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Some Inequalities of Kato Type for Sequences of Operators in Hilbert Spaces

by

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Abstract

By the use of the celebrated Kato inequality we obtain some new inequalities for n-tuples of bounded linear operators on a complex Hilbert space H. Natural applications for functions defined by power series of normal operators as well as different inequalities concerning the Euclidean norm, the Euclidean radius, the s-1-norms and the s-1-radius of an n-tuple of operators are given as well.

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§1. Introduction

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$.

If P is a positive selfadjoint operator on H, i.e. $\langle Px,x\rangle \geq 0$ for any $x \in H$, then the following inequality is a generalization of the Schwarz inequality in H:

$$|\langle Px,y\rangle|^2 \leq \langle Px,x\rangle\langle Py,y\rangle \tag{1.1}$$

for any $x, y \in H$.

The following inequality is of interest as well (see [12, p. 221]).

Let P be a positive selfadjoint operator on H. Then

$$\|Px\|^2 \leq \|P\|\langle Px,x\rangle \tag{1.2}$$

for any $x \in H$.

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The "square root" of a positive bounded selfadjoint operator on H can be defined as follows (see for instance [12, p. 240]): *If the operator* $A \in \mathcal{B}(H)$ *is selfadjoint and positive, then there exists a unique positive selfadjoint operator* $B := \sqrt{A} \in \mathcal{B}(H)$ *such that* $B^2 = A$. *If* A *is invertible, then so is* B.

If $A \in \mathcal{B}(H)$, then the operator A^*A is selfadjoint and positive. Define the "absolute value" operator by $|A| := \sqrt{A^*A}$.

In 1952, Kato [13] proved the following celebrated generalization of the Schwarz inequality for any bounded linear operator T on H:

$$|\langle Tx, y\rangle|^2 \le \langle (T^*T)^\alpha x, x\rangle \langle (TT^*)^{1-\alpha} y, y\rangle \tag{1.3}$$

for any $x, y \in H$ and $\alpha \in [0,1]$. Utilizing the modulus notation introduced above, we can write (1.3) as follows:

$$|\langle Tx, y\rangle|^2 \le \langle |T|^{2\alpha} x, x\rangle \langle |T^*|^{2(1-\alpha)} y, y\rangle \tag{1.4}$$

for any $x, y \in H$ and $\alpha \in [0,1]$.

It is useful to observe that, if $T = N$, a normal operator, i.e., $NN^* = N^*N$, then the inequality (1.4) can be written as

$$|\langle Nx, y\rangle|^2 \le \langle |N|^{2\alpha} x, x\rangle \langle |N|^{2(1-\alpha)} y, y\rangle, \tag{1.5}$$

and in particular, for selfadjoint operators A we can state it as

$$|\langle Ax, y\rangle| \le \big\| |A|^\alpha x \big\| \, \big\| |A|^{1-\alpha} y \big\| \tag{1.6}$$

for any $x, y \in H$ and $\alpha \in [0,1]$.

If $T = U$, a unitary operator, i.e., $UU^* = U^*U = 1_H$, then (1.4) becomes

$$|\langle Ux, y\rangle| \le \|x\| \|y\|$$

for any $x, y \in H$, which provides a natural generalization for the Schwarz inequality in H.

It is natural to consider symmetric powers in the inequalities above: if we choose $\alpha = 1/2$ in (1.4)–(1.6) then we get, for any $x, y \in H$,

$$|\langle Tx, y\rangle|^2 \le \langle |T| x, x\rangle \langle |T^*| y, y\rangle, \tag{1.7}$$

$$|\langle Nx, y\rangle|^2 \le \langle |N| x, x\rangle \langle |N| y, y\rangle, \tag{1.8}$$

$$|\langle Ax, y\rangle| \le \big\| |A|^{1/2} x \big\| \, \big\| |A|^{1/2} y \big\| \tag{1.9}$$

respectively.

It is also worth observing that, if we take the supremum over $y \in H$ with $\|y\| = 1$ in (1.4) then we get

$$\|Tx\|^2 \le \|T\|^{2(1-\alpha)} \langle |T|^{2\alpha} x, x\rangle \tag{1.10}$$

for any $x \in H$, or equivalently

$$\|Tx\| \le \big\||T|^{\alpha}x\big\|\,\|T\|^{1-\alpha} \tag{1.11}$$

for any $x \in H$.

If we take $\alpha = 1/2$ in (1.10), then we get

$$\|Tx\|^2 \le \|T\|\langle |T|x, x\rangle \tag{1.12}$$

for any $x \in H$, which in the particular case of $T = P$, a positive operator, yields (1.2).

For various interesting generalizations, extensions and related results, see [2]–[11], [14]–[20] and [23].

In this paper we pursue a different path. By the use of Kato's inequality (1.4) and by utilizing only elementary techniques and tools such as the discrete Hölder and Cauchy–Bunyakovsky–Schwarz inequalities we provide some new inequalities for n-tuples of bounded linear operators on a complex Hilbert space H. Natural applications for functions defined by power series of normal operators as well as various inequalities concerning the Euclidean norm, the Euclidean radius, the s-1-norms and the s-1-radius of an n-tuple of operators are given as well.

§2. Vector inequalities

The following vector inequality holds:

Theorem 1. *Let $(T_1, \ldots, T_n) \in \mathcal{B}(H) \times \cdots \times \mathcal{B}(H) =: \mathcal{B}^{(n)}(H)$ be an n-tuple of bounded linear operators on the Hilbert space $(H; \langle\cdot,\cdot\rangle)$ and $(p_1, \ldots, p_n) \in \mathbb{R}_+^{*n}$ an n-tuple of nonnegative weights, not all zero. Then*

$$\sum_{j=1}^{n} p_j |\langle T_j x, y\rangle|^2 \le \Big\langle \sum_{j=1}^{n} p_j |T_j|^2 x, x\Big\rangle^{\alpha} \Big\langle \sum_{j=1}^{n} p_j |T_j^*|^2 y, y\Big\rangle^{1-\alpha} \tag{2.1}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$ and $\alpha \in [0,1]$.

Proof. We must prove the inequalities only in the cases $\alpha \in (0,1)$, since the cases $\alpha = 0$ or $\alpha = 1$ follow directly from the corresponding cases of Kato's inequality.

Utilizing Kato's inequality for the operator T_j, $j \in \{1, \ldots, n\}$, we have

$$\begin{aligned} \sum_{j=1}^{n} p_j |\langle T_j x, y\rangle|^2 &\le \sum_{j=1}^{n} p_j \langle |T_j|^{2\alpha} x, x\rangle \langle |T_j^*|^{2(1-\alpha)} y, y\rangle \\ &\le \sum_{j=1}^{n} p_j \langle |T_j|^2 x, x\rangle^{\alpha} \langle |T_j^*|^2 y, y\rangle^{1-\alpha} \end{aligned} \tag{2.2}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$, where for the last inequality we have used the Hölder–McCarthy inequality $\langle P^r x, x\rangle \le \langle Px, x\rangle^r$ that holds for any positive operator P and any power $r \in (0,1)$.

