W})_{\mathrm{Gal}_k} \arrow[r] \arrow[d, "\wr"] & \pi_1^{\mathrm{ab}}(W) \arrow[r] \arrow[d] & \mathrm{Gal}_k^{\mathrm{ab}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r, dashed] & \pi_1^{\mathrm{ab}}(\overline{Z})_{\mathrm{Gal}_k} \arrow[r] & \pi_1^{\mathrm{ab}}(Z) \arrow[r, shift right] & \mathrm{Gal}_k^{\mathrm{ab}} \arrow[l, dashed, shift right] \arrow[r] & 0
\end{tikzcd}$$

where the injectivity on the left at the bottom is induced by the section given by assumption or via the Pontryagin dual of the Hochschild–Serre 4-term sequence (with $G = \mathrm{Gal}_k$ for brevity)

$$0 \to \mathrm{H}^1(G, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(Z, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(\overline{Z}, \mathbb{Q}/\mathbb{Z})^G \to \mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z}) = 0.$$

With the previous result over separably closed fields and the snake lemma one gets the claim. □

Example 4.3. *Let k be a finite field and $\mathbb{P}_k^{d+1}$ be the projective space. Let $f_1, \dots, f_n$ be homogeneous irreducible polynomials defining smooth and geometrically connected hypersurfaces in $\mathbb{P}_k^{d+1}$ such that $V_+(f_i)$ and $V_+(f_j)$ intersect smoothly and every connected component of $V_+(f_i, f_j)$ is geometrically connected (e.g. contains a k-rational point). Then the reciprocity map ρ^Y is injective for $Y := V_+(f_1 \cdots f_n) \hookrightarrow \mathbb{P}_k^{d+1}$.*

Proof. If $d < 2$, then Y is a union of points or curves, so that $\mathrm{H}_2(\Gamma_Y) = 0$. For $d \geq 2$ we use 4.2 together with $\pi_1^{\mathrm{ab}}(\mathbb{P}_k^{d+1}) = \mathrm{Gal}_k^{\mathrm{ab}}$ and refer to 3.6. □

Note that this example can be used to construct a huge $\mathrm{H}_2(\Gamma_Y)$ and nevertheless a vanishing kernel of the reciprocity map.

Example 4.4 (cf. [MSA99, Example 4.1], [Sug09, Example 3.2]). *Let k be a finite field and $n > 1$ an integer such that* $\gcd(n, 6\,\mathrm{char}(k)) = 1$ *and k contains a primitive n-th root of unity ζ. Let* $\mathbb{P}_k^3 = \mathrm{Proj}(k[T_0, T_1, T_2, T_3])$ *be the projective space and*

$$V := V_+(T_0^n + T_1^n + T_2^n + T_3^n) \hookrightarrow \mathbb{P}_k^3$$

a Fermat surface and consider the free action on V given by

$$\tau : (T_0 : T_1 : T_2 : T_3) \mapsto (T_0 : \zeta T_1 : \zeta^2 T_2 : \zeta^3 T_3).$$

Then $X := V / \langle \tau \rangle$ is a smooth and projective surface. Let

$$L = V_+(T_0 + T_1,\, T_2 + T_3) \quad \textit{and} \quad L' = V_+(T_0 + T_1,\, T_2 + \zeta T_3)$$

be two lines on V and C, C' be their images in X. Then $D := C \cup C'$ is a simple normal crossing divisor on X, and C and C' meet in two k-rational points. Set

$$Y := (X \times_k O) \cup (X \times_k \infty) \cup (D \times_k \mathbb{P}^1_k) \subseteq X \times_k \mathbb{P}^1_k,$$

where $O = (0:1)$ and $\infty = (1:0)$ are rational points on $\mathbb{P}^1_k$. Then Y is a simple normal crossing surface in $X \times_k \mathbb{P}^1_k$ which is projective and geometrically connected over k, and the reciprocity map

$$\rho_Y : \mathrm{CH}_0(Y) \to \pi_1^{\mathrm{ab}}(Y)$$

has $\ker(\rho_Y) \cong \mathbb{Z}/n$. Moreover, $\ker(\rho_{Y\otimes F}) \cong \mathbb{Z}/n$ for every finite field extension $F|k$.

Now let $C \cap C' = \{c_1, c_2\}$ and let Γ_Y be the dual complex to Y associated to a numbering of the irreducible components. Then

$$\mathrm{H}_0(\Gamma_Y, \mathbb{Z}) = \mathbb{Z}, \quad \mathrm{H}_1(\Gamma_Y, \mathbb{Z}) = 0, \quad \mathrm{H}_2(\Gamma_Y, \mathbb{Z}) \cong \mathbb{Z},$$

and therefore for every integer m,

$$\mathrm{H}_0(\Gamma_Y, \mathbb{Z}/m) = \mathbb{Z}/m, \quad \mathrm{H}_1(\Gamma_Y, \mathbb{Z}/m) = 0, \quad \mathrm{H}_2(\Gamma_Y, \mathbb{Z}/m) \cong \mathbb{Z}/m.$$

Proof. For the first statements see [Sug09, Example 3.2]. For the homology groups we mention the following: Y, $Y^{[0]}$ resp., has four irreducible components:

$$Y_1 = X \times_k O, \quad Y_2 = C \times_k \mathbb{P}^1_k, \quad Y_3 = C' \times_k \mathbb{P}^1_k, \quad Y_4 = X \times_k \infty.$$

There are six connected components in $Y^{[1]}$:

$$\begin{aligned} &Y_{12} = Y_1 \cap Y_2 = C \times_k O, && Y_{13} = Y_1 \cap Y_3 = C' \times_k O,\\ &Y_{23}^1 \text{ in } Y_2 \cap Y_3 : c_1 \times_k \mathbb{P}^1_k, && Y_{23}^2 \text{ in } Y_2 \cap Y_3 : c_2 \times_k \mathbb{P}^1_k,\\ &Y_{24} = Y_2 \cap Y_4 = C \times_k \infty, && Y_{34} = Y_3 \cap Y_4 = C' \times_k \infty. \end{aligned}$$

And there are four connected components in $Y^{[2]}$:

$$\begin{aligned} &Y_{123}^1 \text{ in } Y_1 \cap Y_2 \cap Y_3 : c_1 \times_k O, && Y_{234}^1 \text{ in } Y_2 \cap Y_3 \cap Y_4 : c_1 \times_k \infty,\\ &Y_{123}^2 \text{ in } Y_1 \cap Y_2 \cap Y_3 : c_2 \times_k O, && Y_{234}^2 \text{ in } Y_2 \cap Y_3 \cap Y_4 : c_2 \times_k \infty. \end{aligned}$$

The homology groups $\mathrm{H}_i(\Gamma_Y, \mathbb{Z})$ can then be computed combinatorially. And the homology groups with coefficients in $\mathbb{Z}/m$ can be computed from the homology groups with coefficients in $\mathbb{Z}$ by the universal coefficient theorem, observing that all $\mathrm{H}_i(\Gamma_Y, \mathbb{Z})$ are torsion-free. □

4.4 shows that the groups $\mathrm{H}_2(\Gamma_Y, \mathbb{Z})$ and $\mathrm{H}_2(\Gamma_Y, \hat{\mathbb{Z}})$ are torsion-free, but the reciprocity map and the reciprocity map modulo n have kernel $\mathbb{Z}/n$. Therefore the kernel is not given by the torsion part of $\mathrm{H}_2(\Gamma_Y, \mathbb{Z})$ or $\mathrm{H}_2(\Gamma_Y, \hat{\mathbb{Z}})$.

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