Now, making use of the weighted discrete Hölder inequality

$$\sum_{j=1}^{n} p_j a_j b_j \le \Big(\sum_{j=1}^{n} p_j a_j^p\Big)^{1/p} \Big(\sum_{j=1}^{n} p_j b_j^q\Big)^{1/q}, \qquad p, q > 1,\ \frac{1}{p} + \frac{1}{q} = 1,$$

where $(a_1, \dots, a_n), (b_1, \dots, b_n) \in \mathbb{R}_+^n$, and choosing $a_j = \langle |T_j|^2 x, x\rangle^\alpha$, $b_j = \langle |T_j^*|^2 y, y\rangle^{1-\alpha}$, $p = 1/\alpha$ and $q = 1/(1-\alpha)$, we get

$$\begin{aligned}
(2.3)\qquad & \sum_{j=1}^{n} p_j \langle |T_j|^2 x, x\rangle^\alpha \langle |T_j^*|^2 y, y\rangle^{1-\alpha} \\
&\le \Big\{\sum_{j=1}^{n} p_j [\langle |T_j|^2 x, x\rangle^\alpha]^{1/\alpha}\Big\}^\alpha \Big\{\sum_{j=1}^{n} p_j [\langle |T_j^*|^2 y, y\rangle^{1-\alpha}]^{1/(1-\alpha)}\Big\}^{1-\alpha} \\
&= \Big\{\sum_{j=1}^{n} p_j \langle |T_j|^2 x, x\rangle\Big\}^\alpha \Big\{\sum_{j=1}^{n} p_j \langle |T_j^*|^2 y, y\rangle\Big\}^{1-\alpha} \\
&= \Big\langle \sum_{j=1}^{n} p_j |T_j|^2 x, x\Big\rangle^\alpha \Big\langle \sum_{j=1}^{n} p_j |T_j^*|^2 y, y\Big\rangle^{1-\alpha}
\end{aligned}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

From (2.2) and (2.3) we deduce the desired inequality (2.1). □

Remark 1. For $y = x$ the inequality (2.1) becomes the following simpler result that is useful for deriving numerical radius inequalities:

$$\begin{aligned}
(2.4)\qquad \sum_{j=1}^{n} p_j |\langle T_j x, x\rangle|^2 &\le \Big\langle \sum_{j=1}^{n} p_j |T_j|^2 x, x\Big\rangle^\alpha \Big\langle \sum_{j=1}^{n} p_j |T_j^*|^2 x, x\Big\rangle^{1-\alpha} \\
&\le \Big\langle \sum_{j=1}^{n} p_j [\alpha |T_j|^2 + (1-\alpha)|T_j^*|^2] x, x\Big\rangle
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

Let $(N_1, \dots, N_n) \in \mathcal{B}^{(n)}(H)$ be an n-tuple of normal operators on the Hilbert space $(H; \langle \cdot, \cdot\rangle)$. Then Theorem 1 yields the following result that can be utilized to obtain various inequalities for functions of normal operators defined by power series:

$$(2.5)\qquad \sum_{j=1}^{n} p_j |\langle N_j x, y\rangle|^2 \le \Big\langle \sum_{j=1}^{n} p_j |N_j|^2 x, x\Big\rangle^\alpha \Big\langle \sum_{j=1}^{n} p_j |N_j|^2 y, y\Big\rangle^{1-\alpha}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$, $\alpha \in [0,1]$ and $(p_1, \dots, p_n) \in \mathbb{R}_+^{*n}$.

In particular, (2.5) implies the following inequality for normal operators:

$$\sum_{j=1}^{n} p_j |\langle N_j x, x\rangle|^2 \le \left\langle \sum_{j=1}^{n} p_j |N_j|^2 x, x\right\rangle \tag{2.6}$$

for any $x \in H$ with $\|x\| = 1$.

The following result provides upper bounds for the sum $\sum_{j=1}^{n} p_j |\langle T_j x, y\rangle|$ and has important consequences in refining the fundamental triangle inequality for the operator norm.

Theorem 2. *Under the assumptions of Theorem 1 we have*

$$\sum_{j=1}^{n} p_j |\langle T_j x, y\rangle| \le \left\langle \sum_{j=1}^{n} p_j |T_j|^{2\alpha} x, x\right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j |T_j^*|^{2(1-\alpha)} y, y\right\rangle^{1/2} \tag{2.7}$$

for any $x, y \in H$.

Proof. From Kato's inequality for T_j, $j \in \{1, \dots, n\}$, we have

$$\sum_{j=1}^{n} p_j |\langle T_j x, y\rangle| \le \sum_{j=1}^{n} p_j \langle |T_j|^{2\alpha} x, x\rangle^{1/2} \langle |T_j^*|^{2(1-\alpha)} y, y\rangle^{1/2} \tag{2.8}$$

for any $x, y \in H$.

Now, making use of the weighted discrete Cauchy–Bunyakovsky–Schwarz inequality

$$\sum_{j=1}^{n} p_j a_j b_j \le \left(\sum_{j=1}^{n} p_j a_j^2\right)^{1/2} \left(\sum_{j=1}^{n} p_j b_j^2\right)^{1/2}$$

where $(a_1, \dots, a_n), (b_1, \dots, b_n) \in \mathbb{R}_+^n$, and choosing $a_j = \langle |T_j|^{2\alpha} x, x\rangle^{1/2}$ and $b_j = \langle |T_j^*|^{2(1-\alpha)} y, y\rangle^{1/2}$, we get

$$\begin{aligned} &\sum_{j=1}^{n} p_j \langle |T_j|^{2\alpha} x, x\rangle^{1/2} \langle |T_j^*|^{2(1-\alpha)} y, y\rangle^{1/2} \\ &\quad \le \left\{\sum_{j=1}^{n} p_j [\langle |T_j|^{2\alpha} x, x\rangle^{1/2}]^2\right\}^{1/2} \left\{\sum_{j=1}^{n} p_j [\langle |T_j^*|^{2(1-\alpha)} y, y\rangle^{1/2}]^2\right\}^{1/2} \\ &\quad = \left\{\sum_{j=1}^{n} p_j \langle |T_j|^{2\alpha} x, x\rangle\right\}^{1/2} \left\{\sum_{j=1}^{n} p_j \langle |T_j^*|^{2(1-\alpha)} y, y\rangle\right\}^{1/2} \\ &\quad = \left\langle \sum_{j=1}^{n} p_j |T_j|^{2\alpha} x, x\right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j |T_j^*|^{2(1-\alpha)} y, y\right\rangle^{1/2} \end{aligned} \tag{2.9}$$

for any $x, y \in H$. ∎

Remark 2. One of possible vector-valued extensions of (2.7) is as follows:

$$\sum_{j=1}^{n} p_j |\langle T_j x, x\rangle| \leq \left\langle \sum_{j=1}^{n} p_j |T_j|^{2\alpha} x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j |T_j^*|^{2(1-\alpha)} x, x \right\rangle^{1/2} \leq \left\langle \sum_{j=1}^{n} p_j \left[\frac{|T_j|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2} \right] x, x \right\rangle \tag{2.10}$$

for any $x \in H$.

Remark 3. The case of symmetric powers in (2.7), when $\alpha = 1/2$, is of interest since it yields the simpler result

$$\sum_{j=1}^{n} p_j |\langle T_j x, y\rangle| \leq \left\langle \sum_{j=1}^{n} p_j |T_j| x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j |T_j^*| y, y \right\rangle^{1/2} \tag{2.11}$$

for any $x, y \in H$.

In particular, from (2.10) we derive

$$\sum_{j=1}^{n} p_j |\langle T_j x, x\rangle| \leq \left\langle \sum_{j=1}^{n} p_j |T_j| x, x \right\rangle^{1/2} \langle \sum_{j=1}^{n} p_j |T_j^*| x, x \rangle^{1/2} \leq \left\langle \sum_{j=1}^{n} p_j [\frac{|T_j| + |T_j^*|}{2}] x, x \right\rangle \tag{2.12}$$

for any $x \in H$.

Let $(N_1, \ldots, N_n) \in \mathcal{B}^{(n)}(H)$ be an n-tuple of normal operators. Then from Theorem 2 we have

$$\sum_{j=1}^{n} p_j |\langle N_j x, y\rangle| \leq \left\langle \sum_{j=1}^{n} p_j |N_j|^{2\alpha} x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j |N_j|^{2(1-\alpha)} y, y \right\rangle^{1/2} \tag{2.13}$$

for any $x, y \in H$. In particular,

$$\sum_{j=1}^{n} p_j |\langle N_j x, x\rangle| \leq \left\langle \sum_{j=1}^{n} p_j |N_j|^{2\alpha} x, x \right\rangle^{1/2} \left\langle \sum_{j=1}^{n} p_j |N_j|^{2(1-\alpha)} x, x \right\rangle^{1/2} \leq \left\langle \sum_{j=1}^{n} p_j \left[\frac{|N_j|^{2\alpha} + |N_j|^{2(1-\alpha)}}{2} \right] x, x \right\rangle \tag{2.14}$$

for any $x \in H$.

§3. Functional inequalities

For any power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ the power series $f_A(z) := \sum_{n=0}^{\infty} |a_n| z^n$ has the same radius of convergence. Obviously if all coefficients $a_n \geq 0$, then $f_A = f$. For more information on this transform, see also [22, p. 246].

For example, if

$$
(3.1)\qquad
\begin{aligned}
f(z) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^n = \ln \frac{1}{1+z}, \qquad z \in D(0,1);\\
g(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \cos z, \qquad z \in \mathbb{C};\\
h(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sin z, \qquad z \in \mathbb{C};\\
l(z) &= \sum_{n=0}^{\infty} (-1)^n z^n = \frac{1}{1+z}, \qquad z \in D(0,1),
\end{aligned}
$$

then

$$
(3.2)\qquad
\begin{aligned}
f_A(z) &= \sum_{n=1}^{\infty} \frac{1}{n} z^n = \ln \frac{1}{1-z}, \qquad z \in D(0,1);\\
g_A(z) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh z, \qquad z \in \mathbb{C};\\
h_A(z) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = \sinh z, \qquad z \in \mathbb{C};\\
l_A(z) &= \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \qquad z \in D(0,1).
\end{aligned}
$$

The following result is a functional generalization of Kato's inequality (1.5) for normal operators.

Theorem 3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function defined by a power series with real coefficients and convergent on the open disk $D(0,R) := \{z \in \mathbb{C} : |z| < R\}$, $R > 0$. If N is a normal operator on the Hilbert space H and for some $\alpha \in (0,1)$ we have $\|N\|^{2\alpha}, \|N\|^{2(1-\alpha)} < R$, then*

$$
(3.3)\qquad |\langle f(N)x, y\rangle| \le \langle f_A(|N|^{2\alpha})x, x\rangle^{1/2} \langle f_A(|N|^{2(1-\alpha)})y, y\rangle^{1/2}
$$

for any $x, y \in H$.

In particular, if $\|N\| < R$, then

$$
(3.4)\qquad |\langle f(N)x, y\rangle| \le \langle f_A(|N|)x, x\rangle^{1/2} \langle f_A(|N|)y, y\rangle^{1/2}
$$

for any $x, y \in H$.

Proof. If N is a normal operator, then for any $j \in \mathbb{N}$ we have

$$
|N^j|^2 = (N^*N)^j = |N|^{2j}.
$$

Now, utilizing the inequality (2.13) we can write

$$\left|\left\langle\sum_{j=0}^{n} a_j N^j x, y\right\rangle\right| \le \sum_{j=0}^{n} |a_j|\,|\langle N^j x, y\rangle| \le \left\langle\sum_{j=0}^{n}|a_j|\,|N^j|^{2\alpha}x,x\right\rangle^{1/2}\left\langle\sum_{j=0}^{n}|a_j|\,|N^j|^{2(1-\alpha)}y,y\right\rangle^{1/2} = \left\langle\sum_{j=0}^{n}|a_j|(|N|^{2\alpha})^j x,x\right\rangle^{1/2}\left\langle\sum_{j=0}^{n}|a_j|(|N|^{2(1-\alpha)})^j y,y\right\rangle^{1/2} \tag{3.5}$$

for any $x, y \in H$ and $n \in \mathbb{N}$.

Since $\|N\|^{2\alpha}, \|N\|^{2(1-\alpha)} < R$, it follows that the series $\sum_{j=0}^{\infty}|a_j|(|N|^{2\alpha})^j$ and $\sum_{j=0}^{\infty}|a_j|(|N|^{2(1-\alpha)})^j$ are absolutely convergent in $\mathcal{B}(H)$, and by taking the limit as $n \to \infty$ in (3.5) we deduce the desired result (3.3). □

Remark 4. Assume that f, R, N and α are as in Theorem 3. If we take the supremum in (3.3) over $y \in H$ with $\|y\| = 1$, then we get

$$\|f(N)x\| \le \langle f_A(|N|^{2\alpha})x,x\rangle^{1/2}\|f_A(|N|^{2(1-\alpha)})\|^{1/2} \tag{3.6}$$

for any $x \in H$, which yields the operator norm inequality

$$\|f(N)\| \le \|f_A(|N|^{2\alpha})\|^{1/2}\|f_A(|N|^{2(1-\alpha)})\|^{1/2}. \tag{3.7}$$

If we take $y = x$ in (3.3), then we get

$$|\langle f(N)x,x\rangle| \le \langle f_A(|N|^{2\alpha})x,x\rangle^{1/2}\langle f_A(|N|^{2(1-\alpha)})x,x\rangle^{1/2} \le \left\langle\left[\frac{f_A(|N|^{2\alpha}) + f_A(|N|^{2(1-\alpha)})}{2}\right]x,x\right\rangle \tag{3.8}$$

for any $x \in H$. This implies the following inequalities for the numerical radius:

$$w(f(N)) \le \begin{cases} \|f_A(|N|^{2\alpha})\|^{1/2}\|f_A(|N|^{2(1-\alpha)})\|^{1/2}; \\ \left\|\dfrac{f_A(|N|^{2\alpha}) + f_A(|N|^{2(1-\alpha)})}{2}\right\|. \end{cases} \tag{3.9}$$

Making use of the examples in (3.1) and (3.2) we get the following vector inequalities, for any $x, y \in H$:

$$|\langle \ln(1_H + N)^{-1}x,y\rangle| \le \langle \ln(1_H - |N|^{2\alpha})^{-1}x,x\rangle^{1/2}\langle \ln(1_H - |N|^{2\alpha})^{-1}y,y\rangle^{1/2}, \quad \|N\| < 1;$$

$$|\langle (1_H + N)^{-1}x,y\rangle| \le \langle (1_H - |N|^{2\alpha})^{-1}x,x\rangle^{1/2}\langle (1_H - |N|^{2\alpha})^{-1}y,y\rangle^{1/2}, \quad \|N\| < 1;$$

$$|\langle \sin(N)x,y\rangle| \le \langle \sinh(|N|^{2\alpha})x,x\rangle^{1/2}\langle \sinh(|N|^{2(1-\alpha)})y,y\rangle^{1/2} \quad \text{for any } N;$$

$$|\langle \cos(N)x,y\rangle| \le \langle \cosh(|N|^{2\alpha})x,x\rangle^{1/2}\langle \cosh(|N|^{2(1-\alpha)})y,y\rangle^{1/2} \quad \text{for any } N.$$

We also have, for instance, the following norm inequalities:

$$\|\sin(N)\| \le \|\sinh(|N|^{2\alpha})\|^{1/2}\|\sinh(|N|^{2(1-\alpha)})\|^{1/2},$$
$$\|\cos(N)\| \le \|\cosh(|N|^{2\alpha})\|^{1/2}\|\cosh(|N|^{2(1-\alpha)})\|^{1/2}$$

for any normal operator N, and

$$\|\ln(1_H + N)^{-1}\| \le \|\ln(1_H - |N|^{2\alpha})^{-1}\|^{1/2}\|\ln(1_H - |N|^{2\alpha})^{-1}\|^{1/2}$$

for N with $\|N\| < 1$.

If we utilize the following power series representations with nonnegative coefficients:

$$\begin{aligned}
\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) &= \sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}, \quad z \in D(0,1);\\
\sin^{-1}(z) &= \sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)}{\sqrt{\pi}(2n+1)n!}z^{2n+1}, \quad z \in D(0,1);\\
\tanh^{-1}(z) &= \sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}, \quad z \in D(0,1);\\
{}_2F_1(\alpha,\beta,\gamma,z) &= \sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}z^{n}, \quad \alpha,\beta,\gamma > 0,\ z \in D(0,1),
\end{aligned} \tag{3.10}$$

where Γ is the Gamma function, then we get the following vector inequalities:

$$\begin{aligned}
&|\langle \exp(N)x,y\rangle| \le \langle \exp(|N|^{2\alpha})x,x\rangle^{1/2}\langle \exp(|N|^{2(1-\alpha)})y,y\rangle^{1/2},\\
&\left|\left\langle \ln\left(\frac{1_H+N}{1_H-N}\right)x,y\right\rangle\right|\\
&\qquad \le \left\langle \ln\left(\frac{1_H+|N|^{2\alpha}}{1_H-|N|^{2\alpha}}\right)x,x\right\rangle^{1/2}\left\langle \ln\left(\frac{1_H+|N|^{2(1-\alpha)}}{1_H-|N|^{2(1-\alpha)}}\right)y,y\right\rangle^{1/2},\\
&|\langle \sin^{-1}(N)x,y\rangle|\\
&\qquad \le \langle \sin^{-1}(|N|^{2\alpha})x,x\rangle^{1/2}\langle \sin^{-1}(|N|^{2(1-\alpha)})y,y\rangle^{1/2},\\
&|\langle \tanh^{-1}(N)x,y\rangle|\\
&\qquad \le \langle \tanh^{-1}(|N|^{2\alpha})x,x\rangle^{1/2}\langle \tanh^{-1}(|N|^{2(1-\alpha)})y,y\rangle^{1/2},\\
&|\langle {}_2F_1(\alpha,\beta,\gamma,N)x,y\rangle|\\
&\qquad \le \langle {}_2F_1(\alpha,\beta,\gamma,|N|^{2\alpha})x,x\rangle^{1/2}\langle {}_2F_1(\alpha,\beta,\gamma,|N|^{2(1-\alpha)})y,y\rangle^{1/2},
\end{aligned} \tag{3.11}$$

for any $x, y \in H$. The first inequality in (3.11) holds for any normal operator N while the others require the assumption $\|N\| < 1$.

We also have the norm inequalities

$$\begin{aligned}\|\exp(N)\| &\le \|\exp(|N|^{2\alpha})\|^{1/2}\|\exp(|N|^{2(1-\alpha)})\|^{1/2},\\ \|\cosh(N)\| &\le \|\cosh(|N|^{2\alpha})\|^{1/2}\|\cosh(|N|^{2(1-\alpha)})\|^{1/2},\\ \|\sinh(N)\| &\le \|\sinh(|N|^{2\alpha})\|^{1/2}\|\sinh(|N|^{2(1-\alpha)})\|^{1/2},\end{aligned}$$

for any normal operator N, and

$$\left\|\ln\left(\frac{1_H+N}{1_H-N}\right)\right\| \le \left\|\ln\left(\frac{1_H+|N|^{2\alpha}}{1_H-|N|^{2\alpha}}\right)\right\|^{1/2}\left\|\ln\left(\frac{1_H+|N|^{2(1-\alpha)}}{1_H-|N|^{2(1-\alpha)}}\right)\right\|^{1/2}$$

for N with $\|N\|<1$.

A similar result is the following:

Theorem 4. *Let $f(z)=\sum_{n=0}^{\infty}a_n z^n$ be a function defined by a power series with real coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. If N is a normal operator on the Hilbert space H, and $z\in\mathbb{C}$ with $|z|^2, |z|\,\|N\|, \|N\|^2<R$, then*

$$|\langle f(zN)x,y\rangle|^2 \le f_A(|z|^2)\langle f_A(|N|^2)x,x\rangle^{\alpha}\langle f_A(|N|^2)y,y\rangle^{1-\alpha} \tag{3.12}$$

for any $x,y\in H$ and $\alpha\in[0,1]$.

In particular,

$$|\langle f(zN)x,y\rangle|^2 \le f_A(|z|^2)\langle f_A(|N|^2)x,x\rangle^{1/2}\langle f_A(|N|^2)y,y\rangle^{1/2}. \tag{3.13}$$

Proof. By the Cauchy–Bunyakovsky–Schwarz inequality we have

$$\left|\left\langle\sum_{j=0}^{n}a_j z^j N^j x,y\right\rangle\right|^2 \le \sum_{j=0}^{n}|a_j|\,|z|^{2j}\sum_{j=0}^{n}|a_j|\,|\langle N^j x,y\rangle|^2 \tag{3.14}$$

for any $n\in\mathbb{N}$ and $x,y\in H$.

Utilizing (2.5) we also have

$$\begin{aligned}\sum_{j=0}^{n}|a_j|\,|\langle N^j x,y\rangle|^2 &\le \left\langle\sum_{j=0}^{n}|a_j|\,|N^j|^2 x,x\right\rangle^{\alpha}\left\langle\sum_{j=0}^{n}|a_j|\,|N^j|^2 y,y\right\rangle^{1-\alpha}\\ &= \left\langle\sum_{j=0}^{n}|a_j|\,|N|^{2j} x,x\right\rangle^{\alpha}\left\langle\sum_{j=0}^{n}|a_j|\,|N|^{2j} y,y\right\rangle^{1-\alpha}\end{aligned} \tag{3.15}$$

for any $n\in\mathbb{N}$ and $x,y\in H$.

By making use of (3.14) and (3.15) we get

$$\left|\left\langle\sum_{j=0}^{n} a_j z^j N^j x, y\right\rangle\right|^2 \leq \sum_{j=0}^{n} |a_j|\,|z|^{2j}\left\langle\sum_{j=0}^{n} |a_j|\,|N|^{2j} x, x\right\rangle^{\alpha}\left\langle\sum_{j=0}^{n} |a_j|\,|N|^{2j} y, y\right\rangle^{1-\alpha} \tag{3.16}$$

for any $n \in \mathbb{N}$ and $x, y \in H$.

Since the series $\sum_{j=0}^{\infty} |a_j||N|^{2j}$ is absolutely convergent, letting $n \to \infty$ in (3.16) yields the desired result (3.12). □

Remark 5. Assume that f, R, z, N and α are as in Theorem 4. If we take the supremum in (3.12) over $y \in H$ with $\|y\| = 1$, then we get

$$\|f(zN)x\|^2 \leq f_A(|z|^2)\langle f_A(|N|^2)x, x\rangle^{\alpha}\|f_A(|N|^2)\|^{1-\alpha} \tag{3.17}$$

for any $x \in H$, which implies the operator norm inequality

$$\|f(zN)\|^2 \leq f_A(|z|^2)\|f_A(|N|^2)\|. \tag{3.18}$$

If we take $y = x$ in (3.12), then we get

$$|\langle f(zN)x, x\rangle|^2 \leq f_A(|z|^2)\langle f_A(|N|^2)x, x\rangle \tag{3.19}$$

for any $x \in H$.

From (3.12) we get the vector inequalities

$$\begin{aligned}
|\langle \exp(zN)x, y\rangle|^2 &\leq \exp(|z|^2)\langle \exp(|N|^2)x, x\rangle^{\alpha}\langle \exp(|N|^2)y, y\rangle^{1-\alpha},\\
|\langle \sin(zN)x, y\rangle|^2 &\leq \sinh(|z|^2)\langle \sinh(|N|^2)x, x\rangle^{\alpha}\langle \sinh(|N|^2)y, y\rangle^{1-\alpha},\\
|\langle \cos(zN)x, y\rangle|^2 &\leq \cosh(|z|^2)\langle \cosh(|N|^2)x, x\rangle^{\alpha}\langle \cosh(|N|^2)y, y\rangle^{1-\alpha},
\end{aligned}$$

for any normal operator N, any complex number z and any $x, y \in H$.

We also have, for instance, from (3.18) the following norm inequalities:

$$\begin{aligned}
\|\exp(zN)\|^2 &\leq \exp(|z|^2)\|\exp(|N|^2)\|,\\
\|\sin(zN)\|^2 &\leq \sinh(|z|^2)\|\sinh(|N|^2)\|,
\end{aligned}$$

for any normal operator N and any complex number z.

Similar results can be stated for other functions.

§4. Applications for the Euclidean norm

In [21], the author has introduced the following norm on $\mathcal{B}^{(n)}(H)$:

$$\|(T_1,\dots,T_n)\|_e := \sup_{(\lambda_1,\dots,\lambda_n)\in\mathbb{B}_n} \|\lambda_1 T_1 + \cdots + \lambda_n T_n\|, \tag{4.1}$$

where $(T_1,\dots,T_n)\in\mathcal{B}^{(n)}(H)$ and $\mathbb{B}_n := \{(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n \mid \sum_{j=1}^n |\lambda_j|^2 \le 1\}$ is the Euclidean closed ball in $\mathbb{C}^n$.

It is clear that $\|\cdot\|_e$ is a norm on $\mathcal{B}^{(n)}(H)$ and for any $(T_1,\dots,T_n)\in\mathcal{B}^{(n)}(H)$ we have

$$\|(T_1,\dots,T_n)\|_e = \|(T_1^*,\dots,T_n^*)\|_e,$$

where T_j^* is the adjoint operator of T_j, $j\in\{1,\dots,n\}$. We call this the *Euclidean norm* of the n-tuple $(T_1,\dots,T_n)\in\mathcal{B}^{(n)}(H)$.

It has been shown in [21] that the following basic inequality holds:

$$\frac{1}{\sqrt{n}}\Big\|\sum_{j=1}^n |T_j^*|^2\Big\|^{1/2} \le \|(T_1,\dots,T_n)\|_e \le \Big\|\sum_{j=1}^n |T_j^*|^2\Big\|^{1/2} \tag{4.2}$$

for any $(T_1,\dots,T_n)\in\mathcal{B}^{(n)}(H)$, and the constants $1/\sqrt{n}$ and 1 are best possible.

In the same paper [21] the author has introduced the *Euclidean operator radius* of $(T_1,\dots,T_n)$ by

$$w_e(T_1,\dots,T_n) := \sup_{\|x\|=1}\Big(\sum_{j=1}^n |\langle T_j x, x\rangle|^2\Big)^{1/2} \tag{4.3}$$

and proved that $w_e(\cdot)$ is a norm on $\mathcal{B}^{(n)}(H)$ and satisfies the double inequality

$$\frac{1}{2}\|(T_1,\dots,T_n)\|_e \le w_e(T_1,\dots,T_n) \le \|(T_1,\dots,T_n)\|_e \tag{4.4}$$

for each $(T_1,\dots,T_n)\in\mathcal{B}^{(n)}(H)$.

As pointed out in [21], the Euclidean numerical radius also satisfies the double inequality

$$\frac{1}{2\sqrt{n}}\Big\|\sum_{j=1}^n |T_j^*|^2\Big\|^{1/2} \le w_e(T_1,\dots,T_n) \le \Big\|\sum_{j=1}^n |T_j^*|^2\Big\|^{1/2} \tag{4.5}$$

for any $(T_1,\dots,T_n)\in\mathcal{B}^{(n)}(H)$, and the constants $1/(2\sqrt{n})$ and 1 are best possible.

In [1], by utilizing the concept of *hypo-Euclidean norm* on $H^n := H\times\cdots\times H$ we obtained the following representation for the Euclidean norm:

Proposition 1. *For any* $(T_1, \ldots, T_n) \in \mathcal{B}^{(n)}(H)$, *we have*

$$\|(T_1, \ldots, T_n)\|_e = \sup_{\|y\|=\|x\|=1} \Big(\sum_{j=1}^{n} |\langle T_j y, x\rangle|^2\Big)^{1/2}. \tag{4.6}$$

The following different lower bound for the Euclidean operator norm $\|\cdot\|_e$ was also obtained in [1]:

Proposition 2. *For any* $(T_1, \ldots, T_n) \in \mathcal{B}^{(n)}(H)$, *we have*

$$\|(T_1, \ldots, T_n)\|_e \geq \frac{1}{\sqrt{n}} \|T_1 + \cdots + T_n\|. \tag{4.7}$$

Utilizing some techniques based on the Boas–Bellman and Bombieri type inequalities we obtained in [1] the following upper bounds:

Proposition 3. *For any* $(T_1, \ldots, T_n) \in \mathcal{B}^{(n)}(H)$, *we have*

$$\|(T_1, \ldots, T_n)\|_e^2 \leq \begin{cases} \max\limits_{1\leq j\leq n} \{\|T_j\|^2\} + \Big[\sum\limits_{1\leq j\neq k\leq n} w^2(T_k^* T_j)\Big]^{1/2}; \\ \max\limits_{1\leq j\leq n} \{\|T_j\|^2\} + (n-1) \max\limits_{1\leq j\neq k\leq n} \{w(T_k^* T_j)\}; \\ \Big[\max\limits_{1\leq j\leq n} \{\|T_j\|^2\} \Big\|\sum\limits_{j=1}^{n} |T_j|^2\|^2 \\ \qquad + \max\limits_{1\leq j\neq k\leq n} \{\|T_j\|\,\|T_k\|\} \sum\limits_{1\leq j\neq k\leq n} w(T_k T_j^*)\Big]^{1/2} \end{cases} \tag{4.8}$$

and

$$\|(T_1, \ldots, T_n)\|_e^2 \leq \begin{cases} \max\limits_{1\leq j\leq n} \Big\{\sum\limits_{k=1}^{n} w(T_k^* T_j)\Big\}; \\ \Big[\sum\limits_{j,k=1}^{n} w^2(T_k^* T_j)\Big]^{1/2}; \\ n \max\limits_{1\leq j\leq n} \Big[\sum\limits_{k=1}^{n} w^2(T_k^* T_j)\Big]^{1/2}; \\ n\Big[\sum\limits_{j=1}^{n} \max\limits_{1\leq k\leq n} \{w^2(T_k^* T_j)\}\Big]^{1/2}. \end{cases} \tag{4.9}$$

Now we can provide one more upper bound for the Euclidean norm:

Proposition 4. *For any* $(T_1, \ldots, T_n) \in \mathcal{B}^{(n)}(H)$, *we have*

$$\|(T_1, \ldots, T_n)\|_e^2 \leq \Big\|\sum_{j=1}^{n} |T_j|^2\Big\|^{\alpha} \Big\|\sum_{j=1}^{n} |T_j^*|^2\Big\|^{1-\alpha} \tag{4.10}$$

and

$$w_e^2(T_1,\dots,T_n) \le \sup_{\|x\|=1}\left[\left\langle\sum_{j=1}^n |T_j|^2 x, x\right\rangle^{\alpha}\left\langle\sum_{j=1}^n |T_j^*|^2 x, x\right\rangle^{1-\alpha}\right] \le \begin{cases}\left[\left\|\sum_{j=1}^n |T_j|^2\right\|\right]^{\alpha}\left[\left\|\sum_{j=1}^n |T_j^*|^2\right\|\right]^{1-\alpha}; \\ \left\|\sum_{j=1}^n [\alpha|T_j|^2+(1-\alpha)|T_j^*|^2]\right\|,\end{cases} \tag{4.11}$$

for any $\alpha \in [0,1]$.

Proof. Utilizing the vector inequality (2.1) and taking the supremum over $\|y\| = \|x\| = 1$ we have

$$\|(T_1,\dots,T_n)\|_e^2 \le \left[\sup_{\|x\|=1}\left\langle\sum_{j=1}^n |T_j|^2 x, x\right\rangle\right]^{\alpha}\left[\sup_{\|y\|=1}\left\langle\sum_{j=1}^n |T_j^*|^2 y, y\right\rangle\right]^{1-\alpha} \tag{4.12}$$

for any $\alpha \in [0,1]$, and since

$$\sup_{\|x\|=1}\left\langle\sum_{j=1}^n |T_j|^2 x, x\right\rangle = \left\|\sum_{j=1}^n |T_j|^2\right\| \quad \text{and} \quad \sup_{\|y\|=1}\left\langle\sum_{j=1}^n |T_j^*|^2 y, y\right\rangle = \left\|\sum_{j=1}^n |T_j^*|^2\right\|,$$

we get from (4.12) the desired result (4.10).

Now from the first inequality in (2.4) we have

$$\begin{aligned} w_e^2(T_1,\dots,T_n) &\le \sup_{\|x\|=1}\left[\left\langle\sum_{j=1}^n |T_j|^2 x, x\right\rangle^{\alpha}\left\langle\sum_{j=1}^n |T_j^*|^2 x, x\right\rangle^{1-\alpha}\right] \\ &\le \left[\sup_{\|x\|=1}\left\langle\sum_{j=1}^n |T_j|^2 x, x\right\rangle\right]^{\alpha}\left[\sup_{\|x\|=1}\left\langle\sum_{j=1}^n |T_j^*|^2 x, x\right\rangle\right]^{1-\alpha} \\ &= \left[\left\|\sum_{j=1}^n |T_j|^2\right\|\right]^{\alpha}\left[\left\|\sum_{j=1}^n |T_j^*|^2\right\|\right]^{1-\alpha} \end{aligned} \tag{4.13}$$

and from the second inequality in (2.4) we also have

$$\begin{aligned} w_e^2(T_1,\dots,T_n) &\le \sup_{\|x\|=1}\left\langle\sum_{j=1}^n [\alpha|T_j|^2+(1-\alpha)|T_j^*|^2]x, x\right\rangle \\ &= \left\|\sum_{j=1}^n [\alpha|T_j|^2+(1-\alpha)|T_j^*|^2]\right\| \end{aligned} \tag{4.14}$$

for any $\alpha \in [0,1]$.

Utilizing (4.13) and (4.14) we get (4.11). □

Remark 6. The case when $\alpha = 1/2$ provides the inequalities

$$\|(T_1, \dots, T_n)\|_e^2 \le \Big\|\sum_{j=1}^n |T_j|^2\Big\|^{1/2} \Big\|\sum_{j=1}^n |T_j^*|^2\Big\|^{1/2} \tag{4.15}$$

and

$$\begin{aligned} w_e^2(T_1, \dots, T_n) &\le \sup_{\|x\|=1} \Big[\Big\langle \sum_{j=1}^n |T_j|^2 x, x\Big\rangle^{1/2} \Big\langle \sum_{j=1}^n |T_j^*|^2 x, x\Big\rangle^{1/2}\Big] \\ &\le \begin{cases} \Big[\Big\|\sum_{j=1}^n |T_j|^2\Big\|\Big]^{1/2} \Big[\Big\|\sum_{j=1}^n |T_j^*|^2\Big\|\Big]^{1/2}; \\ \Big\|\sum_{j=1}^n \Big[\dfrac{|T_j|^2 + |T_j^*|^2}{2}\Big]\Big\|. \end{cases} \end{aligned} \tag{4.16}$$

§5. Applications for s-1-norm and s-1-numerical radius

We can introduce the *s-p-norm* of the n-tuple $(T_1, \dots, T_n) \in \mathcal{B}^{(n)}(H)$ by

$$\|(T_1, \dots, T_n)\|_{s,p} := \sup_{\|y\|=\|x\|=1} \Big(\sum_{j=1}^n |\langle T_j y, x\rangle|^p\Big)^{1/p}. \tag{5.1}$$

This is indeed a norm, since by the Minkowski inequality we have

$$\begin{aligned} &\|(T_1, \dots, T_n) + (V_1, \dots, V_n)\|_{s,p} \\ &= \sup_{\|y\|=\|x\|=1} \Big(\sum_{j=1}^n |\langle T_j y, x\rangle + \langle V_j y, x\rangle|^p\Big)^{1/p} \\ &\le \sup_{\|y\|=\|x\|=1} \Big[\Big(\sum_{j=1}^n |\langle T_j y, x\rangle|^p\Big)^{1/p} + \Big(\sum_{j=1}^n |\langle V_j y, x\rangle|^p\Big)^{1/p}\Big] \\ &\le \sup_{\|y\|=\|x\|=1} \Big(\sum_{j=1}^n |\langle T_j y, x\rangle|^p\Big)^{1/p} + \sup_{\|y\|=\|x\|=1} \Big(\sum_{j=1}^n |\langle V_j y, x\rangle|^p\Big)^{1/p} \\ &= \|(T_1, \dots, T_n)\|_{s,p} + \|(V_1, \dots, V_n)\|_{s,p}, \end{aligned} \tag{5.2}$$

which proves the triangle inequality. The other properties of the norm are obvious.

For $p = 2$ we get

$$\|(T_1, \dots, T_n)\|_{s,2} = \|(T_1, \dots, T_n)\|_e.$$

We are interested in this section in the case $p = 1$, that is,

$$\|(T_1, \dots, T_n)\|_{s,1} := \sup_{\|y\|=\|x\|=1} \sum_{j=1}^n |\langle T_j y, x\rangle|.$$

Since for any $x, y \in H$ we have $\sum_{j=1}^{n} |\langle T_j y, x\rangle| \geq |\langle \sum_{j=1}^{n} T_j y, x\rangle|$, by the properties of the supremum we get the basic inequality

$$\Big\|\sum_{j=1}^{n} T_j\Big\| \leq \|(T_1, \ldots, T_n)\|_{s,1} \leq \sum_{j=1}^{n} \|T_j\|. \tag{5.3}$$

Similarly, we can also introduce the *s-p-numerical radius* of $(T_1, \ldots, T_n) \in \mathcal{B}^{(n)}(H)$ by

$$w_{s,p}(T_1, \ldots, T_n) := \sup_{\|x\|=1} \Big(\sum_{j=1}^{n} |\langle T_j x, x\rangle|^p\Big)^{1/p}, \tag{5.4}$$

which for $p = 2$ reduces to the Euclidean operator radius introduced previously. We observe that the s-p-numerical radius is also a norm on $\mathcal{B}^{(n)}(H)$ for $p \geq 1$, and for $p = 1$ it satisfies the basic inequality

$$w\Big(\sum_{j=1}^{n} T_j\Big) \leq w_{s,1}(T_1, \ldots, T_n) \leq \sum_{j=1}^{n} w(T_j). \tag{5.5}$$

Proposition 5. *For any $(T_1, \ldots, T_n) \in \mathcal{B}^{(n)}(H)$, we have*

$$\|(T_1, \ldots, T_n)\|_{s,1} \leq \Big\|\sum_{j=1}^{n} |T_j|^{2\alpha}\Big\|^{1/2} \Big\|\sum_{j=1}^{n} |T_j^*|^{2(1-\alpha)}\Big\|^{1/2} \tag{5.6}$$

for all $\alpha \in [0, 1]$, and in particular, the following refinement of the triangle inequality holds for the operator norm:

$$\begin{aligned}\Big\|\sum_{j=1}^{n} T_j\Big\| &\leq \|(T_1, \ldots, T_n)\|_{s,1} \\ &\leq \Big\|\sum_{j=1}^{n} |T_j|\Big\|^{1/2} \Big\|\sum_{j=1}^{n} |T_j^*|\Big\|^{1/2} \\ &\leq \frac{1}{2}\Big[\Big\|\sum_{j=1}^{n} |T_j|\Big\| + \Big\|\sum_{j=1}^{n} |T_j^*|\Big\|\Big] \leq \sum_{j=1}^{n} \|T_j\|.\end{aligned} \tag{5.7}$$

Proof. Utilizing the vector inequality (2.7) and taking the supremum over $\|y\| = \|x\| = 1$ we have

$$\begin{aligned}&\|(T_1, \ldots, T_n)\|_{s,1} \\ &\quad \leq \Big\{\sup_{\|x\|=1} \Big\langle \sum_{j=1}^{n} |T_j|^{2\alpha} x, x\Big\rangle\Big\}^{1/2} \Big\{\sup_{\|y\|=1} \Big\langle \sum_{j=1}^{n} |T_j^*|^{2(1-\alpha)} y, y\Big\rangle\Big\}^{1/2}\end{aligned} \tag{5.8}$$

and since

$$\sup_{\|x\|=1}\left\langle\sum_{j=1}^{n}|T_j|^{2\alpha}x,x\right\rangle=\left\|\sum_{j=1}^{n}|T_j|^{2\alpha}\right\|$$

and

$$\sup_{\|y\|=1}\left\langle\sum_{j=1}^{n}|T_j^*|^{2(1-\alpha)}y,y\right\rangle=\left\|\sum_{j=1}^{n}|T_j^*|^{2(1-\alpha)}\right\|,$$

from (5.8) we get the desired inequality (5.6).

The inequality (5.7) follows from (5.6). □

The case of normal operators provides a simpler bound:

Corollary 1. *Let $(N_1,\dots,N_n)\in\mathcal{B}^{(n)}(H)$ be an n-tuple of normal operators. Then*

$$\|(N_1,\dots,N_n)\|_{s,1}\le\left\|\sum_{j=1}^{n}|N_j|^{2\alpha}\right\|^{1/2}\left\|\sum_{j=1}^{n}|N_j|^{2(1-\alpha)}\right\|^{1/2}\tag{5.9}$$

for any $\alpha\in[0,1]$, and in particular,

$$\left\|\sum_{j=1}^{n}N_j\right\|\le\|(N_1,\dots,N_n)\|_{s,1}\le\left\|\sum_{j=1}^{n}|N_j|\right\|\le\sum_{j=1}^{n}\|N_j\|.\tag{5.10}$$

The above results provide an interesting criterion of convergence of series in the Banach algebra $\mathcal{B}(H)$.

Criterion 1. *Let $\{T_j\}_{j\in\mathbb{N}}$ be a sequence of operators in $\mathcal{B}(H)$. If there exists an $\alpha\in(0,1)$ such that the series $\sum_{j=0}^{\infty}|T_j|^{2\alpha}$ and $\sum_{j=0}^{\infty}|T_j^*|^{2(1-\alpha)}$ are convergent in the Banach algebra $\mathcal{B}(H)$, then $\sum_{j=0}^{\infty}T_j$ is convergent in $\mathcal{B}(H)$ and*

$$\left\|\sum_{j=0}^{\infty}T_j\right\|\le\left\|\sum_{j=0}^{\infty}|T_j|^{2\alpha}\right\|^{1/2}\left\|\sum_{j=0}^{\infty}|T_j^*|^{2(1-\alpha)}\right\|^{1/2}.$$

In particular, the convergence of $\sum_{j=0}^{\infty}|T_j|$ and $\sum_{j=0}^{\infty}|T_j^*|$ implies the convergence of $\sum_{j=0}^{\infty}T_j$ in $\mathcal{B}(H)$, and

$$\left\|\sum_{j=0}^{\infty}T_j\right\|\le\left\|\sum_{j=0}^{\infty}|T_j|\right\|^{1/2}\left\|\sum_{j=0}^{\infty}|T_j^*|\right\|^{1/2}.$$

The following result for the s-1-numerical radius may be stated as well:

Proposition 6. *For any* $(T_1, \ldots, T_n) \in \mathcal{B}^{(n)}(H)$, *we have*

$$w_{s,1}(T_1, \ldots, T_n) \le \sup\Big[\Big\langle \sum_{j=1}^{n} p_j |T_j|^{2\alpha} x, x\Big\rangle^{1/2} \Big\langle \sum_{j=1}^{n} p_j |T_j^*|^{2(1-\alpha)} x, x\Big\rangle^{1/2}\Big] \le \begin{cases} \Big\| \sum_{j=1}^{n} |T_j|^{2\alpha} \Big\|^{1/2} \Big\| \sum_{j=1}^{n} |T_j^*|^{2(1-\alpha)} \Big\|^{1/2}; \\ \Big\| \sum_{j=1}^{n} \frac{|T_j|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2} \Big\| \end{cases} \tag{5.11}$$

for each $\alpha \in [0, 1]$, *and, in particular,*

$$w\Big(\sum_{j=1}^{n} T_j\Big) \le w_{s,1}(T_1, \ldots, T_n) \le \begin{cases} \Big\| \sum_{j=1}^{n} |T_j| \Big\|^{1/2} \Big\| \sum_{j=1}^{n} |T_j^*| \Big\|^{1/2}; \\ \Big\| \sum_{j=1}^{n} \frac{|T_j| + |T_j^*|}{2} \Big\|. \end{cases} \tag{5.12}$$

Remark 7. We observe that due to the inequality

$$\frac{1}{2}\Big\| \sum_{j=1}^{n} T_j \Big\| \le w\Big(\sum_{j=1}^{n} T_j\Big) \le \Big\| \sum_{j=1}^{n} \frac{|T_j|^{2\alpha} + |T_j^*|^{2(1-\alpha)}}{2} \Big\|, \tag{5.13}$$

the convergence of the series $\sum_{k=0}^{\infty} [|T_k|^{2\alpha} + |T_k^*|^{2(1-\alpha)}]$ in $\mathcal{B}(H)$ for some $\alpha \in (0, 1)$ suffices for the convergence of $\sum_{k=0}^{\infty} T_k$, which is a slight improvement of the result from Criterion 1.

The case $\alpha = 1/2$ produces the simpler inequality of interest for the numerical radius of a sum:

$$\frac{1}{2}\Big\| \sum_{j=1}^{n} T_j \Big\| \le w\Big(\sum_{j=1}^{n} T_j\Big) \le \frac{1}{2}\Big\| \sum_{j=1}^{n} [|T_j| + |T_j^*|] \Big\|. \tag{5.14}$$

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